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OPERATOR MATRICES AND REACTION-DIFFUSION SYSTEMS

(Conferenza tenuta il 23 novembre 1989)

ABSTRACT. — We show how the recent « matrix theory » for unbounded operator matrices can be used in order to discuss linear reaction-diffusion systems. In particular we obtain information on the existence of a dominant eigenvalue and on the asymptotic behavior of the solutions.

1. - UNBOUNDED OPERATOR MATRICES.

Formally any matrix $\mathcal{A} := (A_{ij})_{n \times n}$ where the entries A_{ij} are linear operators from a Banach space E_j into another Banach space E_i yields a linear operator on the product space $\mathcal{E} := E_1 \times ... \times E_n$ by defining

$$\mathcal{A} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} := \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix},$$

where $g_i := \sum_{j=1}^{n} A_{ij} f_j$ for $(f_1, ..., f_n) \in \mathcal{E}$. Some of the basic matrix operations such as addition and multiplication remain valid, but others such as the formation of « determinant » or « trace » do not make sense anymore. In addition, if the entries A_{ij} are allowed to

(*) This paper has been written during a visit at Tulane University, New Orleans. The author gratefully acknowledges the kind hospitality of J. A. Goldstein and partial support from a National Science Foundation grant.

be unbounded operators, as is necessary for most applications, new difficulties occur. For instance it is not clear and by no means trivial how to choose an appropriate domain $D(\mathcal{A})$ for the operator associated to the formal mapping above.

Among the many things that may go wrong for unbounded operator matrices let me mention one additional example.

EXAMPLE. - Let A be a closed unbounded operator with domain $D(A)$ on a Banach space E. Consider

$$\mathcal{A} := \begin{pmatrix} 0 & Id \\ Id & A \end{pmatrix}$$

with domain $D(\mathcal{A}) := E \times D(A)$ which again is a closed operator on $E \times E$. Since the entries of $\mathcal{A}$ commute, one might expect — as in arbitrary commutative rings — that $\mathcal{A}$ is invertible in $\mathcal{E}$ (i.e. $0 \notin \sigma(\mathcal{A})$) if and only if det $\mathcal{A}$ is invertible in E. Since det $\mathcal{A} = -Id$ the second statement holds but $\mathcal{A}$ has only a formal inverse

$$\mathcal{A}^{-1} := \begin{pmatrix} -A & Id \\ Id & 0 \end{pmatrix},$$

which is not a bounded operator.

In a series of papers and in collaboration with P. Charissiadis, K. J. Engel and A. Holderrieth we have tried to develop a systematic theory for unbounded operator matrices (see the references). In this note it is shown how these results can be applied to a concrete system of linear evolution equations yielding detailed information on the qualitative behavior of the solutions of this system.

2. - A LINEAR REACTION-DIFFUSION SYSTEM.

Reaction-diffusion systems are important and quite difficult equations having numerous applications and a rapidly growing theory. From the huge literature we only mention [Sm] and the recent article by Amann [A]. Since it is our intention to make evident the basic ideas from our « matrix theory » we consider a very simple linear system as it occurs e.g. in [D-L] or [H-M] or as the linearization of certain nonlinear systems (e.g. in [M]).

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and take coefficients $a_{ij} \in \mathbb{C}$, $b_{ij} \in C(\bar{\Omega})$ for $1 \le i, j \le n$. If Δ denotes the Laplacian (or a more general elliptic partial differential operator) then we define the following system

$$(S_1) \qquad \frac{d}{dt} u_i(x,t) = \sum_{j=1}^{n} a_{ij} \, \Delta u_j(x,t) + \sum_{j=1}^{n} b_{ij}(x) \, u_j(x,t)$$

for $1 \le i \le n$, $x \in \Omega$ and $t \ge 0$. We assume Dirichlet boundary conditions

$$u_i(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega$$

and initial values

$$u_i(x,0) = f_i(x) \quad \text{for} \quad x \in \Omega .$$

In order to apply our general theory to this special system we rewrite it in matrix form.

