In Polytopes, Small Balls about Some Vertex Minimize Perimeter

By *Frank Morgan*

ABSTRACT. In (the surface of) a convex polytope P^n in $\mathbb{R}^{n+1}$, for small prescribed volume, geodesic balls about some vertex minimize perimeter.

1. Introduction

Even in smooth Riemannian manifolds, there are relatively few examples of explicitly known regions which minimize perimeter for prescribed volume (see [16, 13.2], [12]). One general result is that for small volume, a perimeter-minimizing region is a nearly round ball where the scalar curvature is large ([14], [19, Theorem 2.2], or in 3D [23, Theorem 18], with [4]).

For singular ambients, one result in general dimensions is that in smooth cones of positive Ricci curvature, geodesic balls about the apex are perimeter minimizing [20, Corollary 3.9]. Examples of Cotton et al. [8, 3] include the surface of the cube in $\mathbb{R}^3$, with perimeter-minimizing regions as pictured in Figure 1. In particular, for small prescribed area, geodesic balls about a vertex minimize perimeter. The analogous result in higher dimensions has remained open (see [20, Remark 2.3]). Our Theorem 3.8 proves that:

> In a polytope P^n in $\mathbb{R}^{n+1}$, for small prescribed volume, geodesic balls about some vertex minimize perimeter.

(Note that by "polytope" we refer to the *boundary* of the compact, convex solid body.) For $n \geq 3$, P^n has singularities of positive dimension, and the perimeter-minimizing spheres pass through these singularities.

1.1. The proof

The proof has the following elements. Let C be the tangent cone at a vertex, with link a spherical polytope Q^{n-1} in $\mathbb{S}^n$.

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(1)   By the Levy-Gromov isoperimetric inequality, the isoperimetric profile for Q is bounded below by the isoperimetric profile for $\mathbb{S}^{n-1}$. This step requires smoothing (Lemma 2.1) and approximation.

(2)   By an isoperimetric comparison Theorem 3.2 for products and cones (Ros after Barthe and Maurey), balls about the apex of C are perimeter minimizing (Theorem 3.3).

(3)   For perimeter-minimizing surfaces in P^n with prescribed volumes approaching zero, strong control of mean curvature after Almgren [1] (Lemma 3.5) is obtained by considering a limit of renormalizations to unit volume. A volume concentration argument shows that the limit is nonzero.

(4)   It follows from (3) that the sum of the diameters of components of perimeter-minimizing regions is small (Lemma 3.6). The proof involves a generalization of lower mass bounds for minimizers of elliptic integrands to allow volume constraints. As a result we may assume that components lie in a neighborhood of the vertices.

(5)   By (2), we may assume that they consist of balls about vertices. A single ball is best.
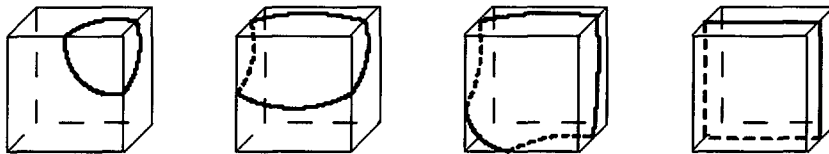

FIGURE 1   Perimeter-minimizing regions in the surface of the cube [3, Figure 1].

1.2. Solid polytopes

Our results also hold in the less singular category of *solid* polytopes (see Section 3.11).

2. Smoothing

The following lemma will be used in the proof of Theorem 3.3.

Lemma 2.1 (Smoothing Lemma).   *Given a (convex) polytope Q^{n-1} in $\mathbb{S}^n$, there are smooth convex hypersurfaces Q_ε which converge smoothly to Q off the singular set with volumes converging to that of Q.*

Proof.   The corresponding result in $\mathbb{R}^n$ is standard and easy (see, e.g., [7, Note 1.3]). Indeed, fix an origin in $\mathbb{R}^n$ inside Q, take the convex function F which is the fraction of the distance from the origin to Q along a straight line, let F_ε be a smoothing of F by symmetric convolution, and let $Q_\varepsilon = \{F_\varepsilon = 1\}$. For convolution, start with a nice smooth symmetric function φ supported in $(0, 1)$ with integral 1 and let $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)/\varepsilon$.

