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NBUFR closure under the formation of coherent systems

Wolfgang Gohout, Ingo Kuhnert

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Closure under the operation of forming coherent systems of independent components is a desirable property for a class of life distributions. We show that the NBUFR class possesses this property.

1 Introduction

Nonparametric classes of distributions enjoy a certain popularity in reliability theory since there often is a lack of sufficient data for choosing an appropriate parametric distribution model. The best-known classes in this context are IFR, IFRA, NBU and NBUE, though many more have been proposed. A recent overview is given in Kuhnert/Gohout (1993).

Amongst others, closure properties under various reliability operations such as convolution, mixture or formation of systems are of interest. Closure under the formation of coherent systems is a particularly desirable property as it permits the assessment of system life length based on knowledge of component life lengths. For these, information is generally more readily obtainable than for the often complex and/or novel system.

Of the four classes mentioned above, only IFRA and NBU are closed under the formation of systems (Barlow/Proschan 1975, p.85 and p.182). The only further classes known to us which possess this property are NBU-t_0 (Hollander/Park/Proschan 1986) and NBUFRA (Loh 1984). In the following, we are going to show that the NBUFR class of life distributions also belongs to this special group of distribution classes.

2 Notation and Definitions

T, T_i	nonnegative random variable describing the life length of a system / component i, $i = 1, \ldots, n$
$F, f,\ F_i, f_i$	distribution and density function of T / T_i
R, R_i	survival function, i.e. reliability of the system / component i
g	system failure rate
x_i	state of component i (1 = working, 0 = failed)
$\mathbf{x}$	vector of component states $(x_1, x_2, \ldots, x_n)$
$(0_i, \mathbf{x}_{\bar{i}})$	$(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$
$(1_i, \mathbf{x}_{\bar{i}})$	$(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$
$\phi(\mathbf{x})$	system structure function (1 = working, 0 = failed)
	ϕ is said to be coherent if it is non-decreasing in each argument x_i and not constant in any argument. For given ϕ, $\{\mathbf{x} \mid \phi(\mathbf{x}) = 1\}$ is the set of path vectors of the system.

The following definition is offered by Desphande/Kochar/Singh (1986), though originality is claimed by Abouammoh/Ahmed (1988):

Definition:
A distribution F is NBUFR (New better than used in failure rate) if

$$g(0) \leq g(t), \quad 0 \leq t < \infty.$$

This is a straightforward variation of the NBAFR class proposed by Loh (1984). Desphande/Kochar/Singh (1986) rename NBAFR to NBUFRA, which better suits the general nomenclature of distribution classes:

Definition:
A distribution F is NBUFRA (New better than used in failure rate average) if

$$g(0) \leq \bar{g}(t) = 1/t \int_0^t g(u)\, du, \quad 0 \leq t < \infty.$$

Dual classes of negative aging can be defined by reversing the direction of inequality in the above definitions. In a certain sense, NBUFR is related to NBUFRA as is IFR to IFRA (cf. Kuhnert/Gohout (1993) for details).

3 Main Result

Theorem: The NBUFR class is closed under the formation of coherent systems of independent components.

Proof:
The reliability of a system may be computed as

$$R(t) \quad = \quad \sum_{\mathbf{x}\in\{0,1\}^n} \phi(\mathbf{x}) \cdot \prod_{i=1}^{n} R_i^{x_i}(t)\, F_i^{1-x_i}(t),$$

that is, by summing the probability for all component state vectors which enable the system to function (Barlow/Proschan 1975, p. 25). Introducing the abbreviation

$$\mathrm{P}_t(\mathbf{x}) := \phi(\mathbf{x}) \cdot \prod_{i=1}^{n} R_i^{x_i}(t)\, F_i^{1-x_i}(t)$$

gives

$$R(t) \quad = \quad \sum_{\mathbf{x}\in\{0,1\}^n} \mathrm{P}_t(\mathbf{x}) \quad = \quad \sum_{\{\mathbf{x}|\phi(\mathbf{x})=1\}} \mathrm{P}_t(\mathbf{x}). \tag{1}$$

$\mathrm{P}_t(\mathbf{x})$ is the probability that at time t, the component state vector takes on the value $\mathbf{x}$ if $\mathbf{x}$ is a path vector, and zero if not.

For the sake of brevity, the time variable t will be omitted as long as no confusion can arise. Nevertheless, it should be clear that f, F, R, g, f_i, F_i and R_i are functions of t. The density function of the time to system failure results by applying the chain rule of differentiation:

$$\begin{aligned}
f(t) &= -\frac{d}{dt}R(t) \\
&= \sum_{\mathbf{x}\in\{0,1\}^n} \sum_{i=1}^{n} \left(x_i R_i^{x_i-1} f_i F_i^{1-x_i} - (1-x_i) F_i^{-x_i} f_i R_i^{x_i} \right) \cdot \phi(\mathbf{x}) \prod_{j\neq i} R_j^{x_j} F_j^{1-x_j} \\
&= \sum_{\mathbf{x}\in\{0,1\}^n} \sum_{i=1}^{n} (2x_i - 1) \cdot f_i \cdot \phi(\mathbf{x}) \prod_{j\neq i} R_j^{x_j} F_j^{1-x_j} \\
&= \sum_{i=1}^{n} \sum_{\{\mathbf{x}|\phi(\mathbf{x})=1\}} (2x_i - 1) \cdot \frac{f_i}{R_i} \cdot \left(\frac{R_i}{F_i}\right)^{1-x_i} \cdot \mathrm{P}_t(\mathbf{x}) \\
&= \sum_{i=1}^{n} \sum_{\mathbf{x}\in C_i \cup D_i^0 \cup D_i^1} (2x_i - 1) \cdot \frac{f_i}{R_i} \cdot \left(\frac{R_i}{F_i}\right)^{1-x_i} \cdot \mathrm{P}_t(\mathbf{x}) \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad C_i &:= \{\mathbf{x}|\phi(\mathbf{x}) = 1 \;\wedge\; x_i = 1 \;\wedge\; \phi(0_i, \mathbf{x}_{\bar{i}}) = 0\}, \\
D_i^0 &:= \{\mathbf{x}|\phi(\mathbf{x}) = 1 \;\wedge\; x_i = 0 \;\wedge\; \phi(0_i, \mathbf{x}_{\bar{i}}) = 1\}, \\
D_i^1 &:= \{\mathbf{x}|\phi(\mathbf{x}) = 1 \;\wedge\; x_i = 1 \;\wedge\; \phi(0_i, \mathbf{x}_{\bar{i}}) = 1\}.
\end{aligned}$$

C_i, D_i^0, D_i^1 are mutually exclusive and $C_i \cup D_i^0 \cup D_i^1 = \{\mathbf{x} | \phi(\mathbf{x}) = 1\}$ for each component i. Note that C_i is the set of path vectors in which component i is functioning critically, that is, failure of i will cause the system to fail, D_i^1 is the set of path vectors in which i is functioning uncritically, and D_i^0 is the set of path vectors in which i is failed, thus of course being uncritical as well.

From monotonicity of ϕ, it follows that

$$(0_i, \mathbf{x}_{\bar{i}}) \in D_i^0 \;\Leftrightarrow\; (1_i, \mathbf{x}_{\bar{i}}) \in D_i^1.$$

Those terms in (2) belonging to $(0_i, \mathbf{x}_{\bar{i}}) \in D_i^0$ and $(1_i, \mathbf{x}_{\bar{i}}) \in D_i^1$ compensate each other exactly, and so do the sums over D_i^0 and D_i^1:

$$\begin{aligned}
& \sum_{\mathbf{x}\in D_i^0} (2x_i - 1) \cdot \frac{f_i}{R_i} \cdot \left(\frac{R_i}{F_i}\right)^{1-x_i} \cdot \mathrm{P}_t(\mathbf{x}) \;+\; \sum_{\mathbf{x}\in D_i^1} (2x_i - 1) \cdot \frac{f_i}{R_i} \cdot \left(\frac{R_i}{F_i}\right)^{1-x_i} \cdot \mathrm{P}_t(\mathbf{x}) \\
=\; & \sum_{\mathbf{x}\in D_i^0} -\frac{f_i}{R_i} \cdot \left(\frac{R_i}{F_i}\right)^{1} \cdot \mathrm{P}_t(0_i, \mathbf{x}_{\bar{i}}) \;+\; \sum_{\mathbf{x}\in D_i^1} \frac{f_i}{R_i} \cdot \left(\frac{R_i}{F_i}\right)^{0} \cdot \mathrm{P}_t(1_i, \mathbf{x}_{\bar{i}}) \\
=\; & \sum_{\mathbf{x}\in D_i^0} -\frac{f_i}{F_i} \cdot \mathrm{P}_t(0_i, \mathbf{x}_{\bar{i}}) \;+\; \sum_{\mathbf{x}\in D_i^1} \frac{f_i}{R_i} \cdot \mathrm{P}_t(1_i, \mathbf{x}_{\bar{i}}) \\
=\; & \sum_{\mathbf{x}\in D_i^0} -\frac{f_i}{F_i} \cdot \mathrm{P}_t(0_i, \mathbf{x}_{\bar{i}}) \;+\; \sum_{\mathbf{x}\in D_i^1} \frac{f_i}{R_i} \cdot \mathrm{P}_t(0_i, \mathbf{x}_{\bar{i}}) \cdot \frac{R_i}{F_i} \\
=\; & 0.
\end{aligned}$$

Thus

$$\begin{aligned}
f(t) &= \sum_{i=1}^{n} \sum_{\mathbf{x}\in C_i} (2x_i - 1) \cdot \frac{f_i}{R_i} \cdot \left(\frac{R_i}{F_i}\right)^{1-x_i} \cdot \mathrm{P}_t(\mathbf{x}) \\
&= \sum_{i=1}^{n} \sum_{\mathbf{x}\in C_i} \frac{f_i(t)}{R_i(t)} \cdot \mathrm{P}_t(\mathbf{x}). \qquad (3)
\end{aligned}$$