Consider Δ with Dirichlet boundary conditions as a closed operator on $E := L^2(\Omega)$ with domain $D(\Delta)$ and identify the functions b_{ij} with the corresponding multiplication operators $f \mapsto b_{ij} f$ on E. On the product space

$$\mathcal{E} := L^2(\Omega) \times \dots \times L^2(\Omega)$$

we then study the Cauchy problem

$$(S_2) \qquad \dot{u}(t) = \mathcal{A} u(t), \qquad u(0) = (f_1, \dots, f_n),$$

where $\mathcal{A} := (A_{ij})_{n \times n} = (a_{ij}\Delta + b_{ij})_{n \times n}$ and $u(t) \in \mathcal{E}$. The domain $D(\mathcal{A})$ of $\mathcal{A}$ is obtained by observing that the b_{ij}'s are all bounded and therefore $D(\mathcal{A})$ coincides with $D(\mathcal{A}_0)$ where $\mathcal{A}_0 := (a_{ij}\Delta)_{n \times n}$. Hence $\mathcal{A}$ with domain

$$D(\mathcal{A}) := \{ (f_1, \dots, f_n) \in \mathcal{E} : \sum_{j=1}^{n} a_{ij} f_j \in D(\Delta) \text{ for } i = 1, \dots, n \}$$

is a closed, densely defined, linear operator on $\mathcal{E}$ (see [E1]). We show that the operator theoretical properties of $\mathcal{A}$ determine existence and qualitative behavior of the solutions of (S_1) and (S_2). We start by studying the spectrum of $\mathcal{A}$.

3. - Spectral theory, I.

If the coefficients b_{ij} are supposed to be constant then the entries A_{ij} commute. Therefore we obtain $\sigma(\mathcal{A})$ from the spectral mapping theorem in [E-N], [E1]. In the non-constant situation it is much more difficult and new tools are necessary. We start by looking at the 2×2-case.

It was I. Schur who observed that (scalar) 2×2-block matrices $\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ are invertible if and only if the so-called « *Schur complement* » $A_{22} - A_{21} A_{11}^{-1} A_{12}$ is invertible provided A_{11}^{-1} exists (see e.g. [L-T], p. 46). This idea can be generalized to our operator matrix context (see [N3], Sect. 2) and can be applied to the computation of $\sigma(\mathcal{A})$.

Proposition. - *Let $\mathcal{A} = (A_{ij})_{2\times 2}$ be the above operator matrix and assume that $\sigma(A_{11})$ is known. Then for $\lambda \notin \sigma(A_{11})$ we have $\lambda \in \sigma(\mathcal{A})$ if and only if $\lambda \in \sigma(A_{22} + A_{21}(\lambda - A_{11})^{-1} A_{12})$.*

We have thus reduced the problem of determining the spectrum of the operator $\mathcal{A}$ in the product space $\mathcal{E}$ to a problem for the « *characteristic operator function* » in the factor space E_2. In the following corollary we give an explicit matrix representation for the resolvent $R(\lambda, \mathcal{A}) := (\lambda - \mathcal{A})^{-1}$ in the remaining resolvent set $\varrho(\mathcal{A}) \setminus \sigma(A_{11})$.

Corollary. - *For $\lambda \in \varrho(\mathcal{A}) \setminus \sigma(A_{11})$ one has*

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} R(\lambda, A_{11})(Id + A_{12} R(\lambda) A_{21} R(\lambda, A_{11})) & R(\lambda, A_{11}) A_{12} R(\lambda) \\ R(\lambda) A_{21} R(\lambda, A_{11}) & R(\lambda) \end{pmatrix},$$

where $R(\lambda) := R(\lambda, A_{22} + A_{21} R(\lambda, A_{11}) A_{12})$.

This representation for $R(\lambda, \mathcal{A})$ allows to extend the above arguments to bigger matrices. Consider the 3×3 matrix $\mathcal{A} = (A_{ij})_{3\times 3}$ as a 2×2 block matrix whose upper left entry is $\mathcal{A}_2 := (A_{ij})_{2\times 2}$. Then $\lambda \in \sigma(\mathcal{A}) \setminus (\sigma(\mathcal{A}_2) \cup \sigma(A_{11}))$ if and only if λ is in the spectrum of the corresponding Schur complement $A_{33} + (A_{31}\, A_{32})\, R(\lambda, \mathcal{A}_2) \begin{pmatrix} A_{23} \\ A_{13} \end{pmatrix}$.

