Statistical Papers 29, 143–150 (1988)

Statistical Papers
Statistische Hefte

Preservation properties for the mean residual life ordering

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Received: Sept. 28, 1987; revised version: Jan. 5, 1988

Abstract. Several preservation results for the mean residual life (m r) ordering are given. In particular, we show that the mr-ordering is preserved under convolutions, mixtures and weak convergence.

1. Introduction and Summary. Recently, Alzaid (1987) introduced a partial ordering among life distributions, based on their mean residual life (MRL) functions, studied its properties and demonstrated its usefulness in reliability, biometry, actuarial studies and demography.

The MRL function (see Section 2 for exact definition) plays an important role in statistical literature. Bryson and Siddiqui (1969), Barlow and Proschan (1975) and Hollander and Proschan (1981) and Hollander and Proschan (1975) have used the MRL function as a notion of ageing. Muth (1980) used the MRL function as a measure of memory. Bhattacharjee (1982) has characterized the class of MRL functions and sequences.

Other orderings related to the mean residual life ordering are the hazard rate (see, e.g., Penedo and Ross, 1980, Whitt,1980, Keilson ans Sumita, 1982 and Ross, 1983) and the likelihood ratio (see, e.g., Lehmann, 1955, Karlin, 1957 and Karlin and Rubin, 1965) orderings.

In this paper we develop preservation properties of the mean residual life ordering under commonly occuring operations in statistics, such as convolution, mixtures and convergence in distributions. Examples of how these results can be useful in recognizing situations when the random variables are mean residual life ordered are mentioned. We also point out that similar results hold for the hazard rate ordering.

2. Preliminaries. In this section we present definitions, notation, and basic facts used throughout the paper.

We use "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing".

Let X and Y be two nonnegative random variables with F and G as their respective distribution functions. Let $\bar{F}(t) = 1 - F(t)$. We will assume that $\bar{F}(o) = \bar{G}(o) = 1$ in all cases.

2.1. Definition. The mean residual lifetime (MRL) corresponding to the random variable X is $\mu_X(t) = E(X-t \mid X \geq t)$.

2.2. Definition. X is said to have a smaller mean residual life than does Y, written $X \underset{mr}{\leq} Y$, if

(2.1) $$E(X \mid X \geq t) \leq E(Y \mid Y \geq t) \quad \text{for all } t \geq o,$$

or equivalently

(2.2a) $$\mu_X(t) \leq \mu_Y(t) \quad \text{for all } t \geq o.$$

We shall sometimes, without confusion, write (2.2a) in the form

(2.2b) $$\mu_F(t) \leq \mu_G(t) \quad \text{for all } t \geq o.$$

Note that, (2.1) is equivalent to saying

(2.3) $$\int_t^\infty \bar{F}(x)dx / \int_t^\infty \bar{G}(x)dx \quad \text{is decreasing in } t \text{ for all } t \geq o$$

(see, Alzaid, 1987).

2.3. Definition. The random variable X has a smaller hazard rate than Y, written $X \underset{H}{\leq} Y$, if

(2.4) $$\bar{F}(x) / \bar{G}(x) \quad \text{is increasing in } x \text{ for all } x \geq o.$$

2.4. Definition. X is said to be smaller than Y in the sense of likelihood ratio ordering, written $X \underset{lr}{\leq} Y$, if $f(x) / g(y)$ is decreasing in x whenever defined, where f and g are the respective densities of X and Y.

2.5. Definition. A probability vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is said to be smaller than the probability vector $\underline{\beta} = (\beta_1,\ldots,\beta_n)$, in the sense of discrete likelihood ratio order, denoted by

$$\alpha \le_{dlr} \beta, \quad \frac{\beta_i}{\alpha_i} \le \frac{\beta_j}{\alpha_j} \quad \text{for all } 1 \le i \le j \le n.$$

2.6. Definition. A function $g : R \to [o,\infty)$ is said to be log-concave if $g(x_1,y_1)g(x_2,y_2) - g(x_1,y_2)\, g(x_2, y_1) \ge o$ whenever $x_1 < x_2$, $y_1 < y_2$.

3. Main Results. In this section we present preservation results for the mean residual life ordering. We point out that similar results hold for both the hazard rate ordering and the likelihood ratio ordering.

We begin by showing that the mean residual life ordering is preserved under weak limits in distributions.

3.1. Theorem. The mean residual life ordering ($\le_{mr}$) preserves the weak convergence property.

Proof. Suppose $\{F_n\}$ and $\{G_n\}$ converge weakly to F and G and that $F_n \le_{mr} G_n$. Then if y is a continuity point of both F and G, it follows that $\mu_F(y) \le \mu_G(y)$. Thus, $\mu_F(y) > \mu_G(y)$ is possible only if y is a discontinuity point of either F or G. Such discontinuity points are at most countable, so there exist continuity points x_n of F and G for which $x_n \downarrow y$ as $n\to\infty$. Consequently, appealing to the right-continuity property of distribution functions

$$\mu_F(y) = \lim_{n\to\infty} \mu_F(x_n) \le \lim_{n\to\infty} \mu_G(x_n) = \mu_G(y),$$

whence a contradiction.

