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On the run length of a Shewhart chart for correlated data

W. Schmid

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We consider an extension of the classical Shewhart control chart to correlated data which was introduced by Vasilopoulos/Stamboulis (1978).
Inequalities for the moments of the run length are given under weak conditions. It is proved analytically that the average run length (ARL) in the in-control state of the correlated process is larger than that in the case of independent variables. The exact ARL is calculated for exchangeable normal variables and autoregressive processes (AR). Moreover, we compare this chart with residual charts. Especially, in the case of an AR(1)-process with positive coefficient, it turns out that the out-of-control ARL of the modified Shewhart chart is smaller than that of the Shewhart chart for the residuals.

Keywords. statistical process control, Shewhart chart, run length, correlated data.

1. Introduction

A lot of statistical control charts have been introduced in the literature, e.g. the Shewhart, EWMA, CUSUM charts (cf. Montgomery (1991)). The most widely used control chart is without doubt the standard Shewhart chart.

Let $\{Y_t\}$ denote the in-control process and $\{X_t\}$ the out-of-control process. Now a variety of departures from the in-control state are possible. Here we consider exclusively the case of an abrupt step-like shift; i.e. it is assumed that

$$X_t = Y_t + a\, I_{\{q,q+1,..\}}(t), \tag{1.1}$$

where $q \in \mathbb{N}$ denotes the position and a the size of the change. I_A stands for the indicator function of the set A.

In the classical Shewhart chart for the case of known parameters the process is out-of-control at time t, if $|X_t - \mu_0| > c\,\sigma$, where $\mu_0 := E(Y_t)$, $\sigma^2 := Var(Y_t)$ and $c > 0$ is a suitable constant usually taken to be 3.

A fundamental assumption in most statistical process control methods is that the observations are independent. However, this condition is not fulfilled for many data sets (see e.g. Berthouex et al. (1978), Notohardjono/Ermer (1986), MacGregor/ Harris (1990)).

In the last years several authors have discussed the impact of a correlation structure on the behaviour of the Shewhart, EWMA and CUSUM charts, e.g. Berthouex et al. (1978), Bagshaw/Johnson (1974/5), Yashchin (1989), Harris/Ross (1991). If it is assumed that the random variables are independent, when in fact they are correlated, it has been shown via simulations (e.g. Montgomery/Mastrangelo (1991), Maragah/Woodall (1992)) that the classical 3σ control limits in Shewhart charts are not suitable due to the high frequency of false alarms. This is not surprising since e.g. in the case of a linear process, i.e. $Y_t = \sum_{v=-\infty}^{\infty} s_v \varepsilon_{t-v}$ with $\sum_{v=-\infty}^{\infty} |s_v| < \infty$, , $s_0 = 1$ and $\{\varepsilon_t\}$ a white noise process with $E(\varepsilon_t) = 0$ and $Var(\varepsilon_t) = \sigma^2$, we have $Var(Y_t) = \sigma^2 \sum_{v=-\infty}^{\infty} s_v^2$; but the variance of Y_t in the case of independent variables is only σ^2, i.e. it is smaller. Consequently a linear process has a greater variance and thus reaches a given bound earlier, provided that the same critical values are used, i.e. the in-control average run length (ARL) is smaller.

However it is not true that the in-control ARL is always shorter in the presence of autocorrelation, as some authors remarked, even if the variance of the correlated and independent process is the same. An example of such a process is given in section 3.

In order to overcome these problems residual charts have been proposed, i.e. classical control charts are applied on the residuals of the process (e.g. Berthouex et al. (1978), Harris/Ross (1991), Montgomery/Mastrangelo (1991), MacGregor/Harris (1993). If the residuals are independent, then these charts behave the same as under the standard conditions, i.e. the Shewhart chart applied to the residuals is more suitable to detect large shifts than EWMA and CUSUM residual charts, while the contrary is valid for small shifts.

Unfortunately, as was illustrated by Harris/Ross (1991) and Ryan (1991), this strategy may be extremely inefficient. This can be seen immediately. If $\{Y_t\}$ is an

AR(p) - process, i.e. $Y_t = \sum_{v=1}^{p} \alpha_v Y_{t-v} + \varepsilon_t$, then the residuals are given by

$$X_t - \sum_{v=1}^{p} \alpha_v X_{t-v} = \varepsilon_t + a\,(1 - \sum_{v=1}^{p} \alpha_v) \qquad \text{for} \quad t \geq p+q.$$

In relation to an independent sample the impact of the shift is suppressed, if $\sum_{v=1}^{p} \alpha_v > 0$, else it is overweighted. Consequently all residual charts behave worse for an AR(1) - modell with $\alpha_1 > 0$ than in the independent case.

However, the residual charts suffer under further disadvantages. For many processes residuals are not independent, especially if the parameters are unknown and are replaced by estimators. Moreover, the practical calculation of the residuals of an ARIMA process may be difficult, especially if the parameters are unknown and have to be estimated. Furthermore the restriction on the residuals leads to a loss in information, since, due to the starting problems, no changes can be detected among the first p observations of an AR(p) - process.

In this paper we choose another procedure. In the following we consider a direct extension of the standard Shewhart chart to the case of correlated data. As already remarked above the reason for the different behaviour lies in the fact that under the assumption of independence the wrong normalizing variance is used, or in other words, the variance of the process is "estimated" badly. Thus it seems to be natural to use the standard deviation $\sqrt{\gamma_0}$ of the correlated process $\{Y_t\}$ for normalization, i.e. one concludes that the process is out-of-control at time t, if

$$|X_t - \mu_0| > c\sqrt{\gamma_0}\,. \tag{1.2}$$

This chart was introduced by Vasilopoulos/Stamboulis (1978). It will be called modified Shewhart chart in the following. The extension to the case that at each time not only one but several observations are measured is obvious. The authors calculated the variance of the mean in the in-control state and gave curves for the modified quality control factors for an AR(2) - process. However, they did not make any statement about the run length of the chart, which is given by $N := \inf\{n \in \mathbb{N} : |X_n - \mu_0| > c\sqrt{\gamma_0}\}$ and $\inf \emptyset := \infty$. This is the main aim of the present paper.