In $\mathbb{S}^n$, we may assume that Q lies in the northern hemisphere H, where distance from the north pole is a convex function. Take the convex function F on H which is the fraction of the distance from the north pole to Q along a geodesic. Choose $\zeta \le \varepsilon$ such that on a ζ neighborhood of $\{F = 1 - \varepsilon\}$, $1 - 2\varepsilon \le F \le 1 - \varepsilon$. Let F_ε be the convolution of F with φ_ζ, which is convex. Let $Q_\varepsilon = \{F_\varepsilon = 1 - 2\varepsilon\}$. $\square$

3. Isoperimetric regions

3.1. Definitions

The cone C over a subset M of the unit sphere in Euclidean space is defined as

$$C = \{tx : x \in M, t \geq 0\} .$$

For a general smooth Riemannian manifold M, the cone (minus the negligible apex) may be defined as the warped product $(0, \infty) \times_f M$ with $f(t) = t$, i.e., as $(0, \infty) \times M$ with the metric

$$ds^2 = dt^2 + t^2 \, dm^2 .$$

A *density* on a Riemannian manifold is a positive function used to weight volume and perimeter (see [17]). Although, we will use only constant densities, Theorem 3.2 holds for variable smooth densities. For the more general context of "mm spaces" (metric spaces with measures), see Gromov [11].

The *isoperimetric profile* $I(M)(V)$ of a space M gives the minimum (or infimum) perimeter required to enclose volume V.

The following useful theorem of Ros says that isoperimetric comparison of a manifold M_2 with the sphere is preserved under product with another manifold M_1. It follows earlier comparisons with Gauss space by Barthe and Maurey. The idea is Schwarz symmetrization: Compare a region in a product $M_1 \times M_2$ with the region in $M_1 \times \mathbb{S}^n$ obtained by replacing vertical slices (contained in M_2) with $(n-1)$-balls in $\mathbb{S}^n$.

Theorem 3.2. (Isoperimetric Comparison Theorem for Warped Products [23, Theorem 3.7]). *Let M_1, M_2 be smooth Riemannian manifolds with densities, with $|M_2| = |\mathbb{S}^n|$. Consider smooth warped products $M_1 \times_f \mathbb{S}^n$, $M_1 \times_f M_2$. If $I(M_2) \geq I(\mathbb{S}^n)$, then $I(M_1 \times_f M_2) \geq I(M_1 \times_f \mathbb{S}^n)$. In particular, if for some region $R \subset M_1$ the product $R \times \mathbb{S}^n$ is minimizing, so is $R \times M_2$, uniquely if $I(M_2) > I(\mathbb{S}^n)$ except at the endpoints.*

Remark. Although, Ros does not consider *warped* products, the proof is the same (and the uniqueness statement follows immediately from the proof).

Theorem 3.2 provides an immediate proof of a theorem of Morgan and Ritoré. Our generalization in part B requires our Smoothing Lemma 2.1.

Theorem 3.3.

(A) [20, Corollary 3.9]. *For $n \geq 3$, let C be a cone over a smooth, closed, connected Riemannian manifold M^{n-1} of Ricci curvature at least $n - 2$. Then geodesic balls about the apex are perimeter minimizing for given volume, uniquely unless $C = \mathbb{R}^n$.*

(B) *Likewise, for $n \geq 2$, let C be the cone over any (convex) spherical geodesic polytope $Q^{n-1} \subset \mathbb{S}^n$. Then geodesic balls about the apex are perimeter minimizing.*

Remark. A similar, trivial result is that in $\mathbb{R}^n$ modulo a finite group Γ of isometries leaving a point p fixed, geodesic balls about p are perimeter minimizing, equivalent to the obvious statement that among Γ-invariant objects in $\mathbb{R}^n$, geodesic balls about p are perimeter minimizing. This observation applies to any perimeter-minimizing geodesic ball about a point p in any Riemannian manifold.

