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Tests of fit for exponentiality based on a characterization via the mean residual life function

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Abstract

We study two new omnibus goodness of fit tests for exponentiality, each based on a characterization of the exponential distribution via the mean residual life function. The limiting null distributions of the test statistics are the same as the limiting null distributions of the Kolmogorov-Smirnov and Cramér-von Mises statistics proposed when testing the simple hypothesis that the distribution of the sample variables is uniform on the interval [0, 1].

Key words and phrases: Exponential distribution; Cramér-von Mises statistic; Kolmogorov-Smirnov statistic, mean residual life function.

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1 Introduction

As evidenced by the recent papers of El Aroui (1996), Gupta and Richards (1997), Gwanyama (1997), Klefsjoe and Westberg (1996) and Nikitin (1996), there is a continued interest in the problem of testing that a random sample comes from an exponential distribution. For an overview of goodness of fit tests for exponentiality, see Ascher (1990), D'Agostino and Stephens (1986), Doksum and Yandell (1984), or Spurrier (1984).

This paper takes a new approach to the problem of testing for exponentiality. This approach is based on a characterization of the exponential distribution via the mean residual life function. To be specific, let $X_1, \ldots, X_n$ be independent copies of a nonnegative random variable X with unknown distribution function $F(x) = P(X \leq x), x \geq 0$. It is well-known that if X has finite positive mean the distribution of X is exponential, i.e.

$$F(x) = 1 - \exp(-\lambda x), \quad x \geq 0,$$

for some $\lambda > 0$, if and only if the mean residual life function is constant, i.e., if

$$E(X - z|X > z) = E(X) \quad \text{for each} \quad z > 0. \tag{1}$$

Now, (1) is easily seen to be equivalent to

$$E(\min(X, z)) = E(X)F(z) \quad \text{for each} \quad z > 0. \tag{2}$$

Since under the assumptions $X \geq 0$ and $0 < E(X) < \infty$, (2) is a characteristic property of the class $\{Exp(\lambda) : \lambda > 0\}$ of exponential distributions, there is the following natural approach to assess exponentiality. Put $\overline{X} = n^{-1}\sum_{k=1}^{n} X_k$ and $U_k = X_k/\overline{X}, k = 1, \ldots, n$. If the distribution of X is $Exp(\lambda)$ for some λ, the distribution of $n^{-1}(U_1, \ldots, U_n)$ is the (singular) Dirichlet distribution $D(1, \ldots, 1)$ on the simplex, see Gupta and Richards (1997). Moreover, for large n the random variables $U_1, \ldots, U_n$ behave approximately like n independent exponential variables with mean 1. In view of (2), the latter observation motivates the use of the Kolmogorov-Smirnov type statistic

$$L_n = \sqrt{n} \sup_{z \geq 0} \left| \frac{1}{n} \sum_{k=1}^{n} \min(U_k, z) - \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}(U_k \leq z) \right| \tag{3}$$

and the Cramér-von Mises type statistic

$$G_n = n \int_0^\infty \left(\frac{1}{n} \sum_{k=1}^{n} \min(U_k, z) - \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}(U_k \leq z) \right)^2 e^{-z}\, dz. \tag{4}$$

Clearly, when the hypothesis

$$H_0 \ : \ \text{the law of } X \text{ is } Exp(\lambda) \text{ for some } \lambda > 0$$

is true, the distributions of L_n and G_n do not depend on the unknown parameter λ. Denoting by $U_{1:n} \leq \ldots \leq U_{n:n}$ the order statistics of the variables $U_1, \ldots, U_n$ introduced above, and defining a sum over an empty index set as zero, the statistic L_n can be written in the form

