Statistical Papers 38, 445-452 (1997)

Statistical Papers

Estimation of critical points in the mixture inverse Gaussian model

Ramesh C. Gupta, Olcay Akman

Received: April 9, 1996; revised version: June 1996

The maximum likelihood estimation for the critical points of the failure rate and the mean residual life function are presented in the case of mixture inverse Gaussian model. Several important data sets are analyzed from this point of view. For each of the data sets, Bootstrapping is used to construct confidence intervals of the critical points.

KEYWORDS and PHRASES: *Failure rate, mean residual life function, length biased inverse Gaussian, Birnbaum Saunders' model*

1. INTRODUCTION

The distribution used most frequently in the modeling of failure time is the Weibull. However, its use is limited by the fact that its hazard rate, while it may be increasing or decreasing, must be monotonic, whatever the values of its parameters. This may be inappropriate where the equipment is such that its hazard rate reaches a peak after some finite period, and then slowly declines. Such a situation arises in the case of several life distributions; for example, inverse Gaussian, log-logistic, Birnbaum Saunders and Burr. The time when the hazard rate reaches its peak (critical point) can help the reliability analysts determine the duration of the burn-in process and thus is an important parameter in reliability studies.

In this paper we consider a random variable X_p whose distribution is a mixture of inverse Gaussian distribution (IGD) and length biased inverse Gaussian distribution (LBIGD) as follows:

Let $f_p(x) = (1-p)\, f_x(x) + p f_x^*(x),\ 0 \le p \le 1$ (1.1)

where

$$f_x(x) = (\lambda/2\pi x^3)^{1/2} \exp\{-\lambda(x-\mu)^2/2\mu^2 x\},\ x > 0, \lambda > 0, \mu >$$
$$= 0 \qquad \text{otherwise}$$

and $f_x^*(x) = x\, f_x(x)/\mu$, *where* $0 < \mu = \mathrm{E}(X) < \infty$.

(1.1) represents a rich family of distributions for different values of the parameter p. For p = 0, it gives the distribution of the original random variable X and for p = 1 it gives the LBIGD. In addition, p = 1/2 yields Birnbaum and Saunders' model (1969) which was derived from a model of fatigue growth. Thus we believe that this model will fit wide variety of data sets. Its failure rate and mean residual life functions have been studied by Gupta and Akman (1995a). For a recent review of length biased distribution and its applications, the reader is referred to Gupta and Kirmani (1990). Note that a model similar to (1.1) was studied by Jorgensen et.al. (1990).

The failure rate of the mixture model (1.1) increases initially and then decreases. On the other hand, the mean residual life functions (MRLF) exhibits a reverse behavior; in otherwords it decreases initially and then increases. For a general discussion of such a behavior, see Gupta and Akman (1995b). In this paper we shall obtain the Bootstrap estimates of both the turning points and provide their confidence intervals. Several important data sets are analyzed from this point of view. It may be pointed out that some attempt in this direction has been made by Hsieh (1990) in the case of IGD and by Chang and Tang (1993) in the case of Birnbaum Saunders' model. Thus our results are more general and incorporate most of the special cases including the cases p = 0 and p = 1.

The organization of the paper is as follows: Section 2 contains the equations for obtaining the critical points of the failure rate and the MRLF. In section 3, we analyze several well known data sets to estimate the critical points.

2. THE CRITICAL POINTS

The failure rate of the model (1.1) is given by

$$r_p(t) = \frac{(\lambda/2\pi t^3)^{1/2}(1-p+pt/\mu)\exp\{-\lambda/2\mu^2 t)(t-\mu)^2\}}{\Phi(-\alpha(t)) - (1-2p)e^{2\lambda/\mu}\Phi(\beta(t))}, \tag{2.1}$$

where $\alpha(t) = \sqrt{\lambda/t}\,(t/\mu-1)$, $\beta(t) = -\sqrt{\lambda/t}\,(t/\mu+1)$ and Φ is the cumulative distribution function of a standard normal. The MRLF of the model is given by

$$\begin{aligned} m_p(t) &= \Big[(\mu-t+p\mu^2/\lambda)\,\Phi(-\alpha(t)) \\ &\quad + (1-2p)(\mu+t-p\mu^2/\lambda)\,e^{2\lambda/\mu}\,\Phi(\beta(t)) \\ &\quad + 2p\cdot\frac{1}{\sqrt{2\pi}}\sqrt{t\mu^2/\lambda}\,\exp\left\{-\frac{1}{2}(\lambda/t)(1-t/\mu)^2\right\}\Big] \\ &\quad \cdot\left[\Phi(-\alpha(t)-(1-2p)e^{2\lambda/\mu}\Phi(\beta(t))\right]^{-1} \end{aligned} \tag{2.2}$$

