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Jump Problem and Removable Singularities for Monogenic Functions

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ABSTRACT. In this article the jump problem for monogenic functions (Clifford holomorphicity) on the boundary of a Jordan domain in Euclidean spaces is investigated. We shall establish some criteria that imply the uniqueness of the solution in terms of a natural analogue of removable singularities in the plane to $\mathbb{R}^{n+1}$ ($n \geq 2$). Sufficient conditions to extend monogenically continuous Clifford algebra valued functions across a hypersurface are proved.

1. Introduction

Let γ be a closed Jordan curve in $\mathbb{C}$ which bounds a bounded domain Δ_+ and its complement $\Delta_- = \mathbb{C} \setminus (\Delta_+ \cup \gamma)$. The jump problem on γ involves seeking a function $\phi(z)$ holomorphic in $\overline{\mathbb{C}} \setminus \gamma$ from the boundary condition

$$\phi^+(t) - \phi^-(t) = f(t),\ t \in \gamma;\ \phi^-(\infty) = 0\,. \tag{1.1}$$

Here $\phi^+(t)$ and $\phi^-(t)$ represent the continuous limit values of ϕ at a point t as this point is approached from Δ_+ and from Δ_-, respectively, and $f(t)$ is a continuous function (jump function) specified on γ. This formulation of the jump problem is termed continuous, i.e., the solutions including their boundary values on Γ are continuous.

The question of looking at a natural multidimensional analogue of such a boundary value problem is closely connected with the problem of generalizing holomorphic functions theory in the plane to higher-dimension.

This article deals with a higher dimensional analogue of the jump problem (1.1) stated within the Clifford analysis setting.

It is well known that Clifford analysis has become, in recent years, a powerful mathematical tool for the treatment of boundary value problems in domains over Euclidean spaces of higher dimension.

With the help of the Clifford analytic methods the literature has achieved essential results in solving some type of principal boundary value problems, which have a lot of applications in mathematical physics and engineering.

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Our main interest lies in the study of the existence of the solution of the problem, but also we analyze sufficient conditions to guarantee the uniqueness of the solution under certain hypotheses. To this end we first derive a Dolzhenko type theorem for a class of monogenic functions in Euclidean spaces, i.e., within the framework of Clifford Analysis. The result thus obtained implies several versions of a higher dimensional Painlevé Theorem.

In [14] Dolzhenko gave a geometrical characterization of sets of removable singularities for classes of holomorphic functions in the plane, in particular, for functions belonging to the Hölder class. Dolzhenko's theorem includes the century-old Painlevé theorem which characterizes geometrically removable sets for continuous functions in $\mathbb{C}$.

2. Preliminaries

The real Clifford algebra associated with $\mathbb{R}^n$ endowed with the Euclidean metric is the minimal enlargement of $\mathbb{R}^n$ to a real linear associative algebra $\mathbb{R}_{0,n}$ with identity such that $x^2 = -|x|^2$, for any $x \in \mathbb{R}^n$.

It thus follows that if $\{e_j\}_{j=1}^n$, is the standard basis of $\mathbb{R}^n$, then we must have that $e_ie_j+e_je_i = -2\delta_{ij}$. Every element $a \in \mathbb{R}_{0,n}$ is of the form $a = \sum_{A\subseteq N} a_Ae_A$, $N = \{1, \ldots, n\}$, $a_A \in \mathbb{R}$, where $e_\emptyset = e_0 = 1$, $e_{\{j\}} = e_j$, and $e_A = e_{\beta_1}\cdots e_{\beta_k}$ for $A = \{\beta_1, \ldots, \beta_k\}$ where $\beta_j \in \{1, \ldots, n\}$ and $\beta_1 < \cdots < \beta_k$. The conjugation is defined by $\overline{a} := \sum_A a_A\overline{e}_A$, where

$$\overline{e}_A := (-1)^k e_{i_k}\cdots e_{i_2}e_{i_1}, \text{ if } \quad e_A = e_{i_1}e_{i_2}\cdots e_{i_k} .$$

Put $\mathbb{R}_{0,n}^{(k)} = \operatorname{span}_{\mathbb{R}}(e_A : |A| = k)$. Then clearly $\mathbb{R}_{0,n}^{(k)}$ is a subspace of $\mathbb{R}_{0,n}$ (the k-vectors in this class) and

$$\mathbb{R}_{0,n} = \sum_{k=0}^{n} \oplus\mathbb{R}_{0,n}^{(k)} .$$

The projection operator of $\mathbb{R}_{0,n}$ on $\mathbb{R}_{0,n}^{(k)}$ is denoted by $[\,]_k$ and $\mathbb{R}$ and $\mathbb{R}^n$ will be identified with $\mathbb{R}_{0,n}^{(0)}$ and $\mathbb{R}_{0,n}^{(1)}$, respectively.

