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Some asymptotic properties in INAR(1) processes with Poisson marginals

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The first-order integer-valued autoregressive(INAR(1)) process with Poisson marginal distributions is considered. It is shown that the sample autocovariance function of the model is asymptotically normally distributed. We derive asymptotic distribution of Yule-Walker type estimators of parameters. It turns out that our Yule-Walker type estimators are better than the conditional least squares estimators proposed by Klimko and Nelson(1978) and Al-Osh and Alzaid(1987). Also, we study the relationship between the model and $M/M/\infty$ queueing system.

1 Introduction

In the past several years, discrete valued stationary processes have been the object of several articles. A great deal of research has been devoted to some discrete model building. McKenzie(1986, 1988) investigated some properties of negative binomial and geometric and Poisson ARMA models. Al-Osh and Alzaid(1987) introduced what they have called integer-valued first-order autoregressive(INAR(1)) model and dealt with discrete time stationary processes with Poisson marginal distributions. Recently, Al-Osh and Aly(1992) and Aly and Bouzar(1994) suggested some advanced integer-valued ARMA models.

The INAR(1) model is of the following form

$$X_n = \alpha * X_{n-1} + W_n, \quad n = 0, \pm 1, \pm 2, \cdots \tag{1.1}$$

where X_n is a Poisson random variable with parameter θ for all n and $\alpha * X = \sum_{i=1}^{X} B_i(\alpha)$, where $B_i(\alpha)$ is a sequence of independent identically distributed binary random variables with $P(B_i(\alpha) = 1) = 1 - P(B_i(\alpha) = 0) = \alpha$ which is independent of X and W_n. As the usual continuos AR(1) model, we assume that X_{n-1} and W_n are independent. Then one can easily show that $\alpha * X_{n-1}$ and W_n are both Poisson random variables with parameters $\alpha\theta$ and $(1-\alpha)\theta$, respectively and the marginal distribution of model (1.1) can be expressed as follows:

$$X_n \stackrel{\mathrm{d}}{=} \sum_{i=0}^{\infty} \alpha^i * W_{n-i}. \tag{1.2}$$

Define X_n to be the number of counts at time n in a system. Let $\mathcal{W}_n$ be a set of objects and W_n be the number of objects in $\mathcal{W}_n$. For example, $\mathcal{W}_n$ and W_n might be the wating line and the number of wating line, respectively, in a queueing system at time n. Then we can also define (1.2) in another way: $\alpha^i * W_{n-i} = \sum_{j=1}^{W_{n-i}} Y_{j,i}^{(n-i)}$ with $Y_{j,k}^{(n)}$ defined by:

$$Y_{j,k}^{(n)} = \begin{cases} 1, & \text{if } j\text{th element of } \mathcal{W}_n \text{ is present in the system at} \\ & \quad \text{time } n+k \text{ with probability } \alpha^i \\ 0, & \text{otherwise.} \end{cases}$$

And

$$P[Y_{j,i_1}^{(n)} = 1, Y_{j,i_2}^{(n)} = 1, \ldots, Y_{j,i_k}^{(n)} = 1] = \alpha^{i_1}\alpha^{i_2-i_1}\ldots\alpha^{i_k-i_{k-1}} = \alpha^{i_k}. \tag{1.3}$$

The α^{i_k} can be interpreted as the probability that an element of $\mathcal{W}_n$ will be element of X_{n+i_k}(this element might be present in the system during times preceding $n + i_k$ as well).

The purpose of this paper is to derive the asymptotic behavior of sample mean and sample autocovariance functions and to investigate the limiting distribution of a certain estimator of parameter (α, θ) in the model (1.1). As an application, in section 3, we present the methods for estimation of parameters in the model (1.1) and compare our results for estimators of α and θ to those of Al-Osh and Alzaid(1987) and Klimko and Nelson(1978) by the variances of estimators. Also, we study the relationship between the model (1.1) and $M/M/\infty$ queueing process in equilibrium.