Because of the complexity of the expressions, we proceed as follows:

Define $\eta_p(t) = -f_p'(t)/f_p(t)$

It can be verified that

$$\frac{d}{dt}\ln r_p(t) = -\eta_p(t) + r_p(t)$$

So the critical point t_p^* of the failure rate is a solution of the equation

$$\eta_p(t) = r_p(t) \tag{2.3}$$

or t_p^* is a solution of the equation

$$r_p(t) = \frac{\lambda}{2\mu^2} + \frac{3}{2t} - \frac{\lambda}{2t^2} - \frac{p}{\mu(1-p)+pt} \tag{2.4}$$

Equation (2.4) cannot be solved for t_p^* explicitly. So in order to obtain a MLE of t_p^*, we substitute the MLE's of the parameters λ, μ and p from a particular data set and solve for t_p^* numerically.

The MLE's of μ, λ and p are obtained by solving

$$p\mu^3 \tilde{x}^{-1} + \mu^2(1-2p) - \mu(1-p)\,\bar{x} = 0 \qquad (2.5)$$

and

$$\lambda = (\tilde{x}^{-1} + \bar{x}/\mu^2 - 2/\mu) \qquad (2.6)$$

where $\tilde{x}$ is the harmonic mean.
For details, see Gupta and Akman (1995a). It should be noted that the derivative of the log-likelihood function with respect to p results in a polynomial of degree (n+1) whose solution requires unnecessary heavy computations when n is large. For fixed p, the above equation (2.5) and (2.6) yield unique solutions for μ and λ. We then search for the values of μ, λ and p which maximize the likelihood function over 1000 values of $p \subset [0,1]$. In order to obtain the critical point k_p^* of the MRLF, we proceed as follows:
The MRLF of any nonnegative random variable X is given by

$$m(t) = E(X - t \mid X > t) = \int_t^{\infty} \bar{F}(x)\,dx / \bar{F}(t)$$

where $\bar{F}(t)$ is the survival function. The failure rate and the MRLF are connected by the relation

$$r(t) = \frac{1 + m'(t)}{m(t)}$$

So in our case, the critical point k_p^* of the MRLF is given by the solution of the equation

$$r_p(t)\,m_p(t) = 1 \qquad (2.7)$$

Since k_p^* cannot be obtained explicitly, we solve (2.7) numerically using the MLE's of the parameters obtained from a particular data set. Note that it has been shown by Gupta and Akman (1995b) that $k_p^* < t_p^*$. This helps us in defining the range of k_p^*.

3. Some Examples

In order to illustrate our methods, four data sets were analyzed. Data Set I has been fitted to the IGD (p = 0) by Chhikara and Folks (1989). It consists of the number of million revolutions before failure for each of the 23 ball bearings used in the life test.

DATA SET I

17.88	28.92	33.00	41.52	42.12	45.60	48.48
51.84	51.96	54.12	55.56	67.80	68.64	68.64
68.88	84.12	93.12	98.64	105.12	105.84	127.92
128.04	173.40					

Data Set II contains 101 observations and has been fitted to the Birnbaum Saunders' (p = 1/2) model by Engelhardt and Bain (1981). It pertains to the fatigue life of 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second.

DATA SET II

70	90	96	97	99	100	103	104	104	105
107	108	108	108	109	109	112	112	113	114
114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130
130	130	131	131	131	131	131	132	132	132
133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142
142	142	144	144	145	146	148	148	149	151
151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196
212									

Data Set III has been fitted to the LBIGD (p=1), see Jorgensen, et.al. (1990). It represents successive failure intervals for the air-conditioning system of aircraft #7913 in Proschan (1963).

DATA SET III

97, 51, 11, 4, 141, 18, 142, 68, 77, 80, 1, 16, 106, 206, 82, 54, 31, 216, 46, 111, 39, 63, 18, 191, 18, 163, 24.

