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Anisotropic Triebel-Lizorkin Spaces with Doubling Measures

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ABSTRACT. We introduce and study anisotropic Triebel-Lizorkin spaces associated with general expansive dilations and doubling measures on $\mathbb{R}^n$ with the use of wavelet transforms. This work generalizes the isotropic methods of dyadic φ-transforms of Frazier and Jawerth to nonisotropic settings.

We extend results involving boundedness of wavelet transforms, almost diagonality, smooth atomic and molecular decompositions to the setting of doubling measures. We also develop localization techniques in the endpoint case of $p = \infty$, where the usual definition of Triebel-Lizorkin spaces is replaced by its localized version. Finally, we establish nonsmooth atomic decompositions in the range of $0 < p \leq 1$, which is analogous to the usual Hardy space atomic decompositions.

1. Introduction and statements of main results

Many areas of analysis involve the study of specific function spaces. In harmonic analysis, the well-known scale of L^p spaces is augmented by the Hardy spaces, the space BMO, and various forms of Lipschitz spaces. Despite inherent differences in the original definitions many of these spaces are closely related and can be studied from a unified perspective by the Littlewood-Paley theory. This gives rise to the study of Besov and Triebel-Lizorkin spaces which form a unifying class of function spaces containing many well-known classical function spaces such as Lebesgue spaces L^p, Hardy spaces H^p, and Hardy-Sobolev spaces.

There were several efforts of extending classical function spaces arising in harmonic analysis from Euclidean spaces to other domains and nonisotropic settings. The usual isotropic dilations can be replaced by more complicated nonisotropic dilation structures as in the study of parabolic Hardy spaces of Calderón and Torchinsky [10, 11] or Hardy spaces on homogeneous groups of Folland and Stein [17]. The nonisotropic variants of Triebel-Lizorkin and Besov spaces for diagonal dilations have been studied by Besov et al [1], Schmeisser and Triebel [32, 33, 34, 35, 36], and Farkas [14]. The other direction is the study of weighted function spaces associated with general Muckenhoupt A_∞ weights. This direction of research for Besov and Triebel-Lizorkin

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spaces was carried over by Bui, Paluszyńskiet al [7, 8, 9] and Rychkov [30]. One should also note that a significant portion of the theory of function spaces can also be done on the large class of spaces of homogeneous type introduced by Coifman and Weiss [13]; for example, see [25, 26, 27]. However, this high level of generality imposes restrictions on possible values of the index p, i.e., $p > 1 - \delta$ for some possibly small $\delta > 0$.

Several aspects of the above mentioned developments can be extended to a larger class (than previously considered diagonal setting) of nonisotropic dilation structures associated with expansive dilations. In the context of Hardy spaces this goal was achieved by the author in [2], where it was demonstrated that significant portion of a real-variable isotropic H^p theory extends to such anisotropic setting. Analogous extensions to anisotropic Triebel-Lizorkin spaces with A_∞ weights and anisotropic Besov spaces with doubling measures were done in [3, 5]. These studies show that the isotropic methods of dyadic φ-transforms of Frazier and Jawerth [18, 20] can be extended to nonisotropic setting associated with general expansive dilations. Among other things proved in [3, 5], weighted anisotropic Triebel-Lizorkin and Besov spaces are characterized by their wavelet transform coefficients and smooth atomic and molecular decompositions of these spaces are established.

It is commonly known that Triebel-Lizorkin spaces are much harder to work with than Besov spaces due to their particular structure. For these reasons weighted Triebel-Lizorkin spaces are often studied with A_∞ weights instead of more general doubling weights as in the case of Besov spaces. The goal of this work is to show that one can also build a coherent theory of weighted anisotropic Triebel-Lizorkin spaces associated with expansive dilations and doubling weights further generalizing the results of [5, 20]. More specifically, this article:

- Extends results from [5, 20] involving boundedness of wavelet transforms, almost diagonality, smooth atomic and molecular decompositions to the setting of doubling measures,
- develops necessary localization techniques for the endpoint case $p = \infty$,
- establishes nonsmooth atomic decompositions (analogous to the Hardy space atomic decompositions) in the range $0 < p \leq 1$.

In addition, a subsequent work [4] continues this direction of research by showing duality and real and complex interpolation results for $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces. In what follows, we summarize the results obtained in this article.

In this work we study function spaces on $\mathbb{R}^n$ associated with an expansive dilation A, that is an $n \times n$ real matrix all of whose eigenvalues λ satisfy $|\lambda| > 1$. The starting point is the Littlewood-Paley decomposition asserting that any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ can be decomposed as

$$f = \sum_{j\in\mathbb{Z}} \varphi_j * f, \qquad \text{where} \quad \varphi_j(x) = |\det A|^j \varphi(A^j x)\,,$$

with the convergences in $\mathcal{S}'$ (modulo polynomials). Here, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is a test function as in Lemma 2.13. Given a smoothness parameter $\alpha \in \mathbb{R}$, an integrability exponent $0 < p < \infty$, and a summability exponent $0 < q \leq \infty$, we introduce the *anisotropic Triebel-Lizorkin space* $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ norm as

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}} = \left\| \left(\sum_{j\in\mathbb{Z}} \left(|\det A|^{j\alpha} |f * \varphi_j| \right)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty\,. \tag{1.1}$$

Here, μ is a doubling measure respecting the action of A. That is,

$$\mu(B_{\rho_A}(x, 2r)) \leq C\mu(B_{\rho_A}(x, r)) \qquad \text{for all} \quad x \in \mathbb{R}^n,\ r > 0\,,$$

where the balls $B_{\rho_A}(x,r)$ are defined with respect to a quasi-norm ρ_A associated with A. Later we show that this definition is independent of the choice of φ satisfying natural support conditions (3.2) and (3.3).

The corresponding discrete Triebel-Lizorkin sequence space $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$, which is indexed by the collection of dilated cubes

$$\mathcal{Q} = \left\{A^{-j}\big([0,1]^n + k\big) : j \in \mathbb{Z},\ k \in \mathbb{Z}^n\right\},$$

such that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} = \left\| \left(\sum_{Q\in\mathcal{Q}} \big(|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q\big)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty . \tag{1.2}$$

Here, $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of the dilated cube Q.

Suppose that (φ, ψ) is an admissible pair of dual frame wavelets as in Definition 2.12. The corresponding wavelet systems consisting of translates and dilates of φ and ψ are customarily denoted by $\{\varphi_Q : Q \in \mathcal{Q}\}$ and $\{\psi_Q : Q \in \mathcal{Q}\}$, resp. Following Frazier and Jawerth, we define the φ-transform, which maps the distribution f to the sequence of its wavelet coefficients $S_\varphi f = \{\langle f, \varphi_Q\rangle\}_{Q\in\mathcal{Q}}$. For any sequence $s = \{s_Q\}_{Q\in\mathcal{Q}}$ of complex numbers, we define formally the inverse φ-transform, which maps s to a distribution $T_\psi s = \sum_{Q\in\mathcal{Q}} s_Q \psi_Q$. Then, the following generalization of the fundamental result of Frazier and Jawerth [5, 20] holds.

Theorem 1.1. *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and a μ is a doubling measure. The φ-transform $S_\varphi : \dot{\mathbf{F}}_p^{\alpha,q} \to \dot{\mathbf{f}}_p^{\alpha,q}$, and the inverse φ-transform $T_\psi : \dot{\mathbf{f}}_p^{\alpha,q} \to \dot{\mathbf{F}}_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\varphi$ is the identity on $\dot{\mathbf{F}}_p^{\alpha,q}$.*

One should emphasize that in the endpoint case of $p = \infty$, the definitions (1.1) and (1.2) must be replaced by their localized versions (3.8) and (3.9), respectively, which were originally introduced in the dyadic case in [20]. This is far more than a cosmetic change. A substantial portion of this work deals with the case of $p = \infty$, which requires special considerations. As a consequence of Theorem 1.1, we deduce that $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces are complete quasi-normed spaces with equivalent norms independent of the choice of a test function φ.

Once Theorem 1.1 is established, we study operators on $\dot{\mathbf{F}}_p^{\alpha,q}$ by transferring them with the use of wavelet transforms to the corresponding sequence spaces $\dot{\mathbf{f}}_p^{\alpha,q}$. Since $\dot{\mathbf{f}}_p^{\alpha,q}$ norms depend only on the magnitude of coefficients, consequently, the analysis on the sequence space level is much easier than in the original space $\dot{\mathbf{F}}_p^{\alpha,q}$. In particular, in Section 4 we study a very useful class of almost diagonal operators on $\dot{\mathbf{f}}_p^{\alpha,q}$, which was originally introduced by Frazier and Jawerth [20]. We show that the expected boundedness result holds also for $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces with doubling weights by generalizing a result in [5]. As an application, in Section 5 we extend smooth atomic and molecular decompositions results in [5, 20] to the setting of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces with doubling weights.

In Section 6 we establish nonsmooth atomic decompositions of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces which are analogous to the usual Hardy space atomic decompositions. The main advantage of such decompositions is that coefficients are controlled by ℓ_p norms, rather than harder to deal $\dot{\mathbf{f}}_p^{\alpha,q}$ norms as in the case of smooth atomic decompositions. In the next section we identify unweighted $\dot{\mathbf{F}}_p^{0,2}(\mathbb{R}^n, A)$ spaces with the anisotropic Hardy spaces H_A^p for $0 < p < \infty$ in the context of expansive dilations A. The last section contains the proofs of several lemmas required in the proof of Theorem 1.1.

2. Some background tools

We start by recalling basic definitions and properties of the Euclidean spaces associated with general expansive dilations.

2.1. Quasi-norms for expansive dilations

Definition 2.1. We say that a real $n \times n$ matrix is *expansive* if all of its eigenvalues satisfy $|\lambda| > 1$. A quasi-norm associated with an expansive matrix A is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$ satisfying

$$\begin{aligned} &\rho_A(x) > 0, && \text{for} \quad x \neq 0, \\ &\rho_A(Ax) = |\det A| \rho_A(x) && \text{for} \quad x \in \mathbb{R}^n , \\ &\rho_A(x + y) \leq H(\rho_A(x) + \rho_A(y)) && \text{for} \quad x, y \in \mathbb{R}^n , \end{aligned} \tag{2.1}$$

where $H \geq 1$ is a constant.

In the standard dyadic case $A = 2\mathrm{Id}$, a quasi-norm ρ_A satisfies $\rho_A(2x) = 2^n \rho_A(x)$ instead of the usual scalar homogeneity. In particular, $\rho_A(x) = |x|^n$ is an example of a quasi-norm for $A = 2\mathrm{Id}$, where $| \cdot |$ represent the Euclidean norm in $\mathbb{R}^n$. One can show that all quasi-norms associated to a fixed dilation A are equivalent, see [2, Lemma 2.4]. Moreover, it is possible to choose a quasi-norm ρ_A such that ρ_A-balls $\{x \in \mathbb{R}^n : \rho_A(x) < r\}$ are convex.

We also need to introduce some convenient notation.

Definition 2.2. Suppose A is expansive matrix and $\sigma(A)$ is its spectrum. If A is diagonalizable over $\mathbb{C}$, let

$$\lambda_- := \min_{\lambda \in \sigma(A)} |\lambda|, \qquad \lambda_+ := \max_{\lambda \in \sigma(A)} |\lambda| .$$

Otherwise, let λ_- and λ_+ be any positive real numbers such that $1 < \lambda_- < \min_{\lambda \in \sigma(A)} |\lambda|$ and $\max_{\lambda \in \sigma(A)} |\lambda| < \lambda_+ < |\det A|$. Define

$$\zeta_+ := \frac{\ln \lambda_+}{\ln |\det A|}, \qquad \zeta_- := \frac{\ln \lambda_-}{\ln |\det A|} .$$

The parameters ζ_- and ζ_+ measure the eccentricity of a dilation A. In general, we have $0 < \zeta_- \leq 1/n \leq \zeta_+ < 1$. For example, in the standard dyadic case $A = 2\mathrm{Id}$, we have $\zeta_- = \zeta_+ = 1/n$.

Definition 2.3. Let $\mathcal{B}$ be the collection of all ρ_A*-balls*

$$B_{\rho_A}(x, r) = \{y \in \mathbb{R}^n : \rho_A(x - y) < r\}, \qquad x \in \mathbb{R}^n,\ r > 0 .$$

Let $\mathcal{Q}$ be the collection of all *dilated cubes*

$$\mathcal{Q} = \{Q = A^j([0, 1]^n + k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$$

adapted to the action of a dilation A. Obviously, if $A = 2\mathrm{Id}$ we obtain the usual collection of *dyadic cubes*. Let

$$x_Q = A^j k, \qquad Q = A^j([0, 1]^n + k) \in \mathcal{Q},$$

be the "lower-left corner" of Q. The scale of a ball $B = B_{\rho_A}(x_0, r) \in \mathcal{B}$ is defined as

$$\text{scale}(B) = \lfloor \log_{|\det A|} r \rfloor .$$

The scale of a dilated cube $Q = A^j([0,1]^n + k) \in \mathcal{Q}$ is defined as $\text{scale}(Q) = j$. Alternatively, $\text{scale}(Q) = \log_{|\det A|} |Q|$.

By renormalizing ρ_A, it is convenient to assume that $|B_{\rho_A}(x,1)| = 1$. Consequently,

$$\left|B_{\rho_A}\left(x, |\det A|^j\right)\right| = |\det A|^j \qquad \text{for any } j \in \mathbb{Z} .$$

Therefore,

$$|\det A|^{\text{scale}(B)} \leq |B| \leq |\det A|^{\text{scale}(B)+1} ,$$

and

$$|Q| \leq |B| \leq |\det A||Q| \qquad \text{for any} \quad Q \in \mathcal{Q},\ B \in \mathcal{B} \text{ with } \quad \text{scale}(Q) = \text{scale}(B) .$$

Note that for any $Q \in \mathcal{Q}$,

$$\text{diam}_{\rho_A}(Q) := \sup\{\rho_A(y_1 - y_2) : y_1, y_2 \in Q\} = |Q| \, \text{diam}_{\rho_A}\left([0,1]^n\right) = C|Q| . \qquad (2.2)$$

The following concept is very useful in the study of the localized norms of $\dot{\mathbf{F}}_\infty^{\alpha,q}$ spaces.

Definition 2.4. The *tent* $\mathcal{T}(P)$ over $P \in \mathcal{Q}$ is defined as

$$\mathcal{T}(P) = \{Q \in \mathcal{Q} : |Q \cap P| > 0 \text{ and } \text{scale}(Q) \leq \text{scale}(P)\} .$$

2.2. Doubling measures for expansive dilations

Definition 2.5. We say that a nonnegative Borel measure μ on $\mathbb{R}^n$ is ρ_A*-doubling* if there exists $\beta = \beta(\mu) > 0$ such that

$$\mu(B_{\rho_A}(x, |\det A| r)) \leq |\det A|^\beta \mu(B_{\rho_A}(x, r)) \qquad \text{for all} \quad x \in \mathbb{R}^n,\ r > 0 . \qquad (2.3)$$

The smallest such β is called a doubling constant of μ.

Remark 2.6. We remark that ρ_A-doubling measure μ does not have to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. For an example of a measure μ on $\mathbb{R}$, which is doubling and singular with respect to Lebesgue measure see [6]. Moreover, it is not hard to show that the doubling constant β is always ≥ 1.

We also remark that any weight w in A_∞ (with respect to a quasi-distance ρ_A) defines a ρ_A-doubling measure μ by $d\mu = w\,dx$, see [5, Definition 2.2]. Hence, by working with doubling measures instead of A_∞ weights we will generalize the results about Triebel-Lizorkin spaces in [5]. To achieve this we will work with a weighted Hardy-Littlewood maximal function. This step is necessary due to the collapse of weighted norm inequalities, and in particular, weighted vector-valued Fefferman-Stein inequality outside A_∞ class.

For any Borel measurable function f define its Hardy-Littlewood maximal function $M_{\rho_A} f$ with respect to ρ_A-doubling measure μ by

$$M_{\rho_A} f(x) = \sup_{x \in B \in \mathcal{B}} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y) .$$

It is easy to verify that we have the following fact. For rudimentary facts about spaces of homogeneous type we refer the reader to [13, 22, 25].

Proposition 2.7. *$(\mathbb{R}^n, \rho_A, \mu)$ is a space of homogeneous type, where ρ_A is a quasi-norm associated with an expansive dilation A, and μ is a ρ_A-doubling measure on $\mathbb{R}^n$.*

As a consequence, the Fefferman-Stein vector-valued inequality holds in our setting.

Theorem 2.8. *Suppose that $1 < p < \infty$, $1 < q \le \infty$, and μ is a ρ_A-doubling measure. Then there exists a constant C such that*

$$\left\| \left(\sum_i |M_{\rho_A} f_i|^q \right)^{1/q} \right\|_{L^p(\mu)} \le C \left\| \left(\sum_i |f_i|^q \right)^{1/q} \right\|_{L^p(\mu)}$$

holds for any $(f_i)_i \subset L^p(\mu)$.

We will also need several results about doubling measures and families $\mathcal{B}$ and $\mathcal{Q}$. For $Q = A^j([0,1]^n + k) \in \mathcal{Q}$, define its center $c_Q = A^j(k + (1/2, \dots, 1/2))$.

Lemma 2.9. *Given families of dilated balls $\mathcal{B}$ and dilated cubes $\mathcal{Q}$, there exist $C_{\mathcal{B}}, C_{\mathcal{Q}} > 0$ such that:*
(a) *For any $Q \in \mathcal{Q}$ we have*

$$B_0 \subset Q \subset B_1, \qquad \text{where } B_0 = B_{\rho_A}\big(c_Q, |Q||\det A|^{-C_{\mathcal{B}}}\big),\ B_1 = B_{\rho_A}\big(c_Q, |Q||\det A|^{C_{\mathcal{B}}}\big),$$

(b) *for any $B \in \mathcal{B}$, the collection*

$$\mathcal{Q}_B = \{Q \in \mathcal{Q} : Q \cap B \ne \emptyset,\ \mathrm{scale}(Q) = \mathrm{scale}(B)\}$$

has at most $C_{\mathcal{Q}}$ elements. Furthermore,

$$\mu(Q) \le C\mu(B) \qquad \text{for all} \quad Q \in \mathcal{Q}_B .$$

The proof of Lemma 2.9 is quite elementary, and hence, it is skipped. As a corollary of doubling of μ, (2.2), and Lemma 2.9 we have

$$\mu(Q) \asymp \mu(B_{\rho_A}(c_Q, |Q|)) \asymp \mu(B_{\rho_A}(x_Q, |Q|)) \qquad \text{for all} \quad Q \in \mathcal{Q} . \tag{2.4}$$

Proposition 2.10. *Suppose that μ is ρ_A-doubling measure. Then:*
(a) *For every $\eta > 0$ there exists a constant $c > 0$ such that*

$$j \in \mathbb{Z},\ k_0, k_1 \in \mathbb{R}^n,\ |k_0 - k_1| < \eta \implies \mu\big(A^j([0,1]^n + k_0)\big) \le c\mu\big(A^j([0,1]^n + k_1)\big) .$$

(b) *For fixed $x_0 \in \mathbb{R}^n$, let $P_j \in \mathcal{Q}$ be such that $\mathrm{scale}(P_j) = j$ and $x_0 \in P_j$. Then*

$$\lim_{j \to \infty} \mu(P_j) = \infty .$$

Proposition 2.10 is a simple consequence of the doubling property of μ. Indeed, by (2.4) we can replace each occurrence of $\mu(Q)$ by $\mu(B_{\rho_A}(x_Q, |Q|))$ with a gain of a multiplicative constant. Finally, we will need a slight variation of [3, Lemma 4.1].