2 Asymptotic Distributions of Sample Autocovariance Function

The estimators which we shall use for the autocovariance function $\gamma(p)$ and the autocorrelation function $\rho(p) = \gamma(p)/\gamma(0)$ from observations of $X_1, \ldots, X_n$ are

$$\hat{\gamma}(p) = n^{-1} \sum_{t=1}^{n-p} (X_t - \bar{X}_n)(X_{t+p} - \bar{X}_n), \quad p = 0, 1, 2, \cdots$$

with $\bar{X}_n = n^{-1} \sum_{t=1}^{n} X_t$ and

$$\hat{\rho}(p) = \hat{\gamma}(p)/\hat{\gamma}(0),$$

respectively. For any integer h, the autocovariance function of an INAR(1) process $\{X_n\}$ defined as (1.1) from Al-Osh and Alzaid(1987) is

$$\gamma(\pm h) = Cov(X_n, X_{n\pm h}) = \alpha^{|h|}\theta$$

and hence the process $\{X_n\}$ has the nonnegative autocorrelation function $\rho(\pm h) = \alpha^{|h|}$.

After some extremely tedious calculation using (1.2) and (1.3), we obtain that for $h, p, q \geq 0$ and $\mu = E(X_t) = \theta$,

$$E(X_t X_{t+p}) = \theta^2 + \theta\alpha^p, \tag{2.1}$$

$$E(X_t X_{t+p} X_{t+p+h}) = \theta^3 + \theta^2(\alpha^p + \alpha^{p+h} + \alpha^h) + \theta\alpha^{p+h} \tag{2.2}$$

and

$$
\begin{aligned}
& E(X_t-\mu)(X_{t+p}-\mu)(X_{t+p+h}-\mu)(X_{t+p+h+q}-\mu) \\
= \;& E(W_t)\sum_{i=0}^{\infty}\alpha^{i+p+h+q} \\
+ \;& \gamma(p)\gamma(q)+\gamma(p+h)\gamma(h+q)+\gamma(p+h+q)\gamma(h). \qquad (2.3)
\end{aligned}
$$

Define $\tilde{\gamma}(h) = n^{-1}\sum_{t=1}^{n}(X_t-\mu)(X_{t+h}-\mu)$. Then we have the following results.

Lemma 2.1. If $\{X_t\}$ is the INAR(1) process and $X_t \stackrel{\text{d}}{=} \sum_{i=0}^{\infty}\alpha^i * W_{t-i}$, where $\{W_t\}$ is a Poisson random variable with parameter $(1-\alpha)\theta$, then

$$
\text{(i)} \quad \lim_{n\to\infty} nVar(\bar{X}_n) = \theta\frac{1+\alpha}{1-\alpha}
$$

$$
\text{and} \quad \text{(ii)} \quad \lim_{n\to\infty} nCov(\bar{X}_n, \tilde{\gamma}(h)) = \begin{cases} \frac{2\theta}{1-\alpha}, & \text{if } h=0, \\ \theta\alpha^h\left(\frac{1+\alpha}{1-\alpha}+h\right), & \text{if } h\ge 1. \end{cases}
$$

Proof. To show (i), by (2.1)

$$
\begin{aligned}
E(\bar{X}_n^2) &= E\Big(\frac{1}{n^2}\sum_{t=1}^{n}X_t\sum_{s=1}^{n}X_s\Big) \\
&= \frac{1}{n^2}\Big[nE(X_t^2)+2\sum_{p=1}^{n-1}(n-p)E(X_tX_{t+p})\Big] \\
&= \frac{1}{n^2}\Big[n^2\theta^2+n\theta+2\sum_{p=1}^{n-1}(n-p)\theta\alpha^p\Big].
\end{aligned}
$$

Thus

$$
Var(\bar{X}_n) = \frac{1}{n}\left(\theta+\frac{2}{n}\sum_{p=1}^{n-1}(n-p)\theta\alpha^p\right).
$$

This shows (i) by letting $n \to \infty$ after multiplying n to the both sides of the above equation. Observe for (ii) that

$$
\begin{aligned}
&E\Big(\frac{1}{n^2}\sum_{t=1}^{n} X_t \sum_{s=1}^{n}(X_s-\theta)(X_{s+h}-\theta)\Big) \\
&= \frac{1}{n}\sum_{p=0}^{n-1}\Big(1-\frac{p}{n}\Big)E\left(X_t(X_{t+p}-\theta)(X_{t+p+h}-\theta)\right) \\
&+ \frac{1}{n}\sum_{p=1}^{h-1}\Big(1-\frac{p}{n}\Big)E\left((X_t-\theta)X_{t+p}(X_{t+h}-\theta)\right) \\
&+ \frac{1}{n}\sum_{p=h}^{n-1}\Big(1-\frac{p}{n}\Big)E\left((X_t-\theta)(X_{t+h}-\theta)X_{t+p}\right). \qquad (2.4)
\end{aligned}
$$

