NORMAN DANCER

University of Sydney

COMPETING SPECIES SYSTEMS WITH DIFFUSION AND LARGE INTERACTIONS

Conferenza tenuta il 31 marzo 1995

ABSTRACT. We discuss non-negative solutions of a Lotka-Volterra competing species system which includes the effect of diffusion. We discuss when the populations coexist, and secondly the behaviour of the system when the interaction between the systems are large. The limiting problems here raise interesting questions for scalar equations.

In this paper, we discuss the system

$$\begin{aligned} -\Delta u &= u(a-u-cv) \\ -\Delta v &= v(d-v-eu) \quad \text{in } \Omega \\ u = v &= 0 \qquad\qquad \text{on } \partial\Omega\,. \end{aligned} \tag{1}$$

We are interested in non-negative solutions. Here Ω is a bounded domain in R^n with smooth boundary and $a, c, d, e > 0$. In fact, we will mainly be interested in the case where c, e are large. The system arises for the time independent solutions of the parabolic system

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= u(a-u-cv) \\ s\frac{\partial v}{\partial t} - \Delta v &= v(d-v-eu) \quad \text{in } \Omega \times [0,\infty) \\ u = v &= 0 \qquad\qquad \text{on } \partial\Omega \times [0,\infty)\,. \end{aligned} \tag{2}$$

Here $s > 0$. This system is a simple two species Lotka-Volterra population model where we incorporate the effect of the diffusion of the species in the domain: $u(x)$ represents the population of the species u at the position x while $v(x)$ represents the population of the species v at x. We will usually take Dirichlet boundary conditions though much of the theory also applies to Neumann boundary conditions. We will comment explicitly in a few cases where there are noticeable differences. Note that since u and v represent populations it is natural to assume they are non negative. It turns out that this simple system has surprisingly rich behaviour. Note that c and e being large corresponds to the two species each interacting strongly with the other species.

We say that a non-negative solution of (1) is stable if it is stable as a solution of the parabolic system (2). Technically, we should specify the space the equation is defined on. For example, we could use $L^p(\Omega) \oplus L^p(\Omega)$ for suitably large but finite p. Instability and asymptotic stability are defined analogously.

We will always assume that $a, d > \lambda_1$, where λ_1 denotes the smallest eigenvalue of $-\Delta$ on Ω for Dirichlet boundary conditions. The reason for this assumption is that otherwise (1) and (2) are rather uninteresting. For example, if $a \leq \lambda_1$, it is easy to show that any non-negative solution of (1) satisfies $u \equiv 0$ while any solution of (2) with non-negative initial values satisfies $u(t) \to 0$ as $t \to \infty$. For both (1) and (2) it is then quite easy to obtain a rather complete understanding of v as well.

If we look for solutions of (1) with $v = 0$, it is easy to see that

$$\begin{aligned} -\Delta u &= u(a-u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega\,. \end{aligned} \tag{3}$$

It is well known and easy to prove that there is a unique non-trivial non-negative solution of (3) denoted by $\bar{u}$. (This uses $a > \lambda_1$.) Thus the non-negative solutions with second component zero are $(0,0)$ and $(\bar{u}, 0)$. By a similar argument if we look for non-negative solutions with first component zero, we obtain exactly two $(0,0)$ and $(0,\bar{v})$. The solutions $(0,0)$, $(\bar{u},0)$ and $(0,\bar{v})$ are usually called semitrivial solutions. It is easy to use the maximum principle to show that any other non-negative solution (u, v) of (1) has the property that $u(x) > 0$ and $v(x) > 0$ on all of Ω. We call these positive solutions. They correspond to populations where both species coexist. In general, it seems difficult to say when a positive solution exists, when it is unique and what the dynamical behaviour of (2) is, though a great deal is known. See [4] and [5] and [22] where many further references can be found.

For the remainder of this paper, we consider the special case when c and e are both large. In this case it is known that there is a positive solution and the question is on the number, stability and asymptotic behaviour of the positive solutions. Note that one can prove the existence of a positive solution by applying degree theory in the cone K of non-negative functions in $L^p(\Omega)\oplus L^p(\Omega)$. The key point is to note that, if c and e are large, $(\bar{u},0)$ and $(0,\bar{v})$ are stable solutions of (2) and this can be used to calculate the indices of $(\bar{u},0)$ and $(0,\bar{v})$ as fixed points of (1) in K. Here we use the main result in [8] to calculate the indices of $(\bar{u},0)$ and $(0,\bar{v})$ relative to K. It turns out that degree theory in cones is a very useful tool for studying the existence of positive solutions of (1) (cp [5]).