It has to be emphasized that this chart can be applied, if e.g. the in-control process is stationary and, in contrary to residual charts, it is not necessary to confine oneselves to ARMA models.

In section 2 we analyse the behaviour of the ARL for a large family of stochastic processes, which cover the case of a stationary Gaussian process. An upper bound for the in-control ARL is given (Theorem 1). Furthermore it is shown that the in-control ARL under "a variety of dependence structures" is larger than in the case of independent variables, if the same constant c is chosen (Theorem 2).

In section 3 we calculate the ARL for exchangeable normal variables and in section 4 for AR-processes. Via simulations we have compared the ARL of the modified Shewhart chart with that of the Shewhart resp. EWMA chart for the residuals. The underlying in-control process was an AR(1)-process. Each chart was calibrated such that the in-control ARL is always the same. It has shown that the modified Shewhart chart is better than the Shewhart chart applied to the residuals, if the coefficient of the AR(1)-process is positive, but it is less efficient in the case $\alpha_1 < 0$. The same behaviour was observed with respect to an EWMA chart with smoothing constant $\lambda = 0.75$.

Now in practice the autocovariance γ_0 will be unknown and we have to estimate this parameter. A suitable estimator is given by $\hat{\gamma}_{0,n} = \frac{1}{n} \sum_{v=1}^{n} (x_v - \overline{x}_n)^2$, provided that n is sufficiently large. Here $x_1, \dots, x_n$ denotes a realization of $\{X_t\}$ and $\overline{x}_n$ stands for the mean. Box/Jenkins (1976, p.33) recommend to choose n greater equal to 50. Thus we have to estimate γ_0 using observations of the in-control process. This can be done by making use of values which have been classified to be in-control (e.g. prerun).

In section 4.3 it is analysed how the ARL of the modified Shewhart chart and that of some residual charts changes, if the parameter γ_0 is estimated. It turns out that all charts react extremely sensible on deviations from the exact parameters.

2. Bounds for the ARL in the case of correlated data

In this section we derive bounds for the ARL for rather general stochastic processes, e.g. processes with elliptically contoured marginal distribution functions, positively lower orthant-dependent random variables, etc. A famous theorem of Kolmogorov says that such processes exist, provided that the family of distribution functions is consistent (see e.g. Brockwell/Davis (1991, p.11)). All of these results are true, if the in-control process is a stationary Gaussian process whose autocovariances converge to zero, if the lag tends to infinity.

Now

$$P_{a,q}(N > k) = P_{a,q}\left(\max_{1 \le n \le k} |X_n - \mu_0| \le c\sqrt{\gamma_0}\right)$$

$$= P_0\left(\max_{1 \le n \le l} |Y_n - \mu_0| \le c\sqrt{\gamma_0},\ -c\sqrt{\gamma_0} - a \le Y_n - \mu_0 \le c\sqrt{\gamma_0} - a \quad \forall q \le n \le k\right),$$

where $l = \min\{q-1, k\}$. Consequently $P_{a,q}(N > k) = P_{-a,q}(N > k)$, if the random vectors $(Y_q - \mu_0, \ldots, Y_k - \mu_0)$ and $-(Y_q - \mu_0, \ldots, Y_k - \mu_0)$ have the same distribution. Here the symbol $P_{a,q}$ means that the probability is calculated with respect to the model (1.1) and P_0 means that no change has arisen. By analogy we write $E_{a,q}$ and E_0 for the expectation.

A k-dimensional random variable Y is said to have an elliptically contoured distribution, if its density function is of the form

$$(\det C)^{-1/2}\, g((y-\mu)^T C^{-1} (y-\mu)), \qquad y, \mu \in \mathbb{R}^k, \tag{2.1}$$

where C is positive definite and $g: [0,\infty) \to [0,\infty)$ is nonincreasing.

The most important member of this family is the multivariate normal distribution which can be obtained by choosing $g(u) = (2\pi)^{-k/2}\exp(-u/2)$. Other members are e.g. the multivariate t-distribution and the multivariate Cauchy distribution.

Furthermore let $S_\tau^{(v)}$, $\tau \ge v \ge 1$ denote the Stirling numbers of the second kind, i.e. the number of partitioning a set of τ elements into v non-empty subsets.

Theorem 2.1. Let $Y_t, t \in \mathbb{N}$, be a Gaussian process. The random variables Y_t, $t \in \mathbb{N}$, are assumed to be identically distributed with $E(Y_t) = \mu_0$ and $Var(Y_t) = \gamma_0$. Furthermore let $C_k := (Cov(Y_i, Y_j))_{i,j=1,\ldots,k}$ be regular for all $k \in \mathbb{N}$ and

$\delta := \inf\{\delta_k : k \in \mathbb{N}\} > 0$, where $\delta_k^2 := \det C_k / (\gamma_0 \det C_{k-1})$ and $C_0 := I$. It follows for $x \in \mathbb{N}$ and with $z := \Phi(c/\delta) - \Phi(-c/\delta)$

$$E_0(N^x) \le \sum_{v=1}^{x} S_x^{(v)} v!\, z^{v-1}/(1-z)^v \le x!/(z(1-z)^x) < \infty .$$

Particularly

$$E_0(N) \le 1/(1-z) \qquad \text{and} \qquad E_0(N^2) \le (1+z)/(1-z)^2.$$

Proof. Using Das Gupta et al. (1972, Theorem 3.2) we obtain that

$$P_0(|Y_1-\mu_0| \le c\sqrt{\gamma_0}, \dots, |Y_k-\mu_0| \le c\sqrt{\gamma_0}) \le \prod_{i=1}^{k} P_0(|Y_i-\mu_0| \le c\sqrt{\gamma_0}/\delta_i)$$

$$\le \prod_{i=1}^{k} P_0(|Y_i-\mu_0| \le c\sqrt{\gamma_0}/\delta) = z^k .$$

Consequently

$$E_0(N^x) = \sum_{i=0}^{\infty} ((i+1)^x - i^x)\, P_0(N > i) \le (1-z) \sum_{i=1}^{\infty} i^x z^{i-1}. \tag{2.2}$$

Since for $|z| < 1$ (see Hansen (1975, p.142))

$$\sum_{i=1}^{\infty} i^x z^i = \sum_{v=1}^{x} S_x^{(v)} v!\, z^v / (1-z)^{v+1} , \tag{2.3}$$

the first inequality follows. By making use of the recurrence relation $S_{\tau+1}^{(v)} = v S_\tau^{(v)} + S_\tau^{(v-1)}$ for $\tau \ge v \ge 1$ (e.g. Abramowitz/Stegun (1984, p.368)), the second inequality can be proved by induction.