Lemma 2.11. *Suppose that μ is ρ_A-doubling measure and $\delta \in \mathbb{R}$. Then, there exist $L, C > 0$ such that*

$$\sum_{Q \in \mathcal{Q},\ \mathrm{scale}(Q)=j} \frac{\mu(Q)^\delta}{(1+\rho_A(x_Q)/\max(1,|Q|))^L} \le C|\det A|^{(2\beta|\delta|+1)|j|} \qquad \textit{for all} \quad j \in \mathbb{Z}\,. \quad (2.5)$$

Proof. We claim that for any $P, Q \in \mathcal{Q}$, $P = A^j([0,1]^n + k)$, $Q = A^j([0,1]^n + l)$, $k, l \in \mathbb{Z}^n$, we have

$$\mu(Q) \le C(1+\rho_A(k-l))^\beta \mu(P)\,. \quad (2.6)$$

Indeed, (2.6) is a consequence of (2.4) and

$$\begin{aligned}\mu\big(B_{\rho_A}(A^j k, |\det A|^j)\big) &\le \mu\big(B_{\rho_A}(A^j l, H|\det A|^j(1+\rho_A(k-l))\big)\\ &\le C(1+\rho_A(k-l))^\beta \mu\big(B_{\rho_A}(A^j l, |\det A|^j)\big)\,,\end{aligned}$$

since μ is ρ_A-doubling measure. Suppose that $j \ge 0$. By (2.6) we have for $L > \beta|\delta| + 1$,

$$\begin{aligned}\sum_{Q \in \mathcal{Q},\ \mathrm{scale}(Q)=j} \frac{\mu(Q)^\delta}{(1+\rho_A(x_Q)/|Q|)^L} &\le C\mu\big(A^j([0,1]^n)\big)^\delta \sum_{k \in \mathbb{Z}^n} (1+\rho_A(k))^{\beta|\delta|-L}\\ &\le C|\det A|^{j\beta\max(\delta,0)} \mu\big([0,1]^n\big)^\delta\,.\end{aligned}$$

Likewise, suppose that $j < 0$. Then for $L > \beta|\delta| + 1$,

$$\begin{aligned}\sum_{Q \in \mathcal{Q},\ \mathrm{scale}(Q)=j} \frac{\mu(Q)^\delta}{(1+\rho_A(x_Q))^L} &\le C\mu\big(A^j([0,1]^n)\big)^\delta \sum_{k \in \mathbb{Z}^n} \frac{(1+\rho_A(k))^{\beta|\delta|}}{\big(1+\rho_A(k)|\det A|^j\big)^L}\\ &\le C\mu\big(A^j([0,1]^n)\big)^\delta |\det A|^{-j\beta|\delta|} \sum_{k \in \mathbb{Z}^n} \big(1+\rho_A(A^j k)\big)^{\beta|\delta|-L}\\ &\le C|\det A|^{j(\beta\min(\delta,0)-\beta|\delta|-1)} \mu\big([0,1]^n\big)^\delta\,.\end{aligned}$$

In the last step we used that for $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$, such that

$$\sum_{k \in \mathbb{Z}^n} \big(1+\rho_A(A^j k)\big)^{-1-\varepsilon} \le C|\det A|^{-j} \qquad \text{for all} \quad j \le 0\,.$$

Combining the above estimates yields (2.5). □

2.3. Wavelet transforms for expansive dilations

Definition 2.12. We say that (φ, ψ) is an *admissible pair of dual frame wavelets* if φ, ψ are test functions in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}\hat{\varphi},\ \operatorname{supp}\hat{\psi} \subset [-\pi,\pi]^n \setminus \{0\} \quad (2.7)$$

$$\sum_{j \in \mathbb{Z}} \overline{\hat{\varphi}\big((A^*)^j \xi\big)}\hat{\psi}\big((A^*)^j \xi\big) = 1 \qquad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}\,, \quad (2.8)$$

where A^* is the adjoint (transpose) of A. Here,

$$\operatorname{supp}\hat{\varphi} = \overline{\{\xi \in \mathbb{R}^n : \hat{\varphi}(\xi) \ne 0\}}\,,$$

and the Fourier transform of f is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\langle x,\xi\rangle}\, dx\,.$$

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we define its *wavelet system* as

$$\varphi_Q(x) = |\det A|^{j/2}\varphi\big(A^j x - k\big), \qquad Q = A^{-j}\big([0,1]^n + k\big) \in \mathcal{Q}\,. \tag{2.9}$$

It is not hard to show that the conditions (2.7), (2.8) imply that (φ, ψ) is a pair of dual frame wavelets in $L^2(\mathbb{R}^n)$. This means that the wavelet systems $\{\varphi_Q : Q \in \mathcal{Q}\}$ and $\{\psi_Q : Q \in \mathcal{Q}\}$ are Bessel sequences, i.e., there exists a constant $C > 0$ such that

$$\sum_{Q\in\mathcal{Q}} |\langle f, \varphi_Q\rangle|^2,\ \sum_{Q\in\mathcal{Q}} |\langle f, \psi_Q\rangle|^2 \le C\|f\|_{L^2}^2 \qquad \text{for all} \quad f \in L^2\big(\mathbb{R}^n\big)\,, \tag{2.10}$$

and we have the reconstruction formula

$$f = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle \psi_Q, \qquad \text{for all} \quad f \in L^2\big(\mathbb{R}^n\big)\,, \tag{2.11}$$

where the above series converges unconditionally in L^2.

The above formula has a counterpart in the form of the reproducing identity (2.15) valid for tempered distributions modulo polynomials $\mathcal{S}'/\mathcal{P}$. For the basic properties of this space we refer to [28, Section 3.3] or [33, Section 5.1]. Here, we only recall that $\mathcal{S}'/\mathcal{P}$ can be identified with the space of all continuous functionals on the closed subspace $\mathcal{S}_0(\mathbb{R}^n)$ of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ given by

$$\mathcal{S}_0\big(\mathbb{R}^n\big) = \left\{\varphi \in \mathcal{S} : \int \varphi(x) x^\alpha\, dx = 0 \quad \text{for all multi-indices} \quad \alpha\right\}. \tag{2.12}$$

Lemmas 2.13 and 2.14 show that any distribution $f \in \mathcal{S}'/\mathcal{P}$ admits the Littlewood-Paley decomposition and the wavelet reproducing formula adapted to an expansive dilation A. Both of these results are anisotropic modifications of their well-known dyadic analogues, see [18, 20, 21]. For the proof of these formulas we refer the reader to [5].

Lemma 2.13. *Suppose that A is an expansive matrix and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is such that*

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}\big((A^*)^j \xi\big) = 1 \qquad \textit{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}\,, \tag{2.13}$$

and supp $\hat{\varphi}$ *is compact and bounded away from the origin. Then for any $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$f = \sum_{j\in\mathbb{Z}} \varphi_j * f\,, \tag{2.14}$$

where $\varphi_j(x) = |\det A|^j \varphi(A^j x)$, and the convergence is in $\mathcal{S}'/\mathcal{P}$.

Lemma 2.14. *If $\varphi, \psi \in \mathcal{S}'(\mathbb{R}^n)$ satisfy* (2.7), (2.8), *then*

$$f = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle \psi_Q, \qquad \textit{for any} \quad f \in \mathcal{S}'/\mathcal{P}\,, \tag{2.15}$$

where the convergence of the above series, as well as the equality, is in $\mathcal{S}'/\mathcal{P}$. More precisely, there exists a sequence of polynomials $\{P_k\}_{k=1}^{\infty} \subset \mathcal{P}$ and $P \in \mathcal{P}$ such that

$$f = \lim_{k\to\infty} \left(\sum_{Q\in\mathcal{Q},\ |\det A|^{-k}\le|Q|\le|\det A|^k} \langle f, \varphi_Q\rangle\psi_Q + P_k \right) + P ,$$

with convergence in $\mathcal{S}'$.

3. Anisotropic $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces with doubling measures and the case $p = \infty$

In this section we extend the class of anisotropic Triebel-Lizorkin spaces studied in [5] to the setting of doubling measures and the endpoint case of $p = \infty$. In the case of $0 < p < \infty$ the usual definition is perfectly satisfactory. However, in the endpoint case we adopt a localized definition of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces which was originally introduced in the dyadic case by Frazier and Jawerth [20]. We show that the resulting spaces are well defined quasi-Banach spaces and they can be characterized by the magnitude of wavelet coefficients.

We start by recalling the usual definition of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces in the range $0 < p < \infty$.

Definition 3.1. For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and a ρ_A-doubling measure μ, we define the *anisotropic Triebel-Lizorkin space* $\dot{\mathbf{F}}_p^{\alpha,q} = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}} = \left\| \left(\sum_{j\in\mathbb{Z}} \left(|\det A|^{j\alpha}|f * \varphi_j|\right)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty , \tag{3.1}$$

where $\varphi_j(x) = |\det A|^j \varphi(A^j x)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (3.2), (3.3)

$$\operatorname{supp}\hat{\varphi} := \overline{\{\xi \in \mathbb{R}^n : \hat{\varphi}(\xi) \neq 0\}} \subset [-\pi,\pi]^n \setminus \{0\} , \tag{3.2}$$

$$\sup_{j\in\mathbb{Z}} \left|\hat{\varphi}\left((A^*)^j\xi\right)\right| > 0 \qquad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\} . \tag{3.3}$$

To emphasize the dependence on φ we will use the notation $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)(\varphi)$ for (3.1). Later we will show that this definition is independent of φ.

The discrete Triebel-Lizorkin sequence space $\dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)$ is defined as the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} = \left\| \left(\sum_{Q\in\mathcal{Q}} \left(|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q\right)^q \right)^{1/q} \right\|_{L^p(\mu)} < \infty , \tag{3.4}$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of the dilated cube Q.

It is known that the naive definition of the space $\dot{\mathbf{F}}_p^{\alpha,q}$ using the norm (3.1) when $p = \infty$ is unsatisfactory. Indeed, Triebel [33, p. 46] remarks that when $p = \infty$ the norm (3.1) is dependent of the choice of the function φ. Moreover, Frazier and Jawerth [20, Section 5] point out that one should expect to have $\dot{\mathbf{F}}_\infty^{0,2} \approx BMO$, which is not the case for the naive definition of $\dot{\mathbf{F}}_\infty^{0,2}$. To overcome this problem Frazier and Jawerth [20] had proposed a localized definition of the norm when $p = \infty$ by considering averages only over small scales. This approach works well for isotropic theory and the goal of this section is to show that it also works for general expansive dilations.

3.1. Localized definition in the case $p = \infty$

Definition 3.2. For $\alpha \in \mathbb{R}$, $0 < q \leq \infty$, and a ρ_A-doubling measure μ, we define the *anisotropic Triebel-Lizorkin space* $\dot{\mathbf{F}}_\infty^{\alpha,q} = \dot{\mathbf{F}}_\infty^{\alpha,q}(\mathbb{R}^n, A, \mu)$ as the collection of all $f \in \mathcal{S}'/\mathcal{P}$ such that,

$$\|f\|_{\dot{\mathbf{F}}_\infty^{\alpha,q}(\mathbb{R}^n,A,\mu)} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{j=-\text{scale}(P)}^{\infty} \left(|\det A|^{j\alpha} |f * \varphi_j(x)|\right)^q d\mu(x) \right)^{1/q} < \infty\,, \quad (3.5)$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (3.2) and (3.3). To emphasize the dependence on φ we will use the notation $\dot{\mathbf{F}}_\infty^{\alpha,q}(\mathbb{R}^n, A, \mu)(\varphi)$ for (3.5). Later we will show that this definition is independent of φ.

The sequence space, $\dot{\mathbf{f}}_\infty^{\alpha,q} = \dot{\mathbf{f}}_\infty^{\alpha,q}(A, \mu)$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}(A,\mu)} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{Q\in\mathcal{Q},\ \text{scale}(Q)\leq\text{scale}(P)} \left(|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x)\right)^q d\mu(x) \right)^{1/q} < \infty\,. \quad (3.6)$$

Naturally, if $q = \infty$, then (3.5) and (3.6) are interpreted as

$$\|f\|_{\dot{\mathbf{F}}_\infty^{\alpha,\infty}} = \sup_{j\in\mathbb{Z}} |\det A|^{j\alpha} \|f * \varphi_j\|_\infty < \infty, \qquad \|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,\infty}} = \sup_{Q\in\mathcal{Q}} |Q|^{-\alpha-1/2} |s_Q| < \infty\,. \quad (3.7)$$

In other words, when $p = q = \infty$, the spaces $\dot{\mathbf{F}}_\infty^{\alpha,\infty}$ and $\dot{\mathbf{f}}_\infty^{\alpha,\infty}$ coincide with Besov spaces $\dot{\mathbf{B}}_\infty^{\alpha,\infty}$ and $\dot{\mathbf{b}}_\infty^{\alpha,\infty}$, resp., and there is no need for localization.

Remark 3.3. For the sake of simplicity it is convenient to consider the spaces $\dot{\mathbf{F}}_\infty^{\alpha,q}$ and $\dot{\mathbf{f}}_\infty^{\alpha,q}$, where the averaging process takes places with respect to the Lebesgue measure instead of μ. More precisely, we consider the unweighted spaces $\dot{\mathbf{F}}_\infty^{\alpha,q}$ and $\dot{\mathbf{f}}_\infty^{\alpha,q}$ defined by the norms

$$\|f\|_{\dot{\mathbf{F}}_\infty^{\alpha,q}} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{|P|} \int_P \sum_{j=-\text{scale}(P)}^{\infty} \left(|\det A|^{j\alpha} |f * \varphi_j(x)|\right)^q dx \right)^{1/q} < \infty\,, \quad (3.8)$$

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{|P|} \int_P \sum_{Q\in\mathcal{Q},\ \text{scale}(Q)\leq\text{scale}(P)} \left(|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x)\right)^q dx \right)^{1/q} < \infty\,. \quad (3.9)$$

There is a much deeper reason why we may insist on the above unweighted definitions. This is because one can show that the norms (3.5) and (3.6) do not depend effectively on the choice of μ, as long as $d\mu = w\,dx$ for some $w \in A_\infty$, see [4, Corollary 3.5]. Consequently, not much generality is gained by the introduction of μ in the case when $p = \infty$. Since this is a very nontrivial fact we will stick to more general norms as in Definition 3.2 in this article.

Remark 3.4. In the case when the family of dilated cubes $\mathcal{Q}$ is *nested*, i.e.,

$$Q, P \in \mathcal{Q} \text{ and } |Q \cap P| > 0 \implies P \subset Q \text{ or } Q \subset P$$

the tent $\mathcal{T}(P) = \{Q \in \mathcal{Q} : Q \subset P\}$ and the definition (3.9) overlaps with the usual dyadic definition of $\dot{\mathbf{f}}_\infty^{\alpha,q}$ by Frazier and Jawerth in [20]. In this case we simply have

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}} = \sup_{P\in\mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{Q\subset P} \left(|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x)\right)^q d\mu(x) \right)^{1/q}. \quad (3.10)$$

In the case when the family of dilated balls $\mathcal{Q}$ is not nested, the norm (3.10) is obviously dominated by (3.6). However, it does not seem that the norms (3.6) and (3.10) are equivalent for a general dilation A, e.g., consider $A = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$. In order to circumvent this problem one could modify the definition of the collection of dilated cubes. Take any $\delta > 0$, and define

$$\tilde{\mathcal{Q}} = \left\{ Q = A^j\left([-\delta, 1+\delta]^n\right) : j \in \mathbb{Z}, k \in \mathbb{Z}^n \right\}.$$

Then, it is not difficult to see that by replacing $\mathcal{Q}$ by $\tilde{\mathcal{Q}}$, we get equivalent norms for discrete spaces $\dot{\mathbf{f}}_p^{\alpha,q}$ for $p < \infty$; the proof boils down to the vector-valued Fefferman-Stein inequality. Moreover, it is possible to show that the norms (3.6) and (3.10) are in fact equivalent after this replacement. We will skip the proof of this fact, since it is not used elsewhere in this article.

Remark 3.5. For $q < \infty$, we can perform integration in (3.6) to obtain

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}} = \sup_{P \in \mathcal{Q}} \left(\frac{1}{\mu(P)} \sum_{|Q| \le |P|} \left(|Q|^{-\alpha-1/2}|s_Q|\right)^q \mu(Q \cap P) \right)^{1/q}.$$

Then it is not difficult to see using Proposition 2.10 that we have the equivalence of norms

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}} \asymp \sup_{P \in \mathcal{Q}} \left(\frac{1}{\mu(P)} \sum_{Q \in \mathcal{T}(P)} \left(|Q|^{-\alpha-1/2}|s_Q|\right)^q \mu(Q) \right)^{1/q}, \tag{3.11}$$

where $\mathcal{T}(P)$ is the tent over P.

To confirm these observations we will prove the following lemma.

Lemma 3.6. *Suppose μ is ρ_A-doubling measure with a doubling constant β. Then, there exists a constant $C > 0$ such that for any integer $M \ge 0$ and for any $f \in \dot{\mathbf{F}}_\infty^{\alpha,q}$*

$$\sup_{P \in \mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{j=-\mathrm{scale}(P)-M}^{\infty} \left(|\det A|^{j\alpha} |f * \varphi_j(x)|\right)^q d\mu(x) \right)^{1/q} \le C |\det A|^{\beta M/q} \|f\|_{\dot{\mathbf{F}}_\infty^{\alpha,q}}. \tag{3.12}$$

Moreover, for any $s \in \dot{\mathbf{f}}_\infty^{\alpha,q}$

$$\sup_{P \in \mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{Q \in \mathcal{T}_M(P)} \left(|Q|^{-\alpha} |s_Q| \tilde{\chi}_Q(x)\right)^q d\mu(x) \right)^{1/q} \le C |\det A|^{\beta M/q} \|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}}, \tag{3.13}$$

where

$$\mathcal{T}_M(P) = \{Q \in \mathcal{Q} : \mathrm{scale}(Q) \le \mathrm{scale}(P) + M\}.$$

Proof. The key to proving (3.12) and (3.13) is the observation that the collection of dilated balls $\mathcal{Q}$ in (3.5) and (3.6) can be replaced by the family of dilated balls $\mathcal{B}$. In fact, a more general result holds.