Proof of Theorem 3.3.   Renormalize M with constant density to make $|M| = |\mathbb{S}^{n-1}|$. By the curvature hypothesis, the Levy-Gromov isoperimetric inequality ([10, 2.2], or [23, Section 2.5]) says that the isoperimetric profiles satisfy

$$I(M) \geq I\left(\mathbb{S}^{n-1}\right) . \tag{3.1}$$

Note that C is the warped product $(0, \infty) \times_f M$ with $f(t) = t$, and that $(0, \infty) \times_f \mathbb{S}^{n-1}$ is just $\mathbb{R}^n$. Hence, by Theorem 3.2, since balls about the origin are perimeter minimizing in $\mathbb{R}^n$, balls about the apex are perimeter minimizing in C. If there is another minimizer, then equality holds nontrivially in Equation (3.1), and $M = \mathbb{S}^{n-1}$, i.e., $C = \mathbb{R}^n$.

Likewise, given any (convex) spherical geodesic polytope $Q^{n-1} \subset \mathbb{S}^n$ with $n \geq 3$, given a volume V, use Lemma 2.1 to obtain a smoothing Q_ε with sectional curvature at least 1 and hence Ricci curvature at least $n - 2$ (and normalized by a constant density to have the same volume as $\mathbb{S}^{n-1}$). Then as before

$$I\left(C_\varepsilon\right) \geq I\left(\mathbb{R}^n\right) . \tag{3.2}$$

Hence,

$$I(C)(V) \geq \limsup I\left(C_\varepsilon\right)(V) \geq \liminf I\left(C_\varepsilon\right)(V) \geq I\left(\mathbb{R}^n\right)(V) ,$$

and balls around the apex are perimeter minimizing.

For the easier case $n = 2$, when all cones are isometric to circular cones, geodesic balls about the apex are well known to be uniquely perimeter minimizing. See [12, Section 8]; alternatively, apply Theorem 3.2 after noting that since $|Q| \leq |\mathbb{S}^1|$, after renormalization $I(Q) \geq I(\mathbb{S}^1)$. $\square$

The following Lemma 3.4 is useful in obtaining uniform estimates in the proof of the main Theorem 3.8.

Lemma 3.4.   *Fix a simplex Δ in $\mathbb{R}^n$. For each integer $N \geq 2$, slice Δ by N equally spaced hyperplanes parallel to each face, the first containing the face, the last through the opposite vertex. Among the many pieces of Δ which arise under such slicing for all such N, at most n shapes occur (up to translation and homothety).*

Remark.   In $\mathbb{R}^2$, slicing a triangle yields two shapes: The triangle and its central inversion (an upside-down triangle). In $\mathbb{R}^3$, slicing a tetrahedron yields three shapes: The tetrahedron, its inversion, and an octahedron (fully truncated tetrahedron).

Proof of Lemma 3.4.   By a linear transformation, we may assume that Δ is regular, with unit normals $a_1 + \cdots + a_{n+1} = 0$, normalized so that

$$a_i \bullet a_j = -1 , \quad \text{except that} \quad a_i \bullet a_i = n . \tag{3.3}$$

Such a regular simplex Δ is given by

$$\Delta = \{-1 < a_i \bullet x < 0 , \quad \text{except} \quad 0 < a_{n+1} \bullet x < 1\} .$$