$$\begin{aligned} L_n &= \sqrt{n} \max_{s=0,1,\ldots,n-1} \sup_{U_{s:n} \leq z < U_{s+1:n}} \left| \frac{1}{n}(U_{1:n} + \ldots + U_{s:n}) + z\left(1 - \frac{s}{n}\right) - \frac{s}{n} \right| \\ &= \sqrt{n} \max\left(\max_{s=0,1,\ldots,n-1} \left[\frac{1}{n}(U_{1:n} + \ldots + U_{s:n}) + U_{s+1:n}\left(1 - \frac{s}{n}\right) - \frac{s}{n} \right], \right. \\ &\qquad \left. \max_{s=1,\ldots,n-1} \left[\frac{s}{n} - \frac{1}{n}(U_{1:n} + \ldots + U_{s:n}) - U_{s:n}\left(1 - \frac{s}{n}\right) \right] \right). \end{aligned} \tag{5}$$

Similarly, putting $A_0 = 0$ and $A_s = \frac{1}{n}\sum_{j=1}^{s}(U_{j:n} - 1)$ for $s = 1, \ldots, n$ gives

$$\begin{aligned} G_n &= n \sum_{s=0}^{n-1} \int_{U_{s:n}}^{U_{s+1:n}} \left(\frac{1}{n}(U_{1:n} + \ldots + U_{s:n}) + z\left(1 - \frac{s}{n}\right) - \frac{s}{n} \right)^2 e^{-z}\, dz \\ &= n \sum_{s=0}^{n-1} \Bigg[A_s^2 \left(\exp(-U_{s:n}) - \exp(-U_{s+1:n})\right) \\ &\quad + 2A_s\left(1 - \frac{s}{n}\right)\left(\exp(-U_{s:n})(1 + U_{s:n}) - \exp(-U_{s+1:n})(1 + U_{s+1:n})\right) \\ &\quad + \left(1 - \frac{s}{n}\right)^2 \exp(-U_{s:n})(2 + 2U_{s:n} + U_{s:n}^2) \\ &\quad - \left(1 - \frac{s}{n}\right)^2 \exp(-U_{s+1:n})(2 + 2U_{s+1:n} + U_{s+1:n}^2) \Bigg]. \end{aligned} \tag{6}$$

The distributions of the statistics L_n and G_n only depend on that of the random vector $(U_{1:n}, \ldots, U_{n:n})$. Interestingly, the distribution of $(U_{1:n}, \ldots, U_{n:n})$ is the same for all random vectors $(X_1, \ldots, X_n)$ having a multivariate Liouville distribution, see Gupta and Richards (1997). Therefore changing from a sample $X_1, \ldots, X_n$ of independent and identically distributed exponential variables to jointly multivariate Liouville distributed variables $X_1, \ldots, X_n$ does not change the distributions of the statistics L_n and G_n. As was observed by Gupta and Richards this invariance property is shared by various other goodness of fit statistics for exponentiality.

The representation

$$G_n = \frac{1}{n}\sum_{k,\ell=1}^{n}\left[2 - 3e^{-\min(U_k,U_\ell)} - 2\min(U_k,U_\ell)(e^{-U_k}+e^{-U_\ell}) + 2e^{-\max(U_k,U_\ell)}\right],$$

which may be obtained directly from (4), shows that G_n is a V-statistic with an estimated parameter, the estimator being $1/\overline{X}$. The asymptotic distribution theory of statistics of this type was treated by De Wet and Randles (1987). We shall not use their results, but exploit the fact that under H_0 the transformed variables

$$T_s = \sum_{k=1}^{s} \frac{n-k+1}{n}(U_{k:n} - U_{k-1:n}), \quad s = 1,\ldots,n-1, \tag{7}$$

behave like the order statistics of $n-1$ independent [0,1] uniform variables. In that way, the asymptotic null distributions of L_n and G_n may be obtained from the limit theorems of the classical Kolmogorov-Smirnov and Cramér-von Mises statistics proposed when testing the simple hypothesis that the underlying distribution is uniform over the interval $[0,1]$. The behavior of the transformed variables $T_1,\ldots,T_{n-1}$ in (7) is a characteristic property of the exponential distribution. In this spirit, Seshadri, M. Csörgő and Stephens (1969) and M. Csörgő (1974) treated the hypothesis H_0 with the Kolmogorov-Smirnov and Cramér-von Mises goodness of fit tests for uniformity based on $T_1,\ldots,T_{n-1}$.