Suppose that $\{F_j(x) : j \in \mathbb{Z}\}$ is a collection of Borel measurable functions on $\mathbb{R}^n$ with nonnegative values. Then we claim that we have the equivalence of the norms

$$\sup_{P \in \mathcal{Q}} \frac{1}{\mu(P)} \int_P \sum_{j=-\mathrm{scale}(P)}^{\infty} F_j(x)\, d\mu(x) \asymp \sup_{B \in \mathcal{B}} \frac{1}{\mu(B)} \int_B \sum_{j=-\mathrm{scale}(B)}^{\infty} F_j(x)\, d\mu(x). \tag{3.14}$$

Indeed, to prove the lower bound in (3.14) take any $B \in \mathcal{B}$. By Lemma 2.9 we know that

$$\mathcal{Q}_B = \{P \in \mathcal{Q} : P \cap B \neq \emptyset,\ \text{scale}(P) = \text{scale}(B)\}$$

has at most $C_{\mathcal{Q}}$. Therefore,

$$\frac{1}{\mu(B)} \int_B \sum_{j=-\text{scale}(B)}^{\infty} F_j(x)\, d\mu(x) \le C_{\mathcal{Q}} C \sum_{P \in \mathcal{Q}_B} \frac{1}{\mu(P)} \int_P \sum_{j=-\text{scale}(P)}^{\infty} F_j(x)\, d\mu(x)$$

since $B \subset \bigcup_{P \in \mathcal{Q}_B} P$ and $\mu(P) \le C\mu(B)$ for $P \in \mathcal{Q}_B$. Conversely, to prove the upper bound in (3.14) take any $P \in \mathcal{Q}$ and let

$$B_0 = B_{\rho_A}(x_P, |P||\det A|^{-C_{\mathcal{B}}}),\ B_1 = B_{\rho_A}(x_P, |P||\det A|^{C_{\mathcal{B}}}) \in \mathcal{B}\,.$$

Then, by Lemma 2.9

$$\frac{1}{\mu(P)} \int_P \sum_{j=-\text{scale}(P)}^{\infty} F_j(x)\, d\mu(x) \le |\det A|^{2\beta C_{\mathcal{B}}} \frac{1}{\mu(B_1)} \int_{B_1} \sum_{j=-\text{scale}(B_1)}^{\infty} F_j(x)\, d\mu(x)$$

since $\mu(B_1) \le |\det A|^{2\beta C_{\mathcal{B}}} \mu(B_0) \le |\det A|^{2\beta C_{\mathcal{B}}} \mu(P)$, which proves (3.14).

Take any $P \in \mathcal{Q}$, and define the ball $B_2 = B_{\rho_A}(x_P, |\det A|^{M+C_{\mathcal{B}}}|P|) \in \mathcal{B}$, where $M \ge 0$. Then, using (3.14)

$$\begin{aligned}
\frac{1}{\mu(P)} \int_P \sum_{j=-\text{scale}(P)-M}^{\infty} F_j(x)\, d\mu(x) &\le C \frac{1}{\mu(B_1)} \int_{B_1} \sum_{j=-\text{scale}(B_1)-M}^{\infty} F_j(x)\, d\mu(x) \\
&\le C|\det A|^{\beta M} \frac{1}{\mu(B_2)} \int_{B_2} \sum_{j=-\text{scale}(B_2)}^{\infty} F_j(x)\, d\mu(x) \qquad (3.15) \\
&\le C|\det A|^{\beta M} \sup_{Q \in \mathcal{Q}} \frac{1}{\mu(Q)} \int_Q \sum_{j=-\text{scale}(Q)}^{\infty} F_j(x)\, d\mu(x)\,,
\end{aligned}$$

where the constant C is independent of M.

Hence, choosing $F_j(x) = (|\det A|^{j\alpha}|f * \varphi_j(x)|)^q$, (3.14) yields

$$||f||_{\dot{\mathbf{F}}_{\infty}^{\alpha,q}(\mathbb{R}^n,A,\mu)} \asymp \sup_{B \in \mathcal{B}} \left(\frac{1}{\mu(B)} \int_B \sum_{j=-\text{scale}(B)}^{\infty} \left(|\det A|^{j\alpha}|f * \varphi_j(x)|\right)^q d\mu(x) \right)^{1/q}. \qquad (3.16)$$

Moreover, (3.15) yields (3.12).

Likewise, choosing $F_j(x) = (\sum_{Q \in \mathcal{Q},\ \text{scale}(Q)=-j} |Q|^{-\alpha}|s_Q|\tilde{\chi}_Q(x))^q$, (3.14) yields

$$\begin{aligned}
\|s\|_{\dot{\mathbf{f}}_{\infty}^{\alpha,q}(A,\mu)} &= \sup_{P \in \mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{j=-\infty}^{\text{scale}(P)} \sum_{Q \in \mathcal{Q},\ \text{scale}(Q)=j} \left(|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q(x)\right)^q d\mu(x) \right)^{1/q} \\
&= \sup_{P \in \mathcal{Q}} \left(\frac{1}{\mu(P)} \int_P \sum_{j=-\text{scale}(P)}^{\infty} F_j(x)\, d\mu(x) \right)^{1/q} \\
&\asymp \sup_{B \in \mathcal{Q}} \left(\frac{1}{\mu(B)} \int_B \sum_{j=-\text{scale}(B)}^{\infty} F_j(x)\, d\mu(x) \right)^{1/q} \qquad (3.17) \\
&= \sup_{B \in \mathcal{B}} \left(\frac{1}{\mu(B)} \int_B \sum_{Q \in \mathcal{Q},\ \text{scale}(Q) \le \text{scale}(B)} \left(|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q(x)\right)^q d\mu(x) \right)^{1/q}.
\end{aligned}$$

Moreover, a direct calculation shows that (3.15) yields (3.13). □

3.2. Wavelet transforms for $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$

Our next goal is to establish boundedness of φ-transforms for Triebel-Lizorkin spaces for the entire range of parameters $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$ including the special case of $p = \infty$. As a consequence of this result we will deduce two other fundamental results:
–The definition of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces is independent of the choice of a test function φ;
–the completeness of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces.

Definition 3.7. Suppose that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ are such that supp $\hat{\varphi}$, supp $\hat{\psi}$ are compact and bounded away from the origin. Recall that the φ-transform S_φ, often called the *analysis transform*, is the map taking each $f \in \mathcal{S}'/\mathcal{P}$ to the sequence $S_\varphi f = \{(S_\varphi f)_Q\}_{Q\in\mathcal{Q}}$ defined by $(S_\varphi f)_Q = \langle f, \varphi_Q\rangle$. Here, we follow the convention $\langle f, \varphi\rangle = f(\overline{\varphi})$ for $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. The inverse φ-transform, T_ψ, often called the *synthesis transform*, is the map taking the sequence $s = \{s_Q\}_{Q\in\mathcal{Q}}$ to $T_\psi s = \sum_{Q\in\mathcal{Q}} s_Q \psi_Q$.

To see that T_ψ is well-defined for any $s \in \dot{\mathbf{f}}_p^{\alpha,q}$, we will prove the following lemma.

Lemma 3.8. *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, μ is ρ_A-doubling measure, and $\psi \in \mathcal{S}_0(\mathbb{R}^n)$, where $\mathcal{S}_0(\mathbb{R}^n)$ is given by (2.12). Then for any $s \in \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)$, $T_\psi s = \sum_{Q\in\mathcal{Q}} s_Q \psi_Q$ converges in $\mathcal{S}'/\mathcal{P}$. Moreover, the synthesis transform $T_\psi : \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu) \to \mathcal{S}'/\mathcal{P}$ is continuous.*

Proof. Take any $\phi \in \mathcal{S}_0(\mathbb{R}^n)$. We will use the following elementary estimate: For any $L > 0$ there exist constants $N, C > 0$ such that

$$|\langle\psi_Q, \phi_P\rangle| \le C\|\psi\|_N \|\phi\|_N \left(1 + \frac{\rho_A(x_Q - x_P)}{\max(|P|, |Q|)}\right)^{-L} \min\left(\frac{|Q|}{|P|}, \frac{|P|}{|Q|}\right)^L \quad \text{for all} \quad Q, P \in \mathcal{Q}. \tag{3.18}$$

Here, the constant C depends only on $L > 0$ and $\|\phi\|_N = \sup_{x\in\mathbb{R}^n} \sup_{|\gamma|\le N} (1 + |x|)^N |\partial^\gamma \phi(x)|$ is a norm in $\mathcal{S}(\mathbb{R}^n)$. The estimate (3.18) can be proved directly using decay, smoothness, and vanishing moments of $\phi, \psi \in \mathcal{S}_0(\mathbb{R}^n)$. Alternatively, (3.18) follows immediately from the almost diagonal estimates established in [5]. Indeed, modulo a multiplicative constant $c > 0$ the wavelet systems $\{\psi_Q/c\}_{Q\in\mathcal{Q}}$ and $\{\phi_Q/c\}_{Q\in\mathcal{Q}}$ form families of smooth analysis and synthesis molecules in the sense of Definition 5.1 of arbitrary smoothness, decay, and number of vanishing moments. Moreover, the constant $c > 0$ depends linearly on the norms $\|\psi\|_N$ and $\|\phi\|_N$ for some sufficiently large N. Consequently, by [5, Lemma 5.1] the matrix $\{\langle\psi_Q, \phi_P\rangle\}_{Q,P\in\mathcal{Q}}$ is almost diagonal on $\dot{\mathbf{f}}_{p'}^{\alpha',q'}$ for every range of parameters $\alpha' \in \mathbb{R}$, $0 < p', q' < \infty$. A quick inspection of almost diagonal condition, see Definition 4.1, yields (3.18).

Take any $s \in \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)$. By (3.4) we have for $0 < p < \infty$,

$$|s_Q| \le \|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} |Q|^{\alpha+1/2} \mu(Q)^{-1/p} \qquad \text{for all} \quad Q \in \mathcal{Q}.$$

Likewise, by (3.6) we have for $p = \infty$,

$$|s_Q| \le \|s\|_{\dot{\mathbf{f}}_\infty^{\alpha,q}} |Q|^{\alpha+1/2} \qquad \text{for all} \quad Q \in \mathcal{Q}.$$

Applying (3.18) for $P = [0, 1]^n$ yields

$$|\langle\psi_Q, \phi\rangle| \le C\|\phi\|_N \left(1 + \frac{\rho_A(x_Q)}{\max(1, |Q|)}\right)^{-L} \min\left(|Q|, |Q|^{-1}\right)^L \qquad \text{for} \quad Q \in \mathcal{Q},$$

where the constant C is independent of ϕ and Q. Combining the above estimates with Lemma 2.11 yields

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}} |s_Q||\langle\psi_Q,\phi\rangle| \\
&\le C||\phi||_N||s||_{\dot{\mathbf{f}}_p^{\alpha,q}} \sum_{j\in\mathbb{Z}} \sum_{\mathrm{scale}(Q)=j} |Q|^{\alpha+1/2} \min\left(|Q|,|Q|^{-1}\right)^L \frac{\mu(Q)^{-1/p}}{(1+\rho_A(x_Q)/\max(1,|Q|))^L} \\
&\le C||\phi||_N||s||_{\dot{\mathbf{f}}_p^{\alpha,q}} \sum_{j\in\mathbb{Z}} |\det A|^{j(\alpha+1/2)+|j|(2\beta/p+1)-|j|L} \le C||\phi||_N||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}
\end{aligned}$$

for sufficiently large $L > 0$. Hence, the series $T_\psi s = \sum_{Q\in\mathcal{Q}} s_Q\psi_Q$ converges in $\mathcal{S}'/\mathcal{P}$. That is, if we define $T_\psi s$ by

$$\langle T_\psi s,\phi\rangle = \sum_{Q\in\mathcal{Q}} s_Q\langle\psi_Q,\phi\rangle \qquad \text{for all} \quad \phi\in\mathcal{S}_0(\mathbb{R}^n)\,,$$

then we have

$$|\langle T_\psi s,\phi\rangle| \le C||\phi||_N||s||_{\dot{\mathbf{f}}_p^{\alpha,q}} \qquad \text{for all} \quad \phi\in\mathcal{S}_0(\mathbb{R}^n)\,.$$

This shows the continuity of T_ψ and completes the proof of the lemma. □

Definition 3.9. Given a sequence $s = \{s_Q\}_Q$ and $r,\lambda > 0$, define its majorant sequence $s^*_{r,\lambda} = \{(s^*_{r,\lambda})_Q\}_Q$ by

$$\left(s^*_{r,\lambda}\right)_Q = \left(\sum_{P\in\mathcal{Q},\ |P|=|Q|} |s_P|^r/\left(1+|Q|^{-1}\rho_A(x_Q-x_P)\right)^\lambda\right)^{1/r}.$$

Clearly, we always have $|s_Q| \le (s^*_{r,\lambda})_Q$ for any $Q\in\mathcal{Q}$.

In order to prove the boundedness of S_φ and T_ψ, we need the following two lemmas which are generalizations of their dyadic analogues shown by Frazier and Jawerth [20]. Lemma 3.10 was already shown in [5] when $p < \infty$ and $d\mu = w\,dx$ with $w \in A_\infty$. The proofs of Lemmas 3.10 and 3.11 can be found in Section 8.

Lemma 3.10. *Suppose $\alpha\in\mathbb{R}$, $0 < p,q \le \infty$, and μ is ρ_A-doubling measure with a constant β. Then for any $r > 0$ and $\lambda > \beta\max(1,r/q,r/p)$, there is a constant $C > 0$ such that*

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \le \left\|s^*_{r,\lambda}\right\|_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \le C\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \qquad \textit{for all} \quad s=\{s_Q\}_Q\,.$$

Lemma 3.11. *Suppose $\varphi\in\mathcal{S}(\mathbb{R}^n)$ is such that $\operatorname{supp}\hat\varphi$ is compact and bounded away from the origin. For any $f\in\mathcal{S}'/\mathcal{P}$ and $\gamma\in\mathbb{N}$ define the sequences $\sup(f) = \{\sup_Q(f)\}_{Q\in\mathcal{Q}}$ and $\inf(f) = \{\inf_Q(f)\}_{Q\in\mathcal{Q}}$ by setting*

$$\begin{aligned}
\sup{}_Q(f) &= |Q|^{1/2}\sup_{y\in Q}\left|\tilde\varphi_j * f(y)\right| \\
\inf{}_Q(f) &= |Q|^{1/2}\sup\left\{\inf_{y\in P}\left|\tilde\varphi_j * f(y)\right| : \mathrm{scale}(P)=\mathrm{scale}(Q)-\gamma,\ P\cap Q\ne\emptyset\right\},
\end{aligned}$$

where $j = -\mathrm{scale}(Q)$ and $Q\in\mathcal{Q}$.

Suppose that $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$. Then for sufficiently large γ we have

$$||f||_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\tilde{\varphi})} \asymp ||\sup(f)||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \asymp ||\inf(f)||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)}, \tag{3.19}$$

with constants independent of f.

We are now ready to prove the anisotropic version of the fundamental wavelet transform boundedness result of Frazier and Jawerth [20]. In the case of $p < \infty$ and $d\mu = w\,dx$ with $w \in A_\infty$, Theorem 3.12 was already shown in [5] and it remains to prove the case when μ is a ρ_A-general doubling measure or $p = \infty$. However, we will take a slightly different approach than in [5] to accommodate the special case of $p = \infty$. One should add that our argument works without any changes also when $q = \infty$; this case was inadvertently claimed without the proof in [5]. In fact, a result such as Lemma 3.8 is needed there, since sequences with finite support are not dense in $\dot{\mathbf{f}}_p^{\alpha,q}$ when $p = \infty$ or $q = \infty$.

Theorem 3.12. *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure. Assume that $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ are such that $\operatorname{supp}\hat{\varphi}$, $\operatorname{supp}\hat{\psi}$ are compact and bounded away from the origin. Then the operators $S_\varphi : \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)(\tilde{\varphi}) \to \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)$ and $T_\psi : \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu) \to \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)(\varphi)$ are bounded, $\tilde{\varphi}(x) = \overline{\varphi(-x)}$. In addition, if φ, ψ satisfy (2.7), (2.8) then $T_\psi \circ S_\varphi$ is the identity on $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)(\varphi) = \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)(\tilde{\varphi})$.*

Proof. To prove the boundedness of T_ψ, take any $s = \{s_Q\}_Q \in \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)$. We will show that $f = T_\psi s = \sum_Q s_Q \psi_Q$ converges in $\dot{\mathbf{F}}_p^{\alpha,q}$ and we have the bound $||T_\psi s||_{\dot{\mathbf{F}}_p^{\alpha,q}} \le C||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}$. By Lemma 3.8, the series $f = \sum_Q s_Q \psi_Q$ converges in $\mathcal{S}'/\mathcal{P}$. Therefore, the following estimate established in [5, Theorem 3.5] holds for any $\lambda > 1$

$$|f * \varphi_j(x)| \le C \sum_{i=j-M}^{i=j+M} \sum_{\operatorname{scale}(Q)=-i} \left(s^*_{1,\lambda}\right)_Q \tilde{\chi}_Q(x),$$

where M is the smallest integer such that

$$\operatorname{supp}\widehat{\varphi_j} \cap \operatorname{supp}\widehat{\psi_i} = \emptyset \qquad \text{for} \quad |i - j| > M.$$

Consequently, by choosing $\lambda > \beta\max(1, 1/q, 1/p)$, Lemma 3.10 yields the required bound in the case $p < \infty$ by exactly the same argument as in [5, Theorem 3.5]. To deal with the case $p = \infty$, take any $P \in \mathcal{Q}$. Applying (3.13) and Lemma 3.10 with $\lambda > \beta\max(1, 1/q)$, yields

$$\begin{aligned}
&\frac{1}{\mu(P)} \int_P \sum_{j=-\operatorname{scale}(P)}^{\infty} \left(|\det A|^{j\alpha}|f * \varphi_j(x)|\right)^q d\mu(x) \\
&\le C\frac{1}{\mu(P)} \int_P \sum_{j=-\operatorname{scale}(P)}^{\infty} \left(|\det A|^{j\alpha} \sum_{i=j-M}^{i=j+M} \sum_{\operatorname{scale}(Q)=-i} \left(s^*_{1,\lambda}\right)_Q \tilde{\chi}_Q(x)\right)^q d\mu(x) \\
&\le C \sum_{l=-M}^{M} \frac{1}{\mu(P)} \int_P \sum_{j=-\operatorname{scale}(P)}^{\infty} \sum_{\operatorname{scale}(Q)=-j+l} \left(|\det A|^{j\alpha}\left|\left(s^*_{1,\lambda}\right)_Q\right|\tilde{\chi}_Q(x)\right)^q d\mu(x) \\
&= C \sum_{l=-M}^{M} |\det A|^{l\alpha} \frac{1}{\mu(P)} \int_P \sum_{Q\in\mathcal{Q},\ \operatorname{scale}(Q)\le\operatorname{scale}(P)+l} \left(|Q|^{-\alpha}\left|\left(s^*_{1,\lambda}\right)_Q\right|\tilde{\chi}_Q(x)\right)^q d\mu(x) \\
&\le C|\det A|^{M(|\alpha|+\beta)} \left\|s^*_{1,\lambda}\right\|^q_{\dot{\mathbf{f}}_\infty^{\alpha,q}} \le C'\|s\|^q_{\dot{\mathbf{f}}_\infty^{\alpha,q}}.
\end{aligned}$$

Taking the supremum over all $P \in \mathcal{Q}$ shows $\|T_\psi s\|_{\dot{\mathbf{F}}^{\alpha,q}_\infty} \le C'\|s\|_{\dot{\mathbf{f}}^{\alpha,q}_\infty}$ for all $s \in \dot{\mathbf{f}}^{\alpha,q}_\infty$.