Our slicing yields pieces of the form

$$\left\{ -\frac{k_i}{N} < a_i \bullet x < -\frac{k_i - 1}{N} , \quad \text{except} \quad \frac{k_{n+1} - 1}{N} < a_{n+1} \bullet x < \frac{k_{n+1}}{N} : 1 \leq k_i \leq N \right\} ,$$

or by scaling and replacing k_{n+1} by $-k_{n+1} + 1$, more generally

$$S_k = \{-k_i < a_i \bullet x < -k_i + 1 : k_i \in \mathbb{Z}\} . \tag{3.4}$$

By Equation (3.3), translating S_k by a_1 just adds $(-n, 1, 1, \ldots, 1)$ to k, and similarly for each a_i. Hence, translations generate $\{\sum k_i = 0\}$, and the shape of S_k is characterized by $\sum k_i$. By summing Equation (3.4) over i, we see that for nonempty S_k,

$$0 < \sum k_i < n + 1 \,,$$

yielding at most n different shapes, as asserted. □

Lemma 3.5. *Locally, let S_α be a weakly convergent sequence of mD isoperimetric hypersurfaces without boundary, with a nonzero limit. Then there are constants $C_1, \delta > 0$ such that for α large, in any δ-ball B, S_α minimizes area plus $C_1|V|$ in comparison with other surfaces in B with the same boundary enclosing net volume V with S_α.*

Remarks and definitions. This lemma holds in the general context of rectifiable currents [16], with weak convergence in the flat norm or as measures. Isoperimetric surfaces satisfy an equilibrium condition of "weakly constant mean curvature" H, which means that for any smooth family of diffeomorphisms for which the initial rate of change of enclosed volume $dV/dt|_0$ is nonzero, the derivative of area A with respect to volume satisfies

$$\left.\frac{dA}{dV}\right|_0 = mH \,.$$

The conclusion of the lemma is a kind of uniform mean curvature bound. The limit ambient may have singularities, as long as it is a smooth manifold at some point of S.

Proof of Lemma 3.5. Choose $\delta_1 > 0$ small enough so that there are two smooth δ_1-balls about points of the limit S such that any third δ_1-ball B is disjoint from one of them, say B_0. Let Φ_t be a smooth family of diffeomorphisms supported in the interior of B_0 such that for S, $dV/dt|_0 = 1$; let $A_0 = dA/dt|_0$. Choose $t_0 > 0$ so small that for all $-t_0 \le t \le t_0$, $.9 \le dV/dt \le 1.1$ and $|dA/dt| \le |A_0| + 1$. Consequently, we may assume that for all S_α, for all $-.9t_0 \le V \le .9t_0$, such diffeomorphisms of S_α alter enclosed volume with S_α by V and increase at most $|V| (|A_0| + 1)/.9 = C_1 V$. Choose $\delta < \delta_1$ such that a δ-ball has volume less than $.9t_0$.

Now let T be any surface which differs from S_α only in a δ-ball B. Obtain T' by altering T in B_0 so that it bounds net volume 0 with S_α, such that its area satisfies

$$|T'| \le |T| + C_2|V| \,.$$

Since S_α is isoperimetric,

$$|S_\alpha| \le |T'| \le |T| + C_2|V| \,,$$

as desired. □

The following lemma is a modification of a standard "monotonicity" argument (see [16, Ch. 9]).

Lemma 3.6. *Let C be an nD (convex) polytopal cone in $\mathbb{R}^{n+1}$. Given $C_1 > 0$, there is an $a > 0$ such that the following holds. Let S be an $(n-1)$-dimensional surface (rectifiable current) in a closed ball B of radius 1 about one of its points, with boundary contained in ∂B. Suppose that S minimizes area plus $C_1|V|$ in comparison with other surfaces in B with the same boundary enclosing net volume V with S. Then the area of S is at least a.*

Proof. Choose a vector in the interior of the cone. Projection f onto the orthogonal hyperplane Π is a bijection of C with Π, which for some $C_2 > 0$, shrinks distance, area, and volume by a

factor of at most C_2. In particular, there is a $\delta > 0$ such that the projection of a unit ball about a point in C contains a δ-ball about the projection of the point.