Here it is essential that we allowed product spaces made up by different factor spaces (E^2 and E in our case) and that we had an explicit representation for $R(\lambda, \mathcal{A}_2)$.

One might proceed in this way, but clearly the formulas will quickly become very messy.

4. - Well-posedness.

The existence of solutions to our system ($S2$) is guaranteed if (and in a certain sense only if) $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{E}$. For this the bounded entries b_{ij} do not matter and we can assume $\mathcal{A} = (a_{ij}\,\triangle)$. Such operator matrices have been studied in great detail and generality in [N2], [E2] and [E-N2].

We recall from [G] that the Laplace operator $\triangle$ generates on $E = L^2(\Omega)$ an analytic semigroup of angle $\frac{\pi}{2}$. Therefore it follows from [N2], Thm. 2.3 that the generator property of $\mathcal{A}$ is characterized by the location of the eigenvalues of the coefficient matrix $(a_{ij})_{n\times n}$ alone. Using in addition that $\triangle$ has compact resolvent on E we obtain the following result.

Proposition. - *The following assertions are equivalent.*

(*a*) *The operator matrix* $\mathcal{A} = (a_{ij}\,\triangle + b_{ij})_{n\times n}$ *generates a strongly continuous semigroup on* $\mathcal{E}$.

(*b*) *All eigenvalues* λ *of* $(a_{ij})_{n\times n}$ *satisfy* $Re\lambda > 0$ *and* $\lambda = 0$ *is a pole of the resolvent of order at most one.*

In that case the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ *is analytic and* $\mathcal{A}$ *has compact resolvent.*

5. - Positivity.

Once we know that solutions to our Cauchy problem (S_2) exist (i.e., if $\mathcal{A}$ generates a semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ on $\mathcal{E}$) then it is of great importance for theory and applications to know when all solutions corresponding to positive initial values remain positive for all $t \geq 0$. This property is expressed by the fact that the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ consists of positive operators on the Banach lattice $\mathcal{E}$. See [N1].

PROPOSITION. - *For the semigroup* $(e^{t\mathcal{A}})_{t\geq 0}$ *generated by* $\mathcal{A} = (A_{ij})_{n\times n} = (a_{ij}\,\Delta + b_{ij})_{n\times n}$ *on* $\mathcal{E}$ *the following assertions are equivalent.*

(*a*) $0 \leq e^{t\mathcal{A}}$ *for all* $t \geq 0$.

(*b*) $0 \leq e^{tA_{ii}}$ *for all* $t \geq 0$ *and* $0 \leq A_{ij}$ *for* $i \neq j$.

(*c*) $0 \leq a_{ii}$ *and* b_{ii} *real valued for* $i = 1, \ldots, n$ *and* $a_{ij} = 0$, $b_{ij} \geq 0$ *for* $i \neq j$.

Proof. - The equivalence of (a) and (b) has been shown in [N2] and [N-C] under more general assumptions. For the remaining equivalence we observe first that $A_{ii} = a_{ii}\,\Delta + b_{ii}$ generates a positive semigroup on E if and only if $a_{ii} \geq 0$ and b_{ii} is real valued (see [N1]: the generator of a positive semigroup is « real »). Since the differential operator Δ never maps all positive functions in its domain into the cone of positive functions we conclude that $0 \leq A_{ij}$ if and only if $a_{ij} = 0$ and $b_{ij} \geq 0$. Hence (b) and (c) are equivalent. ■

REMARK. - Reaction-diffusion systems satisfying condition (c) are called cooperative systems. We have shown that these conditions are necessary and sufficient for positivity.

6. - SPECTRAL THEORY, II.

From now on we assume that the conditions from Section 5 implying positivity are satisfied. Then it is known (see [N1]) that $\mathcal{A}$ and the corresponding semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ possess a quite rich spectral theory. These geometric properties of $\sigma(\mathcal{A})$ will turn out to be more useful than the attempt in Section 3 to compute $\sigma(\mathcal{A})$ precisely. We now gather some qualitative information on $\sigma(\mathcal{A})$ which follows from general operator and semigroup theory.