By (2.1) and (2.2), the equation (2.4) is $\frac{1}{n}\sum_{p=0}^{n-1}\Big(1-\frac{p}{n}\Big)(\theta\alpha^{p+h} + \theta\gamma(h)) + \frac{1}{n}\sum_{p=1}^{h-1}\Big(1-\frac{p}{n}\Big)(\theta\alpha^{h}+\theta\gamma(h)) + \frac{1}{n}\sum_{p=h}^{n-1}\Big(1-\frac{p}{n}\Big)(\theta\alpha^{p}+\theta\gamma(h))$.

Hence $\lim_{n\to\infty} nCov(\bar{X}_n, \tilde{\gamma}(h)) = \theta\alpha^h\Big(\frac{1+\alpha}{1-\alpha}+h\Big), \quad \text{if } h \geq 1.$

Lemma 2.2. Let $\{X_t\}$ be the same form as in Lemma 2.1. Then, for $p \geq q$,

$$
\lim_{n\to\infty} nCov(\tilde{\gamma}(p), \tilde{\gamma}(q))
$$

$$
= \begin{cases}
\frac{\alpha^p}{1-\alpha^2}\theta\Big[(1-\alpha^2)\{p(1+\theta\alpha^{-p}+\theta\alpha^q) \\
\qquad -q(1+\theta\alpha^{-q}-\theta\alpha^q) \quad -\theta\alpha^q\} \\
\qquad +(1+\alpha)^2+2\theta\alpha^q+\theta\alpha^{-q}(1+\alpha^2)\Big], & \text{if } q \geq 1, \\
\frac{\alpha^p}{1-\alpha^2}\theta\Big[(1-\alpha^2)(p+2\theta p-\theta) \\
\qquad +(1+\alpha)^2+3\theta+\theta\alpha^2\Big], & \text{if } q = 0.
\end{cases}
$$

Proof. By (2.1), (2.2) and (2.3), one can show that

$$
\begin{aligned}
&E\Big(\frac{1}{n^2}\sum_{t=1}^{n}\sum_{s=1}^{n}(X_t-\theta)(X_{t+p}-\theta)(X_s-\theta)(X_{s+q}-\theta)\Big)\\
&= \gamma(p)\gamma(q)+\frac{1}{n^2}\Big\{\sum_{\xi=0}^{p-q-1}(n-\xi)(\theta\alpha^{p}\\
&\qquad +\gamma(\xi)\gamma(p-\xi-q)+\gamma(p-\xi)\gamma(\xi+q))\\
&\quad +\sum_{\xi=p-q}^{p-1}(n-\xi)(\theta\alpha^{\xi+q}+\gamma(\xi+q)\gamma(p-\xi)+\gamma(\xi)\gamma(\xi-p+q))\\
&\quad +\sum_{\xi=p}^{n-1}(n-\xi)(\theta\alpha^{\xi+q}+\gamma(\xi)\gamma(\xi-p+q)+\gamma(\xi+q)\gamma(\xi-p))\\
&\quad +\sum_{\xi=1}^{q-1}(n-\xi)(\theta\alpha^{\xi+p}+\gamma(\xi)\gamma(\xi-q+p)+\gamma(\xi+p)\gamma(q-\xi))\\
&\quad +\sum_{\xi=q}^{n-1}(n-\xi)(\theta\alpha^{\xi+p}+\gamma(\xi)\gamma(\xi-q+p)+\gamma(\xi+p)\gamma(\xi-q))\Big\}.
\end{aligned}
\tag{2.5}
$$

Subtracting $\gamma(p)\gamma(q)$ from the both sides of (2.5) and letting them $n\to\infty$ after multiplying n since $\gamma(h)=\alpha\theta^h$, we have the results.