Given S and a point in S, let T and p denote their images under f. For $r \leq \delta$, let $T_r = T \cap B(p, r)$, let S_r denote the preimage of T_r, and let $g(r)$ denote the area of T_r. For almost all r, the boundary of T_r has area at most $g'(r)$ (see [16, Chapter 9]). By the isoperimetric theorem for the ball, this same boundary bounds a surface T' of area

$$\left|T'\right| \leq \alpha g'(r)^{(n-1)/(n-2)} ,$$

for some isoperimetric constant α depending only on dimension; together with T_r, T' bounds a region of mass at most

$$\beta_0 \left[g(r) + \alpha g'(r)^{(n-1)/(n-2)} \right]^{n/(n-1)} \leq \beta g(r)^{n/(n-1)} + \beta g'(r)^{n/(n-2)}$$

for some isoperimetric constant β_0 and constant β. The preimage $S' = f^{-1}(T')$ has area

$$\left|S'\right| \leq C_2 \left|T'\right| \leq C_2 \alpha g'(r)^{(n-1)/(n-2)} ;$$

together with S_r, S' bounds a region of mass at most $C_2 \beta g(r)^{n/(n-1)} + C_2 \beta g'(r)^{n/(n-2)}$. By the minimizing property of S,

$$
\begin{aligned}
|S_r| &\leq& |S'| + C_1 C_2 \beta g(r)^{n/(n-1)} + C_1 C_2 \beta g'(r)^{n/(n-2)} \\
&\leq& C_2 \alpha g'(r)^{(n-1)/(n-2)} + C_1 C_2 \beta g(r)^{n/(n-1)} + C_1 C_2 \beta g'(r)^{n/(n-2)} .
\end{aligned}
$$

Since $g(r) \leq |S_r|$, for some $C_3 > 0$,

$$g(r) \left(1 - C_3 g(r)^{1/(n-1)} \right) \leq C_3 \left[g'(r)^{(n-1)/(n-2)} + g'(r)^{n/(n-2)} \right] .$$

We may assume that the coefficient of $g(r)$ for any $r \leq 1/2C_2 < 1$ is at least $1/2$, since otherwise we have a lower bound as desired for $g(r)$. Let

$$A = \left\{ 0 < r < 1 : g'(r) \leq 1 \right\} .$$

We may assume that $|A| \geq 1/2$, since otherwise we immediately obtain a lower bounded as desired for $g(r)$. On A,

$$g(r) \leq C_4 g'(r)^{(n-1)/(n-2)} .$$

Since g is monotonically increasing, integration yields $g(r) \geq C_5 r^{n-1}$ again yielding the desired lower bound for $g(1/2C_1)$ and hence for $|S|$. $\square$

Corollary 3.7.   *Given a polytope P^n in $\mathbb{R}^{n+1}$ and $C_1, \delta > 0$, there is an $a > 0$ such that the following holds. Let S be an $(n - 1)$-dimensional surface in P which in any δ-ball B minimizes area plus $C_1|V|$ in comparison with other surfaces in B with the same boundary enclosing net volume V with S. If the unit ball $B(p, 1)$ about a point p in S lies in the cone at a vertex of P, then the area of $S \cap B(p, 1)$ is at least a.*

Proof.   By scaling we may assume that $\delta = 1$. Lemma 3.6 applies to the cone over each vertex of P. Let a be the minimum of the corresponding a's. $\square$

Gnepp, Ng, and Yoder [8, 3] proved that in the surface of a cube or regular tetrahedron, for small prescribed area, balls around a vertex are perimeter minimizing. The same interesting

question in higher dimensions has remained open (see [20, Remark 2.3]). The following theorem answers the question affirmatively. (Recall that by "polytope" we refer to the *boundary* of the compact, convex solid body.)