The boundedness of S_φ follows immediately from Lemma 3.11. Indeed, suppose that $f \in \dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\tilde{\varphi})$ and $Q = A^{-j}([0,1]^n + k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. Then

$$|(S_\varphi f)_Q| = |\langle f, \varphi_Q\rangle| = |Q|^{1/2}\big|(\tilde{\varphi}_j * f)(x_Q)\big| \le \sup{}_Q(f)\,,$$

and it suffices to invoke (3.19). We remark here that the boundedness of S_φ in the case $p < \infty$ can be shown more directly without the use of Lemma 3.11, see [5]. However, in the case $p = \infty$ this lemma is indispensable.

Finally, if we assume additionally that φ and ψ satisfy (2.7) and (2.8), then by Lemma 2.14, $T_\psi \circ S_\varphi$ is the identity on $\dot{\mathbf{F}}^{\alpha,q}_p$. More precisely, $\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\tilde{\varphi}) \hookrightarrow \dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\varphi)$ is a bounded inclusion. Hence, by reversing the roles of φ and $\tilde{\varphi}$ we have

$$\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\tilde{\varphi}) = \dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\varphi)\,,$$

which completes the proof of Theorem 3.12. □

3.3. Completeness of $\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)$ and canonical representatives in $\mathcal{S}'$

As a corollary of Theorem 3.12 we obtain that the definition of $\dot{\mathbf{F}}^{\alpha,q}_p$ spaces is independent of $\varphi \in \mathcal{S}$. The proof of Corollary 3.13 is identical as the proof of the same result in the range $0 < p < \infty$, see [5, Corollary 3.7].

Corollary 3.13. *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is ρ_A-doubling measure. Then the space $\dot{\mathbf{F}}^{\alpha,q}_p$ is well-defined in the sense that, for any φ^1 and φ^2 satisfying* (3.2) *and* (3.3), *their associated quasi-norms in $\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)(\varphi^i)$, $i = 1, 2$, are equivalent, i.e., there exist constants $C_1, C_2 > 0$ such that*

$$C_1\|f\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)(\varphi^1)} \le \|f\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)(\varphi^2)} \le C_2\|f\|_{\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n,A,\mu)(\varphi^1)}\,. \tag{3.20}$$

Finally, Theorem 3.12 also yields the completeness of $\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)$ spaces.

Corollary 3.14. *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is ρ_A-doubling measure. The inclusion map $i : \dot{\mathbf{F}}^{\alpha,q}_p = \dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu) \hookrightarrow \mathcal{S}'/\mathcal{P}$ is continuous. Moreover, $\dot{\mathbf{F}}^{\alpha,q}_p$ equipped with $\|\cdot\|_{\dot{\mathbf{F}}^{\alpha,q}_p}$ is a complete quasi-normed space.*

Proof. Suppose that φ and ψ satisfy (2.7) and (2.8). By Lemma 3.8 the map $T_\psi : \dot{\mathbf{f}}^{\alpha,q}_p \to \mathcal{S}'/\mathcal{P}$ is continuous and by Theorem 3.12 the map $S_\varphi : \dot{\mathbf{F}}^{\alpha,q}_p \to \dot{\mathbf{f}}^{\alpha,q}_p$ is also continuous. Hence, by Lemma 2.14, $i = T_\psi \circ S_\varphi : \dot{\mathbf{F}}^{\alpha,q}_p \to \mathcal{S}'/\mathcal{P}$ is a continuous inclusion.

Once the continuity of the inclusion map i is established, the completeness of $\dot{\mathbf{F}}^{\alpha,q}_p(\mathbb{R}^n, A, \mu)$ follows by a standard argument using Fatou's Lemma. □

Note that the proof of Corollary 3.14 is much less involved than that of the corresponding result in [5]. This is because the current proof relies on Lemma 3.8 and it is a consequence of the main Theorem 3.12. In [5] the completeness of $\dot{\mathbf{F}}^{\alpha,q}_p$ spaces was established before and in fact it was used in the proof of Theorem 3.12.

When studying smooth molecular decompositions we will need the following result borrowed from [5, 20], which resolves all sorts of issues caused by the fact the elements of $\dot{\mathbf{F}}^{\alpha,q}_p$ are equivalence classes of tempered distributions $\mathcal{S}'/\mathcal{P}$. Proposition 3.15 guarantees the existence of canonical representatives of elements in $\dot{\mathbf{F}}_p^{\alpha,q}$ modulo polynomials of degree $\leq L = \lfloor \alpha/\zeta_- \rfloor$.

Proposition 3.15. *Suppose that $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, and μ is a ρ_A-doubling measure. Let $f \in \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$. For any $\varphi^1 \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{\varphi^1}$ is compact and bounded away from the origin, and (2.13) holds, there exists a sequence of polynomials $\{P_k^1\}_{k=1}^{\infty}$ with $\deg P_k^1 \leq L = \lfloor \alpha/\zeta_- \rfloor$ such that*

$$g^1 := \lim_{k\to\infty} \left(\sum_{j=-k}^{\infty} (\varphi^1)_j * f + P_k^1 \right) \tag{3.21}$$

exists in $\mathcal{S}'$. Moreover, if g^2 is the corresponding limit in (3.21) for some other $\varphi^2 \in \mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{\varphi^2}$ is compact and bounded away from the origin, and (2.13) holds, then

$$g^1 - g^2 \in \mathcal{P} \quad \text{and} \qquad \deg\left(g^1 - g^2\right) \leq L\,. \tag{3.22}$$

Proof. The key estimate in the proof of Proposition 3.15 is that for any $j < 0$ and a multi-index γ we have

$$\sup_{x\in\mathbb{R}^n} \frac{\left|\partial^\gamma \left((\varphi^1)_j * f\right)(x)\right|}{(1+|x|)^N} \leq C |\det A|^{j(|\gamma|\zeta_- - \alpha)} ||f||_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)}\,. \tag{3.23}$$

In the case $p < \infty$, (3.23) is shown exactly in the same way as [5, Proposition 3.8] using [3, Corollary 3.2], which is an improved version of [5, Corollary 3.1], valid in the setting of ρ_A-doubling measures. However, in the case $p = \infty$ we need a minor modification of our argument. Using the same arguments as in [5] for any $j < 0$ and $M \in \mathbb{N}$ we have

$$\begin{aligned}
\sup_{|\gamma|=M} \left\|\partial^\gamma \left((\varphi^1)_j * f\right)\right\|_\infty &\leq C |\det A|^{jM\zeta_-} \sup_{|\gamma|=M} \left\|\left(\partial^\gamma \varphi^1\right)_j * f\right)\right\|_\infty \\
&\leq C |\det A|^{j(M\zeta_- - \alpha)} \sup_{|\gamma|=M} ||f||_{\dot{\mathbf{F}}_\infty^{\alpha,\infty}(\mathbb{R}^n,A,\mu)(\partial^\gamma \varphi_1)} \\
&\leq C |\det A|^{j(M\zeta_- - \alpha)} ||f||_{\dot{\mathbf{F}}_\infty^{\alpha,\infty}} \leq C |\det A|^{j(M\zeta_- - \alpha)} ||f||_{\dot{\mathbf{F}}_\infty^{\alpha,q}}\,,
\end{aligned}$$

where in the last step we used [4, Corollary 3.7]. This shows that the crucial estimate (3.23) holds also for $p = \infty$ and the rest of the proof is identical as in [5, Proposition 3.8] and hence it is skipped. □

As a corollary of Lemma 2.14 and Proposition 3.15, we have the following.

Corollary 3.16. *Let $f \in \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$. For any admissible pair of dual frame wavelets (φ^1, ψ^1), there exists a sequence of polynomials $\{P_k^1\}_{k=1}^{\infty}$ with $\deg P_k^1 \leq L = \lfloor \alpha/\zeta_- \rfloor$, such that*

$$g^1 := \lim_{k\to\infty} \left(\sum_{Q\in\mathcal{Q},\ |\det A|^{-k} \leq |Q| \leq |\det A|^k} \langle f, (\varphi^1)_Q \rangle (\psi^1)_Q + P_k^1 \right) \tag{3.24}$$

exists in $\mathcal{S}'$. Moreover, if g^2 is the corresponding limit in (3.24) for some other such pair (φ^2, ψ^2), then (3.22) holds.

4. Almost diagonal operators

In this section we probe the boundedness of almost diagonal operators on $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$. Almost diagonal operators were introduced in the dyadic case by Frazier and Jawerth [20] with the aim of proving boundedness results for operators in $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces. That is, one can always translate a problem of a boundedness of an operator on $\dot{\mathbf{F}}_p^{\alpha,q}$ to the equivalent problem in the corresponding wavelet domain $\dot{\mathbf{f}}_p^{\alpha,q}$ by using Theorem 3.12. Since operators on sequence spaces are in general more tractable, this approach results in greater simplicity.

We start by recalling the definition of almost diagonal operators in the setting of expansive dilations. Since we deal with a more general situation than in [5] it is compulsory to adjust the definition of the decay parameter J which depends on the doubling constant of μ instead of the regularity of a weight $w \in A_\infty$.

Definition 4.1. Suppose $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure. Let $J = \beta \max(1, 1/p, 1/q)$. We say that an operator $\mathcal{A}$, with an associated matrix $\{a_{QP}\}_{Q,P\in\mathcal{Q}}$, where $a_{QP} = (\mathcal{A}e^P)_Q$, is an *almost diagonal* operator on $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$, if there exists an $\epsilon > 0$ such that,

$$\sup_{Q,P\in\mathcal{Q}} |a_{QP}|/\kappa_{QP}(\epsilon) < \infty \tag{4.1}$$

where

$$\kappa_{QP}(\epsilon) = \left(\frac{|Q|}{|P|}\right)^{\alpha+1/2}\left(1+\frac{\rho_A(x_Q-x_P)}{\max(|P|,|Q|)}\right)^{-J-\epsilon} \min\left[\left(\frac{|Q|}{|P|}\right)^{\epsilon}, \left(\frac{|P|}{|Q|}\right)^{J+\epsilon}\right].$$

Theorem 4.2. *Suppose $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and μ is a ρ_A-doubling measure. An almost diagonal operator $\mathcal{A}$ is bounded as a linear operator on $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$.*

Proof. By a standard rescaling argument it suffices to prove Theorem 4.2 in the case $\alpha = 0$, see [5, Theorem 4.1].

First, we consider the case $\min(p,q) > 1$, which implies that $J = \beta$ in Definition 4.1. In addition we also assume that $p < \infty$. Let $\mathcal{A}$ be an almost diagonal operator on $\dot{\mathbf{f}}_p^{0,q}$ with matrix $\{a_{QP}\}_{Q,P}$ satisfying condition (4.1). We write $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$, with

$$(\mathcal{A}_0 s)_Q = \sum_{P\in\mathcal{Q},\ |P|\ge|Q|} a_{QP}s_P \qquad \text{and} \qquad (\mathcal{A}_1 s)_Q = \sum_{P\in\mathcal{Q},\ |P|<|Q|} a_{QP}s_P$$

for $s = \{s_P\}_P \in \dot{\mathbf{f}}_p^{0,q}$. For $Q \in \mathcal{Q}$, $\operatorname{scale}(Q) = j$, and $x \in Q$, we have

$$\begin{aligned}
|(\mathcal{A}_1 s)_Q| &\le C \sum_{|P|<|Q|} \kappa_{QP}(\epsilon)|s_P| \le C \sum_{|P|<|Q|} \left(\frac{|P|}{|Q|}\right)^{-1/2+J+\epsilon} \frac{|s_P|}{\left(1+|Q|^{-1}\rho_A(x_P-x_Q)\right)^{\beta+\epsilon}} \\
&= C\sum_{i<j} |\det A|^{(i-j)(-1/2+\beta+\epsilon)} \sum_{\operatorname{scale}(P)=i} \frac{|s_P|}{\left(1+|Q|^{-1}\rho_A(x_P-x_Q)\right)^{\beta+\epsilon}} \\
&\le C\sum_{i<j} |\det A|^{(j-i)(1/2-\epsilon)} M_{\rho_A}\Bigg(\sum_{\operatorname{scale}(P)=i} |s_P\chi_P|\Bigg)(x)
\end{aligned}$$

using Lemma 8.1 with $a = r = 1$ and $\lambda = \beta + \epsilon$. Hence, we have

$$\sum_{\operatorname{scale}(Q)=j} |(\mathcal{A}_1 s)_Q \tilde{\chi}_Q|^q \le C\Bigg(\sum_{i<j} |\det A|^{(i-j)\epsilon} M_{\rho_A}\Bigg(\sum_{\operatorname{scale}(P)=i} |s_P\tilde{\chi}_P|\Bigg)\Bigg)^q.$$

Therefore, by Minkowski's inequality for ℓ^q spaces

$$\|\mathcal{A}_1 s\|_{\dot{\mathbf{f}}_p^{0,q}} \leq C \left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{i<0} |\det A|^{i\epsilon} M_{\rho_A}\left(\sum_{\text{scale}(P)=j-i} |s_P \tilde{\chi}_P| \right)\right)^q \right)^{1/q} \right\|_{L^p(\mu)}$$

$$\leq C \left\| \left(\sum_{j\in\mathbb{Z}} \left(M_{\rho_A}\left(\sum_{\text{scale}(P)=j} |s_P \tilde{\chi}_P| \right)\right)^q \right)^{1/q} \right\|_{L^p(\mu)} .$$

By Theorem 2.8 we conclude that

$$\|\mathcal{A}_1 s\|_{\dot{\mathbf{f}}_p^{0,q}} \leq C \left\| \left(\sum_P |s_P \tilde{\chi}_P|^q \right)^{1/q} \right\|_{L^p(\mu)} = C \|s\|_{\dot{\mathbf{f}}_p^{0,q}} .$$

To show the corresponding estimate for $\mathcal{A}_0$, we apply the same argument as for $\mathcal{A}_1$ using the condition

$$\kappa_{QP}(\epsilon) \leq C \left(\frac{|Q|}{|P|} \right)^{1/2+\epsilon} \left(1 + |P|^{-1} \rho_A(x_P - x_Q)\right)^{-\beta-\epsilon} .$$

Therefore, both $\mathcal{A}_0$ and $\mathcal{A}_1$ are bounded on $\dot{\mathbf{f}}_p^{0,q}$ and, hence, $\mathcal{A}$ is also bounded when $p < \infty$.

The case $\min(p,q) \leq 1$ can be shown in two ways. One can estimate $\|\mathcal{A}s\|_{\dot{\mathbf{f}}_p^{0,q}}$ directly using Lemma 8.1 for appropriate choices of parameters a, r, λ as in the proof of Lemma 3.10. Alternatively, the case $r = \min(p,q) \leq 1$ can be reduced to the case $r > 1$ as in [20, Theorem 3.3] and [5, Theorem 4.1]. For the sake of completeness we recall this argument.

We observe that $\mathcal{A} = \{a_{QP}\}_{Q,P}$ is almost diagonal on $\dot{\mathbf{f}}_p^{0,q}$, i.e., (4.1) holds for some $\epsilon > 0$ if and only if

$$\mathcal{A}' = \{a'_{QP}\}_{Q,P} = \{|a_{QP}|^r (|Q|/|P|)^{1/2-r/2}\}_{Q,P}$$

is almost diagonal on $\dot{\mathbf{f}}_{p/r}^{0,q/r}$, i.e., (4.1) holds for $\{a'_{QP}\}_{Q,P}$ and $\epsilon' = r\epsilon$. Hence, we can pick an $\tilde{r} < r$ so close to r that the almost diagonal condition (4.1) still holds with $r = \min(p,q)$ replaced by $\tilde{r}$. This means that $p/\tilde{r} > 1$, $q/\tilde{r} > 1$, and that the matrix

$$\tilde{\mathcal{A}} = \{\tilde{a}_{QP}\}_{Q,P} = \left\{ |a_{QP}|^{\tilde{r}} \left(\frac{|Q|}{|P|} \right)^{1/2-\tilde{r}/2} \right\}_{Q,P}$$

satisfies the almost diagonal condition (4.1) on $\dot{\mathbf{f}}_{p/\tilde{r}}^{0,q/\tilde{r}}$ for a smaller value of $\tilde{\epsilon}$ than $\epsilon' = r\epsilon$, since $\tilde{J} = \beta \max(1, \tilde{r}/p, \tilde{r}/q) = \beta$. Indeed, we have

$$|\tilde{a}_{QP}| \leq C \left(\frac{|Q|}{|P|} \right)^{1/2} \left(1 + \frac{\rho_A(x_Q - x_P)}{\max(|Q|,|P|)} \right)^{-\beta\tilde{r}/r - \tilde{r}\epsilon} \min\left[\left(\frac{|Q|}{|P|} \right)^{\tilde{r}\epsilon}, \left(\frac{|P|}{|Q|} \right)^{\beta\tilde{r}/r + \tilde{r}\epsilon} \right] .$$

Given $s \in \dot{\mathbf{f}}_p^{0,q}$, define $t = \{t_Q\}_Q$ by $t_Q = |Q|^{1/2-\tilde{r}/2} |s_Q|^{\tilde{r}}$. Then

$$\|t\|_{\dot{\mathbf{f}}_{p/\tilde{r}}^{0,q/\tilde{r}}}^{1/\tilde{r}} = \left\| \left(\sum_{Q\in\mathcal{Q}} \left(|Q|^{1/2-\tilde{r}/2} |s_Q|^{\tilde{r}} \tilde{\chi}_Q \right)^{q/\tilde{r}} \right)^{\tilde{r}/q} \right\|_{L^{p/\tilde{r}}(\mu)}^{1/\tilde{r}} = \|s\|_{\dot{\mathbf{f}}_p^{0,q}} . \tag{4.2}$$

The equality (4.2) also holds for $p = \infty$, where the localized definition (3.6) is used instead of (3.4). By the $\tilde{r}$-inequality, we have

$$|(\mathcal{A}s)_Q| \leq \left(\sum_P |a_{QP}|^{\tilde{r}} |s_P|^{\tilde{r}} \right)^{1/\tilde{r}} = \left(|Q|^{\tilde{r}/2-1/2} \sum_P |\tilde{a}_{QP}| |t_P| \right)^{1/\tilde{r}} = \left(|Q|^{\tilde{r}/2-1/2} (\tilde{\mathcal{A}}t)_Q \right)^{1/\tilde{r}} .$$

Hence, using (4.2) twice

$$\|\mathcal{A}s\|_{\dot{\mathbf{f}}_p^{0,q}} \le \|\tilde{\mathcal{A}}t\|^{1/\tilde{r}}_{\dot{\mathbf{f}}^{0,q/\tilde{r}}_{p/\tilde{r}}} \le C\|t\|^{1/\tilde{r}}_{\dot{\mathbf{f}}^{0,q/\tilde{r}}_{p/\tilde{r}}} = C\|s\|_{\dot{\mathbf{f}}_p^{0,q}}$$

since $\tilde{\mathcal{A}}$ is bounded on $\dot{\mathbf{f}}^{0,q/\tilde{r}}_{p/\tilde{r}}(A,\mu)$, by Theorem 4.2 in the already shown case $\min(p,q) > 1$.