Data Set IV pertains to the failure intervals of aircraft 8044 of the Proschan's (1963) data. This data set was fitted by Gupta and Akman (1995a) to the model (1.1), assuming p is unknown.

DATA SET IV

487, 18, 100, 7, 98, 5, 85, 91, 43, 230, 3, 130.
The estimates of t_p^* and k_p^* and the corresponding variances were obtained by Bootstrapping for the four data sets and are presented in Table 1.

Bootstrap Estimates

DATA SET	t_p^*	Var t_p^*	k_p^*	Var k_p^*
I	131.970	1.223	105.862	1.418
II	53.970	1.446	14.019	1.099
III	60.400	0.401	44.863	1.121
IV	2.987	0.776	1.094	0.306

TABLE 1

The above values were obtained as follows: For each data set, MLE's are obtained for the parameters. These estimates are used in the expressions of $\eta_p(t)$ *and* $r_p(t)$ and equation (2.3) is solved for t_p^*. The MLE's of the parameters are then substituted in the model. This is now considered as a true model and 1000 samples of the same size are generated from this model. From each generated sample, estimates of the parameters are obtained which in turn yield an estimate of t_p^* by solving equation (2.3) as before. The mean of such 1000 t_p^*'s is given as t_p^* and variance of these t_p^*s is given in the next column of Table 1. Similar procedure was used to obtain k_p^* by solving equation (2.7).

The results in Table 1 are used to obtain 95% percentile confidence intervals, see Hall (1992) and Efron and Tibshirani (1993) for details. The results are as follows:

Bootstrap Confidence Intervals

DATA SET	C.I. for t_p^*	C.I. for k_p^*
I	(125.101, 137.820)	(81.700, 112.657)
II	(47.000, 56.811)	(9.993, 15.148)
III	(51.021, 64.995)	(35.130, 47.614)
IV	(2.551, 4.000)	(0.976, 1.339)

TABLE 2

The above intervals are obtained as follows:

As explained above, once 1000 t_p^*'s are obtained, they are ordered. Then 25th and 975th observations are taken as the lower and upper limits of the intervals.

REFERENCES

Birnbaum, Z.W. and Saunders, S.C. (1969). A new family of life distributions. J. Appl. Prob. 6, 319-327.

Chang, D.S. and Tang, L.C. (1993). Reliability Bounds and Critical Time for the Birnbaum-Saunders Distribution. IEEE Transactions on Reliability. V. 42 No. 3, 464-469.

Chhikara, R.S. and Folks, L. (1989). The Inverse Gaussian Distribution Theory, Methodology and Applications. Marcel Dekker, Inc., New York.

Englehardt, M., and Bain, L.J. and Wright, F.T. (1981). Inferences on the parameters of the Birnbaum-Saunders fatigue life distribution based on maximum likelihood estimation. Technometrics 23, 251-256.

Efron B. and Tibshirani, R.J. (1993). An Introduction to the Bootstrap. Chapman and Hall, New York.

Gupta, R.C. and Akman, O. (1995a). On the Reliability Studies of a weighted inverse Gaussian model. Journal of Statistical Planning and Inference, 48, 69-83.

Gupta, R.C. and Akman, O. (1995b). Mean residual life function for certain type of non-monotonic ageing. Stochastic Models, 11(1), 219-225.

Gupta, R.C. and Kirmani, S. (1990). The Role of Weighted Distributions in Stochastic Modeling. Commun. Statist. Theory and Meth. 19(9) 3147-3167.

Hall, P. (1992). The Bootstrap and Edgeworth Expansion. Springer-Verlag, New York.

Hsieh, H.K. (1990). Estimating the Critical Time of the Inverse Gaussian Hazard Rate. IEEE Transactions on Reliability. V. 39, No. 3, 342-345.

Jorgensen, B., Seshadri, V. and Whitmore, G.A. (1991). On the mixture of the inverse Gaussian distribution with its complementary reciprocal. Scand. J. Statist. 18, 77-89.

Proschan, F. (1963). Theoretical Explanation of observed decreasing failure rate. Technometrics 5, 375-383.

Ramesh C. Gupta
Department of Mathematics and Statistics
University of Maine
Orono, ME 04469-5752, USA

Olcay Akman
Department of Mathematics and Statistics
Utah State University
Logan, UT 84322-3900, USA