Theorem 3.8. *For $n \geq 1$, let P^n be a polytope in $\mathbb{R}^{n+1}$. For small prescribed volume V, geodesic balls about some vertex are perimeter minimizing.*

Remark. Except for polyhedra in $\mathbb{R}^3$ ($n = 2$), our proof does not provide uniqueness, because of the use of smoothing and approximation. Of course uniqueness fails for polygons ($n = 1$).

Proof. Let R be a perimeter-minimizing region in P of small volume V. We will need some concentration of volume. Partition P into simplices and slice each simplex into small polytopes K_i by N equally spaced hyperplanes parallel to each face. By Lemma 3.4, there are only finitely many shapes for the K_i (independent of N, up to homothety) and for some positive constant β (independent of N) we may choose N such that

$$\beta V \leq |K_i| \leq V .$$

We claim that *there is a constant $\delta > 0$ depending only on P such that some K satisfies*

$$|R \cap K| \geq \delta V . \tag{3.5}$$

To prove this claim, first note that the least area A for small volume V satisfies $A \leq C V^{(n-1)/n}$. We may assume that $\delta < 1/2$ and $|R \cap K_i| \leq |K_i|/2$. Because there are only finitely many shapes for the K_i, by the relative isoperimetric inequality [16, 12.3 (1)], there is a single constant γ such that

$$|\partial R \cap K_i| \geq \gamma |R \cap K_i|^{(n-1)/n} \geq \frac{\gamma |R \cap K_i|}{\max |R \cap K_j|^{1/n}} .$$

Summing over i yields

$$A \geq \gamma V / \max |R \cap K_j|^{1/n} ,$$

$$\max |R \cap K_j|^{1/n} \geq \gamma V / A \geq \gamma V^{1/n} / C .$$

Hence, some K satisfies

$$|R \cap K| \geq \delta V$$

with $\delta = \gamma^n / C^n$, proving the claim (3.5).

To complete the proof of the theorem, let R_α be a sequence of perimeter-minimizing regions with V_α small and approaching 0. For each R_α use the claim to choose K_α with $\beta V_\alpha \leq |K_\alpha| \leq V_\alpha$ and

$$|R_\alpha \cap K_\alpha| \geq \delta V_\alpha \geq \delta |K_\alpha| .$$

Rescale up to $1 = V_\alpha \geq |K_\alpha| \geq \beta$ with K_α centered at the origin. By compactness [16, 9.1], we may assume that the R_α converge to a perimeter-minimizing region R with $1 \geq |R| \geq \beta\delta > 0$ lying in the tangent cone at a vertex or the most complicated type of point that stays within a bounded distance of the origin. By Lemma 3.5, there are constants $C_1, \delta_1 > 0$ such that for α large, in any δ_1-ball B, $S_\alpha = \partial R_\alpha$ minimizes area plus $C_1 |V|$. By Corollary 3.7, the sum of the diameters of components of S_α and hence of the R_α is bounded. Since the polytope is being scaled up, by translation of components, we may assume that R_α is contained in the union of disjoint balls about the vertices. By Theorem 3.3 (B), we may assume that R_α consists of balls about vertices v. Since for such a ball, $A = c_v V^t$ with $t = (n-2)/(n-1) < 1$, by the concavity of V^t, a single ball is best. $\qquad\square$

Remark 3.9. A more general but more technical version of the concentration argument at the beginning of the proof of Theorem 3.8 uses bi-Lipschitz equivalence with $\mathbb{R}^n$ as in the proof of Lemma 3.6. The small polytopes K_i may then be replaced by cubes in $\mathbb{R}^n$. This approach obviates the need for our cute Lemma 3.4.

Remark 3.10 (Nonconvex polytopes). Theorem 3.8 fails for general "nonconvex polytopes." For example, consider two tall skinny pyramids in $\mathbb{R}^3$, attached at the apices. By Theorem 3.8, for small volume, in each pyramid separately a ball about the apex is perimeter minimizing. Hence, in their union, those same sets are perimeter minimizing. But balls about the common apex are unions of such sets and have more perimeter.