The following result shows that the mean residual life ordering is preserved under convolutions.

3.2. Theorem. Let X_1, X_2 and Y be three nonnegative random variables, where Y is independent of both X_1 and X_2, also let Y have density g. Then $X_1 \le_{mr} X_2$ and g is log-concave imply that $X_1 + Y \le_{mr} X_2 + Y$.

Proof. We have to show that

$$\frac{\int_0^\infty\int_0^\infty g(t-u)P(X_1>x+u)\,dx\,du}{\int_0^\infty\int_0^\infty g(t-u)P(X_2>x+u)\,dx\,du} \geq \frac{\int_0^\infty\int_0^\infty g(s-u)P(X_1>x+u)\,dx\,du}{\int_0^\infty\int_0^\infty g(s-u)P(X_2>x+u)\,dx\,du}$$

for all $0 \leq s \leq t$, or equivalently,

$$(3.1)\qquad \begin{vmatrix} \int_0^\infty\int_0^\infty g(s-u)P(X_2>x+u)\,dx\,du & \int_0^\infty\int_0^\infty g(s-u)P(X_1>x+u)\,dx\,du \\ \int_0^\infty\int_0^\infty g(t-u)P(X_2>x+u)\,dx\,du & \int_0^\infty\int_0^\infty g(t-u)P(X_1>x+u)\,dx\,du \end{vmatrix} \geq 0.$$

Next, by the well known basic composition formula (Karlin 1968, p.17), the left side of (3.1) is equal to

$$\int\int_{u_1<u_2} \begin{vmatrix} g(s-u_1) & g(s-u_2) \\ g(t-u_1) & g(t-u_2) \end{vmatrix} \begin{vmatrix} \int_0^\infty P(X_2>x+u_1)\,dx & \int_0^\infty P(X_1>x+u_1)\,dx \\ \int_0^\infty P(X_2>x+u_2)\,dx & \int_0^\infty P(X_1>x+u_2)\,dx \end{vmatrix} du_1\,du_2.$$

The conclusion now follows if we note that the first determinant is nonnegative since g is log-concave, and that the second determinant is nonnegative since $X_1 \leq_{mr} X_2$.

3.3 Corollary. If $X_1 \leq_{mr} Y_1$ and $X_2 \leq_{mr} Y_2$ where X_1 is independent of X_2 and Y_1 is independent of Y_2, then the following statements hold:

(i) If X_1 and Y_2 have log-concave densities, then $X_1+X_2 \leq_{mr} Y_1+Y_2$.

(ii) If X_2 and Y_1 have log-concave densities, then $X_1+X_2 \leq_{mr} Y_1+Y_2$.

Proof. The following chain of inequalities establish (i):

$$X_1 + X_2 \leq_{mr} X_1 + Y_2 \leq_{mr} Y_1 + Y_2.$$

The proof of (ii) is similar.

Let $\underline{\alpha} = (\alpha_1,\dots,\alpha_n)$ be less ordered than $\underline{\beta} = (\beta_1,\dots,\beta_n)$, in the sense of the discrete likelihood ratio ordering. We shall compare the distribution functions of

$$Y_1 = \alpha_1 X_1 + \dots + \alpha_n X_n \quad \text{and} \quad Y_2 = \beta_1 X_1 + \dots + \beta_n X_n$$

3.4. Theorem. Let $X_1,\dots, X_n$ be a collection of random variables with corresponding distribution functions $F_1,\dots,F_n$ such that $X_1 \leq_{mr} X_2 \leq_{mr} \dots \leq_{mr} X_n$, and let $\underline{\alpha} = (\alpha_1,\dots,\alpha_n)$ and $\underline{\beta} = (\beta_1,\dots,\beta_n)$ be two probability vectors with $\underline{\alpha} \leq_{dlr} \underline{\beta}$.

Then

$$\sum_{i=1}^{n} \alpha_i X_i \le_{mr} \sum_{i=1}^{n} \beta_i X_i .$$

<u>**Proof.**</u> We need to establish

$$\frac{\int_0^\infty \sum_{i=1}^{n} \beta_i \bar{F}_i(t+x)\,dx}{\int_0^\infty \sum_{i=1}^{n} \alpha_i \bar{F}_i(t+x)\,dt} \le \frac{\int_0^\infty \sum_{i=1}^{n} \beta_i \bar{F}_i(t+y)\,dt}{\int_0^\infty \sum_{i=1}^{n} \alpha_i \bar{F}_i(t+y)\,dt} \tag{3.2}$$

for all $0 \le x \le y$.