Note that in the case of independent and identically distributed random variables $E_0(N^x)$ is equal to the first upper bound given in Theorem 2.1, since $\delta^2 = \delta_k^2 = \mathrm{Var}(Y_1)/\gamma_0 = 1$.

Corollary. Let Y_t, $t \in \mathbb{Z}$, be a (weakly) stationary process with $E(Y_t) = \mu_0$ and autocovariance function $\{\gamma_t\}$. Suppose that

$$\gamma_0 > 0 \quad , \quad \gamma_t \to 0 \quad \text{as} \quad t \to \infty. \tag{2.4}$$

Then $\{\delta_v\}$ is nonincreasing and for $k \in \mathbb{N}$

$$\lim_{v\to\infty} \gamma_0 \delta_v^2 = \lim_{v\to\infty} E(Y_v - \hat{Y}_v)^2 = \sigma^2 := E(Y_k - \hat{Y}_k^*)^2 ,$$

where $\hat{Y}_k$ denotes the best linear one-step predictor of $1, Y_1, \dots, Y_{k-1}$ for Y_k and $\hat{Y}_k^*$ the best linear predictor of $1, Y_t$, $t < k$ for Y_k.
Thus we may choose $\delta = \sigma/\sqrt{\gamma_0}$ in Theorem 2.1.

Proof. This result follows immediately with Brockwell/Davis (1991, Corollary 5.1.1, Ex. 2.18) and the definition of the best linear predictor. Note that σ^2 does not depend on k.

In the case of independent and identically distributed random variables we have $\sigma^2 = \mathrm{Var}(Y_k) = \gamma_0$.
It must be emphasized that the assumption (2.4) is rather weak and is satisfied for all causal ARMA processes. This condition ensures that C_k is regular for all $k \in \mathbb{N}$.
Theorem 2.1 can be extended to other stochastic processes having elliptically contoured marginal distributions.

Remark. Let $Y_t, t \in \mathbb{N}$, be a stochastic process. Assume that for all $k \in \mathbb{N}$ the k-dimensional random variable $(Y_1, \dots, Y_k)^T$ has an elliptically contoured distribution function and that the random variables Y_t, $t \in \mathbb{N}$, are identically distributed with $E(Y_t) = \mu_0$ and $\mathrm{Var}(Y_t) = \gamma_0$. If the corresponding matrix C_k (see (2.1)) is regular for all k, and $\delta := \inf\{\delta_k : k \geq 1\} > 0$, where $\delta_k^2 := \det C_k / (\gamma_0 \det C_{k-1})$ and $C_0 := I$, then we get with the same arguments as in the proof of Theorem 2.1

$$E_0(N^{\varkappa}) \leq \sum_{i=0}^{\infty} ((i+1)^{\varkappa} - i^{\varkappa}) \, P_{0,I}(|Y_1 - \mu_0| \leq c/\delta, \dots, |Y_i - \mu_0| \leq c/\delta), \qquad (2.5)$$

where the symbol $P_{0,I}$ indicates that C_k is the identity matrix.
If we consider the multivariate t distribution with ν degrees of freedom, $P_{0,I}$ on the right side of (2.5) is equal to (see Tong (1990, p.214))

$$2\nu \int_0^{\infty} x \, (\Phi(cx/\delta) - \Phi(-cx/\delta))^i \, h_\nu(\nu x^2) \, dx,$$

where h_ν denotes the density of the central χ^2-distribution function with ν degrees of freedom (symbol χ_ν^2) and Φ the standard normal distribution function.

Thus, e.g.,

$$E_0(N) \le \nu \int_0^\infty \frac{x}{1-\Phi(cx/\delta)} h_\nu(\nu x^2)\,dx < \infty, \text{ if } \nu > c^2/\delta^2 .$$

Next we compare the in-control ARL of a process with dependence structure with that of independent variables. A lower bound for $E_0(N)$ is derived. We consider classes of distributions which are positively dependent. The random variables $Y_1,\dots,Y_n$ are said to be positively lower orthant-dependent (PLOD), if (cf Tong (1990, p.93))

$$P(Y_1 \le y_1,\dots,Y_n \le y_n) \ge \prod_{i=1}^{n} P(Y_i \le y_i) \quad \text{for all } y_1,\dots,y_n \in \mathbb{R}. \qquad (2.6)$$

Theorem 2.2. Let $Y_t, t \in \mathbb{N}$, be a stochastic process. Assume that the random variables Y_t, $t \in \mathbb{N}$, are identically distributed with $E(Y_t) = \mu_0$ and $Var(Y_t) = \gamma_0 \in (0,\infty)$. If the random variables $|Y_1-\mu_0|,\dots,|Y_n-\mu_0|$ satisfy (2.6) for all n, then it follows for $\varkappa \in \mathbb{N}$ that

$$E_0(N^\varkappa) \ge \sum_{\nu=1}^{\varkappa} S_\varkappa^{(\nu)} \nu!\,(F(c)-F(-c-0))^{\nu-1}/(1-(F(c)-F(-c-0)))^\nu , \qquad (2.7)$$

particularly $E_0(N) \ge 1/(1-(F(c)-F(-c-0)))$,

where F denotes the distribution of the standardized variable $(Y_i-\mu_0)/\sqrt{\gamma_0}$.

Proof. Let $c^* := c\sqrt{\gamma_0}$. (2.6) implies that

$$P_0(N>k) = P_0(|Y_1-\mu_0| \le c^*, \dots, |Y_k-\mu_0| \le c^*)$$

$$\ge \prod_{i=1}^{k} P_0(|Y_i-\mu_0| \le c^*) = (F(c)-F(-c-0))^k .$$

Thus we obtain

$$E_0(N^\varkappa) \ge \sum_{i=0}^{\infty} ((i+1)^\varkappa - i^\varkappa)(F(c)-F(-c-0))^i .$$

Using (2.3) the result follows at once.