For $n = 2$, Theorem 3.8 holds for nonconvex polyhedra locally homeomorphic to $\mathbb{R}^2$, for which the tangent cones are all isometric to circular cones. The only additional ingredient for the proof is that in a cone with cone angle greater than 2π, Euclidean discs (away from the apex) are perimeter minimizing. This fact follows, for example, from Theorem 3.2.

For $n \geq 3$, Theorem 3.8 fails even for nonconvex polytopes homeomorphic to the 3-sphere. For example, let K^2 be the surface of a unit cube in $\mathbb{R}^3 = \{x \in \mathbb{R}^4 : x_4 = 1\}$, surmounted by a tall skinny tetrahedron with apex p. Note that in the cone $0 \times K^2$ over K^2, for small volume, a ball about a point q in $0 \times \{p\}$ has less perimeter than a ball about 0, the apex of the cone, because of the small solid angle. To get the counterexample polytope P^3, truncate the cone at $\{x_4 = 1\}$ and surmount it on the surface of a big hypercube in $\mathbb{R}^4$. A ball about q still has less perimeter than a ball about any vertex of P^3, essentially because the exterior of K in $\mathbb{R}^3$ has no small solid angles.

Remark 3.11 (Solid polytopes). The results of this article apply to the nicer category of solid polytopes $P^n \subset \mathbb{R}^n$, which have no interior singularities. Theorem 3.3 with uniqueness and proof go over without smoothing to solid spherical polytopes $Q^{n-1} \subset \mathbb{S}^{n-1}$, a result apparently first proved (without uniqueness) by Lions and Pacella ([15, Theorem. 1.1], see also [18, Remark after Theorem 10.6]). Likewise, Theorem 3.8 with uniqueness and proof go over to solid polytopes $P^n \subset \mathbb{R}^n$.

Theorem 3.2 also has the following general consequence.

Theorem 3.12. *Let M^n be a compact C^1 submanifold of $\mathbb{R}^{n+1}$ (with or without boundary), or polytope (convex or not, solid or boundary, but compact), or any compact Lipschitz neighborhood retract. Then in the cone over a small homothetic copy of M, balls about the apex are isoperimetric.*

Proof. The isoperimetric profile for M satisfies

$$I(M)(V) \geq cV^{(n-1)/n}$$

for $0 \leq V \leq |M|/2$ ([6, 4.4.2 (2)]; see [16, 12.3]). Hence, the isoperimetric profile for a small homothetic copy M' of M, renormalized with constant density to make $|M'| = |\mathbb{S}^n|$, satisfies

$$I\big(M'\big) \geq I\big(\mathbb{S}^n\big) , \tag{3.6}$$

with equality only at the endpoints. Hence, by Theorem 3.2, in the cone over M', balls about the apex are uniquely perimeter minimizing. $\square$

Remark. Similar results were already known for products $M \times \mathbb{R}$ [5, 2.11] and apparently $M \times \mathbb{R}^n$ [9]. Such results follow from Theorem 3.2 for $M \times \mathbb{R}$ and $M \times \mathbb{S}^1$ by comparison with results on $\mathbb{S}^n \times \mathbb{R}$ [21] and $\mathbb{S}^n \times \mathbb{S}^1$ [22].

V. Bayle pointed out to me that Berard et al. [2, Theorem 2] give explicit sufficient conditions on the scaling down of M for Equation (3.6) and hence Theorem 3.12 to hold, in terms of the Ricci curvature, the diameter, and the dimension of M.

Conjecture 3.13 (M. Hutchings [13, Conjecture 3]). *Let M, N be compact Riemannian manifolds. Then for fixed volume fraction, in the product of a small homothetic copy of M with N, an isoperimetric region is the product of M with an isoperimetric region in N.*

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References

[1] Almgren, Jr., F. J. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints, *Mem. Amer. Math. Soc.* **4**(165), (1976).