Multiplying by the denominators and canceling out equal terms shows that (3.2) is equivalent to

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_i \alpha_j \int_0^\infty \bar{F}_i(u+x)\,du \int_0^\infty \bar{F}_j(v+y)\,dv$$

$$\le \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_i \alpha_j \int_0^\infty \bar{F}_i(u+y)\,du \int_0^\infty \bar{F}_j(v+x)\,dv,$$

or

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j > i}}^{n} [\beta_i \alpha_j \int_0^\infty \bar{F}_i(u+x)\,du \int_0^\infty \bar{F}_j(v+y)\,dv + \beta_j \alpha_i \int_0^\infty \bar{F}_j(u+x)\,du \int_0^\infty \bar{F}_j(v+y)\,dv]$$

$$\le \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j > i}}^{n} [\beta_i \alpha_j \int_0^\infty \bar{F}_i(v+y)\,dv \int_0^\infty \bar{F}_j(u+x)\,du + \beta_j \alpha_i \int_0^\infty \bar{F}_j(v+y)\,dv \int_0^\infty \bar{F}_i(u+x)\,du].$$

Now, for each fixed pair (i,j) with $i<j$ we have

$$\beta_i \alpha_j \int_0^\infty \bar{F}_i(v+y)\,dv \int_0^\infty \bar{F}_j(x+u)\,du$$

$$+ \beta_j \alpha_i \int_0^\infty \bar{F}_j(v+y)\,dv \int_0^\infty \bar{F}_i(u+x)\,du$$

$$- \beta_i \alpha_j \int_0^\infty \bar{F}_i(x+u)\,du \int_0^\infty \bar{F}_j(y+v)\,dv$$

$$- \beta_j \alpha_i \int_0^\infty \bar{F}_j(x+u)\,du \int_0^\infty \bar{F}_i(y+v)\,dv$$

$$= (\beta_i \alpha_j - \beta_j \alpha_i)\ [\int_0^\infty \bar{F}_i(y+v)\,dy \int_0^\infty \bar{F}_j(x+u)\,dx$$

$$- \int_0^\infty \bar{F}_i(x+u)\,dx \int_0^\infty \bar{F}_j(y+v)\,dv] ,$$

which is nonnegative because both terms are nonnegative by assumption. This completes the proof.

In any attempt to construct new mean residual life ordered random variables from known ones, the following theorem might be used.

<u>3.5. Theorem.</u> If $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are sequences of independent random variables with $X_i \leq_{mr} Y_i$ and X_i, Y_i have log-concave densities for all i, then

$$\sum_{i=1}^{n} X_i \leq_{mr} \sum_{i=1}^{n} Y_i \qquad (n=1,2,\ldots).$$

<u>Proof.</u> We shall prove the theorem by induction. Clearly, the result is true for n=1. Assume that the result is true for p=n−1, i.e.,

$$\sum_{i=1}^{n-1} X_i \leq_{mr} \sum_{i=1}^{n-1} Y_i \quad . \tag{3.3}$$

Note that each of the two sides of (3.3) has log-concave density (see, e.g., Karlin, 1968, p.128). Appealing to Corollary 3.3, the result follows.

<u>3.6. Remark.</u> Similar results hold if the mean residual life ordering is replaced by the hazard rate ordering in Theorem 3.2 and its corollary, Theorem 3.4 and Theorem 3.5.

To demonstrate the usefulness of the above results in recognizing mean residual life ordered random variables, we consider the following

<u>3.7. Example.</u> Let $X_{\underline{\lambda}}$ denote the convolution of n exponential distributions with parameters $\lambda_1, \ldots, \lambda_n$ respectively. Assume without loss of generality that $\lambda_1 \leq \ldots \leq \lambda_n$. Since exponential densities are log-concave, Theorem 3.5 implies that $X_{\underline{\lambda}} \leq_{mr} X_{\underline{\mu}}$ whenever $\lambda_i \geq \mu_i$ for i=1,...,n.

<u>3.8. Example.</u> Let $X_{\underline{\lambda}}$ be as described in Example 3.7. An application of Theorem 3.4 immediately yields $\sum_{i=1}^{n} \alpha_i X_{\lambda_i} \leq_{mr} \sum_{i=1}^{n} \beta_i X_{\lambda_i}$ for every two probability vectors $\underline{\alpha}$ and $\underline{\beta}$ such that $\underline{\alpha} \leq_{dlr} \underline{\beta}$. Another application of Theorem 3.4 is contained in:

<u>3.9. Example.</u> Let $X_{\underline{\lambda}}$ and $X_{\underline{\mu}}$ be as given in Example 3.7. For $o \leq q \leq p \leq 1$ and $p+q=1$, we have

$$p\,X_{\underline{\lambda}} + q\,X_{\underline{\mu}} \leq_{mr} q\,X_{\underline{\lambda}} + p\,X_{\underline{\mu}} \,.$$

It is remarkable that the above example can be generalized to higher dimensions, with obvious modifications in $\underline{\alpha}$ and $\underline{\beta}$.

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