Note that the quantity on the right side of (2.7) is equal to the $\varkappa$-th moment of the run length for a sequence of independent random variables. Thus Theorem 2.2 states that in the case of correlated variables (in the sense described above) the $\varkappa$-th moment of the run length is larger.

The condition (2.6) in Theorem 2.2 is e.g. satisfied for the multivariate normal distribution (see Tong (1990, p.154)) and for the multivariate t distribution (see Tong (1990,p.208)). A more detailed investigation of the proof of Theorem 2.2 shows that it is sufficient to demand $P(\max_{1\le i\le n} Y_i \le x) \ge \prod_{i=1}^{n} P(Y_i \le x)$ for all $x \in \mathbb{R}$.

Usually the quantity c is determined such that $E_0(N)$ is equal to a given value ξ, i.e. $c = c_\xi$ is a solution of $E_0(N) = \xi$. We denote the solution in the case of independent variables by $c_\xi^{(0)}$. N_ξ and $N_\xi^{(0)}$ shall denote the corresponding run lengths.

If the assumptions of Theorem 2.2 are satisfied, it follows that $c_\xi^{(0)} \ge c_\xi$, since

$$E_0(\inf\{n \in \mathbb{N}: |X_n| > c_\xi^{(0)} \sqrt{\gamma_0}\}) \ge E_0(N_\xi^{(0)}) = \xi = E_0(N_\xi).$$

Thus, if c is determined via tables for independent variables, the resulting in-control ARL is larger than ξ, provided that the variables are correlated.

Up to now we have only discussed the behaviour of the ARL in the in-control state. Next we give a result for the out-of-control situation. In the following we shall always assume that out-of-control means that a change-point is present (see model (1.1)).

Theorem 2.3. Let $\{Y_t\}$ be a stochastic process. Assume that $E(Y_t) = \mu_0$ and $Var(Y_t) = \gamma_0 \in (0, \infty)$ for all $t \in \mathbb{N}$. Furthermore let all marginal distributions of $\{Y_t - \mu_0\}$ have a continuous density f which is symmetric about the origin (i.e. $f(x) = f(-x)$ for all x) and, additionally, let f be unimodal (cf Tong (1980, p.51)), i.e. that $\{x: f(x) \ge \lambda\}$ is convex for all $\lambda > 0$. Moreover, let $x \in \mathbb{N}$. Then it follows that for all a, q

$$E_{a,q}(N^x) \le E_0(N^x)$$

and that for q fixed $E_{a,q}(N^x)$ is a nonincreasing function in $|a|$.

Proof. Let $c^* := c\sqrt{\gamma_0}$, $Z_i := Y_i - \mu_0$ and $\underline{Z}_k := (Z_1, \dots, Z_k)^T$.

It follows with Tong (1980, Theorem 4.1.1) that for $|a| \ge |\tilde{a}|$

$$P_{a,q}(N > k) = P_{a,q}(|Z_1| \le c^*, .., |Z_l| \le c^*, |Z_q + a| \le c^*, \dots, |Z_k + a| \le c^*)$$

$$= P_0(-c^* \le Z_1 \le c^*, \dots, -c^* \le Z_l \le c^*, -c^* - a \le Z_q \le c^* - a, \dots, -c^* - a \le Z_k \le c^* - a)$$

$$= P_0(\underline{Z}_k \in [-c^*, c^*]^k + a\,\underline{a}_q) \le P_0(\underline{Z}_k \in [-c^*, c^*]^k + \tilde{a}\,\underline{a}_q)$$

$$\le P_0(\underline{Z}_k \in [-c^*, c^*]^k) = P_0(N > k) ,$$

where $\underline{a}_q \in \mathbb{R}^k$ denotes a vector whose first $(q-1)$ components are all equal to 0 and the other ones are all equal to -1. Using (2.2) we obtain the desired result.

3. Exchangeable normal variables

The determination of the exact ARL for correlated processes is usually rather difficult. In this section we consider exchangeable random variables. It is possible to give an explicit expression for the ARL of exchangeable normal variables, i.e. for stationary Gaussian processes with autocovariance function γ_v equal to a constant ρ for all $v \neq 0$. Thus these variables are independent, if and only if ρ is equal to zero.

We are interested how the present correlation structure influences the run length of the process.

Theorem 3.1. Let $\{Y_t\}$ be a Gaussian process with $E(Y_t) = \mu_0$ and $Var(Y_t) = \gamma_0 \in (0,\infty)$ for all t and with $Corr(Y_i, Y_j) = \rho \in [0,1)$ for all $i \neq j$. Let $n = \Phi'$. We get for $k \ge 1$

$$P_{a,q}(N > k) = \int_{-\infty}^{\infty} \zeta(z;\rho)^l \; \zeta_a(z;\rho)^{\max\{0,k-q+1\}} \; n(z)\, dz$$

with $\zeta_a(z;\rho) := \Phi((c - \sqrt{\rho}\, z - a/\sqrt{\gamma_0})/\sqrt{1-\rho}) - \Phi((-c - \sqrt{\rho}\, z - a/\sqrt{\gamma_0})/\sqrt{1-\rho})$ and $\zeta(z;\rho) := \zeta_0(z;\rho)$. Consequently for $x \in \mathbb{N}$

$$E_{a,1}(N^x) = \sum_{v=1}^{x} S_x^{(v)}\, v! \int_{-\infty}^{\infty} \frac{\zeta_a(z;\rho)^{v-1}}{(1-\zeta_a(z;\rho))^v}\, n(z)\, dz \qquad \text{and}$$

$$E_{a,q}(N) = \int_{-\infty}^{\infty} \frac{1}{1-\zeta(z;\rho)}\, n(z)\, dz + \int_{-\infty}^{\infty} \zeta(z;\rho)^{q-1} \left(\frac{1}{1-\zeta_a(z;\rho)} - \frac{1}{1-\zeta(z;\rho)} \right) n(z)\, dz.$$