There is an intimate connection between their statistics and L_n and G_n (or even more pronounced with the statistic G_n' introduced below). This connection is revealed in Section 2. The limiting null distributions of L_n and G_n turn out to be the same as the asymptotic null distributions of the Kolmogorov-Smirnov and the Cramér-von Mises statistic, respectively, when testing for uniformity in the unit interval. It will also be shown in Section 2 that the corresponding level α tests that reject H_0 for large values of L_n and G_n, respectively, are consistent against any fixed alternative distribution. Thus, there are two new omnibus tests available for the composite hypothesis of exponentiality. Some empirical power values of these tests obtained by simulation are presented in Section 3.

Among the multitude of available tests for exponentiality (see, e.g., Ascher (1990) or D'Agostino and Stephens (1986)), the procedures under discussion belong to the group of tests that use a characterization approach to goodness of fit. Emphasizing characterization procedures, O'Reilly and Stephens (1982) discuss a systematic approach to goodness of fit tests for composite hypotheses.

2 Asymptotic results

The following result shows that the test statistics L_n and G_n, although involving the estimator $1/\overline{X}$ which is "hidden" in $U_1, \ldots, U_n$, have standard limiting null distributions.

Theorem 1. *For a standard Brownian bridge* $\{B(t), 0 \leq t \leq 1\}$, *put* $L = \max_{0 \leq t \leq 1} |B(t)|$ *and* $G = \int_0^1 B(t)^2 dt$. *If the hypothesis of exponentiality is true, then*

a) $\lim_{n\to\infty} P(L_n \leq x) = P(L \leq x),\ x \geq 0,$

b) $\lim_{n\to\infty} P(G_n \leq x) = P(G \leq x),\ x \geq 0.$

Proof. Since the distributions of L_n and G_n do not depend on the parameter λ of the underlying exponential distribution, assume without loss of generality that the random variables X_k have the distribution function $F(x) = 1 - \exp(-x), x \geq 0$ Let $V_{1:n-1} \leq \ldots \leq V_{n-1:n-1}$ be the order statistics of a sample of $n-1$ independent and identically distributed uniform $[0,1]$ variables $V_1, \ldots, V_{n-1}$. Putting $V_{0:n} = 0$ and $V_{n:n-1} = 1$, the proof is easily done by remembering that

$$\left(\frac{n-k+1}{n}(U_{k:n} - U_{k-1:n}),\ 1 \leq k \leq n\right) \stackrel{\mathcal{D}}{=} \left(\frac{X_s}{\sum_{k=1}^{n} X_k}\ ,\ 1 \leq s \leq n\right)$$

$$\stackrel{\mathcal{D}}{=} (V_{s:n-1} - V_{s-1:n-1}\ ,\ 1 \leq s \leq n),$$

where $\stackrel{\mathcal{D}}{=}$ means equality of distributions. Recalling T_s from (7) then gives

$$\begin{aligned} L_n &= \sqrt{n}\max\left(\max_{s=0,1,\ ,n-1}\left[\frac{1}{n}(U_{1\cdot n} + \ldots + U_{s.n}) + U_{s+1:n}\left(1-\frac{s}{n}\right) - \frac{s}{n}\right],\right. \\ &\qquad \left.\max_{s=1,\ldots,n-1}\left[\frac{s}{n} - \frac{1}{n}(U_{1:n} + \ldots + U_{s:n}) - U_{s:n}\left(1-\frac{s}{n}\right)\right]\right) \qquad (8) \\ &= \sqrt{n}\max\left(\max_{s=0,\ldots,n-1}\left[T_{s+1} - \frac{s}{n}\right],\ \max_{s=1,\ldots,n-1}\left[\frac{s}{n} - T_s\right]\right) \\ &\stackrel{\mathcal{D}}{=} \sqrt{n}\max\left(\max_{s=0,\ldots,n-1}\left[V_{s+1:n-1} - \frac{s}{n}\right],\ \max_{s=1,\ldots,n-1}\left[\frac{s}{n} - V_{s:n-1}\right]\right). \end{aligned}$$