Combining (1) and (3), the system failure rate is

$$g(t) = \frac{f(t)}{R(t)} = \frac{\sum_{i=1}^{n} \sum_{\mathbf{x}\in C_i} \frac{f_i(t)}{R_i(t)} \cdot \mathrm{P}_t(\mathbf{x})}{\sum_{\{\mathbf{x}|\phi(\mathbf{x})=1\}} \mathrm{P}_t(\mathbf{x})} = \sum_{i=1}^{n} \frac{\sum_{\mathbf{x}\in C_i} \frac{f_i(t)}{R_i(t)} \cdot \mathrm{P}_t(\mathbf{x})}{\sum_{\mathbf{x}\in C_i \cup D_i^0 \cup D_i^1} \mathrm{P}_t(\mathbf{x})}. \qquad (4)$$

Since the T_i are NBUFR,

$$\begin{aligned}
g(t) &\geq \sum_{i=1}^{n} \frac{\sum_{\mathbf{x}\in C_i} f_i(0) \cdot \mathrm{P}_t(\mathbf{x})}{\sum_{\mathbf{x}\in C_i \cup D_i^0 \cup D_i^1} \mathrm{P}_t(\mathbf{x})} \\
&= \sum_{i=1}^{n} f_i(0) \cdot \left(1 - \frac{\sum_{\mathbf{x}\in D_i^0 \cup D_i^1} \mathrm{P}_t(\mathbf{x})}{\sum_{\mathbf{x}\in C_i \cup D_i^0 \cup D_i^1} \mathrm{P}_t(\mathbf{x})}\right) \\
&\geq \sum_{i=1}^{n} f_i(0) \cdot (1 - \phi(0_i, \mathbf{1}_{\bar{i}})). \qquad (5)
\end{aligned}$$

The last inequality is a consequence of the fact that

$$\frac{\sum_{\mathbf{x}\in D_i^0 \cup D_i^1} P_t(\mathbf{x})}{\sum_{\mathbf{x}\in C_i \cup D_i^0 \cup D_i^1} P_t(\mathbf{x})} \leq 1 \tag{6}$$

and $\phi(0_i, \mathbf{1}_{\bar{i}}) = 1$ unless i is a one–component critical cut. But in this case, $\phi(0_i, \mathbf{1}_{\bar{i}}) = 0$ implies $D_i^0 \cup D_i^1 = \{\mathbf{x} | \phi(0_i, \mathbf{x}_{\bar{i}}) = 1\} = \emptyset$ by monotonicity, and thus the expression in (6) is zero as well.

On the other hand, evaluating (4) for $t = 0$ gives

$$g(0) = \sum_{i=1}^{n} \sum_{\mathbf{x}\in C_i} f_i(0) \cdot P_0(\mathbf{x}) \tag{7}$$

since $R(0) = R_i(0) = 1$. Now,

$$P_0(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} = \mathbf{1} \\ 0 & \text{else} \end{cases}$$

so that

$$\begin{aligned} g(0) &= \sum_{i=1}^{n} \sum_{\mathbf{x}\in C_i \cap \{\mathbf{1}\}} f_i(0) \\ &= \sum_{i=1}^{n} f_i(0) \cdot (1 - \phi(0_i, \mathbf{1}_{\bar{i}})), \end{aligned} \tag{8}$$

since for each i, it follows from $\phi(0_i, \mathbf{1}_{\bar{i}}) = 0$ that $\mathbf{1} \in C_i$ and from $\phi(0_i, \mathbf{1}_{\bar{i}}) = 1$ obviously $\mathbf{1} \notin C_i$.

The relationship (8) has a certain intuitive appeal. Note that a system made up of independent components will not suffer from multiple component failures, i.e. several component failures at the same point in time. A system failure at $t = 0$ therefore may only be encountered if a single component fails at that instant and if, in addition, this single component failure leads to a system failure, that is, this component is a one-element minimal cut. (8) is the sum of the "probabilities" for component failures of this kind.

Putting together (5) and (8) gives

$$g(t) \quad \geq \quad \sum_{i=1}^{n} f_i(0) \cdot (1 - \phi(0_i, \mathbf{1}_{\bar{i}})) \quad = \quad g(0),$$

as claimed in the theorem above.

As may have been expected, the dual class of negative aging (NWUFR) is not closed under the formation of systems. To see this, a counter-example can easily be constructed from a parallel system of two components with iid exponential failure times.

Some further properties of the NBUFR class should be noted.

1. NBUFR implies NBUFRA and NBURFR (New better than used in renewal failure rate) as given in Abouammoh/Ahmed (1992), thus a system composed of NBUFR components belongs to these classes as well. NBUFR itself contains the smaller and better-known classes IFR, IFRA and NBU.

2. The class of NBUFR distributions is – as are all classes describing positive aging – not closed under mixture (Kuhnert/Gohout 1993). It is, however, closed under convolution (Abouammoh/Ahmed 1988).

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PD Dr. Dr. Wolfgang Gohout, Dr. Ingo Kuhnert
Universität Gießen, FB 02
Professur für Statistik und Ökonometrie
Licher Str. 64
35394 Gießen