Furthermore

$$Var_{a,1}(N) = 2 \int_{-\infty}^{\infty} \frac{\zeta_a(z;\rho)}{(1-\zeta_a(z;\rho))^2}\, n(z)\,dz + \int_{-\infty}^{\infty} \frac{1}{1-\zeta_a(z;\rho)}\, n(z)\,dz - \left(\int_{-\infty}^{\infty} \frac{1}{1-\zeta_a(z;\rho)}\, n(z)\,dz \right)^2.$$

Proof. It follows with Tong (1990, Theorem 5.3.1) that

$$P_{a,q}(N>k) = P_0(-c \le (Y_n-\mu_0)/\sqrt{\gamma_0} \le c \quad \forall\, 1 \le n \le l,$$

$$-c - a/\sqrt{\gamma_0} \le (Y_n-\mu_0)/\sqrt{\gamma_0} \le c - a/\sqrt{\gamma_0} \quad \forall\, q \le n \le k)$$

$$= P_0\Big(\bigcap_{n=1}^{l}\{-c \le \sqrt{1-\rho}\, Z_n + \sqrt{\rho}\, Z_0 \le c\} \cap$$

$$\bigcap_{n=q}^{k}\{-c - a/\sqrt{\gamma_0} \le \sqrt{1-\rho}\, Z_n + \sqrt{\rho}\, Z_0 \le c - a/\sqrt{\gamma_0}\}\Big),$$

where $Z_0, \dots, Z_n$ denote independent and $N_{0,1}$ -distributed random variables,

$$= \int_{-\infty}^{\infty} P_0\Big(\bigcap_{n=1}^{l}\{-c-\sqrt{\rho}\, z \le \sqrt{1-\rho}\, Z_n \le c - \sqrt{\rho}\, z\} \cap$$

$$\bigcap_{n=q}^{k}\{-c-\sqrt{\rho}\, z - a/\sqrt{\gamma_0} \le \sqrt{1-\rho}\, Z_n \le c - \sqrt{\rho}\, z - a/\sqrt{\gamma_0}\} \,\Big|\, Z_0 = z\Big)\, n(z)\, dz$$

$$= \int_{-\infty}^{\infty} \zeta(z;\rho)^{l}\, \zeta_a(z;\rho)^{\max\{0,\, k-q+1\}}\, n(z)\, dz.$$

We obtain with the equality in (2.2) and the Theorem of B. Levi

$$E_{a,1}(N^{x}) = \int_{-\infty}^{\infty} (1-\zeta_a(z;\rho))\, n(z) \sum_{i=1}^{\infty} i^{x}\, \zeta_a(z;\rho)^{i-1}\, dz\,.$$

Applying (2.3) the equality is proved. Furthermore

$$E_{a,q}(N) = \sum_{k=0}^{\infty} P_{a,q}(N>k) = \sum_{k=0}^{q-2} P_{a,q}(N>k) + \int_{-\infty}^{\infty} \zeta(z;\rho)^{q-1} \sum_{k=0}^{\infty} \zeta_a(z;\rho)^{k}\, n(z)\, dz.$$

$$= \sum_{k=0}^{q-2} P_{a,q}(N>k) + \int_{-\infty}^{\infty} \frac{\zeta(z;\rho)^{q-1}}{1-\zeta_a(z;\rho)}\, n(z)\, dz$$

and thus the theorem is proved.

In the following we choose c as the $(1-\alpha/2)$-quantile of the standard normal distribution, i.e. $c = \Phi^{-1}(1-\alpha/2)$.

Table 1 shows that the in-control ARL as well as the in-control variance of the run length increase, if the correlation coefficient ρ increases, while the skewness of the run length decreases.

Table 1. Expectation, standard deviation and skewness of the run length under the null hypothesis "no change-point" for various values of ρ ($\gamma_0 = 1$, $\alpha = 0.01$)

ρ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
$E_0(N)$	100.0	115.1	155.3	242.6	461.2	1189.9	5230.0	67179.0
$\sqrt{Var_0(N)}$	98.5	122.8	184.9	325.0	702.3	2090.1	10829.0	169538.0
skewness	0.49	0.43	0.38	0.34	0.31	0.27	0.24	0.20

It can be seen from Table 2 what happens, if the process $\{Y_t\}$ is out-of-control. It is also discussed how the ARL varies for different positions of the change-point. At first glance these results are surprising for q large. However, since the probability of false alarms increases, if q increases, it may occur that the out-of-control ARL is smaller than q.

Table 2. $E_{a,q}(N)$ for various values of a and q ($\gamma_0 = 1$, $\rho = 0.3$, $\alpha = 0.01$)

q / a	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
1	242.58	184.57	84.86	27.45	8.35	3.25	1.78	1.28	1.10
2	242.58	184.30	84.50	27.65	9.07	4.17	2.75	2.27	2.09
3	242.58	184.05	84.18	27.89	9.81	5.09	3.71	3.24	3.06
5	242.58	183.56	83.64	28.49	11.33	6.91	5.62	5.16	4.99
10	242.58	182.49	82.74	30.46	15.24	11.39	10.24	9.83	9.67
50	242.58	178.57	87.90	53.06	44.86	42.69	41.98	41.71	41.61

4. AR-Processes

In this section we determine the run length of an AR(p)-process. We confine ourselves to the case q = 1. Although our expression for the ARL is not suitable for a numerical evaluation, it permits more insight into the properties of the modified Shewhart chart for such type of processes.