Finally, the case $p = \infty$ and $q > 1$ requires a special argument due to localized definition of $\dot{\mathbf{f}}_p^{\alpha,q}$ spaces. A direct approach is quite complicated since it must involve local estimates as in the proof of Lemma 3.10 when $p = \infty$. Instead, it is easier to apply the duality argument using the results established in [4]. By [4, Corollary 3.5], the spaces $\dot{\mathbf{f}}_\infty^{\alpha,q}(A,\mu)$ do not depend on the choice of μ, at least when $d\mu = w\,dx$ for some $w \in A_\infty$. Hence, without much loss of generality we can restrict ourselves to the unweighted case. By [4, Corollary 4.5] we have the duality $(\dot{\mathbf{f}}_1^{0,q'})^* \approx \dot{\mathbf{f}}_\infty^{\alpha,q}$, where $1/q + 1/q' = 1$. Define the transpose of $\mathcal{A}$ by $\mathcal{A}' = \{\overline{a_{PQ}}\}_{Q,P\in\mathcal{Q}}$. Since $\mathcal{A}$ is almost diagonal on $\dot{\mathbf{f}}_\infty^{\alpha,q}$ by the symmetry and $J = 1$, so is $\mathcal{A}'$ on $\dot{\mathbf{f}}_1^{0,q'}$. By the already shown case $\mathcal{A}'$ is bounded on $\dot{\mathbf{f}}_1^{0,q'}$. Hence, its adjoint operator $(\mathcal{A}')^* = \mathcal{A}$ is bounded on $(\dot{\mathbf{f}}_1^{0,q'})^* \approx \dot{\mathbf{f}}_\infty^{\alpha,q}$. Note that the identification $(\mathcal{A}')^* = \mathcal{A}$ follows from the duality pairing given by the usual scalar product of sequences indexed by $\mathcal{Q}$. This completes the proof of Theorem 4.2. □

5. Smooth atomic and molecular decompositions

In this section we extend smooth atomic and molecular decompositions of Frazier and Jawerth [20] to the setting of expansive dilations and doubling measures. The corresponding results for A_∞ weights in the case $p < \infty$ were shown in [5] and here we describe the necessary modifications which are needed for these arguments to work.

We start by recalling the definitions of smooth molecules.

Definition 5.1. Suppose $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure with doubling constant β. Let $J = \beta\max(1, 1/p, 1/q)$ and $N = \max(\lfloor (J - \alpha - 1)/\zeta_- \rfloor, -1)$.

We say that $\Psi_Q(x)$ is a *smooth synthesis molecule* for $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ supported near $Q \in \mathcal{Q}$ with $\mathrm{scale}(Q) = -j$ and $j \in \mathbb{Z}$, if there exist $M > J$ such that

$$\left|\partial^\gamma\left[\Psi_Q(A^{-j}\cdot)\right](x)\right| \le \frac{|\det A|^{j/2}}{\left(1+\rho_A(x - A^j x_Q)\right)^M} \quad \text{for} \quad |\gamma| \le \lfloor \alpha/\zeta_- \rfloor + 1\,, \tag{5.1}$$

$$|\Psi_Q(x)| \le \frac{|\det A|^{j/2}}{\left(1+\rho_A\left(A^j(x - x_Q)\right)\right)^{\max(M,(M-\alpha)\zeta_+/\zeta_-)}}, \tag{5.2}$$

$$\int x^\gamma \Psi_Q(x)\,dx = 0 \qquad \text{for} \quad |\gamma| \le N\,. \tag{5.3}$$

We say that $\Phi_Q(x)$ is a *smooth analysis molecule* for $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ supported near $Q \in \mathcal{Q}$ with $\mathrm{scale}(Q) = -j$ and $j \in \mathbb{Z}$, if there exists $M > J$ such that

$$\left|\partial^\gamma\left[\Phi_Q(A^{-j}\cdot)\right](x)\right| \le \frac{|\det A|^{j/2}}{\left(1+\rho_A(x - A^j x_Q)\right)^M} \quad \text{for} \quad |\gamma| \le N+1\,, \tag{5.4}$$

$$|\Phi_Q(x)| \le \frac{|\det A|^{j/2}}{\left(1+\rho_A\left(A^j(x - x_Q)\right)\right)^{\max(M,1+\alpha\zeta_+/\zeta_-+M-J)}}, \tag{5.5}$$

$$\int x^\gamma \Phi_Q(x)\,dx = 0 \qquad \text{for} \quad |\gamma| \le \lfloor \alpha/\zeta_- \rfloor\,. \tag{5.6}$$

We say that $\{\Phi_Q\}_{Q\in\mathcal{Q}}$ is a *family of smooth synthesis (analysis) molecules*, if each Φ_Q is a smooth synthesis (analysis) molecule supported near Q.

Remark 5.2. Note that the above definition of smooth molecules is identical with [5, Definition 5.1]. The only exception is the method of determining the decay parameter J, and consequently, the vanishing moment parameter N. Recall from [5] that when $w \in A_\infty$, the decay parameter defined by $J = \max(1, r_0/p, 1/q)$, where $r_0 = \inf\{r : w \in A_r\}$, coincides with the decay parameter in the definition of almost diagonal operators in [5, Definition 4.1]. Therefore, in both situations the decay parameter J originates in the same way.

The key ingredient in proving smooth molecular decompositions is the following lemma, which is a nonisotropic variant of [20, Corollary B.3].

Lemma 5.3. *Suppose that $\{\Phi_Q\}_Q$ and $\{\Psi_Q\}_Q$ are families of smooth analysis and synthesis molecules for $\dot{\mathbf{F}}_p^{\alpha,q}$, respectively. Then the matrix $\{a_{QP}\}$, given by $a_{QP} = \langle\Psi_P, \Phi_Q\rangle$, is almost diagonal on $\dot{\mathbf{f}}_p^{\alpha,q}$. More precisely, there exist $C > 0$ and $\epsilon > 0$, such that*

$$|\langle\Psi_P, \Phi_Q\rangle| \le C\kappa_{QP}(\epsilon) \qquad \textit{for all} \quad Q, P \in \mathcal{Q}\,.$$

In the setting of nonexpansive dilations and A_∞ weights Lemma 5.3 was proved in [5]. In fact, a close inspection of this argument shows that given any $J \ge 1$, $\alpha \in \mathbb{R}$, and families of functions $\{\Psi_Q\}$ and $\{\Phi_Q\}$ satisfying (5.1)–(5.6), the matrix $\{\langle\Psi_P, \Phi_Q\rangle\}_{Q,P}$ satisfies almost diagonality estimate (4.1) for some $\epsilon > 0$. Therefore, Remark 5.2 shows that Lemma 5.3 holds in the current setting of ρ_A-doubling measures.

As a consequence of Lemma 5.3 we obtain the following result.

Theorem 5.4 (Smooth Molecular Analysis and Synthesis). *Suppose that A is an expansive matrix, $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure. Then there exists a constant $C > 0$, such that:*

(i) *If $\{\Psi_Q\}_Q$ is a family of smooth synthesis molecules for $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$, then*

$$\Big\| \sum_{Q\in\mathcal{Q}} s_Q\Psi_Q \Big\|_{\dot{\mathbf{F}}_p^{\alpha,q}} \le C\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} \qquad \textit{for all} \quad s = \{s_Q\}_Q \in \dot{\mathbf{f}}_p^{\alpha,q}(A, \mu)\,.$$

(ii) *If $\{\Phi_Q\}_Q$ is a family of smooth analysis molecules, then*

$$\|\{\langle f, \Phi_Q\rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha,q}} \le C\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}} \qquad \textit{for all} \quad f \in \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)\,.$$

The proof of Theorem 5.4 follows along the lines of the corresponding results in [5] with the use of Lemma 5.3. The biggest technical difficulty in the proof of the above theorem is to justify the meaningfulness of the pairing $\langle f, \Phi_Q\rangle$ since $f \in \dot{\mathbf{F}}_p^{\alpha,q}$ is an equivalence class in $\mathcal{S}'/\mathcal{P}$, and Φ_Q may not even belong to $\mathcal{S}$. However, the usual pairing procedure as in [5, Lemma 5.7] works.

Lemma 5.5. *Suppose that $f \in \dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ and Φ_Q is a smooth analysis molecule for $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ supported near $Q \in \mathcal{Q}$. Then for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (2.7) and (2.8), the series*

$$\langle f, \Phi_Q\rangle := \sum_{j\in\mathbb{Z}} \big\langle\tilde{\varphi}_j * \psi_j * f, \Phi_Q\big\rangle = \sum_{P\in\mathcal{Q}} \langle f, \varphi_P\rangle\langle\psi_P, \Phi_Q\rangle \tag{5.7}$$

converges absolutely and its value is independent of the choice of φ and ψ satisfying (2.7) *and* (2.8).

The proof of Lemma 5.5 is exactly the same as that of [5, Lemma 5.7] and uses Proposition 3.15 and Corollary 3.16, and hence it is skipped. Finally, we assert that the elements of $\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n, A, \mu)$ admit smooth atomic decompositions.

Definition 5.6. A function $a_Q(x)$ is said to be a *smooth atom* supported near a cube $Q = A^{-j}([0,1]^n + k) \in \mathcal{Q}$ if it satisfies

$$\operatorname{supp} a_Q \subset A^{-j}\big([-\delta_0, 1+\delta_0]^n + k\big)\,, \tag{5.8}$$

where $\delta_0 > 0$ is some fixed constant, and

$$\big|\partial^\gamma\big[a_Q(A^{-j}\,\cdot\,)\big](x)\big| \le |Q|^{-1/2} \qquad \text{for} \quad |\gamma| \le \tilde{K}\,, \tag{5.9}$$

$$\int_{\mathbb{R}^n} x^\gamma a_Q(x)\,dx = 0 \qquad \text{for} \quad |\gamma| \le \tilde{N}\,, \tag{5.10}$$

where $\tilde{N} \ge N$ is the same as in Definition 5.1 and $\tilde{K} \ge \max(\lfloor \alpha/\zeta_- \rfloor + 1, 0)$. Recall that

$$N = \max(\lfloor (J - \alpha - 1)/\zeta_- \rfloor, -1) \quad \text{where } J = \beta \max(1, 1/p, 1/q).$$

We say that $\{a_Q\}_{Q\in\mathcal{Q}}$ is a *family of smooth atoms*, if each function a_Q is a smooth atom supported near Q.

Theorem 5.7 (Smooth Atomic Decomposition). *Suppose that A is an expansive matrix, $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and μ is a ρ_A-doubling measure. For any $f \in \dot{\mathbf{F}}_p^{\alpha,q}$ there exists a family of smooth atoms $\{a_Q\}$ and a sequence of coefficients $s = \{s_Q\} \in \dot{\mathbf{f}}_p^{\alpha,q}$, such that,*

$$f = \sum_{Q\in\mathcal{Q}} s_Q a_Q, \qquad \textit{and} \qquad \|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} \le C\|f\|_{\dot{\mathbf{F}}_p^{\alpha,q}}\,, \tag{5.11}$$

where the above series converges unconditionally in $\dot{\mathbf{F}}_p^{\alpha,q}$. Conversely, for any family of smooth atoms $\{a_Q\}$,

$$\Big\|\sum_Q s_Q a_Q\Big\|_{\dot{\mathbf{F}}_p^{\alpha,q}} \le C\|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}}\,. \tag{5.12}$$

The proof of Theorem 5.7 uses Theorems 3.12 and 5.4 and is a verbatim copy of the corresponding result in [5]. Hence, it is skipped.

Remark 5.8. At this point, it should be clear that the theory of anisotropic Triebel-Lizorkin spaces introduced in [5] extends to the setting of doubling measures. In particular, the results for inhomogeneous Triebel-Lizorkin spaces can be deduced from the corresponding results for homogeneous $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces by the same arguments as in [5, 20]. Moreover, we conjecture that the results in the inhomogeneous case are valid under a weaker hypothesis of local doubling, i.e., (2.3) holds only for $r < 1$. Indeed, Rychkov [30] extended several results on inhomogeneous Triebel-Lizorkin and Besov spaces to the weighted (but isotropic) setting for the class of local Muckenhoupt weights A_p^{loc}. Hence, it seems very plausible that similar results can be obtained in the nonisotropic setting. However, we will not pursue this direction here.

Remark 5.9. Despite certain gain of generality of this work compared to its predecessor [5], one should emphasize that the results obtained here and there have some fundamental differences. For example, the decay and vanishing moment parameters in the definition of smooth molecules depend on the doubling constant of a measure μ instead of the regularity of a weight $w \in A_\infty$ as in [5]. Consequently, the results of [5] have better quantitative characteristics than the ones obtained here as long as we stay in the realm of A_∞ weights. This is a prize to be paid by studying Triebel-Lizorkin spaces with doubling measures instead of A_∞ weights.

6. Nonsmooth atomic decompositions of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces for $0 < p \le 1$

The goal of this section is to establish a more traditional type of atomic decomposition of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces than Theorem 5.7, where the coefficients in atomic decompositions are controlled by the ℓ^p norms rather than more cumbersome $\dot{\mathbf{f}}_p^{\alpha,q}$ norms. Obviously, there is a prize to pay for this. One must restrict the range to $0 < p \le 1$ and allow less regular atoms in our decompositions.

We will follow a more direct approach to nonsmooth atomic decompositions as described by Grafakos [24] instead of a slightly roundabout approach via real interpolation by Frazier and Jawerth [20, Section 7]. Naturally, we will work on the sequence space level and hence we start by introducing the concept of atoms for $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$ spaces.

Definition 6.1. Suppose that $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and μ is a ρ_A-doubling measure. We say that a sequence $r = \{r_Q\}_Q$ is a p_1-*atom* for $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$, where $p \le p_1 \le \infty$, if there exists $\bar{Q} \in \mathcal{Q}$ such that

$$r_Q = 0 \qquad \text{if} \quad \text{scale}(Q) > \text{scale}(\bar{Q}) \text{ or } \quad |Q \cap \bar{Q}| = 0\,, \tag{6.1}$$

$$\left\| G^{\alpha,q}(r) \right\|_{L^{p_1}(\mu)} \le \mu(\bar{Q})^{1/p_1 - 1/p}\,, \tag{6.2}$$

where

$$G^{\alpha,q}(r) = \left(\sum_{P \in \mathcal{Q}} \left(|P|^{-\alpha} |r_P| \tilde{\chi}_P \right)^q \right)^{1/q}. \tag{6.3}$$

Remark 6.2. In other words, (6.1) says that the support of an atom r must be located at the tent $\mathcal{T}(\bar{Q})$ over $\bar{Q}$. That is, r_Q could be nonzero only on the cubes $Q \in \mathcal{Q}$ which have nonzero intersection with $\bar{Q}$, $|Q \cap \bar{Q}| > 0$, and lie at scales at most of scale$(\bar{Q})$.

For any $p \le p_1 < p_2 \le \infty$, every p_2-atom r for $\dot{\mathbf{f}}_p^{\alpha,q}$ is also a p_1-atom modulo a multiplicative constant c independent of r, i.e., cr is a p_1-atom for $\dot{\mathbf{f}}_p^{\alpha,q}$. To see this, it suffices to use Hölder's inequality, and observe that the support of $G^{\alpha,q}(r)$ is contained in a dilated ball B with scale(B) controlled by scale$(\bar{Q})$ due to Lemma 2.9. Hence, we will work mostly with ∞-atoms r, which satisfy

$$\left\| G^{\alpha,q}(r) \right\|_{L^\infty} \le \mu(\bar{Q})^{-1/p}\,. \tag{6.4}$$

Note that we adopt a slightly more restrictive definition of ∞-atoms than the original approach of Frazier and Jawerth [20] by following [24, Section 6.6.c]. Indeed, (6.4) is replaced in [20] by

$$||r||_{\dot{\mathbf{f}}_\infty^{\alpha,q}} \le \mu(\bar{Q})^{-1/p}\,.$$

The following concept of order between cubes, introduced by the author in [2], plays an important role in our arguments.

Definition 6.3. We say that a cube $Q \in \mathcal{Q}$ is *stacked below* the cube $P \in \mathcal{Q}$, and write $Q \preccurlyeq P$, if there is a chain of cubes $Q = Q_0, Q_1, \dots, Q_s = P \in \mathcal{Q}$ such that

$$\operatorname{scale}(Q_i) < \operatorname{scale}(Q_{i+1}) \quad \text{and} \quad |Q_i \cap Q_{i+1}| > 0 \quad \text{for all} \quad i = 0, \dots, s-1 .$$

The relation $\preccurlyeq$ induces a partial order in $\mathcal{Q}$.

Remark 6.4. Suppose that $\mathcal{Q}'$ is a subfamily of $\mathcal{Q}$. Let $\max(\mathcal{Q}')$ be the set of maximal elements in $\mathcal{Q}'$ with respect to the relation $\preccurlyeq$. If a subfamily $\mathcal{Q}'$ does not contain arbitrary large cubes, i.e., $\sup_{Q \in \mathcal{Q}'} \operatorname{scale}(Q) < \infty$, then for any cube $Q \in \mathcal{Q}'$ there is always a cube $P \in \max(\mathcal{Q}')$ with $Q \preccurlyeq P$. In general, a maximal cube P is not unique unless, for example, the dilation $A = 2\mathrm{Id}$ and we work with nicely nested dyadic cubes.

We shall need a simple geometric lemma; for the proof see [2, p. 105].

Lemma 6.5. *There is a universal constant $\eta \in \mathbb{N}$ such that whenever we have two cubes $Q, P \in \mathcal{Q}$ with $Q \preccurlyeq P = A^{j_0}([0,1]^n + k_0)$ then*

$$Q \subset \bigcup_{|k-k_0|<\eta} A^{j_0}\big([0,1]^n + k\big) .$$

The main technical result of this section is the following theorem which is a generalization of a result which can be found in [24, Theorem 6.6.5].

Theorem 6.6. *Suppose $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, and μ is a ρ_A-doubling measure. Then for any $s \in \dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$, there exists a sequence of scalars $\{\lambda_j\}$, and ∞-atoms $\{r_j\}$ for $\dot{\mathbf{f}}_p^{\alpha,q}$ such that*

$$s = \sum_j \lambda_j r_j, \quad \textit{and} \quad \Big(\sum_j |\lambda_j|^p\Big)^{1/p} \le C \|s\|_{\dot{\mathbf{f}}_p^{\alpha,q}} , \tag{6.5}$$

for some constant C independent of s.