If c and e are large, the first question to ask is what is the asymptotic behaviour of solutions. For simplicity, we assume that $ce^{-1} \to 1$ as $e \to \infty$ though our methods could handle cases where $ce^{-1} \to \alpha \in [0,\infty]$ as $e \to \infty$. Then it turns out [9] that there are three possibilities for positive solutions. We assume that $(u_{i_1} v_i)$ are positive solutions of (1) for $c = c_i$, $e = e_i$, where $e_i \to \infty$ and $c_i/e_i \to 1$ as $i \to \infty$. Then, after taking subsequences, one of the following holds:

(i) $e_i u_i \to \bar{u}$ and $c_i v_i \to \bar{v}$ in $L^\infty(\Omega)$ as $i \to \infty$ where $(\bar{u},\bar{v})$ is a positive solution of

$$\begin{aligned} -\Delta u &= u(a-v) \\ -\Delta v &= v(d-u) \quad \text{in } \Omega \\ u = v &= 0 \qquad \text{on } \partial\Omega\,. \end{aligned} \tag{4}$$

(ii) $u_i \to w_0^+$ and $v_i \to -w_0^-$ in $L^p(\Omega)$ for all $p < \infty$ as $i \to \infty$ where w_0 is non-trivial and changes sign and w_0 solves

$$\begin{aligned} \Delta w &= aw^+ + dw^- - |w|w \quad \text{in } \Omega \\ w &= 0 \qquad\qquad \text{on } \partial\Omega\,, \end{aligned} \tag{5}$$

(where $w = w^+ + w^-$).

(iii) $e_i\|u_i\|_\infty \to \infty$, $c_i\|v_i\|_\infty \to \infty$, $\|u_i\|_\infty + \|v_i\|_\infty \to 0$, $(\|u_i\|_\infty)^{-1}u_i \to w_1^+$, $(\|v_i\|_\infty)^{-1}v_i \to -w_1^-$ in $L^p(\Omega)$ for all $p < \infty$ as $i \to \infty$ where w_1 is a non-trivial changing sign solution of

$$\begin{aligned} \Delta w &= aw^+ + dw^- && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega , \end{aligned} \tag{6}$$

We discuss each of these types of solutions. Firstly, we discuss sulutions of type (i). We firstly note that there is a converse result. If $(\bar{u}, \bar{v})$ is a non-degenerate (that is, the linearization is invertible) positive solution of (4), then for all large c and e there is a (locally) unique positive solution of (1) with eu near $\bar{u}$ and cv near $\bar{v}$. Note that these solutions of (1) are small solutions. If (6) has only the trivial solution, it can be shown that there is an a priori bound in $L^\infty(\Omega) \oplus L^\infty(\Omega)$ for the positive solutions of (4) and the sum of their fixed point indices is $\text{index}_{L^\infty(\Omega)}(\tilde{B}_1, 0)$ where $\tilde{B}_1 w = (-\Delta)^{-1}(aw^+ + dw^-)$. Here the fixed point indices for the system are indices relative to the natural cone K of non-negative functions in $L^\infty(\Omega) \oplus L^\infty(\Omega)$. This is the first of a number of rather surprising connections between the competing species system with large interactions and so called jumping nonlinearities. The latter have been extensively studied for quite different reasons. Further references on them can be found in [1], [14], [15], [17] and [23]. However, it is unclear if (4) always has a positive solution (for $a, d > \lambda_1$) even if we assume that (6) has only the trivial solution. (Note that $\text{index}_{L^\infty(\Omega)}(\tilde{B}_1, 0)$ may well be zero.) In the case of Neumann boundary conditions, there is a noticeable difference in the theory because the natural analogue of (4) always has a simple constant solution. Note also that (4) may sometimes have several positive solutions. Lastly, the solution of type (i) of (1) can always be shown to be unstable solutions of (2).