Theorem 4.1. Let $\{Y_t\}$ be a causal solution of $Y_t = \sum_{v=1}^{p} \alpha_v Y_{t-v} + \varepsilon_t$. Assume that $P(z) := 1 - \sum_{v=1}^{p} \alpha_v z^v$ has no zeros inside the unit circle and that the random variables ε_t, $t \in \mathbb{Z}$, are independent and identically distributed with $\varepsilon_t \sim N_{0,\sigma^2}$ for all t , where $0<\sigma<\infty$. Then it follows with $\tilde{c} := c\sqrt{\gamma_0}$, $\tilde{\Upsilon} := \min\{p,k\}$, $\tilde{C}_{\tilde{\Upsilon}} := (\mathrm{Cov}(Y_i, Y_j)/\sigma^2)_{i,j=1,..,\tilde{\Upsilon}}$ and $\underline{z}_{\tilde{\Upsilon}} := (z_1, \dots, z_{\tilde{\Upsilon}})^T$ that

$$P_{a,1}(N>k) = \int_{[(-\tilde{c}-a)/\sigma,(\tilde{c}-a)/\sigma]^k} \cdots \int n_{0,\tilde{C}_{\tilde{\Upsilon}}}(\underline{z}_{\tilde{\Upsilon}}) \prod_{v=p+1}^{k} n(z_v - \sum_{i=1}^{p} \alpha_i z_{v-i})\, dz_k \dots dz_1 ,$$

$$E_{a,1}(N) = 1 + \sum_{k=1}^{\infty} \int_{[(-\tilde{c}-a)/\sigma,(\tilde{c}-a)/\sigma]^k} \cdots \int n_{0,\tilde{C}_{\tilde{\Upsilon}}}(\underline{z}_{\tilde{\Upsilon}}) \prod_{v=p+1}^{k} n(z_v - \sum_{i=1}^{p} \alpha_i z_{v-i})\, dz_k \dots dz_1 .$$

Here n denotes the density function of $N_{0,1}$ and $n_{\mu,C}$ the density of the multivariate normal distribution with expectation μ and covariance matrix C.

Proof. Since $P_{a,1}(N>k) = P_0(|Y_1 + a| \le \tilde{c}, \dots, |Y_k + a| \le \tilde{c})$ and an application of the transformation rule for densities shows that for $k \ge p+1$ and with $\alpha_0 := -1$

$$f_{(Y_1,..,Y_k)}(z_1,..,z_k) = f_{(Y_1,\dots,Y_p,\varepsilon_{p+1},..,\varepsilon_k)}(z_1,..,z_p, -\sum_{v=0}^{p} \alpha_v z_{p+1-v}, \dots, -\sum_{v=0}^{p} \alpha_v z_{k-v})$$

$$= n_{0,C_p}(\underline{z}_p) \prod_{v=p+1}^{k} n((z_v - \sum_{i=1}^{p} \alpha_i z_{v-i})/\sigma),$$

the result follows.

Note that $E_0(N)$ does not depend on σ.

Corollary. Let $\{Y_t\}$ be as described in Theorem 4.1 and $p=1$, then the in-control average run length does not change, if α_1 is replaced by $-\alpha_1$.

Proof. We distinguish between k even resp. odd. If k is even, we make the substitution $t_v = -z_v$ for v odd; else, if k is odd, we substitute $t_v = -z_v$ for v even.

Now we confine ourselves to the case of an AR(1) - process.

Theorem 4.2. Let $\{Y_t\}$ be a causal solution of $Y_t = \alpha_1 Y_{t-1} + \varepsilon_t$. Assume that $\alpha_1 \in [0,1)$ and that the random variables ε_t, $t \in \mathbb{Z}$, are independent and identically distributed with $\varepsilon_t \sim N_{0,\sigma^2}$ for all t, where $0<\sigma<\infty$. Then $P_0(N>k)$ is a nondecreasing function in α_1 for all $k \in \mathbb{N}$ fixed. Consequently $E_0(N)$ is also a nondecreasing function in α_1.

Proof. First we observe that $(Y_1/\sqrt{\gamma_0}, \dots, Y_k/\sqrt{\gamma_0})^T \sim \mathfrak{N}(0, (\alpha_1^{|v-i|})_{i,v=1,..,k})$. An explicit formula for the inverse of the covariance matrix can be found in Box/Jenkins (1976, ch.7). Since the inverse matrix is an M-matrix (see Tong (1990,p.78)), the result follows with Theorem 5.1.6 from Tong (1990, p.103).

This result says that the in-control ARL increases, if α_1, i.e. the dependence structure, increases.

Let us consider further properties of the modified Shewhart chart for an AR(1)-process. Setting $Z_t = (1-\alpha_1) X_t$ for $t \geq 1$, it follows that $Z_t = \alpha_1 Z_{t-1} + (1-\alpha_1)(\varepsilon_t + a(1-\alpha_1))$ for $t \geq 2$, i.e. $\{X_t\}$ behaves like an exponentially weighted moving average (EWMA) with smoothing constant $\lambda = 1-\alpha_1$ and head start.

Since $\{X_t\}$ is a Markov process, an approximation of the moments of the run length N can be obtained via the Markov-chain approach (e.g. Lucas/Saccucci (1990) for EWMA control schemes).

The residuals $X_t - \alpha_1 X_{t-1}$ of $\{X_t\}$ are equal to $\varepsilon_t + a(1-\alpha_1)$ for $t \geq 2$ resp. $\varepsilon_1 + a$ for $t = 1$. Consequently the modified Shewhart chart shows a similar behaviour as the EWMA chart with $\lambda = 1-\alpha_1$ applied to the residuals.

We calculated the expectation, the standard deviation and the skewness of the run length by means of simulations. For this we generated independently M = 10 000 realizations of an AR(1)-process. Each one was contaminated according to (1.1). The results are given in Table 3.

Table 3. Expectation, standard deviation and skewness of the run length N for an AR(1)-process ($\alpha_1 = 0.5$, $q = p = 1$, $\alpha = 0.01$, $\varepsilon_t \sim \Phi$, $M = 10\,000$)

a	0	0.5	1.0	1.5	2.0	2.5	3.0
$E_{a,1}(N)$	110.67	68.82	29.93	13.67	7.07	3.98	2.54
$\sqrt{Var_{a,1}(N)}$	109.42	69.30	29.83	13.76	7.07	3.98	2.41
skewness	1.97	2.19	1.90	2.12	1.99	2.22	2.50

The in-control ARL of the process is larger than in the case of independent random variables. The results for $E_{a,1}(N)$ and $\sqrt{Var_{a,1}(N)}$ are nearly the same. The skewness of the run length increases, if the size of the change-point increases.