Proof. Suppose s is an arbitrary element of $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$. As a preliminary step, we wish to replace s by its majorant sequence $s^* = s^*_{r,\lambda}$, where $r > 0$ and $\lambda > \beta \max(1, r/q, r/p)$ are the same as in Lemma 3.10. The advantage of s^* over s is that the sequence s^* is locally almost constant within each scale, and yet, it still belongs to $\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)$. This allows us to remedy some serious difficulties arising from the fact the family of dilated cubes $\mathcal{Q}$ is generally not nested.

For $j \in \mathbb{Z}$, define a function $g_j^{\alpha,q}(s^*)$ by

$$g_j^{\alpha,q}(s^*) = \Big(\sum_{P \in \mathcal{Q},\ \operatorname{scale}(P) \ge j} \big(|P|^{-\alpha} |s_P^*| \tilde{\chi}_P\big)^q \Big)^{1/q} .$$

For the purposes of the proof, it is useful to insist that $P \in \mathcal{Q}$ are of the form $P = A^j([0,1)^n + k)$ and hence dilated cubes are disjoint (versus having common faces) within each scale. Note that for every $x \in \mathbb{R}^n$, $g_j^{\alpha,q}(s^*)(x) \ge g_{j+1}^{\alpha,q}(s^*)(x)$, and

$$\lim_{j\to\infty} g_j^{\alpha,q}\big(s^*\big)(x) = 0, \qquad \lim_{j\to\infty} g_j^{\alpha,q}\big(s^*\big)(x) = G^{\alpha,q}\big(s^*\big)(x) . \tag{6.6}$$

Indeed, (6.6) is obvious when $0 < q < \infty$. When $q = \infty$, one must use that $G^{\alpha,q}(s^*) \in L^p(\mu)$ and the fact that

$$\lim_{j\to\infty} \mu(A^j([0,1]^n)) = \lim_{j\to\infty} \mu(A^j([-1,1]^n)) = \mu(\mathbb{R}^n) = \infty\,, \tag{6.7}$$

which is a consequence of Proposition 2.10.

For $k \in \mathbb{Z}$, we define a family of dilated cubes

$$\mathcal{Q}_k = \{Q \in \mathcal{Q} : \sup_{x\in Q} g^{\alpha,q}_{\mathrm{scale}(Q)}(s^*)(x) > 2^k\}\,.$$

Clearly, $\mathcal{Q}_k \subset \mathcal{Q}_{k+1}$, and by (6.6),

$$\Omega_k := \{x \in \mathbb{R}^n : G^{\alpha,q}(s^*)(x) > 2^k\} \subset \bigcup_{Q\in\mathcal{Q}_k} Q\,. \tag{6.8}$$

Moreover, we claim that there is $m \in \mathbb{N}$, independent of k, such that the converse inclusion holds

$$\bigcup_{Q\in\mathcal{Q}_k} Q \subset \Omega_{k-m}\,. \tag{6.9}$$

Indeed, take any $Q \in \mathcal{Q}_k$ and $x = x(Q) \in Q$ such that $g^{\alpha,q}_j(s^*)(x) > 2^k$, where $j = \mathrm{scale}(Q)$. Then for any $P_1, P_2 \in \mathcal{Q}$, $\mathrm{scale}(P_1) = \mathrm{scale}(P_2) \ge j$, and $|P_1 \cap Q|, |P_2 \cap Q| > 0$, by Lemma 6.5 we have

$$\begin{aligned}
(s^*_{P_1})^r &= \sum_{P\in\mathcal{Q},\ |P|=|P_1|} |s_P|^r\big(1 + |P_1|^{-1}\rho_A(x_{P_1} - x_P)\big)^{-\lambda}\\
&\ge \sum_{P\in\mathcal{Q},\ |P|=|P_2|} |s_P|^r\big(1 + H|P_2|^{-1}\rho_A(x_{P_1} - x_{P_2}) + H|P_2|^{-1}\rho_A(x_{P_2} - x_P)\big)^{-\lambda}\\
&\ge \sup_{|k|<\eta} \big(H + H\rho_A(k)\big)^{-\lambda} \sum_{P\in\mathcal{Q},\ |P|=|P_2|} |s_P|^r\big(1 + |P_2|^{-1}\rho_A(x_{P_2} - x_P)\big)^{-\lambda} = c(s^*_{P_2})^r\,.
\end{aligned}$$

Here, c is the value of supremum above, and η is the same as in Lemma 6.5. Hence,

$$G^{\alpha,q}(y) \ge g^{\alpha,q}_j(s^*)(y) \ge c^{1/r} g^{\alpha,q}_j(s^*)(x) > c^{1/r}2^k > 2^{k-m} \qquad \text{for all} \quad y \in Q\,,$$

where $m \in \mathbb{N}$ is chosen so that $2^m > c^{-1/r}$, which proves (6.9).

Observe that the definition of family $\mathcal{Q}_k$ implies that

$$\Big(\sum_{P\in\mathcal{Q}\setminus\mathcal{Q}_k} \big(|P|^{-\alpha}|s^*_P|\tilde{\chi}_P(x)\big)^q\Big)^{1/q} \le 2^k \qquad \text{for all} \quad x \in \mathbb{R}^n\,. \tag{6.10}$$

Indeed, take any $x_0 \in \mathbb{R}^n$ and suppose that $G^{\alpha,q}(s^*)(x_0) > 2^k$; otherwise, the conclusion is trivial. By (6.6), let $j_0 \in \mathbb{Z}$ be the unique integer such that

$$g^{\alpha,q}_{j_0}(s^*)(x_0) > 2^k \quad \text{and} \quad g^{\alpha,q}_{j_0+1}(s^*)(x_0) \le 2^k\,.$$

For any scale $j \in \mathbb{Z}$, let P_j be the unique dilated cube such that $\mathrm{scale}(P_j) = j$ and $x_0 \in P_j$. Note that $P_j \in \mathcal{Q}_k$ for every $j \le j_0$, and hence

$$\begin{aligned}
\Big(\sum_{j\in\mathbb{Z}} \sum_{P_j\in\mathcal{Q}\setminus\mathcal{Q}_k} \big(|P_j|^{-\alpha}|s^*_{P_j}|\tilde{\chi}_{P_j}(x_0)\big)^q\Big)^{1/q} &= \Big(\sum_{j>j_0} \sum_{P_j\in\mathcal{Q}\setminus\mathcal{Q}_k} \big(|P_j|^{-\alpha}|s^*_{P_j}|\tilde{\chi}_{P_j}(x_0)\big)^q\Big)^{1/q}\\
&\le g^{\alpha,q}_{j_0+1}(s^*)(x_0) \le 2^k\,,
\end{aligned}$$

which shows (6.10).

Let $\mathcal{M}_k = \max(\mathcal{Q}_k \setminus \mathcal{Q}_{k+1})$ be the family of maximal cubes in $\mathcal{Q}_k \setminus \mathcal{Q}_{k+1}$ with respect to the partial order $\preccurlyeq$. We claim that for any cube $P \in \mathcal{Q}_k \setminus \mathcal{Q}_{k+1}$, there is $Q \in \mathcal{M}_k$, such that $P \preccurlyeq Q$. Indeed, take any $Q \in \mathcal{Q}_k \setminus \mathcal{Q}_{k+1}$ with $P \preccurlyeq Q$. Fix $x_0 \in P$, and let $Q' \in \mathcal{Q}$ be such that $x_0 \in Q'$ and $\text{scale}(Q') = \text{scale}(Q)$. Then by Lemma 6.5, $\mu(Q') \le c\mu(Q)$, where c is the same constant as in Proposition 2.10(a). By (6.9), $Q \subset \Omega_{k-m}$ and consequently,

$$\mu(Q') \le c\mu(Q) \le c2^{(m-k)p} \big\| G^{\alpha,q}(s^*) \big\|^p_{L^p(\mu)} = c2^{(m-k)p} \|s^*\|^p_{\dot{\mathbf{f}}^{\alpha,q}_p(A,\mu)} < \infty .$$

Therefore, by Proposition 2.10(b), $\text{scale}(Q')$ must be bounded from above. Consequently,

$$\sup\{\text{scale}(Q) : Q \in \mathcal{Q}_k \setminus \mathcal{Q}_{k+1},\ P \preccurlyeq Q\} < \infty ,$$

which proves the claim.

Let $\mathcal{M}'_k$ be the inflated version of $\mathcal{M}_k$ defined by

$$\mathcal{M}'_k = \big\{P \in \mathcal{Q} : \exists Q \in \mathcal{M}_k,\ \text{scale}(Q) = \text{scale}(P),\ |k - k_0| < \eta ,\\ \text{where} \quad P = A^j\big([0,1)^n + k\big), Q = A^j\big([0,1)^n + k_0\big)\big\} .$$

Let $\{Q_{k,l}\}_{l \in L_k}$ be any enumeration of cubes in $\mathcal{M}'_k$. Lemma 6.5 guarantees that for any $P \in \mathcal{Q}_k \setminus \mathcal{Q}_{k+1}$ there is $Q_{k,l} \in \mathcal{M}'_k$ such that $|P \cap Q_{k,l}| > 0$. Thus, we can inductively define a partition of the family $\mathcal{Q}_k \setminus \mathcal{Q}_{k+1}$ into subfamilies $\{\mathcal{Q}_{k,l}\}_l$ such that

$$P \in \mathcal{Q}_{k,l} \implies \text{scale}(P) \le \text{scale}(Q_{k,l}) \text{ and } \quad |P \cap Q_{k,l}| > 0 . \tag{6.11}$$

Hence, a subfamily $\mathcal{Q}_{k,l}$ consists of a certain portion of cubes in $\mathcal{Q}_k \setminus \mathcal{Q}_{k+1}$ which have nonzero intersection with $Q_{k,l}$ and scales lower than $\text{scale}(Q_{k,l})$. Note that it might happen that some subfamilies $\mathcal{Q}_{k,l}$'s are empty due to the fact that either all cubes in $\mathcal{Q}_k \setminus \mathcal{Q}_{k+1}$ lying below $Q_{k,l}$ were assigned to a different subfamily or there were no such cubes in the first place.

By (6.8) and (6.9), $\{\mathcal{Q}_k \setminus \mathcal{Q}_{k+1}\}_{k \in \mathbb{N}}$ is a partition of the entire family $\mathcal{Q}$. Consequently, $\{\mathcal{Q}_{k,l}\}_{k \in \mathbb{Z}, l \in L_k}$ is also a partition of $\mathcal{Q}$. This partition induces sequences $s^{k,l} = \{s^{k,l}_P\}_{P \in \mathcal{Q}}$ by setting

$$s^{k,l}_P = \begin{cases} s_P & P \in \mathcal{Q}_{k,l} , \\ 0 & \text{otherwise} . \end{cases}$$

Obviously,

$$s = \sum_{k \in \mathbb{Z}} \sum_{l \in L_k} s^{k,l} .$$

By (6.10),

$$G^{\alpha,q}(s^{k,l})(x) \le \Bigg(\sum_{P \in \mathcal{Q}_k \setminus \mathcal{Q}_{k+1}} \big(|P|^{-\alpha} |s^*_P| \tilde{\chi}_P(x)\big)^q \Bigg)^{1/q} \le 2^{k+1} \qquad \text{for all} \quad x \in \mathbb{R}^n . \tag{6.12}$$

Finally, define atoms $\{r^{k,l}\}_{k,l}$ as appropriate normalizations of $\{s^{k,l}\}_{k,l}$,

$$r^{k,l}_P = 2^{-k-1} \mu(Q_{k,l})^{-1/p} s^{k,l}_P \qquad \text{for} \quad P \in \mathcal{Q} .$$

To verify that each $r^{k,l}$ is an ∞-atom for $\dot{\mathbf{f}}^{\alpha,q}_p$ with respect to the cube $Q_{k,l}$ it suffices to use (6.11) and (6.12),

$$\big\| G^{\alpha,q}(r^{k,l}) \big\|_\infty = 2^{-k-1} \mu(Q_{k,l})^{-1/p} \big\| G^{\alpha,q}(s^{k,l}) \big\|_\infty \le \mu(Q_{k,l})^{-1/p} .$$

Clearly,

$$s=\sum_{k\in\mathbb{Z}}\sum_{l\in L_k}\lambda_{k,l}s^{k,l}, \qquad \text{where} \quad \lambda_{k,l}=2^{k+1}\mu(Q_{k,l})^{1/p}.$$

Note that for each $k\in\mathbb{Z}$,

$$\sum_{l\in L_k}\mu(Q_{k,l})=\sum_{Q\in\mathcal{M}'_k}\mu(Q)\le cK\sum_{Q\in\mathcal{M}_k}\mu(Q)\le cK\mu\Bigg(\bigcup_{Q\in\mathcal{Q}_k\setminus\mathcal{Q}_{k+1}}Q\Bigg),$$

where the constant c is the same as in Proposition 2.10(a), and K is the cardinality of $\mathbb{Z}^n\cap B(0,\eta)$. Therefore, by (6.9),

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{l\in L_k}|\lambda_{k,l}|^p&=\sum_{k\in\mathbb{Z}}2^{(k+1)p}\sum_{l\in L_k}\mu(Q_{k,l})\le cK\sum_{k\in\mathbb{Z}}2^{(k+1)p}\mu(\Omega_{k-m})\\&=cK2^{(m+2)p}\frac{1}{p}\sum_{k\in\mathbb{Z}}p2^{kp}\mu\big(\{x:G^{\alpha,q}(s^*)(x)>2^{k+1}\}\big)\\&\le C\int_0^\infty p\lambda^{p-1}\mu\big(\{x:G^{\alpha,q}(s^*)(x)>\lambda\}\big)\,d\lambda\\&=C\|s^*\|^p_{\dot{\mathbf{f}}^{\alpha,q}_p(A,\mu)}\le C'\|s\|^p_{\dot{\mathbf{f}}^{\alpha,q}_p(A,\mu)},\end{aligned}$$

where the last step follows from Lemma 3.10. This shows (6.5) and completes the proof of Theorem 6.6. □

As a corollary, we obtain an atomic characterization of $\dot{\mathbf{f}}^{\alpha,q}_p$-spaces which is a generalization of [20, Theorem 7.2].

Theorem 6.7. *Suppose $0<p\le 1$, $p\le q\le\infty$, $\alpha\in\mathbb{R}$, and μ is a ρ_A-doubling measure. Then for any $p\le p_1\le\infty$,*

$$\|s\|_{\dot{\mathbf{f}}^{\alpha,q}_p}\asymp\inf\left\{\Big(\sum_j|\lambda_j|^p\Big)^{1/p}:s=\sum_j\lambda_jr_j \text{ and each } r_j \text{ is a } p_1\text{-atom for } \dot{\mathbf{f}}^{\alpha,q}_p\right\}. \quad (6.13)$$

Proof. By Remark 6.2, every ∞-atom r is also a p_1-atom (modulo a multiplicative constant). Hence, the lower bound for $\|s\|_{\dot{\mathbf{f}}^{\alpha,q}_p}$ follows immediately from Theorem 6.6. To prove the upper bound one must use

$$\|s+t\|^p_{\dot{\mathbf{f}}^{\alpha,q}_p}\le\|s\|^p_{\dot{\mathbf{f}}^{\alpha,q}_p}+\|t\|^p_{\dot{\mathbf{f}}^{\alpha,q}_p},$$

which is a consequence of p-triangle inequality and Minkowski's inequality with exponent q/p. By Remark 6.2, every p_1-atom r is also a p-atom (modulo a multiplicative constant) and hence, $\|r\|_{\dot{\mathbf{f}}^{\alpha,q}_p}\le C$. If s is as in (6.13), then

$$\|s\|^p_{\dot{\mathbf{f}}^{\alpha,q}_p}=\Big\|\sum_j\lambda_jr_j\Big\|^p_{\dot{\mathbf{f}}^{\alpha,q}_p}\le\sum_j|\lambda_j|^p\|r_j\|^p_{\dot{\mathbf{f}}^{\alpha,q}_p}\le C\sum_j|\lambda_j|^p.$$ □

Obviously, Theorem 6.7 is most interesting when $p_1=\infty$, since it yields atomic decomposition into the most restrictive class of ∞-atoms. Hence, we will restrict ourselves to this case.

Finally, we define nonsmooth atoms for $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces as conglomerates of smooth atoms for $\dot{\mathbf{F}}_p^{\alpha,q}$ with coefficients given by atoms for $\dot{\mathbf{f}}_p^{\alpha,q}$.

Definition 6.8. Suppose that $0 < p \le 1$ and $p \le q \le \infty$. We say that b is a *nonsmooth atom* for $\dot{\mathbf{F}}_p^{\alpha,q}$ if $b = \sum_{Q\in\mathcal{Q}} r_Q a_Q$, where $r = \{r_Q\}_{Q\in\mathcal{Q}}$ is an ∞-atom for $\dot{\mathbf{f}}_p^{\alpha,q}$ for some fixed cube $\bar{Q} \in \mathcal{Q}$, and each a_Q is a smooth atom supported near a cube Q.

Obviously, one can also define nonsmooth atoms for $\dot{\mathbf{F}}_p^{\alpha,q}$ with more general normalizations, where $r = \{r_Q\}_{Q\in\mathcal{Q}}$ is a p_1-atom for $\dot{\mathbf{f}}_p^{\alpha,q}$, instead of an ∞-atom and $p_1 \ge p$. As a consequence of (5.8) and (6.1), observe that there exists a universal constant $R > 0$ such that

$$\operatorname{supp} b \subset A^{-j}\big([-R, R]^n + k\big) \tag{6.14}$$

for any atom b corresponding to $\bar{Q} = A^{-j}([0,1]^n + k)$. By Theorem 5.7, every atom b for $\dot{\mathbf{F}}_p^{\alpha,q}$ satisfies $||b||_{\dot{\mathbf{F}}_p^{\alpha,q}} \le C$ for some universal constant C.

Finally, we can prove nonsmooth atomic decomposition of $\dot{\mathbf{F}}_p^{\alpha,q}$ spaces.

Theorem 6.9. *Suppose that $0 < p \le 1$, $p \le q \le \infty$, $\alpha \in \mathbb{R}$, and μ is a ρ_A-doubling measure. Then for any $f \in \mathcal{S}'/\mathcal{P}$,*

$$||f||_{\dot{\mathbf{F}}_p^{\alpha,q}} \asymp \inf\left\{\Big(\sum_j |\lambda_j|^p\Big)^{1/p} : f = \sum_j \lambda_j b_j \text{ and each } b_j \text{ is a nonsmooth atom for } \dot{\mathbf{F}}_p^{\alpha,q}\right\}. \tag{6.15}$$

Proof. The lower bound of $||f||_{\dot{\mathbf{F}}_p^{\alpha,q}}$ is a direct consequence of smooth atomic decomposition for $\dot{\mathbf{F}}_p^{\alpha,q}$-spaces and Theorem 6.7. Indeed, for any $f \in \dot{\mathbf{F}}_p^{\alpha,q}$ find its decomposition $f = \sum s_Q a_Q$ into smooth atoms $\{a_Q\}$ and group them according to atomic decomposition of $s = \{s_Q\} \in \dot{\mathbf{f}}_p^{\alpha,q}$.