Let

$$
V=\begin{pmatrix} V_{11} & V_{12}\\ V_{12}' & V_{22}\end{pmatrix} \tag{2.6}
$$

be the $(h+1)\times(h+1)$ dimensional matrix with

$$
\begin{aligned}
V_{11} &= \lim_{n\to\infty} nVar(\bar{X}_n),\\
V_{12} &= (\lim_{n\to\infty} nCov(\bar{X}_n,\tilde{\gamma}(p)))', \quad p=1,\ldots,h,\\
\text{and}\quad V_{22} &= \lim_{n\to\infty} nCov(\tilde{\gamma}(p),\tilde{\gamma}(q)), \quad p,q=1,\ldots,h.
\end{aligned}
$$

Then we have the following theorems.

Theorem 2.3. If $\{X_t\}$ is an INAR(1) model and X_t is according to Poisson(θ), then

$$\sqrt{n}\begin{pmatrix}\bar{X}_n\\ \tilde{\gamma}(0)\\ \tilde{\gamma}(1)\\ \vdots\\ \tilde{\gamma}(h)\end{pmatrix} \xrightarrow{\mathrm{d}} N\left(\begin{pmatrix}\theta\\ \gamma(0)\\ \gamma(1)\\ \vdots\\ \gamma(h)\end{pmatrix}, \quad V\right)$$

Proof. We first define a sequence of $(h+2)$ random vectors $\{Z_t\}$ by

$$Z_t' = (X_t^*,\, X_t^*X_t^*,\, X_t^*X_{t+1}^*, \cdots, X_t^*X_{t+h}^*),$$

where $X_t^* = \sum_{i=0}^{m} \alpha^i * W_{t-i}$. Since Z_t is a strictly stationary $(m+h)$ dependent sequence, one can show by m-dependent C.L.T. and the Cramer-Wold device that

$$\sqrt{n}\begin{pmatrix}\bar{X}_n^*\\ \tilde{\gamma}_m(0)\\ \tilde{\gamma}_m(1)\\ \vdots\\ \tilde{\gamma}_m(h)\end{pmatrix} \xrightarrow{\mathrm{d}} N\left(\begin{pmatrix}\theta_m\\ \gamma_m(0)\\ \gamma_m(1)\\ \vdots\\ \gamma_m(h)\end{pmatrix}, \quad V_m\right), \tag{2.7}$$

where $\bar{X}_n^* = \frac{1}{n}\sum_{t=1}^{n} X_t^*$, $\tilde{\gamma}_m(p) = \frac{1}{n}\sum_{t=1}^{n}(X_t^* - \theta_m)(X_{t+p}^* - \theta_m)$, $\theta_m = \theta(1-\alpha^{m+1})$, $\gamma_m(p)$ and V_m such that $\gamma_m(p) \to \gamma(p)$ and $V_m \to V$ as $m \to \infty$. Note that

$$\begin{aligned}
&Var(\sqrt{n}(\bar{X}_n - \bar{X}_n^*))\\
&= nVar\Big(\frac{1}{n}\sum_{j=1}^{n}\sum_{i=m+1}^{\infty} \alpha^i * W_{j-i}\Big)\\
&= \theta\alpha^{m+1}\Big(1 + 2\frac{\alpha(1-\alpha^{n-1})}{1-\alpha}\Big) - \frac{2}{n}\sum_{i=1}^{n-1}\alpha^i i\theta\alpha^{m+1}
\end{aligned}$$

$$\leq \theta\alpha^{m+1}\Big(1+2\frac{\alpha(1-\alpha^{n-1})}{1-\alpha}\Big).$$

Thus we have, for $\varepsilon > 0$,

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup P\left[\sqrt{n}\left|\bar{X}_n^* - \bar{X}_n - \theta_m + \theta\right| > \varepsilon\right] = 0. \tag{2.8}$$

Moreover, from the similar calculation in Lemma 2.2 and the preceding arguments, it can be shown that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup P\Big[\sqrt{n}\,|\tilde{\gamma}_m(p) - \tilde{\gamma}(p) - \gamma_m(p) + \gamma(p)| > \varepsilon\Big] = 0. \tag{2.9}$$

Hence, (2.7)–(2.9) establish the claim by an application of Proposition 6.3.9 in Brockwell and Davis(1987) since $\theta_m \to \theta$ and $V_m \to V$ as $m \to \infty$.