Now the problem arises how various control charts can be compared. For this we calibrated each chart, i.e. we choose the value c such that the ARL in the in-control state is equal to a given constant. Consequently c depends on the parameter α_1. Theorem 4.2 implies that c decreases, if $\alpha_1 \geq 0$ increases. The values c in Table 4 were calculated via simulations. They nearly coincide with those given by Lucas/Saccucci (1990) for EWMA charts.

Table 4. $c = c_{500}(\alpha_1)$ calculated such that $E_0(N) = 500$ ($q = p = 1$, $\varepsilon_t \sim \Phi$, $M = 50\,000$)

α_1	0	± 0.1	± 0.2	± 0.3	± 0.4	± 0.5	± 0.6	± 0.7	± 0.8	± 0.9
$c_{500}(\alpha_1)$	3.089	3.088	3.087	3.085	3.080	3.071	3.054	3.023	2.964	2.822

We compared the modified Shewhart chart (m.Shew.) with the Shewhart chart resp. EWMA chart applied to the residuals (Shewres resp. EWMAres, see Table 5 - 8). In the case of an AR(1) -process the ARL of the Shewhart chart for the residuals is equal to $1 + \frac{1}{1 - (\Phi(c - a(1-\alpha_1)/\sigma) - \Phi(-c - a(1-\alpha_1)/\sigma))}$. Note that the ARL of the modified Shewhart chart is smaller, if $\alpha_1 = 0$ and c is fixed.

Table 5. A comparison of several control charts ($q = p = 1$, $\varepsilon_t \sim \Phi$, $\lambda = 0.75$, $\alpha_1 = -0.5$, $M = 50\,000$, in - control ARL = 500)

a	0	0.5	1.0	1.5	2.0	2.5	3.0
m.Shew.	500.22	235.00	74.21	26.70	10.94	5.09	2.79
Shewres	500.00	103.96	18.87	5.99	3.15	2.34	2.09
EWMAres	498.42	63.04	10.86	4.41	2.87	2.33	2.10

Table 6. A comparison of several control charts ($q = p = 1$, $\varepsilon_t \sim \Phi$, $\lambda = 0.75$, $\alpha_1 = 0.3$, $M = 50\,000$, in - control ARL = 500)

a	0	0.5	1.0	1.5	2.0	2.5	3.0
m.Shew.	499.16	218.28	64.80	22.82	9.74	4.82	2.80
Shewres	500.00	298.04	118.54	49.28	22.95	12.09	7.20
EWMAres	499.87	234.44	74.53	27.78	13.03	7.48	5.03

Table 7. A comparison of several control charts ($q = p = 1$, $\varepsilon_t \sim \Phi$, $\lambda = 0.75$, $\alpha_1 = 0.6$, $M = 50\,000$, in-control ARL = 500)

a	0	0.5	1.0	1.5	2.0	2.5	3.0
m.Shew.	500.50	282.09	107.74	44.98	21.29	11.05	6.20
Shewres	500.00	412.52	262.90	154.71	91.32	55.51	35.00
EWMAres	498.36	367.85	196.24	101.10	54.83	31.53	19.59

Table 8. A comparison of several control charts ($q = p = 1$, $\varepsilon_t \sim \Phi$, $\lambda = 0.75$, $\alpha_1 = 0.9$, $M = 50\,000$, in - control ARL = 500)

a	0	0.5	1.0	1.5	2.0	2.5	3.0
m.Shew.	499.25	441.10	314.66	208.97	138.77	92.77	63.91
Shewres	500.00	493.59	475.20	447.15	412.52	374.46	335.64
EWMAres	500.89	489.55	460.60	418.24	368.63	318.43	272.28

Table 5 - 8 show that the modified Shewhart chart provides the best results, if α_1 is not negative. The Shewhart and the EWMA chart for the residuals behave extremely bad for a large coefficient, e.g. for $\alpha_1 \geq 0.6$. However, it can also be seen that the out-of-control ARL of the modified Shewhart chart is large, if α_1 is near 1. For a negative coefficient the residual charts turn out to be better (cf Table 5). This result is not surprising as indicated in the introduction.
While the out-of-control ARL of the modified Shewhart chart increases, if $|\alpha_1|$ increases, that of the residual charts increases, if α_1 increases. They provide better results as in the case of independent variables for a negative coefficient and worse for $\alpha_1 > 0$. The smallest out-of-control ARL for the modified Shewhart chart is obtained for an independent sample.

Up to now we assumed that all parameters of the process $\{Y_t\}$ are known. However, in most cases this will not be satisfied.
In the following we demand that a realization of another AR(p) - process $\{Y'_t\}$ is known, where $\{Y_t\}$ and $\{Y'_t\}$ have the same distribution and $\{Y_t\}$ and $\{Y'_t\}$ are independent. Using the realization of $\{Y'_t\}$ we calculate estimators for the parameters of $\{Y_t\}$. This assumption is frequently made in statistical process

control. In many applications such data sets are available (e.g. prerun).

Let $\hat{p}$, $\hat{\underline{\alpha}}_{\hat{p}}$ and $\hat{\sigma}$ denote suitable estimators of p, $\underline{\alpha}_p := (\alpha_1,..,\alpha_p)^T$ and σ (see e.g. Brockwell/Davis (1991)). An estimator of γ_0 can be obtained e.g. by solving the Yule - Walker - equations.

Now we calculate the estimates p^*, $\underline{\alpha}^*_{p^*}$, σ^* and γ_0^* for a given realization of $\{Y_t'\}$. Using these estimates we determine $c = c_\xi(p^*, \underline{\alpha}^*_{p^*}, \sigma^*)$ such that $E_0^*(\inf\{n \in \mathbb{N}: |X_n| > c\sqrt{\gamma_0^*}\}) = \xi$, where ξ denotes a given constant. Here E_0^* denotes the expectation taken with respect to an AR(p^*) - process with coefficients $\underline{\alpha}^*_{p^*}$ and σ^*.