We now discuss solutions of type (iii). There is a problem here (and this also affects cases (i) and (ii)). (6) is not well understood though there are many partial results. See the references mentioned above. For example, it has been conjectured that $A_0 = \{(a, d) \in R^2 : (6 \text{ has a non-trivial solution}\}$ has empty interior. (This has been proved for generic domains in [13].) Assuming this, we see that case (iii) is an unusual case. Note however that as we vary a or d so that $\text{index}_{L^\infty(\Omega)}(\tilde{B}_1, 0)$ changes solutions of this type must occur. (This then corresponds to a change of solutions from type (i) to type (ii)). It can be shown [11] that solutions of this type are always unstable.

Lastly, we consider solutions of type (ii). This is the most interesting case. Firstly, note that solutions of type (ii) correspond to the two species u and v largely segregating to mostly "live" on different parts of the domain. This seems to have biological interest. We first consider when (5) has changing sign solutions. To do this, we need to consider some results on the structure of A_0. It was proved in [1] (see also [16]) that there is a continuous strictly decreasing curve T in $(\lambda_1, \infty) \times (\lambda_1, \infty)$ such that $(\lambda_2, \lambda_2) \in T, T$ contains points with a arbitrarily large, T symmetric for reflection in the line $a = d$, $T \subseteq A_0$ and A_0 does not intersect the component W of $((\lambda_1, \infty) \times (\lambda_1, \infty)) \backslash T$ containing $\left(\frac{1}{2}(\lambda_1 + \lambda_2), \frac{1}{2}(\lambda_1 + \lambda_2)\right)$. Geometrically, W is the set of points in $(\lambda_1, \infty) \times (\lambda_1, \infty)$ "below" T. Then it can be proved [9] that (5) has a changing sign solution if and only if $(a, d) \in (\lambda_1, \infty) \times (\lambda_1, \infty) \backslash (W \cup T)$ (geometrically, (a, d) is "above" T). This is a variational argument. The proof shows that if $(a, d) \notin A_0$, zero is a solution of mountain pass type of (5) if and only if $(a, d) \in W$. (Here we mean mountain pass in the sense of [19].) The solutions that we find here are obtained by variational arguments and hence it is not clear they will necessarily yield solutions of (1). (Note that (5) has a variational structure but (1) does not seem to.) However, if (5) has only finitely many solutions and if (a, d) is above T, a mountain pass type argument implies that (5) has a changing sign solution $\tilde{w}$ of fixed point index -1. One can then use a degree argument to prove that if c, e are both large and ce^{-1} is close to 1, then there is a positive solution (u, v) of (1) with u close to $\tilde{w}^+$ and v close to $-\tilde{w}^-$ in $L^p(\Omega)$. It is also (locally) unique if $\tilde{w}$ is non-degenerate. These arguments do not use that $\tilde{w}$ is of mountain-pass type but only that it has non-zero degree. Note that, by a result in [13] (which is an improvement of [25]), for generic Ω, all non trivial solutions of (5) are non-degenerate. Note also that one could use Morse theory arguments to show that (5) has at least two non-trivial changing sign solutions in many cases (as in [10]). Thus we usually obtain multiple solutions of (1) in this way. Finally note that if $a = d$ is large a result of Clark [3] implies that (5) has many changing sign solutions and thus, usually in his case, (1) has many positive solutions (for c, e large, c/e close to 1).

A more interesting and difficult question is when (5) has a stable changing sign solution (stable for the natural corresponding parabolic equation). This is of interest because, if w is a non-degenerate solution of (5), then the corresponding solution of (1) of type (ii) (for c, e large and c/e near 1) are stable (as solutions of (2)) if and only if w is stable. This is proved in [11]. Note that this is the only way in which we obtain stable positive solutions of (1) for c, e large (and c/e close to 1). Thus it is of considerable interest to decide when (5) has a stable sign changing solution. This seems a difficult problem. Here, for once, it is interesting to study the case where c and e are large and c/e is close to $\alpha \in [0, \infty]$. In this case there is a completely analogous theory where the limiting equation (5) is replaced by

$$-\Delta u = au^+ du^- - \alpha(u^+)^2 + (u^-)^2 \quad \text{in } \Omega \tag{7}$$

with Dirichlet boundary conditions. (The equation needs to be changed slightly if $\alpha = \infty$). If α is large or small (including $\alpha = 0$) it is proved in [12] that (7) has no stable sign changing solution and hence by [11] it follows that (1) has no stable positive solution for c, e large with c/e close to α. It can also be proved that (7) never has a sign changing solution if Ω is a ball or an annulus (and thus never if $n = 1$). There are a number of other results on the non-existence of sign changing stable solutions of (5) in [12]. Unfortunately, they show that some of the standard techniques for obtaining sign changing solutions from positive solutions usually yield unstable solutions.