To prove the upper bound one must use

$$||f+g||^p_{\dot{\mathbf{F}}_p^{\alpha,q}} \le ||f||^p_{\dot{\mathbf{F}}_p^{\alpha,q}} + ||g||^p_{\dot{\mathbf{F}}_p^{\alpha,q}},$$

which is a consequence of p-triangle inequality and Minkowski's inequality with exponent q/p. Since every (nonsmooth) atom b for $\dot{\mathbf{F}}_p^{\alpha,q}$ satisfies $||b||_{\dot{\mathbf{F}}_p^{\alpha,q}} \le C$, the upper bound of $||f||_{\dot{\mathbf{F}}_p^{\alpha,q}}$ follows immediately. □

In the next section we explore the connections between atomic decompositions of Hardy spaces and Theorem 6.9.

7. Identification with anisotropic Hardy spaces

The goal of this section is to identify unweighted $\dot{\mathbf{F}}_p^{0,2}(A, \mathbb{R}^n)$ spaces with the (real) Hardy spaces H_A^p for $0 < p < \infty$ in the context of expansive dilations A. The corresponding isotropic result is well known and boils down to the square function characterization of Hardy spaces, e.g., [24, Theorem 6.4.15].

For various equivalent ways of introducing the usual isotropic Hardy spaces on $\mathbb{R}^n$ we refer to [16, 31]. In the context of expansive dilations A, anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ were studied by the author [2]. There are several equivalent definitions of Hardy spaces using maximal functions or atomic decompositions. Theorem 7.1 establishes the square function characterization stated informally as

$$f \in H^p_A(\mathbb{R}^n) \iff S(f) := \Big(\sum_{j\in\mathbb{Z}} |\varphi_j * f|^2\Big)^{1/2} \in L^p ,$$

where $\varphi \in \mathcal{S}$ satisfies (3.2) and (3.3).

Theorem 7.1. *Suppose that A is an expansive dilation and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies*

$$\operatorname{supp}\hat{\varphi} \subset [-\pi,\pi]^n \setminus \{0\} \qquad \text{and} \qquad \sum_{j\in\mathbb{Z}} \big|\hat{\varphi}\big((A^*)^j\xi\big)\big|^2 = 1 \qquad \text{for all } \xi \neq 0 . \tag{7.1}$$

Then $H^p_A(\mathbb{R}^n) = \dot{\mathbf{F}}^{0,2}_p(\mathbb{R}^n, A)$ for all $0 < p < \infty$. More precisely, any $f \in \dot{\mathbf{F}}^{0,2}_p \subset \mathcal{S}'/\mathcal{P}$ is identified with its canonical representative

$$f = \sum_{Q\in\mathcal{Q}} \langle f, \varphi_Q\rangle \varphi_Q$$

where the series converges in $\mathcal{S}'$ and we have

$$\|f\|_{H^p_A} \asymp \|f\|_{\dot{\mathbf{F}}^{0,2}_p} \qquad \text{for all } f \in \mathcal{S}' . \tag{7.2}$$

Proof. The condition (7.1) assures that φ is a tight frame wavelet associated with A with all vanishing moments. Hence, by [2, Lemma 6.3 and Theorem 6.7 in Ch. 2] for $0 < p \leq 1$ and [2, Lemma 6.10 and Theorem 6.13 in Ch. 2] for $1 < p < \infty$, the following diagram commutes and the maps S_φ and T_φ are bounded

$$\begin{array}{ccccc} & & \dot{\mathbf{f}}^{0,2}_p(A) & & \\ & \overset{S_\varphi}{\nearrow} & & \overset{T_\varphi}{\searrow} & \\ H^p_A(\mathbb{R}^n) & & \xrightarrow{\ \mathrm{Id}\ } & & H^p_A(\mathbb{R}^n) \end{array} . \tag{7.3}$$

Here, S_φ and T_φ are the usual analysis and synthesis transform. Moreover, for any $s \in \dot{\mathbf{f}}^{0,2}_p(A)$, the series $T_\varphi s = \sum_{Q\in\mathcal{Q}} s_Q \varphi_Q$ converges unconditionally in $H^p_A(\mathbb{R}^n)$ and hence in $\mathcal{S}'$. On the other hand, Theorem 3.12 shows that we a similar commutative diagram

$$\begin{array}{ccccc} & & \dot{\mathbf{f}}^{0,2}_p(A) & & \\ & \overset{S_\varphi}{\nearrow} & & \overset{T_\varphi}{\searrow} & \\ \dot{\mathbf{F}}^{0,2}_p(\mathbb{R}^n, A) & & \xrightarrow{\ \mathrm{Id}\ } & & \dot{\mathbf{F}}^{0,2}_p(\mathbb{R}^n, A) . \end{array} \tag{7.4}$$

A priori, Theorem 3.12 says that $T_\varphi s \in \dot{\mathbf{F}}^{0,2}_p$ and hence it is an element of $\mathcal{S}'/\mathcal{P}$. Since the sequence space is the same in both cases, $T_\varphi s$ can be identified with a specific element of H^p_A and hence $\mathcal{S}'$ by (7.3). Combining diagrams (7.3) and (7.4) shows (7.2). □

As an immediate corollary of Theorems 3.12 and 7.1 we have the following.

Corollary 7.2. *Suppose that* $0 < p < \infty$ *and* $\varphi \in \mathcal{S}$ *satisfies* (3.2) *and* (3.3). *Then for any* $f \in \mathcal{S}'$, *there exists a unique polynomial* $P \in \mathcal{P}$ *such that*

$$||f - P||_{H_A^p} \asymp \left\| \left(\sum_{j\in\mathbb{Z}} |\varphi_j * f|^2 \right)^{1/2} \right\|_{L^p} . \tag{7.5}$$

Note that the square function $S(f)$ on right-hand side of (7.5) does not "detect" polynomials, i.e., $S(f) = S(f - P)$. Hence, an appropriate representative in the equivalence class of f in $\mathcal{S}'/\mathcal{P}$ must be chosen to yield a valid member $f - P$ of the Hardy space H_A^p.

Finally, we are ready to discuss Theorem 6.9 in the setting of Hardy spaces. Let $b = \sum r_Q a_Q$ be any nonsmooth atom for $\dot{\mathbf{F}}_p^{0,2}$, $0 < p \le 1$, supported around cube $\bar{Q}$. Then for any $1 < p_0 < \infty$,

$$||b||_{L^{p_0}} \asymp ||b||_{\dot{\mathbf{F}}_{p_0}^{0,2}} \le C||r||_{\dot{\mathbf{f}}_{p_0}^{0,2}} \le C|\bar{Q}|^{1/p_1 - 1/p} .$$

By (6.14) the support of b is contained in enlarged copy of $\bar{Q}$. Since the series $b = \sum r_Q a_Q$ converges in $\dot{\mathbf{F}}_{p_1}^{0,2} = L^{p_1}$ norm, the vanishing moments of a_Q's are inherited by b, i.e.,

$$\int_{\mathbb{R}^n} x^\gamma b(x)\, dx = 0 \qquad \text{for} \quad |\gamma| \le \tilde{N} ,$$

where $\tilde{N} \ge N = \max(\lfloor (J-1)/\zeta_- \rfloor, -1) = \lfloor (1/p - 1)/\zeta_- \rfloor$. Hence, up to a multiplicative constant, a nonsmooth atom b for $\dot{\mathbf{F}}_p^{0,2}$ is a (p, p_1, N)-atom in the setting of anisotropic Hardy spaces, see [2, Definition 4.1]. Therefore, Theorem 6.9 yields the atomic decomposition of H_A^p spaces into L^{p_1}-atoms for $1 < p_1 < \infty$. Furthermore, when $p_1 = \infty$, Theorem 6.9 yields the atomic decomposition into "BMO-atoms" instead of more familiar L^∞-atoms, see [20, Section 7].

8. Proofs of auxiliary results

8.1. Proof of Lemma 3.10

To prove Lemma 3.10 we need two auxiliary lemmas. Lemma 8.1 is a generalization of [5, Lemma 6.2] for maximal functions associated with ρ_A-doubling measures. Lemma 8.2 is a geometric result on the family of dilated cubes $\mathcal{Q}$.

Lemma 8.1. *Suppose* $0 < a \le r < \infty$, $\lambda > \beta r/a$, *and* $i, j \in \mathbb{Z}$. *Then for any sequence* $s = \{s_P\}_P$ *and for each cube* $Q \in \mathcal{Q}$ *with* $\text{scale}(Q) = j$ *we have*

$$\left(\sum_{\text{scale}(P)=i} |s_P|^r \Big/ \left(1 + \frac{\rho_A(x_Q - x_P)}{\max(|P|, |Q|)} \right)^\lambda \right)^{1/r} \tag{8.1}$$

$$\le C|\det A|^{(j-i)_+ \beta/a} \left(M_{\rho_A} \left(\sum_{\text{scale}(P)=i} |s_P|^a \chi_P \right)(x) \right)^{1/a} \qquad \textit{for all} \quad x \in Q ,$$

where the constant C *depends only on* $\lambda - \beta r/a$. *In particular, if* $i = j$, *then*

$$\sum_{\text{scale}(Q)=j} (s_{r,\lambda}^*)_Q \tilde{\chi}_Q \le C \left(M_{\rho_A} \left(\sum_{\text{scale}(Q)=j} |s_Q| \tilde{\chi}_Q \right)^a \right)^{1/a} \tag{8.2}$$

with the same constant C.

Proof. Fix $Q \in \mathcal{Q}$ with $\text{scale}(Q) = j$. Consider the first case when $i \geq j$. Define

$$A_0 = \{P \in \mathcal{Q} : \text{scale}(P) = i \quad \text{and} \quad \rho_A(x_Q - x_P)/|P| \leq 1\},$$
$$A_k = \left\{P \in \mathcal{Q} : \text{scale}(P) = i \quad \text{and} \quad |\det A|^{k-1} < \rho_A(x_Q - x_P)/|P| \leq |\det A|^k\right\} \qquad k \geq 1 .$$

Then

$$\sum_{P\in A_k} \frac{|s_P|^r}{(1+\rho_A(x_Q - x_P)/|P|)^\lambda} \leq C|\det A|^{-k\lambda} \sum_{P\in A_k} |s_P|^r \leq C|\det A|^{-k\lambda}\Big(\sum_{P\in A_k} |s_P|^a\Big)^{r/a}$$
$$= C|\det A|^{-k\lambda}\Big(\int_{\tilde{B}} \sum_{P\in A_k} \frac{|s_P|^a}{\mu(P)}\chi_P\, d\mu\Big)^{r/a},$$

where

$$\bigcup_{P\in A_k} P \subset \tilde{B} := B_{\rho_A}\big(x_Q, 2H|\det A|^{k+i}\big).$$

Since μ is ρ_A-doubling

$$\mu(\tilde{B}) \leq C|\det A|^{\beta k}\mu(P) \qquad \text{for any} \quad P \in A_k,\ k \geq 0 .$$

Hence, by the definition of the maximal operator, we have

$$\sum_{P\in A_k} \frac{|s_P|^r}{(1+\rho_A(x_Q - x_P)/|P|)^\lambda} \leq C|\det A|^{-k\lambda+\beta kr/a}\left(\frac{1}{\mu(\tilde{B})}\int_{\tilde{B}} \sum_{P\in A_k} |s_P|^a\chi_P\, d\mu\right)^{r/a}$$
$$\leq C|\det A|^{-k(\lambda-\beta r/a)}\Big(M_{\rho_A}\Big(\sum_{\text{scale}(P)=i} |s_P|^a\chi_P\Big)(x)\Big)^{r/a}$$

for any $x \in Q \subset \tilde{B}$. Summing over $k \geq 0$, yields (8.1).

In the second case $i < j$, we redefine A_k's by

$$A_0 = \{P \in \mathcal{Q} : \text{scale}(P) = i \quad \text{and} \quad \rho_A(x_Q - x_P)/|Q| \leq 1\},$$
$$A_k = \left\{P \in \mathcal{Q} : \text{scale}(P) = i \quad \text{and} \quad |\det A|^{k-1} < \rho_A(x_Q - x_P)/|Q| \leq |\det A|^k\right\} \qquad k \geq 1 .$$

Then, as before

$$\sum_{P\in A_k} \frac{|s_P|^r}{(1+\rho_A(x_Q - x_P)/|Q|)^\lambda} \leq C|\det A|^{-k\lambda+\beta(k+j-i)r/a}\left(\frac{1}{\mu(\tilde{B})}\int_{\tilde{B}} \sum_{P\in A_k} |s_P|^a\chi_P\right)^{r/a}$$
$$\leq C|\det A|^{(j-i)\beta r/a-k(\lambda-\beta r/a)}\Big(M_{\rho_A}\Big(\sum_{\text{scale}(P)=i} |s_P|^a\chi_P\Big)(x)\Big)^{r/a}$$

for any $x \in Q \subset \tilde{B}$. Here, we used that

$$\bigcup_{P\in A_k} P \subset \tilde{B} := B_{\rho_A}\big(x_Q, 2H|\det A|^{k+j}\big)$$

and

$$\mu(\tilde{B}) \le C|\det A|^{\beta(k+j-i)}\mu(P) \qquad \text{for any} \quad P \in A_k,\ k \ge 0\,.$$

Summing over $k \ge 0$, yields (8.1).

To see (8.2), multiply both sides of (8.1) by $\tilde{\chi}_Q$, and sum over all $Q \in \mathcal{Q}$ with $\operatorname{scale}(Q) = j$,

$$\begin{aligned}\sum_{\operatorname{scale}(Q)=j} \left(s^*_{r,\lambda}\right)_Q \tilde{\chi}_Q &\le C \sum_{\operatorname{scale}(Q)=j} \left(M_{\rho_A}\Big(\sum_{\operatorname{scale}(P)=j} |s_P|\tilde{\chi}_P\Big)^a\right)^{1/a} \chi_Q \\ &= C\left(M_{\rho_A}\Big(\sum_{\operatorname{scale}(P)=j} |s_P|\tilde{\chi}_P\Big)^a\right)^{1/a},\end{aligned}$$

since $\{Q \in \mathcal{Q} : \operatorname{scale}(Q) = j\}$ is a partition of $\mathbb{R}^n$. □

Lemma 8.2. *Suppose that $P = A^{j_0}([0,1]^n + k_0) \in \mathcal{Q}$, where $j_0 \in \mathbb{Z}$, $k_0 \in \mathbb{Z}^n$. Whenever $Q, \tilde{Q} \in \mathcal{Q}$ satisfy*

$$j = \operatorname{scale}(Q) = \operatorname{scale}(\tilde{Q}) \le \operatorname{scale}(P) = j_0 \tag{8.3}$$

and

$$Q \cap P \ne \emptyset, \qquad \tilde{Q} \cap \left(P + A^{j_0}k\right) \ne \emptyset \qquad \text{for some} \quad k \in \mathbb{Z}^n,\ \rho_A(k) > K\,, \tag{8.4}$$

then we have

$$|Q|^{-1}\rho_A(x_Q - x_{\tilde{Q}}) \ge \frac{|\det A|^{j_0-j}\rho_A(k)}{2H}\,. \tag{8.5}$$

Moreover, the constant $K > 0$ is independent of the choice of P, Q, and $\tilde{Q}$.

Proof. Let $K = 2H\sup_{y\in U}\rho_A(y)$, where $U = 2(U_0 - U_0)$ and $U_0 = \bigcup_{l\le 0} A^l([0,1]^n)$. Since U_0, and hence U, are compact we have $K < \infty$. Take any $Q = A^j([0,1]^n + k_1)$ and $\tilde{Q} = A^j([0,1]^n + k_2)$, $j \le j_0$ and $k_1, k_2 \in \mathbb{Z}^n$, satisfying (8.3) and (8.4). Since

$$A^{j-j_0}\left([0,1]^n + k_1\right) \cap \left([0,1]^n + k_0\right) \ne \emptyset, \qquad A^{j-j_0}\left([0,1]^n + k_2\right) \cap \left([0,1]^n + k_0 + k\right) \ne \emptyset$$

we have

$$A^{j-j_0}k_1 - k_0 \in U_0 - U_0, \qquad A^{j-j_0}k_2 - k_0 - k \in U_0 - U_0\,.$$

Hence, $A^{j-j_0}(k_1 - k_2) + k \in U$. Thus,

$$\rho_A\left(A^{j-j_0}(k_1 - k_2)\right) \ge (1/H)\rho_A(k) - \sup_{y\in U}\rho_A(y) = (1/H)\rho_A(k) - K/(2H) \ge \frac{\rho_A(k)}{2H}\,,$$

since $\rho_A(k) > K$. This shows (8.5), since $|Q|^{-1}\rho_A(x_Q - x_{\tilde{Q}}) = \rho_A(k_1 - k_2)$. □

Proof of Lemma 3.10. The case $p < \infty$ is a consequence of Lemma 8.1. Indeed, take any $r > 0$ and $\lambda > \beta\max(1, r/q, r/p)$. If $r < \min(q, p)$, then we set $a = r$. Otherwise, if $r \ge \min(q, p)$, then take a such that $\beta r/\lambda < a < \min(r, q, p)$. It is possible to choose such an a, since $\lambda > \beta\max(1, r/q, r/p)$ implies $\beta r/\lambda < \min(r, q, p)$. In both cases we have that

$$0 < a \le r < \infty, \quad \lambda > \beta r/a, \quad q/a > 1, \quad p/a > 1\,.$$

Therefore, (8.2) in Lemma 8.1 yields

$$\|s^*_{r,\lambda}\|_{\dot{\mathbf{f}}^{\alpha,q}_p(A,\mu)} \le C\left\|\left(\sum_{j\in\mathbb{Z}}\left(M_{\rho_A}\left(\sum_{|Q|=|\det A|^{-j}} |Q|^{-\alpha}|s_Q|\tilde{\chi}_Q\right)^a\right)^{q/a}\right)^{a/q}\right\|^{1/a}_{L^{p/a}(\mu)} .$$

Since $q/a > 1$ and $p/a > 1$, by the Fefferman-Stein vector-valued maximal inequality we can remove M_{ρ_A} from the above estimate (by increasing a constant C) to obtain

$$\|s^*_{r,\lambda}\|_{\dot{\mathbf{f}}^{\alpha,q}_p(A,\mu)} \le C\|s\|_{\dot{\mathbf{f}}^{\alpha,q}_p(A,\mu)} .$$

Next, we consider the case $p = \infty$. Without any loss we can also assume that $0 < q < \infty$, since the proof of the case $p = q = \infty$ is immediate.