Next we show that, under the conditions of Theorem 2.3, $\tilde{\gamma}(p)$ and $\hat{\gamma}(p)$ have the same asymptotic distribution.

Theorem 2.4. When X_t is according to the model (1.1), then

$$\sqrt{n}\begin{pmatrix}\bar{X}_n\\ \hat{\gamma}(0)\\ \hat{\gamma}(1)\\ \vdots\\ \hat{\gamma}(h)\end{pmatrix} \xrightarrow{\mathrm{d}} N\left(\begin{pmatrix}\theta\\ \gamma(0)\\ \gamma(1)\\ \vdots\\ \gamma(h)\end{pmatrix},\quad V\right),$$

where $\hat{\gamma}(h) = n^{-1}\sum_{t=1}^{n-h}(X_t - \bar{X}_n)(X_{t+h} - \bar{X}_n)$, $\gamma(h) = E(X_t - \theta)(X_{t+h} - \theta)$ and V is defined by (2.6).

Proof. From Proposition 7.3.4 in Brockwell and Davis(1987), simple algebra gives, for $0 \leq p \leq h$,

$$\begin{aligned}
&\sqrt{n}\Big(\tilde{\gamma}(p) - \hat{\gamma}(p)\Big) \\
&= \sqrt{n}(\bar{X}_n - \theta)\Big[\frac{1}{n}\sum_{t=1}^{n-p} X_t + \frac{1}{n}\sum_{t=1}^{n-p} X_{t+p} - \Big(1 - \frac{p}{n}\Big) \\
&\times (\bar{X}_n + \theta)\Big] + \frac{1}{\sqrt{n}}\sum_{t=n-p+1}^{n} (X_t - \theta)(X_{t+p} - \theta). \qquad (2.10)
\end{aligned}$$

The last term in (2.10) is $o_p(1)$, since $n^{-1/2}E\Big|\sum_{t=n-p+1}^{n}(X_t - \theta)(X_{t+p} - \theta)\Big| \leq n^{-1/2}p\gamma(0)$ and $n^{-1/2}p\gamma(0) \to 0$ as $n \to \infty$. And the first term in (2.10) is also $o_p(1)$, since $\sqrt{n}(\bar{X}_n - \theta) = O_p(1)$ and $\bar{X} \xrightarrow{P} \theta$ by Theorem 2.3. This completes the proof.

3 Applications

3.1 Estimation of Parameters

In what follows we investigate two methods for estimating the parameters in the INAR(1) model. The methods for estimating the parameters which we mention here are the Yule-Walker type estimation and the conditional least squares estimation . We present the Yule-Walker type estimators of parameters by using the results of section 2. The conditional least squares estimation is first considered by Klimko and Nelson(1978) and applied by Al-Osh and Alzaid(1987) .

3.1.1 The Yule-Walker Type Estimators

From section 2, we can easily obtained various Yule-Walker type estimators for parameters in INAR(1) model (1.1). One of several estimators is

$$\hat{\alpha} = \hat{\rho}(1) = \frac{\sum_{t=1}^{n-1}(X_t - \bar{X}_n)(X_{t+1} - \bar{X}_n)}{\sum_{t=1}^{n}(X_t - \bar{X}_n)^2}$$

and $\hat{\theta} = \bar{X}_n$.

Theorem 3.1 Under the same assumptions given in Theorem 2.4, we have

$$\sqrt{n}\begin{pmatrix} \hat{\theta} \\ \hat{\alpha} \end{pmatrix} \xrightarrow{\mathrm{d}} N\left(\begin{pmatrix} \theta \\ \alpha \end{pmatrix}, \ W\right),$$

where $\hat{\rho}(1) = \dfrac{\hat{\gamma}(1)}{\hat{\gamma}(0)}$, $W = (w_{ij}), i,j = 1,2$, $w_{11} = \theta\dfrac{1+\alpha}{1-\alpha}$, $w_{12} =$ $w_{21} = 0$ and $w_{22} = \dfrac{\alpha(1-\alpha)}{\theta} + (1-\alpha^2)$. And hence $\hat{\theta}$ and $\hat{\alpha}$ are asymptotically independent.