LEMMA 1. - *The resolvent of* $\mathcal{A} = (a_{ij}\,\Delta + b_{ij})_{n\times n}$ *is compact. Therefore* $\sigma(\mathcal{A})$ *is a discrete set of eigenvalues.*

Proof. - The resolvent $R(\lambda, \Delta)$ and hence the resolvent of $\mathrm{diag}(a_{ii}\,\Delta)$ is compact. But this property is preserved under bounded perturbations. ■

LEMMA 2. - *The semigroup* $(e^{t\mathcal{A}})_{t\geq 0}$ *is analytic, hence the spectrum* $\sigma(\mathcal{A})$ *is bounded on imaginary strips of the form* $\{\lambda \in \mathbb{C} : \alpha \leq \mathrm{Re}\lambda \leq \beta\}$.

Proof. - That $(e^{t\mathcal{A}})$ is analytic has been observed above. Therefore the spectrum of $\mathcal{A}$ is contained in some proper sector (see [G]) and obviously bounded on imaginary lines. ■

For the following lemma and later use we introduce the *spectral bound*

$$s(\mathcal{A}) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathcal{A})\}$$

of $\mathcal{A}$.

LEMMA 3. - *The semigroup* $(e^{t\mathcal{A}})$ *is positive, hence its boundary spectrum* $\sigma_+(\mathcal{A}) := \sigma(\mathcal{A}) \cap \{s(\mathcal{A}) + i\mathbb{R}\}$ *is « cyclic », i.e. if* $s(\mathcal{A}) + i\mu \in \sigma(\mathcal{A})$ *then* $s(\mathcal{A}) + ik\mu \in \sigma(\mathcal{A})$ *for all* $k \in \mathbb{Z}$.

Proof. - This Perron-Frobenius type result is due to G. Greiner and can be found in [N1], C-III, Cor. 2.12. ■

Combining all these lemmas we obtain the existence of a *dominant (or: leading) eigenvalue* of $\mathcal{A}$.

PROPOSITION. - *If* $\mathcal{A} = (a_{ij}\,\Delta + b_{ij})_{n\times n}$ *satisfies the positivity assumptions from Section 5 then there exists a real eigenvalue* λ_0 *of* $\mathcal{A}$ *such that*

$$Re\lambda < \lambda_0$$

for all other eigenvalues $\lambda \in \sigma(\mathcal{A})$.

Proof. - From Lemma 3 it follows that $s(\mathcal{A}) \in \sigma(\mathcal{A})$ and therefore $s(\mathcal{A})$ is an eigenvalue by Lemma 1. If there is another eigenvalue $s(\mathcal{A}) + i\mu$ for $\mu \neq 0$ then there are infinitely many on the line $s(\mathcal{A}) + i\mathbb{R}$ contradicting Lemma 2. Hence there are only finitely many eigenvalues in the strip $\{\lambda \in \mathbb{C} : s(\mathcal{A}) - \varepsilon \leq \mathrm{Re}\lambda \leq s(\mathcal{A})\}$ and $\lambda_0 = s(\mathcal{A})$ is dominant. ■

REMARK. - We point out that in this section we did not use the matrix structure but only certain functional analytic properties of the operator $\mathcal{A}$.

7. - STABILITY.

« *Stability* » in our situation means that all solutions of (S_2) converge to zero as t goes to infinity. More precisely we want that

$$\lim_{\to\infty} \|e^{t\mathcal{A}}\| = 0 .$$

By the infinite dimensional analogue of Liapunov's theorem (see e.g. [N1], A-IV, Remark 1.7) this is equivalent to the fact that $s(\mathcal{A}) < 0$. Since $\mathcal{A}$ has a dominant eigenvalue λ_0 (see Section 6) it suffices to determine the sign of λ_0. The following proposition shows how this problem in the product space $\mathcal{E}$ can be reduced to n problems in the factor space E. To that purpose we consider each principal submatrix

$$\mathcal{A}_k := (A_{ij})_{k\times k}$$

as a 2×2 block matrix

$$\mathcal{A}_k = \begin{pmatrix} \mathcal{A}_{k-1} & \mathcal{B}_k \\ \mathcal{C}_k & A_{kk} \end{pmatrix}$$

on $E^k = E^{k-1} \times E$, where $\mathcal{B}_k = \begin{pmatrix} A_{1k} \\ \vdots \\ A_{k-1k} \end{pmatrix}$ and $\mathcal{C}_k = (A_{k1}, \dots, A_{kk-1})$

(see also Section 3). Then we obtain the following characterization of stability.