A comparison of the last expression on the right-hand side with

$$D_{n-1} = \sqrt{n-1}\max\left(\max_{s=0,\ldots,n-1}\left[V_{s+1:n-1} - \frac{s}{n-1}\right],\ \max_{s=1,\ldots,n-1}\left[\frac{s}{n-1} - V_{s:n-1}\right]\right),$$

the Kolmogorov-Smirnov statistic based on $V_1, \ldots, V_{n-1}$, yields the assertion for L_n, because $\lim_{n\to\infty} P(D_{n-1} \le x) = P(L \le x)$ for each $x \ge 0$.

To prove the corresponding assertion for G_n, introduce the related test statistic

$$\begin{aligned}
G'_n &= n \int_0^\infty \left[\frac{1}{n}\sum_{k=1}^n \min(U_k, z) - \frac{1}{n}\sum_{k=1}^n \mathbf{1}(U_k \le z)\right]^2 \left(1 - \frac{1}{n}\sum_{k=1}^n \mathbf{1}(U_k \le z)\right) dz \\
&= n \sum_{s=0}^{n-1} \int_{U_{s:n}}^{U_{s+1:n}} \left[\frac{1}{n}(U_{1:n} + \ldots + U_{s:n}) + z\left(1 - \frac{s}{n}\right) - \frac{s}{n}\right]^2 \left(1 - \frac{s}{n}\right) dz \\
&= \frac{n}{3} \sum_{s=0}^{n-1} \left\{ \left[\frac{1}{n}(U_{1:n} + \ldots + U_{s:n}) + U_{s+1:n}\left(1 - \frac{s}{n}\right) - \frac{s}{n}\right]^3 \right. \\
&\qquad \left. - \left[\frac{1}{n}(U_{1:n} + \ldots + U_{s:n}) + U_{s:n}\left(1 - \frac{s}{n}\right) - \frac{s}{n}\right]^3 \right\} \\
&= \frac{n}{3} \sum_{s=0}^{n-1} \left\{ \left[T_{s+1} - \frac{s}{n}\right]^3 - \left[T_s - \frac{s}{n}\right]^3 \right\} \\
&\overset{\mathcal{D}}{=} \frac{n}{3} \sum_{s=0}^{n-1} \left\{ \left[V_{s+1:n-1} - \frac{s}{n}\right]^3 - \left[V_{s:n-1} - \frac{s}{n}\right]^3 \right\}.
\end{aligned}$$

A comparison with the Cramér-von Mises statistic

$$\begin{aligned}
W_n &= (n-1) \int_0^1 \left(\frac{1}{n-1}\sum_{k=1}^{n-1} \mathbf{1}(V_k \le z) - z\right)^2 dz \\
&= \frac{n-1}{3} \sum_{s=0}^{n-1} \left(\left[V_{s+1:n-1} - \frac{s}{n-1}\right]^3 - \left[V_{s:n-1} - \frac{s}{n-1}\right]^3 \right)
\end{aligned}$$

and the fact that $\lim_{n\to\infty} P(W_{n-1} \le x) = P(G \le x)$, $x \ge 0$, then shows that $\lim_{n\to\infty} P(G'_n \le x) = P(G \le x)$, $x \ge 0$. To finish the proof, note that

$$|G_n - G'_n| \le \frac{U_{n:n}}{n^{1/2}} L_n^2 E_n, \tag{9}$$

where

$$E_n = \sup_{z>0} \sqrt{n} \left| \frac{1}{n}\sum_{k=1}^n \mathbf{1}(U_k \le z) - (1 - e^{-z}) \right|$$

is the Kolmogorov-Smirnov statistic for testing the composite hypothesis of exponentiality. Since L_n and E_n have limit distributions (for E_n, see Stephens, 1976) and