Proof. The proof is straightforward by Theorem 2.4, since $\gamma(0) = \theta$, $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \alpha$ and

$$W = DVD' = \begin{pmatrix} \theta\dfrac{1+\alpha}{1-\alpha} & 0 \\ 0 & \dfrac{\alpha(1-\alpha)}{\theta} + (1-\alpha^2) \end{pmatrix},$$

where V is defined by (2.6) and

$$D = \theta^{-1}\begin{pmatrix} \theta & 0 & 0 \\ 0 & -\alpha & 1 \end{pmatrix}.$$

3.1.2 The Conditional Least Squares Estimators

Al-Osh and Alzaid(1987) applied to the conditional least squares method for estimation of parameters in the INAR(1) model which was developed by Klimko and Nelson(1978). It is based on minimization of the sum of squared deviations about the conditional expectation

$$E(X_t/X_{t-1}) = \alpha X_{t-1} + (1-\alpha)\theta.$$

They estimate α and θ by trying to minimize the conditional sum of squares

$$C_n(\alpha,\theta) = \sum_{t=1}^{n} [(X_t - E(X_t/X_{t-1})]^2$$

with respect to α and θ. The estimators are

$$\hat{\alpha}_c = \frac{\sum_{t=1}^{n} X_t X_{t-1} - n^{-1}(\sum_{t=1}^{n} X_t \sum_{t=1}^{n} X_{t-1})}{\sum_{t=1}^{n} {X_{t-1}}^2 - n^{-1}(\sum_{t=1}^{n} X_{t-1})^2},$$

$$\hat{\theta}_c = (1-\hat{\alpha}_c)n^{-1}(\sum_{t=1}^{n} X_t - \hat{\alpha}_c \sum_{t=1}^{n} X_{t-1}).$$

and by Theorem (3.2) of Klimko and Nelson(1978), $(\hat{\alpha},\hat{\theta})$ are asymptotically normally distributed as

$$\sqrt{n}\begin{pmatrix}\hat{\alpha}_c\\ \hat{\theta}_c\end{pmatrix} \xrightarrow{\mathrm{d}} N\left(\begin{pmatrix}\alpha\\ \theta\end{pmatrix},\quad V^{-1}WV^{-1}\right),$$

where the limiting covariance matrix $V^{-1}WV^{-1}$ is

$$\begin{pmatrix} \frac{\alpha(1-\alpha)}{\theta} + (1-\alpha^2) & -\alpha - 2\theta(1+\alpha) \\ -\alpha - 2\theta(1+\alpha) & \frac{1}{1-\alpha}\theta(4\theta+1)(\alpha+1) \end{pmatrix}.$$

Note that as we expect, the marginal limiting distribution of the conditional least squares estimator $\hat{\alpha}_c$ is the same as that of our estimator $\hat{\alpha}$ but our estimator $\hat{\theta}$ is better than $\hat{\theta}_c$ from the comparison of the variances.

3.2 Relationship between INAR(1) and $M/M/\infty$ Queueing System

The time series processes have already been used for modelling in queuing problems. Steudel and Wu(1977) show that the queue behavior of a uniformly sampled queueing system with a single server and Poisson-exponential activities($M/M/1$) is adequately described by an AR(1) model. McKenzie(1988) investigated that the Poisson AR(1) is in fact the $M/M/\infty$ queueing observed at regularly spaced intervals of time. This was noted also by Steutel et el.(1983). McKenzie(1988) also shows that the Poisson MA(q) is the $M/D/\infty$ queueing system.

In this section, we will investigate the relationship between the INAR(1) model and $M/M/\infty$ queueing process in equilibrium.

Let Q_n be the queue length at time n in a $M/M/\infty$ queueing system with arrival rate μ_1 and service rate μ_2 and $\rho = \mu_1/\mu_2$. Then,

$$Q_n = Q_{n-1} + \lambda_n - \mu_n, \quad n = 1, 2, \ldots \tag{3.1}$$

where Q_{n-1}, λ_n and μ_n represent the number of customers not completing their service at time $n-1$, the number of new customers arriving at a system during the time interval $(n-1, n]$ and the number of all customers completing their service during the time interval $(n-1, n]$, respectively. Since μ_n depends on $Q_{n-1} + \lambda_n$, we may decomposite μ_n as $\mu_{n-1}^Q + \mu_n^\lambda$, where $\mu_{n-1}^Q, \mu_n^\lambda$ depends on only Q_{n-1}