[2] Bérard, P., Besson, G., and Gallot, S. Sur une inégalité isopérimétrique qui généralise celle de Paul Lévy-Gromov [An isoperimetric inequality generalizing the Paul Levy-Gromov inequality], *Invent. Math.* **80**, 295–308, (1985).

[3] Cotton, A., Freeman, D., Gnepp, A., Ng, T. F., Spivack, J., and Yoder, C. The isoperimetric problem on some singular surfaces, *J. Aust. Math. Soc.* **78**, 167–197, (2005).

[4] Druet, O. Sharp local isoperimetric inequalities involving the scalar curvature, *Proc. Amer. Math. Soc.* **130**, 2351–2361, (2002).

[5] Duzaar, F. and Steffen, K. Existence of hypersurfaces with prescribed mean curvature in Riemannian manifolds, *Indiana Univ. Math. J.* **45**, 1045–1093, (1996).

[6] Federer, H. *Geometric Measure Theory,* Springer-Verlag, 1969.

[7] Ghomi, M. Optimal smoothing for convex polytopes, *Bull. London Math. Soc.* **36**, 483–492, (2004).

[8] Gnepp, A., Ng, T. F., and Yoder, C. Isoperimetric domains on polyhedra and singular surfaces, *NSF "SMALL" Undergraduate Research Geometry Group Report,* Williams College, 1998.

[9] Gonzalo, J. Large soap bubbles and isoperimetric regions in the product of Euclidean space with a closed manifold, PhD thesis, Univ. Calif. Berkeley, 1991.

[10] Gromov, M. Isoperimetric inequalities in Riemannian manifolds, Appendix I to Vitali D. Milman and Gideon Schechtman, Asymptotic Theory of Finite Dimensional Normed Spaces, *Lect. Notes in Math.* **1200**, Springer-Verlag, 1986.

[11] Gromov, M. Isoperimetry of waists and concentration of maps, *IHES,* 2002, http://www.ihes.fr/~gromov/topics/topic11.html.

[12] Howards, H., Hutchings, M., and Morgan, F. The isoperimetric problem on surfaces, *Amer. Math. Monthly* **106**, 430–439, (1999).

[13] Hutchings, M. Isoperimetric ignorance: A personal report, email to Frank Morgan, July 16, 2001.

[14] Kleiner, B. cited by Per Tomter, Constant mean curvature surfaces in the Heisenberg group, *Proc. Sympos. Pure Math.* **54**(1), 485–495, (1993).

[15] Lions, P.-L. and Pacella, F. Isoperimetric inequalities for convex cones, *Proc. Amer. Math. Soc.* **109**, 477–485, (1990).

[16] Morgan, F. *Geometric Measure Theory: A Beginner's Guide,* Academic Press, 3rd ed., 2000.

[17] Morgan, F. Manifolds with density, *Notices Amer. Math. Soc.* **52**, 853–858, (2005).

[18] Morgan, F. *Riemannian Geometry: A Beginner's Guide,* A. K. Peters, 1998.

[19] Morgan, F. and Johnson, D. L. Some sharp isoperimetric theorems for Riemannian manifolds, *Indiana Univ. Math. J.* **49**, 1017–1041, (2000).

[20] Morgan, F. and Ritoré, M. Isoperimetric regions in cones, *Trans. Amer. Math. Soc.* **354**, 2327–2339, (2002).

[21]   Pedrosa, R. H. L. The isoperimetric problem in spherical cylinders, *Ann. Global Anal. Geom.* **26**, 333–354, (2004).

[22]   Pedrosa, R. H. L. and Ritoré, M. Isoperimetric domains in the Riemannian product of a circle with a simply connected space form, *Indiana Univ. Math. J.* **48**, 1357–1394, (1999).

[23]   Ros, A. The isoperimetric problem, in *Global Theory of Minimal Surfaces* (Clay Research Institution Summer School, 2001, Hoffman, D., Ed.,) Amer. Math. Soc., 2005.

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267
e-mail: Frank.Morgan@williams.edu