$n^{-1/2}U_{n:n}$ tends to 0 in probability as $n \to \infty$, it follows from (9) that $G_n - G'_n$ converges to 0 in probability as well, which concludes the proof. ■

Seshadri, M. Csörgő and Stephens (1969) used transformation techniques for treating the hypothesis of exponentiality. The n independent variables $X_1, \ldots, X_n$ are transformed to the $n-1$ variables $T_1, \ldots, T_{n-1}$ defined in (7) which under H_0 behave like the order statistics of $n-1$ independent uniform variables. The proof of Theorem 1 shows that when building the Kolmogorov-Smirnov and Cramér-von Mises goodness of fit statistics for testing the hypothesis of uniformity on the basis of these transformed variables, one essentially obtains L_n and G'_n.

The following discussion addresses the problem of consistency of the tests based on L_n and G_n. For a given level of significance $\alpha \in (0,1)$, the hypothesis of exponentiality is rejected if $L_n > \ell_n$ and $G_n > g_n$, respectively, where ℓ_n and g_n are the $(1-\alpha)-$quantiles of the distributions of L_n and G_n under H_0, respectively.

Theorem 2. *Let X have any non-exponential distribution on $[0,\infty)$ with positive, possibly infinite mean μ. Then*

$$\lim_{n\to\infty} P(L_n \le \ell_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} P(G_n \le g_n) = 0.$$

PROOF. If $0 < \mu = E(X) < \infty$, then

$$\frac{1}{\sqrt{n}}\, L_n \longrightarrow \sup_{z\ge 0}\left|\frac{1}{\mu}\, E(\min(X,z)) - P(X \le z)\right| \tag{10}$$

and

$$\frac{1}{n}\, G_n \longrightarrow \int_0^\infty \left[E\left(\min\left(\frac{X}{\mu}, z\right)\right) - P\left(\frac{X}{\mu} \le z\right)\right]^2 e^{-z}\, dz \tag{11}$$

in probability as $n \to \infty$. Both stochastic limits are zero if and only if the distribution of X is exponential.

To prove (10), note that

$$\begin{aligned}\frac{L_n}{\sqrt{n}} &= \sup_{z\ge 0}\left|\frac{1}{n}\sum_{k=1}^{n} \min\left(\frac{X_k}{\overline{X}}, z\right) - \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}\left(\frac{X_k}{\overline{X}} \le z\right)\right| \\ &= \sup_{t\ge 0}\left|\frac{1}{\overline{X}}\, A_n(t) - B_n(t)\right|,\end{aligned}$$

where

$$A_n(t) = \frac{1}{n}\sum_{k=1}^{n}\min(X_k,t), \qquad B_n(t) = \frac{1}{n}\sum_{k=1}^{n}\mathbf{1}(X_k \le t).$$

Letting $\|g\| = \sup_{t\ge 0}|g(t)|$ for a function $g : [0,\infty) \to \mathbb{R}$, and putting $A_0(t) = E(\min(X,t))$, $B_0(t) = P(X \le t)$, it follows that

$$\left\|\frac{1}{\overline{X}}A_n - B_n - \left(\frac{1}{\mu}A_0 - B_0\right)\right\|$$

$$\le \left|\frac{1}{\overline{X}} - \frac{1}{\mu}\right| \|A_n\| + \frac{1}{\mu}\|A_n - A_0\| + \|B_n - B_0\|.$$