PROPOSITION. - *Let* $\mathcal{A} = (A_{ij})_{n\times n} = (a_{ij}\,\Delta + b_{ij})_{n\times n}$ *satisfy the positivity assumptions from Section 5. Then the following assertions are equivalent.*

(*a*) $\lim_{t\to\infty} \|e^{t\mathcal{A}}\| = 0$.

(*b*) $\lambda_0 = s(\mathcal{A}) < 0$.

(*c*) $s(\mathcal{A}_k) < 0$ *for* $k = 1, \dots, n$.

(*d*) $s(A_{11}) < 0$ *and* $s(A_{kk} - \mathcal{C}_k \mathcal{A}_{k-1}^{-1} \mathcal{B}_k) < 0$ *for* $k = 1, \dots, n$.

(*e*) $s(a_{11}\,\Delta + b_{11}) < 0, s(a_{22}\,\Delta + b_{22} - b_{21}(a_{11}\Delta + b_{11})^{-1} b_{12}) < 0, \dots,$ $s(a_{nn}\,\Delta + b_{nn} - \mathcal{C}_n\,\mathcal{A}_{n-1}^{-1}\mathcal{B}_n) < 0$.

Proof. - The equivalence of (a) and (b) is shown in [N1], A-IV, (1.8). Condition (c) clearly is stronger than (b). Since the off-diagonal entries of $\mathcal{A}$ are all positive it follows from the monotonicity of the spectral bound (see [N1], C-II, Lemma 4.10) that $s(\mathcal{A}_k) \leq s(\mathcal{A})$ for all k. Therefore (b) implies (c). The apparently more complicated condition (c) now allows to reduce the problem to the factor space E. In fact it follows via the Schur complement characterization from Section 3 that $s(\mathcal{A}_k) < 0$ if and only if $s(\mathcal{A}_{k-1}) < 0$ and $s(A_{kk} - \mathcal{C}_k \mathcal{A}_{k-1}^{-1} \mathcal{B}_k) < 0$. See [N4] and [C-N] for more details. Hence (d) and (c) are equivalent. Condition (e) is only a more concrete version of (d). ∎

REMARKS. - 1. For complex matrices $\mathcal{A} = (a_{ij})_{n \times n}$ condition **(d)** is equivalent to

$$(d') \qquad (-1)^{k+1} \det \mathcal{A}_k < 0 \quad \text{for} \quad k = 1, \ldots, n.$$

This means that $-\mathcal{A}$ is a so called « *M-matrix* » (see [L-T], Sect. 15.2).

2. If the coefficients b_{ij} in $\mathcal{A} = (a_{ij} \Delta + b_{ij})$ are constant and therefore the entries of $\mathcal{A}$ commute it follows from the spectral mapping theorem for the resolvent (see [N1], A-III, Prop. 2.5) that (d) is equivalent to

(d'') The scalar matrix $A := (a_{ij}\, s(\Delta) + b_{ij})$ satisfies the condition (d').

This again is equivalent to

(d''') The eigenvalues of A have negative real part.

In this case we are thus able to characterize stability for (S_2) in terms of purely finite dimensional conditions.

8. - CONVERGENCE TO EQUILIBRIUM.

In this final section we face the situation when the dominant eigenvalue λ_0 of $\mathcal{A}$ is zero. In that case the corresponding eigenfunctions are invariant under the semigroup $(e^{t\mathcal{A}})_{t \geq 0}$. The most interesting case occurs when this eigenspace is one-dimensional and spanned by a strictly positive element in $\mathcal{E}$. This always holds if the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is *irreducible* on $\mathcal{E}$ (see [N1], C-III, Prop. 3.5). It is therefore quite useful that in our case we are able to characterize irreducibility in terms of an associated scalar matrix.