Take any $r > 0$ and $\lambda > \beta \max(1, r/q)$. Fix a dilated cube $P = A^{j_0}([0,1]^n + k_0) \in \mathcal{Q}$ and let $\bar{P}$ be the union of neighboring dilated cubes to P, i.e.,

$$\bar{P} = \sum_{k\in\mathbb{Z}^n,\ \rho_A(k)\le K} \left(P + A^{j_0}k\right) ,$$

where K is the same as in Lemma 8.2. Define sequences $t = \{t_Q\}_{Q\in\mathcal{Q}}$ and $u = \{u_Q\}_{Q\in\mathcal{Q}}$ by

$$t_Q = \begin{cases} s_Q & Q \subset \bar{P},\ |Q| \le |P| , \\ 0 & \text{otherwise} , \end{cases} \qquad u_Q = s_Q - t_Q .$$

Then we have

$$\left(s^*_{r,\lambda}\right)^r_Q = \left(t^*_{r,\lambda}\right)^r_Q + \left(u^*_{r,\lambda}\right)^r_Q \qquad \text{for all} \quad Q \in \mathcal{Q} ,$$

and hence there is a constant c (dependent on q and r) such that

$$\left(s^*_{r,\lambda}\right)_Q \le c\left(\left(t^*_{r,\lambda}\right)^q_Q + \left(u^*_{r,\lambda}\right)^q_Q\right)^{1/q} \qquad \text{for all} \quad Q \in \mathcal{Q} .$$

Consequently, the estimate of $\dot{\mathbf{f}}^{\alpha,q}_\infty$-norm of $s^*_{r,\lambda}$ will follow from the corresponding bounds on $t^*_{r,\lambda}$ and $u^*_{r,\lambda}$. By the already proved Lemma 3.10 and $p = q < \infty$ we have

$$\begin{aligned}
\frac{1}{\mu(P)}\int_P \sum_{|Q|\le|P|} \left(|Q|^{-\alpha}\left(t^*_{r,\lambda}\right)_Q\tilde{\chi}_Q(x)\right)^q d\mu(x) &\le \frac{1}{\mu(P)}\int_P \sum_{Q\in\mathcal{Q}} \left(|Q|^{-\alpha}\left(t^*_{r,\lambda}\right)_Q\tilde{\chi}_Q(x)\right)^q d\mu(x) \\
= \frac{1}{\mu(P)}\|t^*_{r,\lambda}\|^q_{\dot{\mathbf{f}}^{\alpha,q}_q} \le C\frac{1}{\mu(P)}\|t\|^q_{\dot{\mathbf{f}}^{\alpha,q}_q} &= \frac{C}{\mu(P)}\int_{\mathbb{R}^n} \sum_{Q\subset\bar{P},\ |Q|\le|P|} \left(|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q(x)\right)^q d\mu(x) \\
\le \sum_{P'\subset\bar{P},\ |P'|=|P|} \frac{C}{\mu(P')}\int_{P'} \sum_{|Q|\le|P'|} &\left(|Q|^{-\alpha}|s_Q|\tilde{\chi}_Q(x)\right)^q d\mu(x) \le C\|s\|^q_{\dot{\mathbf{f}}^{\alpha,q}_\infty} .
\end{aligned}$$

To estimate $u^*_{r,\lambda}$ we must invoke Lemma 8.2. Namely, if for $Q, \tilde{Q} \in \mathcal{Q}$, $|Q| = |\tilde{Q}| \le |P|$, $Q \cap P \ne \emptyset$ and $\tilde{Q} \not\subset \bar{P}$, then (8.5) holds. Hence,

$$\frac{1}{\mu(P)}\int_P \sum_{|Q|\le|P|} \left(|Q|^{-\alpha}\left(u^*_{r,\lambda}\right)_Q\tilde{\chi}_Q(x)\right)^q d\mu(x)$$

$$= \frac{1}{\mu(P)} \sum_{j=-\infty}^{j_0} \sum_{\text{scale}(Q)=j} \mu(Q \cap P) \sum_{|\tilde{Q}|=|Q|} \frac{(|Q|^{-\alpha-1/2}|u_{\tilde{Q}}|)^q}{(1+|Q|^{-1}\rho_A(x_Q - x_{\tilde{Q}}))^\lambda}$$

$$\leq \sum_{j=-\infty}^{j_0} \sum_{\substack{k\in\mathbb{Z}^n \\ \rho_A(k)>K}} \sum_{\substack{\text{scale}(\tilde{Q})=j \\ \tilde{Q}\cap(P+A^{j_0}k)\neq\emptyset}} \frac{(2H)^\lambda}{(|\det A|^{j_0-j}\rho_A(k))^\lambda} (|\tilde{Q}|^{-\alpha-1/2}|u_{\tilde{Q}}|)^q$$

$$\leq (2H)^\lambda \sum_{j=-\infty}^{j_0} \sum_{\substack{k\in\mathbb{Z}^n \\ \rho_A(k)>K}} \frac{|\det A|^{(\beta-\lambda)(j_0-j)}}{\rho_A(k)^\lambda} \left[\frac{1}{\mu(P+A^{j_0}k)} \sum_{\substack{\text{scale}(\tilde{Q})=j \\ \tilde{Q}\cap(P+A^{j_0}k)\neq\emptyset}} \mu(\tilde{Q})(|\tilde{Q}|^{-\alpha-1/2}|u_{\tilde{Q}}|)^q \right]$$

$$\leq C||u||^q_{\dot{\mathbf{f}}^{\alpha,q}_\infty} \leq C||s||^q_{\dot{\mathbf{f}}^{\alpha,q}_\infty} .$$

Here, we used that $\mu(P + A^{j_0}k) \leq C|\det A|^{\beta(j_0-j)}\mu(\tilde{Q})$, the expression in the bracket is dominated by $||u||^q_{\dot{\mathbf{f}}^{\alpha,q}_\infty}$ by (3.11), and the fact that the series outside the bracket is finite. Combining the above estimates yields $||s^*_{r,\lambda}||_{\dot{\mathbf{f}}^{\alpha,q}_\infty} \leq C||s||_{\dot{\mathbf{f}}^{\alpha,q}_\infty}$, which completes the proof of Lemma 3.10. □

8.2. Proof of Lemma 3.11

To prove Lemma 3.11 we need the following adaptation of Peetre's mean value inequality, see [20, Lemma A.4].

Lemma 8.3. *Let K be a compact subset of $\mathbb{R}^n$ and $r, \lambda > 0$. Suppose that $f \in \mathcal{S}'$ and $\operatorname{supp} \hat{f} \subset K$. For $\gamma \in \mathbb{N}$, define sequences $\{a_Q\}_{Q\in\mathcal{Q}}$ and $\{b_Q\}_{Q\in\mathcal{Q}}$ by*

$$a_Q = \sup_{y\in Q}|f(y)| \qquad b_Q = \sup\{\inf_{y\in P}|f(y)| : \text{scale}(P) = \text{scale}(Q) - \gamma,\ P \cap Q \neq \emptyset\} . \quad (8.6)$$

Then for sufficiently large γ we have

$$\left(a^*_{r,\lambda}\right)_Q \asymp \left(b^*_{r,\lambda}\right)_Q \qquad \textit{for all} \quad Q \in \mathcal{Q},\ \text{scale}(Q) = 0 , \quad (8.7)$$

with constants independent of f and Q.

Proof. Assume that $Q = [0, 1]^n + k_0$, where $k_0 \in \mathbb{Z}^n$. Initially, we will show that (8.7) holds for $f \in \mathcal{S}$ with $\operatorname{supp} \hat{f} \subset K$. Take any $P \in \mathcal{Q}$ with scale$(P) = 0$. By the mean value theorem

$$a_P \leq b_P + \operatorname{diam}\left(A^{-\gamma}([0,1]^n)\right) \sup_{y\in P}|\nabla f(y)| \leq b_P + c(\lambda_-)^{-\gamma} d_P , \quad (8.8)$$

where $d_P = \sup_{y\in P}|\nabla f(y)|$. In the last step we used that A is expansive, i.e., for $\gamma \geq 0$,

$$\left|A^{-\gamma}x\right| \leq c(\lambda_-)^{-\gamma}|x|, \qquad x \in \mathbb{R}^n .$$

Pick $g \in \mathcal{S}$ such that $\operatorname{supp}\hat{g}$ is compact and $\hat{g}(\xi) = 1$ for $\xi \in K$. Note that $f = f * g$ and for arbitrary $M > 0$, $|\nabla g(z)| \leq C(1 + \rho_A(z))^{-M}$, where $C = C(M) > 0$. Hence, if $r > 1$, then by

Hölder's inequality $1/r + 1/r' = 1$,

$$\begin{aligned} d_P &\le \sup_{x\in P}\int_{\mathbb{R}^n} |f(y)||\nabla g(x-y)|\,dy \\ &\le C\sup_{x\in P}\left(\int_{\mathbb{R}^n} |f(y)|^r(1+\rho_A(x-y))^{-r(M-1)}\,dy\right)^{1/r}\left(\int_{\mathbb{R}^n}(1+\rho_A(x-y))^{-r'}\,dy\right)^{1/r'} \\ &\le C\left(\sum_{L\in\mathcal{Q},\ \mathrm{scale}(L)=0} (a_L)^r(1+\rho_A(x_P-x_L))^{-r(M-1)}\right)^{1/r}. \end{aligned}$$

In the last step we split integration over cubes $L = l + [0,1]^n$, $l \in \mathbb{Z}^n$, and we used the inequality

$$\rho_A(x_P - x_L) \le H^2\left(2\,\mathrm{diam}_{\rho_A}\left([0,1]^n\right) + \inf_{x\in P,\ y\in L}\rho_A(x-y)\right).$$

Hence, taking $M > 2 + \lambda$ yields

$$\begin{aligned} &(d^*_{r,\lambda})_Q \\ &\le C\left(\sum_{\mathrm{scale}(P)=0}\ \sum_{\mathrm{scale}(L)=0}(a_L)^r(1+\rho_A(x_P-x_L))^{-r(M-1)}(1+\rho_A(x_Q-x_P))^{-\lambda}\right)^{1/r} \\ &\le C\left(\sum_{\mathrm{scale}(L)=0}(a_L)^r(1+\rho_A(x_Q-x_L))^{-\lambda}\sum_{\mathrm{scale}(P)=0}(1+\rho_A(x_P-x_L))^{-r(M-1)+\lambda}\right)^{1/r} \\ &\le C(a^*_{r,\lambda})_Q\,. \end{aligned}$$

Here, we used the estimates

$$\sum_{k\in\mathbb{Z}}(1+\rho_A(k))^{-1-\delta} < \infty, \qquad \delta > 0\,,$$

and

$$H(1+\rho_A(x_P-x_L))(1+\rho_A(x_Q-x_P)) \ge (1+\rho_A(x_Q-x_L))\,.$$

Likewise, if $0 < r < 1$, then it suffices to use r-triangle inequality to obtain the same estimate $(d^*_{r,\lambda})_Q \le C(a^*_{r,\lambda})_Q$. Thus, by (8.8),

$$(a^*_{r,\lambda})_Q \le 2^{1/r}\left((b^*_{r,\lambda})_Q + c(\lambda_-)^{-\gamma}(d^*_{r,\lambda})_Q\right) \le 2^{1/r}\left((b^*_{r,\lambda})_Q + cC(\lambda_-)^{-\gamma}(a^*_{r,\lambda})_Q\right).$$

Since $f \in \mathcal{S}$, $(a^*_{r,\lambda})_Q < \infty$. Therefore, by taking sufficiently large γ we have $(a^*_{r,\lambda})_Q \le C(b^*_{r,\lambda})_Q$, where the constant C is independent of f and Q. This shows (8.7) for $f \in \mathcal{S}$, since the converse estimate $(b^*_{r,\lambda})_Q \le (a^*_{r,\lambda})_Q$ is trivial.

To remove the assumption that $f \in \mathcal{S}$, we apply a standard regularization argument. Let $h \in \mathcal{S}$ satisfy $\mathrm{supp}\,\hat{h} \subset B(0,1)$, $\hat{h}(\xi) \ge 0$, and $h(0) = 1$. By the Fourier inversion formula, $|h(x)| \le 1$ for all $x \in \mathbb{R}^n$. For $0 < \delta < 1$, let $f_\delta(x) = f(x)h(\delta x)$. Then $\mathrm{supp}\,\hat{f_\delta} \subset K + B(0,1)$, $f_\delta \in \mathcal{S}$, $|f_\delta(x)| \le |f(x)|$ for all x, and $f_\delta(x) \to f(x)$ uniformly on compact sets as $\delta \to 0$. Applying (8.7) to f_δ and letting $\delta \to 0$, we obtain (8.7) for a general $f \in \mathcal{S}'$. □

Recall that in Lemma 3.11 we require that $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is such that $\operatorname{supp}\hat{\varphi}$ is compact and bounded away from the origin. For any $f \in \mathcal{S}'/\mathcal{P}$ and $\gamma \in \mathbb{N}$ we also recall that the sequence $\inf(f) = \{\inf_Q(f)\}_{Q\in\mathcal{Q}}$ is given by

$$\inf{}_Q(f) = |Q|^{1/2} \sup\left\{ \inf_{y\in P} \left|\tilde{\varphi}_j * f(y)\right| : \operatorname{scale}(P) = \operatorname{scale}(Q) - \gamma,\ P \cap Q \neq \emptyset\right\},$$

where $j = -\operatorname{scale}(Q)$ and $Q \in \mathcal{Q}$. Under these assumptions we have the following lemma.

Lemma 8.4. *Suppose that $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$. Then for any $\gamma \geq 0$ we have*

$$||\inf(f)||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \leq C||f||_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\tilde{\varphi})},$$

where C is independent of $f \in \mathcal{S}'/\mathcal{P}$.

Proof. For fixed $\gamma \geq 0$ define the sequence $s = \{s_P\}$ by

$$s_P = |P|^{1/2} \inf_{y\in P} \left|\tilde{\varphi}_{i-\gamma} * f(y)\right| \qquad \text{for} \quad P \in \mathcal{Q},\ \operatorname{scale}(P) = -i\,.$$

Clearly, we have

$$|Q|^{-1/2} \inf{}_Q(f) = \sup\left\{|P|^{-1/2}|s_P| : P \cap Q \neq \emptyset,\ \operatorname{scale}(P) = \operatorname{scale}(Q) - \gamma\right\}.$$

Fix $j \in \mathbb{Z}$ and $Q \in \mathcal{Q}$ with $\operatorname{scale}(Q) = -j$. Suppose that $P_1, P_2 \in \mathcal{Q}$ are such that

$$\operatorname{scale}(P_1) = \operatorname{scale}(P_2) = -j - \gamma, \quad y_1 \in P_1 \cap Q \neq \emptyset, \quad y_2 \in P_2 \cap Q \neq \emptyset\,. \tag{8.9}$$

Then by (2.2)

$$\rho_A(x_{P_1} - x_{P_2}) \leq H^2(\rho_A(x_{P_1} - y_1) + \rho_A(y_1 - y_2) + \rho_A(y_2 - x_{P_2})) \leq C|Q|\,.$$

Then for any $0 < r < \infty$ and $\lambda > 1$,

$$s_{P_1} \leq (1 + \rho_A(x_{P_1} - x_{P_2})/|P_1|)^{\lambda/r}\left(s^*_{r,\lambda}\right)_{P_2} \leq C|\det A|^{\gamma\lambda/r}\left(s^*_{r,\lambda}\right)_{P_2}\,.$$

Combining this with (8.9) yields

$$\sum_{\operatorname{scale}(Q)=-j} \inf{}_Q(f)\tilde{\chi}_Q \leq C|\det A|^{\gamma\lambda/r} \sum_{P\in\mathcal{Q},\ \operatorname{scale}(P)=-j-\gamma} \left(s^*_{r,\lambda}\right)_P \tilde{\chi}_P\,.$$

Choosing $r > 0$ and $\lambda > \beta\max(1, r/q, r/p)$ as in Lemma 3.10 we have

$$\begin{aligned}
||\inf(f)||_{\dot{\mathbf{f}}_p^{\alpha,q}} &\leq C|\det A|^{\gamma(\lambda/r-\alpha)}\left\|s^*_{r,\lambda}\right\|_{\dot{\mathbf{f}}_p^{\alpha,q}} \leq C|\det A|^{\gamma(\lambda/r-\alpha)}||s||_{\dot{\mathbf{f}}_p^{\alpha,q}}\\
&\leq C|\det A|^{\gamma(\lambda/r-\alpha)}\left\|\left(\sum_{i\in\mathbb{Z}}\left(|\det A|^{i\alpha}|\tilde{\varphi}_{i-\gamma} * f|\right)^q\right)^{1/q}\right\|_{L^p(\mu)}\\
&= C|\det A|^{\gamma\lambda/r}||f||_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\tilde{\varphi})}\,.
\end{aligned}$$

Note that the last estimate works also in the case $p = \infty$ with $\dot{\mathbf{F}}_p^{\alpha,q}$ and $\dot{\mathbf{f}}_p^{\alpha,q}$ norms replaced by their localized analogues $\dot{\mathbf{F}}_\infty^{\alpha,q}$ and $\dot{\mathbf{f}}_\infty^{\alpha,q}$. □

Lemma 3.11 is now a simple consequence of Lemmas 8.3 and 8.4.

Proof of Lemma 3.11. The estimate

$$||f||_{\dot{\mathbf{F}}_p^{\alpha,q}(\mathbb{R}^n,A,\mu)(\tilde{\varphi})} \leq ||\sup(f)||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)}$$

is easily verified from the definitions.

Fix any $Q_0 \in \mathcal{Q}$ with scale$(Q_0) = -j$, $j \in \mathbb{Z}$. Define $g(x) = (\tilde{\varphi}_j * f)(A^{-j}x)$. Note that $\operatorname{supp}\hat{g} \subset K := \operatorname{supp}\hat{\varphi}$. Define sequences $\{a_Q\}$ and $\{b_Q\}$ by (8.6) with f replaced by g. A direct calculation shows that

$$a_{A^jQ} = |Q|^{-1/2}\sup{}_Q(f), \qquad b_{A^jQ} = |Q|^{-1/2}\inf{}_Q(f), \qquad Q \in \mathcal{Q}\,.$$

Hence, by Lemma 8.3 applied to the cube A^jQ_0,

$$\left(\sup(f)^*_{r,\lambda}\right)_{Q_0} = |Q_0|^{1/2}\left(a^*_{r,\lambda}\right)_{A^jQ_0} \leq c|Q_0|^{1/2}\left(b^*_{r,\lambda}\right)_{A^jQ_0} = c\left(\inf(f)^*_{r,\lambda}\right)_{Q_0}\,.$$

Since $Q_0 \in \mathcal{Q}$ is arbitrary, by choosing $r > 0$ and $\lambda > 1$ as in Lemma 3.10 we have

$$||\sup(f)||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)} \leq c||\inf(f)||_{\dot{\mathbf{f}}_p^{\alpha,q}(A,\mu)}\,.$$

Combining the above with Lemma 8.4 completes the proof of Lemma 3.11. □

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