On the other hand, there are three known methods for sometimes constructing sign changing stable positive solutions; by domain variation arguments (as in [7]), by minimization and by singular perturbation (Γ convergence) methods. We discuss briefly the second and third of these.

Rather surprisingly, it can turn out that the global minimizer of the natural functional corresponding to (7) (which is necessarily stable) may sometimes change sign if the non-linearity is not odd. Of course, if this occurs, we have stable changing sign solutions. That this can occur was first observed by Sweers [26] for a closely related problem by choosing smooth domains approximating a domain with a corner. In fact with care, one can show that this can even occur with Ω strongly convex in R^2 with two axes of symmetry. These examples have α small and d large. Note that the global minimum must preserve the symmetries. Sweers (personal communication) pointed out an easy proof of this, which answers a question in [12]. Note that the Neumann problem behaves somewhat different since it is known [2] and [20] that both (1) and (7) (with Neumann boundary conditions) have no stable non-constant positive solutions when Ω is convex.

Another approach to find stable positive solutions of (7) is to study

$$-\epsilon\Delta u = au^+ + du^- - \alpha(u^+)^2 + (u^-)^2 \quad \text{in } \Omega \tag{8}$$

with Dirichlet boundary conditions. Here ϵ is positive and small. Note that this is a special case of (7) (after rescaling). We assume that $\alpha^3\alpha^{-1} = d^3$ (which corresponds to certain areas under the integral of the nonlinearity being equal). We assume this condition for the remainder of this article. (Indeed, if this fails and if Ω is convex, we conjecture that (8) has no stable sign changing solution for small positive ϵ.) In this case Γ convergence ideas ([21] and [12]) imply that there is a natural limit problem as $\epsilon \to 0$ of the form

$$c_1 \operatorname{Per}_\Omega\{x : u(x) = a\alpha^{-1}\} + c_2 H_{n-1}\{x \in \partial\Omega : u(x) = a\alpha^{-1}\}$$

where $\operatorname{Per}_\Omega$ is the perimeter in the sense of [18], we are loking at functions u of bounded variation in Ω such that $u(x) \in \{-d, a\alpha^{-1}\}$ $a.e.$ in Ω and H_{n-1} is $n-1$ dimensional Hausdorff measure. Note that since u is of bounded variation, u has a trace on $\partial\Omega$ by [28]. c_1 and c_2 are constants determined by integrals of the nonlinearity. $c_1 > 0$, $|c_2| < c_1$ and any other c_1 and c_2 can occur (for a suitable nonlinearity).

It can be shown [21] that any isolated local minimizer of this limit problem generates local minimizers of our original problem for small ϵ. These solutions have transition layers near the part of the boundary of $\{x \in \Omega : u(x) = a\alpha^{-1}\}$ in the interior of Ω. Thus it is of considerable interest to study this limit problem. Unfortunately, it does not seem easy. If $n \leq 7$, a result of [27] implies, for a local minimizer, the boundary T of $\{x \in \Omega : u(x) = ax^{-1}\}$ is smooth in the interior of Ω and thus this part of T has mean curvature zero. Thus, if $n = 2$, this boundary consists of straight lines in the interior of Ω. However, even knowing this, it does not seem easy to understand the local minimizers of the limit problem when $n = 2$. This is, in part, a geometric problem (when we look for local minimizers amongst those u where the transition layers are straight lines). The corresponding problem in the Neumann case is much easier to analyze because, in this case, $c_2 = 0$.

References

[1] D'Aujourd'hui, M., *Sur l'ensemble de resonance d'un problem demi-lineaire*, Ecole Polytechnique de Lausanne, preprint.

[2] Casten, R. and Holland, C., *Instability results for reaction diffusion equations with Neumann boundary conditions*, J. Diff. Eqns., **27** (1978), 266–273.

[3] Clark, D., *A variant of the Lusternik-Schnirelman theory*, Indiana U. Math J., **22** (1972), 65–74.

[4] Cosner, C. and Lazer, A., *Stable coexistence in the Volterra-Lotka competition model with diffusion*, SIAM J. Math. Anal., **44** (1984), 1112–1132.