In view of $\|A_n\| \le \overline{X}$, the first term of this upper bound converges to zero almost surely by the strong law of large numbers. For the second term, use a Glivenko-Cantelli type argument to show that $\|A_n - A_0\| \to 0$ a.s. Since $\|B_n - B_0\| \to 0$ a.s. by Glivenko-Cantelli, it follows that

$$\left\|\frac{1}{\overline{X}}A_n - B_n\right\| \underset{n\to\infty}{\longrightarrow} \left\|\frac{1}{\mu}A_0 - B_0\right\| \quad \text{a.s.},$$

which proves (10). Assertion (11) is an immediate consequence of $\|\overline{X}^{-1}A_n - B_n - (\mu^{-1}A_0 - B_0)\| \to 0$ a.s.

For the case $E(X) = \infty$, fix positive constants ε and M, and let $t > 0$ such that $E(\min(X/t, M)) \le \varepsilon/2$. Since $\overline{X} \to \infty$ a.s., we have for sufficiently large n (depending on ω in a set of probability 1) both $\overline{X} \ge t$ and

$$\frac{1}{n}\sum_{k=1}^{n}\min\left(\frac{X_k}{t}, M\right) \le E\left(\min\left(\frac{X}{t}, M\right)\right) + \frac{\varepsilon}{2},$$

whence, for such n,

$$\sup_{0\le z\le M}\frac{1}{n}\sum_{k=1}^{n}\min\left(\frac{X_k}{\overline{X}}, z\right) \le \frac{1}{n}\sum_{k=1}^{n}\min\left(\frac{X_k}{t}, M\right) \le \varepsilon.$$

Consequently,

$$\sup_{0\le z\le M}\left|\frac{1}{n}\sum_{k=1}^{n}\min\left(\frac{X_k}{\overline{X}}, z\right) - \frac{1}{n}\sum_{k=1}^{n}\mathbf{1}\left(\frac{X_k}{\overline{X}} \le z\right)\right| \to 1 \text{ a.s.}$$

for each $M > 0$. Since $L_n/\sqrt{n} \le 1$, it follows that $L_n/\sqrt{n} \to 1$ in probability. Likewise, $G_n/n \to 1$ in probability. In any case,

$$\lim_{n\to\infty} P(L_n \le x) = 0 \quad \text{and} \quad \lim_{n\to\infty} P(G_n \le x) = 0 \quad \text{for each} \quad x \ge 0,$$

which proves the assertion. ■

3 Empirical results

To give an impression of the speed of convergence to the limiting null distributions, TABLE I shows some critical values of L_n and G_n obtained by simulation. The sample sizes are $n = 20$ and $n = 50$, and the significance levels chosen are $\alpha = 0.1, 0.05, 0.025, 0.01$. The entries in TABLE I are based on 100 000 replications. The row denoted by "∞" gives the critical values of the limiting null distributions of the Kolmogorov-Smirnov and the Cramér-von Mises statistic, taken from the Biometrika Tables for Statisticians, Volume 2 (Pearson and Hartley (1972)).

Empirical power values (rounded to the nearest integer) for some alternative distributions (Gamma ($\mathcal{G}$), Weibull ($\mathcal{W}$) and Lognormal distributions ($\mathcal{LN}$) with scale parameter 1 and shape parameter θ, uniform $\mathcal{U}[0,1]$, Half-Normal ($\mathcal{HN}$), Half-Cauchy ($\mathcal{HC}$), χ^2_1, Power distributions ($\mathcal{PW}$) with density $\theta^{-1}x^{1/\theta-1}, 0 < x < 1$, LIFR (linear increasing failure rate) distributions with density $(1+\theta x)\exp(-(x+(\theta/2)x^2)), x > 0$, and JSHAPE (JS) distributions with density $(1+\theta x)^{-1/\theta-1}, x > 0$, are shown in TABLE II. An asterisk denotes power 100%. The significance level is $\alpha = 0.05$, and the sample sizes are $n = 20$ and $n = 50$.