and λ_n, respectively. That is, equation (3.1) can be represented by the following form

$$Q_n = Q_{n-1} - \mu^Q_{n-1} + \lambda_n - \mu^\lambda_n. \tag{3.2}$$

We may define $Z = Q_{n-1} - \mu^Q_{n-1}$ and $Y = \lambda_n - \mu^\lambda_n$. Obviously, from the structure of (3.2), $Z \geq 0$ and $Y \geq 0$ *a.s.* and by Poisson thinning process, Z and Y are independent. Then we can show that the distribution of Z is as follows. Let T_i be the remaining service time for customer i during the time interval $(n-1, n]$. By the memoryless property of the exponential distribution, T_i has an exponential distribution with parameter μ_2. Hence one can show that by independence of T_i for $i = 1, 2, \cdots, Q_{n-1}$,

$$\begin{aligned}
P(Z = k) &= \sum_{m=k}^{\infty} \binom{m}{k} P(T \leq 1)^{m-k} P(T > 1)^k P(Q_{n-1} = m) \\
&= \sum_{m=k}^{\infty} \frac{m!}{(m-k)!k!} (1 - e^{-\mu_2})^{m-k} (e^{-\mu_2})^k \frac{\rho^m e^{-\rho}}{m!} \\
&= \frac{(\rho e^{-\mu_2})^k e^{-\rho e^{-\mu_2}}}{k!}, \quad k = 0, 1, 2, \ldots.
\end{aligned}$$

where $P(Q_{n-1} = m) = \frac{e^{-\rho}\rho^m}{m!}$ is the equilibrium distribution of $M/M/\infty$ queueing system. This implies that Z and Y are independent Poisson random variables with parameters $\rho e^{-\mu_2}$ and $\rho(1 - e^{-\mu_2})$, respectively. Thus the previous arguments imply that the queue length process in $M/M/\infty$ system can be represented as INAR(1) model (1.1) when $\alpha = e^{-\mu_2}$ and $\theta = \rho$.

We now investigate asymptotic behavior of estimators in $M/M/\infty$ queueing process with $\bar{Q}_n = \frac{1}{n}\sum_{i=1}^n Q_i$.

Theorem 3.2 Let Q_n be the queue length at time n of $M/M/\infty$ in equilibrium with arrival rate μ_1 and service rate μ_2. Then an estimator of (μ_1, μ_2) is $(\hat{\mu_1} = \bar{Q}_n\hat{\alpha}, \hat{\mu_2} = -log\hat{\alpha})$ and

$$\begin{pmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \end{pmatrix} \xrightarrow{\mathrm{d}} N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \ \Sigma\right),$$

where $\hat{\alpha} = \dfrac{\sum_{i=1}^{n-1}(Q_i - \bar{Q_n})(Q_{i+1} - \bar{Q_n})}{\sum_{i=1}^{n}(Q_i - \bar{Q_n})^2}$,

$\sum = (\sigma_{ij}), \quad i,j = 1,2, \quad \sigma_{11} = \theta \log\alpha(\dfrac{1+\alpha}{1-\alpha} + 2) + \dfrac{\theta(1-\alpha)}{\alpha^2}(\alpha + \theta(1+\alpha)), \sigma_{12} = \sigma_{21} = \log\alpha + \dfrac{1-\alpha}{\alpha^2}(\alpha + \theta(1+\alpha)), \sigma_{22} = \dfrac{1-\alpha}{\alpha^2\theta}(\alpha + \theta(1+\alpha))$.

Proof. Since $E(Q_n) = \rho = \mu_1/\mu_2$ and $Cov(Q_n, Q_{n-1}) = \alpha = e^{-\mu_2}$, a moment estimator of (μ_1, μ_2) is $(\hat{\mu_1}, \hat{\mu_2})$ and the asymptotic normality of $(\hat{\mu_1}, \hat{\mu_2})$ is immediate by Theorem 3.1.

Remark: From Theorem 3.2, one can easily find the estimator of (μ_1, μ_2) and its asymptotic distribution by only observing equal-time spaced queue lengths in $M/M/\infty$ system.

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