[5] Dancer, E.N., *On the existence and uniqueness of positive solutions for competing species equations with diffusion*, Trans. Amer. Math. Soc., **326** (1991), 829–859.

[6] Dancer, E.N., *On positive solutions of some pairs of differential equations II*, J. Diff. Eqns., **60** (1985), 236–258.

[7] DANCER, E.N., *The effect of domain shape on the number of positive solutions of certain nonlinear equations*, J. Diff. Eqns., **74** (1988), 120–156.

[8] DANCER, E.N., *On the indices of fixed points of mappings in cones and applications*, J. Math. Anal. Appl., **91** (1983), 131–151.

[9] DANCER, E.N. and YIHONG DU, *Competing species equations with diffusion, large interactions and jumping nonlinearities*, J. Diff. Eqns., **114** (1994), 434–475.

[10] DANCER, E.N. and YIHONG DU, *Existence of changing sign solutions for some semilinear problems with jumping nonlinearities at zero*, Proc. Royal Soc. Edimburgh, **124a** (1994), 1165–1176.

[11] DANCER, E.N. and ZONGMING GUO, *Uniqueness and stability for solutions of competing species equations with large interactions*, Applied Nonlinear Anal., **1** (1994), 19–45.

[12] DANCER, E.N. and ZONGMING GUO, *Some remarks on the stability of sign changing solutions*, Tohoku Math. J., **47** (1995), 199–225.

[13] DANCER, E.N., *Generic domain dependence for non-smooth equations and the open set problem for jumping nonlinearities*, Top. Methods nonlinear Anal., **1** (1993), 139–150.

[14] DANCER, E.N., *On the Dirichlet problem for weakly nonlinear elliptic partial differential equations*, Proc. Royal Soc. Edimburgh, **76a** (1977), 283–300.

[15] DANCER, E.N., *On the existence of solutions of certain asymptotically homogeneous problems*, Math. Zeit, **177** (1981), 33–48.

[16] DE FIGUEIREDO, D.G. and GOSSEZ, J.P., *On the first curve of the Fucik spectrum of an elliptic operator*, Diff. Integral Eqns., **7** (1994), 1285–1302.

[17] FUCIK, S., Solvability of nonlinear equations and boundary value problems, Reidel, Dordrecht, 1980.

[18] GIUSTI, E., Minimal surfaces and functions of bounded variation, Birkhauser, Boston, 1982.

[19] HOFER, H., *A note on the topological degree of a critical point of mountain pass type*, Proc. Amer. Mat. Soc., **90** (1984), 309–315.

[20] KISHIMOTO, K. and WEINBERGER, H., *The spatial homogeneity of stable equilibrium of some reaction-diffusion systems on convex domains*, J. Diff. Eqns., **58** (1985), 15–21.

[21] KOHN, R. and STERNBERG, P., *Local minimizers and singular perturbations*, Proc. Royal Soc. Edimburgh, **111a** (1989), 69–94.

[22] KORMAN, P. and LEUNG, A., *On the existence and uniqueness of positive steady states in Lotka-Volterra ecological models with diffusion*, Appl. Anal., **26** (1987), 145–160.

[23] MARINO, A., MICHELETTI, A. and PISTOIA, A., *A non symmetric asymptotically linear elliptic problem*, preprint.

[24] MATANO, H., *Asymptotic behaviour and stability of solutions of semilinear diffusion equations*, Publ. RIMS Kyoto University, **15** (1979), 401–454.

[25] SAUT, J. and TEMAN, R., *Generic properties of nonlinear boundary value problems*, Comm. Partial Diff. Eqns., **4** (1979), 293–319.

[26] SWEERS, G., *A sign changing local minimizer on a convex domain*, pp. 251–258 in Progress in Partial Differential Equations, Ed. C. Bandle, Pitman, Harlow, 1992.

[27] TAMANINI, I., *Regularity results for almost minimal oriented hypersurfaces in R^N*, Quaderni del Dipartimento di Matematica dell'Università di Lecce, 1984.

[28] ZIEMER, W., Weakly differentiable functions, Springer-Verlag, New York, 1989.

Norman Dancer
School of Mathematics and Statistics
University of Sydney
N.S.W. 2006, Australia

Pervenuta in Redazione il 5 marzo 1996