Thus the process stopped at time n, if

$$|X_n| > c_\xi(p^*, \underline{\alpha}^*_{p^*}, \sigma^*)\sqrt{\gamma_0^*} \quad ,$$

where $\gamma_0^* = \gamma_0(p^*, \underline{\alpha}^*_{p^*}, \sigma^*)$. Consequently the run length is equal to

$$\hat{N} := \inf\{n \in \mathbb{N}: |X_n| > c_\xi(p^*, \underline{\alpha}^*_{p^*}, \sigma^*)\sqrt{\gamma_0^*}\}.$$

Theorem 4.3. Suppose that the assumptions of Theorem 4.1 are satisfied. Let $\hat{p}$, $\hat{\underline{\alpha}}_{\hat{p}}$ and $\hat{\sigma}$ denote estimators of p, $\underline{\alpha}_p$ and σ which are calculated via $\{Y_t'\}$. Then it follows with $c_\xi^* := c_\xi(p^*, \underline{\alpha}^*_{p^*}, \sigma^*)\sqrt{\gamma_0^*}$ that

$$P_{a,1}(\hat{N} > k \,|\, \hat{p} = p^*, \hat{\underline{\alpha}}_{p^*} = \underline{\alpha}^*_{p^*}, \hat{\sigma} = \sigma^*)$$

$$= \int_{[(-c_\xi^*-a)/\sigma,(c_\xi^*-a)/\sigma]^k} \cdots \int n_{0,\tilde{C}_{\tilde{1}}}(\underline{z}_{\tilde{1}}) \prod_{v=p+1}^{k} n(z_v - \sum_{i=1}^{p} \alpha_i z_{v-i})\, dz_k .. dz_1 \quad ,$$

$$E_{a,1}(\hat{N} \,|\, \hat{p} = p^*, \hat{\underline{\alpha}}_{p^*} = \underline{\alpha}^*_{p^*}, \hat{\sigma} = \sigma^*)$$

$$= 1 + \sum_{k=1}^{\infty} \int_{[(-c_\xi^*-a)/\sigma,(c_\xi^*-a)/\sigma]^k} \cdots \int n_{0,\tilde{C}_{\tilde{1}}}(\underline{z}_{\tilde{1}}) \prod_{v=p+1}^{k} n(z_v - \sum_{i=1}^{p} \alpha_i z_{v-i})\, dz_k .. dz_1 .$$

Proof. Follows with the same arguments as Theorem 4.1.

A comparison of Theorem 4.1 and 4.3 shows that in the case $\tilde{c} \geq c_\xi^*$ the true ARL is underestimated, otherwise overestimated, provided that c is chosen such that $E_0(N) = \xi$.

In Table 9 - 11 we show what happens, if the parameters are estimated.

Table 9. $E_{a,1}(\hat{N} \mid \hat{p} = 1, \hat{\alpha}_1 = \alpha_1^*, \hat{\sigma} = 1)$ for various values of a and α_1^* and c as in Table 4 (m.Shew., $q = p = 1$, $\alpha_1 = 0.5$, $\varepsilon_t \sim \Phi$, $M = 50000$)

α_1^* / a	0	0.5	1.0	1.5	2.0	2.5	3.0
0.4	298.59	161.82	60.18	24.82	11.72	6.17	3.60
0.45	379.02	197.72	71.19	28.48	13.19	6.83	3.92
0.5	499.90	252.78	87.15	33.70	15.24	7.72	4.35
0.55	709.84	345.93	113.43	42.06	18.39	9.03	4.98
0.6	1213.32	560.48	168.95	59.23	24.43	11.54	6.08

Table 10. $E_{a,1}(\hat{N} \mid \hat{p} = 1, \hat{\alpha}_1 = \alpha_1^*, \hat{\sigma} = 1)$ for various values of a and α_1^* and c as in Table 4 (Shewres, $q = p = 1$, $\alpha_1 = 0.5$, $\varepsilon_t \sim \Phi$, $M = 50000$)

α_1^* / a	0	0.5	1.0	1.5	2.0	2.5	3.0
0.4	469.98	319.17	150.33	70.31	35.35	19.48	11.63
0.45	494.97	351.84	175.87	86.24	44.23	24.67	14.64
0.5	500.00	374.46	202.23	103.98	55.51	31.38	18.87
0.55	494.88	390.09	227.33	124.58	70.11	40.68	24.99
0.6	470.86	390.40	247.69	146.52	86.85	54.83	33.34

Table 11. $E_{a,1}(\hat{N} \mid \hat{p} = 1, \hat{\alpha}_1 = \alpha_1^*, \hat{\sigma} = 1)$ for various values of a and α_1^* and c as in Table 4 (EWMAres, $\lambda = 0.75$, $q = p = 1$, $\alpha_1 = 0.5$, $\varepsilon_t \sim \Phi$, $M = 50000$)

α_1^* / a	0	0.5	1.0	1.5	2.0	2.5	3.0
0.4	354.01	211.38	86.17	38.25	19.22	11.13	7.26
0.45	425.65	262.04	109.91	48.71	24.32	13.74	8.69
0.5	498.87	320.14	140.90	63.50	31.62	17.52	10.92
0.55	565.51	381.41	179.78	83.94	42.41	23.31	14.17
0.6	613.70	442.76	225.22	111.07	57.71	32.26	19.42

The results of Table 9 - 11 are a bit surprising. It can be seen that the estimation of parameters may have a strong influence on the ARL.

The ARL of the modified Shewhart chart increases, if the true parameter value α_1 is overestimated, otherwise it decreases. A similar behaviour can be observed for an EWMA chart for the residuals (Table 11) and for the Shewhart chart for the residuals (Table 10, except the case $a = 0$).

It is remarkable that the in-control ARL of m.Shew. changes dramatically, even if the coefficient α_1 is estimated quite precisely. This effect also occurs for EWMA-res, but it is considerably smaller. In comparison, Shewres reacts nearly robust to deviations in the in-control state.

In spite of this behaviour, it must be emphasized that the out-of-control ARL of m.Shew. is the smallest of all three charts, if $a \geq 1$. This shows that in relation to the other charts, the ARL of m.Shew. decreases more rapidly. As in the case of known parameters EWMAres is better than Shewres, provided that $a \geq 1$.

We think that these simulations show that the influence of parameter estimation on the ARL has to be studied in more detail in future. This is a crucial point, especially in connection with correlated data, which was not treated up to now in literature.

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Wolfgang Schmid

Universität Ulm, Abteilung Stochastik

D - 89069 Ulm