PROPOSITION. - *Let $\mathcal{A} = (a_{ij}\,\Delta + b_{ij})$ satisfy the positivity assumptions from Section 5 with $a_{ii} > 0$. Then the following assertions are equivalent.*

(*a*) *The semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is irreducible in $\mathcal{E}$.*

(*b*) *The matrix $D := (\delta_{ij})$ with $\delta_{ij} := \begin{cases} 1 \text{ if } b_{ij} \neq 0 \\ 0 \text{ if } b_{ij} = 0 \end{cases}$ is irreducible in $\mathbb{R}^n$ (see [L-T], Sect. 15.1 or [Sch], Chap. I).*

Proof (*). - It is well known that Δ, hence $a_{ii}\Delta + b_{ii}$ (see [N1], B-III, Ex. 3-10 and C-III, Prop. 3.3) generate irreducible semigroups on E. Therefore the only closed invariant ideals in $\mathcal{E}$ for the operator matrix $\operatorname{diag}(a_{ii}\Delta + b_{ii})$ are of the form $J := J_1 \times \dots \times J_n$ where $J_i = \{0\}$ or $J_i = E$ for $i = 1, \dots, n$. Since $\mathcal{A}$ is a positive perturbation of $\operatorname{diag}(a_{ii}\Delta + b_{ii})$ it suffices to consider ideals of the above form and we can assume $J_1 = \dots = J_k = \{0\}$ and $J_{k+1} = \dots = J_n = E$ for some k. Such an ideal is $\mathcal{A}$-invariant if and only if $b_{ij} = 0$ for $1 \leq i \leq k < j \leq n$, i.e., if and only if the scalar matrix D is reducible. ■

If $(e^{t\mathcal{A}})_{t\geq 0}$ is irreducible if follows from the classical Krein-Rutman theorem (see [N1], C-III, Prop. 3.5) that the dominant eigenvalue $\lambda_0 = 0$ is a simple pole of the resolvent. It is therefore possible to decompose $\mathcal{E}$ into the fixed space of $(e^{t\mathcal{A}})_{t\geq 0}$, which is one-dimensional, and a $(e^{t\mathcal{A}})$-invariant subspace on which the restricted semigroup has spectral bound (and by [N1], A-IV growth bound) strictly smaller than zero. We therefore conclude that $(e^{t\mathcal{A}})_{t\geq 0}$ converges (in operator norm and exponentially) to a strictly positive projection onto its one-dimensional fixed space. The case $\lambda_0 \neq 0$ can be reduced via rescaling ([N1], A-I, 3.1) to the above situation. The information obtained so far will now be collected in one final theorem and stated in terms of the solutions of the original reaction- diffusion system (S_1).

(*) Due to W. Arendt.

THEOREM. - *Assume that the coefficients in* (S_1) *satisfy* $a_{ii} \geq 0$, b_{ii} *real-valued for* $i = 1, \dots, n$ *and* $a_{ij} = 0$, $b_{ij} \geq 0$ *for* $i \neq j$ *and that*

$$D := (\delta_{ij})_{n \times n} \qquad \text{with} \qquad \delta_{ij} := \begin{cases} 1 & \text{if} \quad b_{ij} \neq 0 \\ 0 & \text{if} \quad b_{ij} = 0 \end{cases}$$

is an irreducible matrix. Then there exist a unique real number λ_0 *and strictly positive functions*

$$\Phi := (\Phi_1, \dots, \Phi_n) \in (L^2(\Omega))^n, \ \Psi := (\Psi_1, \dots, \Psi_n) \in (L^2(\Omega))^n$$

such that for every initial function $f := (f_1, \dots, f_n) \in (L^2(\Omega))^n$ *the solution* $u(t, x) := (u_1(x, t), \dots, u_n(x, t))$ *of* (S_1) *satisfies*

$$\|\cdot\|_2 - \lim_{t \to \infty} e^{-\lambda_0 t} u_i(\cdot, t) = \left(\sum_{j=0}^{n} \int_{\Omega} f_j(x) \, \psi_j(x) \, dx \right) \Phi_i$$

for $i = 1, \dots, n$ *and uniformly on the unit ball of* $(L^2(\Omega))^n$. *In particular, if* $\lambda_0 = 0$ *then all solutions of* (S_1) *converge to a unique, equidistributed equilibrium.*

SUNTO. — In questo lavoro si mostra come la teoria delle matrici con operatori non limitati è utile per lo studio dei sistemi lineari di reazione-diffusione. Si ottengono risultati sull'esistenza di un autovalore dominante e sul comportamento asintotico delle soluzioni.

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