The alternative distributions chosen were also considered by Baringhaus and Henze (1991), who derived estimated powers for various other competitive procedures for testing the hypothesis of exponentiality. Each entry in TABLE II represents the percentage of 10 000 Monte Carlo samples declared significant by the new tests based on L_n and G_n and the classical Cramér-von Mises test, based on

$$W_n = \sum_{k=1}^{n}\left(1 - \exp(-U_{k:n}) - \frac{k-1/2}{n}\right)^2 + \frac{1}{12n}.$$

Some information regarding the limiting null distribution of W_n (mean, variance, percentage points) can be found in Stephens (1976). Roughly, one can say that the tests based on L_n, G_n and W_n behave nearly in the same way. They can clearly be recommended as omnibus procedures for the testing problem under consideration. The two new tests offer the advantage that the test statistics L_n and G_n have standard limiting null distributions.

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TABLE I

Critical values of L_n and G_n

	L_n				G_n			
α	0.1	0.05	0.025	0.01	0.1	0.05	0.025	0.01
$n = 20$	1.202	1.334	1.453	1.600	0.340	0.446	0.554	0.703
$n = 50$	1.206	1.338	1.458	1.609	0.342	0.454	0.566	0.731
∞	1.224	1.358	1.480	1.628	0.347	0.461	0.581	0.743

TABLE II

Percentage of 10000 Monte Carlo samples declared significant; test size $\alpha = 0.05$; sample sizes $n = 20, 50$

Distrib-ution	$n = 20$			$n = 50$			Distrib-ution	$n = 20$			$n = 50$		
	L_n	G_n	W_n	L_n	G_n	W_n		L_n	G_n	W_n	L_n	G_n	W_n
$\mathcal{W}(0.6)$	58	71	70	95	98	98	$\mathcal{LN}(0.7)$	54	54	62	95	93	98
$\mathcal{W}(0.8)$	14	22	20	35	46	43	$\mathcal{LN}(0.8)$	29	28	34	64	61	76
$\mathcal{W}(1.2)$	16	14	14	29	29	28	$\mathcal{LN}(1.0)$	13	16	16	22	26	30
$\mathcal{W}(1.4)$	35	35	35	72	77	74	$\mathcal{LN}(1.5)$	56	66	62	92	95	93
$\mathcal{W}(1.6)$	59	63	61	95	98	97	$\mathcal{HC}$	60	67	63	93	95	93
χ_1^2	37	52	53	82	89	90	JS(0.5)	35	46	41	73	80	76
$\mathcal{PW}(0.8)$	92	93	91	*	*	*	JS(1.0)	76	83	80	98	99	99
$\mathcal{PW}(1.2)$	49	44	41	92	90	86	$\mathcal{U}[0,1]$	73	72	67	99	99	98
$\mathcal{PW}(1.4)$	32	24	24	73	65	62	$\mathcal{G}(0.4)$	62	75	76	97	99	99
$\mathcal{PW}(2.0)$	11	11	19	29	28	48	$\mathcal{G}(0.6)$	22	33	32	55	67	67
$\mathcal{PW}(3.0)$	44	54	64	88	91	96	$\mathcal{G}(0.8)$	6	10	9	13	18	18
LIFR(1)	21	19	18	43	45	40	$\mathcal{G}(1.4)$	16	15	15	31	32	32
LIFR(2)	31	30	28	66	69	63	$\mathcal{G}(1.6)$	25	24	25	53	57	57
LIFR(4)	44	44	42	83	87	83	$\mathcal{G}(1.8)$	36	36	37	72	77	77
LIFR(6)	51	51	49	90	93	90	$\mathcal{G}(2.0)$	46	48	49	86	90	90
LIFR(10)	60	61	59	95	97	95	$\mathcal{G}(2.4)$	64	68	68	98	99	99
$\mathcal{HN}$	24	22	21	52	54	49	$\mathcal{G}(3.0)$	85	89	89	*	*	*

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