In what follows an element $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with

$$x = x_0 + \sum_{j=1}^{n} x_je_j \in \mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)} .$$

Elements of $\mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$ are often called paravectors. Notice that for $x \in \mathbb{R}^{n+1}$, we thus have that

$$x\overline{x} = \overline{x}x = |x|^2 .$$

By means of the conjugation, $\mathbb{R}_{0,n}$ may be endowed with the natural Euclidean norm $|a|^2 = [a\overline{a}]_0$. An algebra norm is defined by taking $|a|_0^2 = 2^n|a|^2$.

We consider functions u defined in some subset Ω of $\mathbb{R}^{n+1}$ with values in $\mathbb{R}_{0,n}$. These functions may be written as

$$u(x) = \sum_A u_A(x)e_A ,$$

where u_A are $\mathbb{R}$-valued functions.

We say that u belongs to some classical class of functions on Ω if each of its components u_A belongs to that class.

In [12] (see also [13]) a theory of monogenic functions with values in Clifford algebras is considered which generalizes in a natural way the theory of holomorphic functions of one complex variable to the $(n+1)$-dimensional Euclidean space. Monogenic functions are null solutions of the generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$:

$$\partial_x := \sum_{j=0}^{n} e_j \partial_{x_j} \,.$$

It is a first order elliptic operator whose left and right fundamental solution is given by

$$e(x) = \frac{1}{\sigma_{n+1}} \frac{\overline{x}}{|x|^{n+1}}, \quad x \in \mathbb{R}^{n+1} \setminus \{0\} \,,$$

where σ_{n+1} is the area of the unit sphere in $\mathbb{R}^{n+1}$. If Ω is open in $\mathbb{R}^{n+1}$ and $u \in C^1(\Omega)$, then u is said to be left (resp. right) monogenic in Ω if $\partial_x u = 0$ (resp. $u\, \partial_x = 0$) in Ω. Furthermore, for a non-open set $\mathbf{E} \subset \mathbb{R}^{n+1}$ we call u monogenic in $\mathbf{E}$ if it is monogenic in some open neighborhood of $\mathbf{E}$.

Notice that the fundamental solution $e(x)$ is both left and right monogenic in $\mathbb{R}^{n+1} \setminus \{0\}$. Other basic examples of monogenic functions are obtained by means of the Cliffordian Cauchy transform. Assume that $\Omega = \Omega_+$ is a bounded domain in $\mathbb{R}^{n+1}$ with a sufficiently smooth boundary $\Gamma := \partial\Omega_+$. Then for each continuous function u in Γ, its Cliffordian Cauchy transform $\mathcal{C}_\Gamma u$ is formally defined by

$$(\mathcal{C}_\Gamma u)(x) := \int_\Gamma e(y-x)\kappa(y)u(y)\, d\mathcal{H}^n(y) \,, \; x \notin \Gamma \,,$$

and its singular version, the singular Cauchy transform $\mathcal{S}_\Gamma$ (also called the Hilbert transform) on Γ is given by

$$(\mathcal{S}_\Gamma u)(x) := 2\int_\Gamma e(y-x)\kappa(y)(u(y)-u(x))\, d\mathcal{H}^n(y) + u(x) \,, \; x \in \Gamma \,.$$

Hereby $\kappa(y)$ is the outward pointing unit normal to Γ at $y \in \Gamma$ defined according to Federer [17] and the integral in $\mathcal{S}_\Gamma$ is taken in the sense of the principal value.

A measure function is an increasing continuous function $h(r)$, $r \geq 0$, such that $h(0) = 0$. The Hausdorff measure of the set $\mathbf{E} \subset \mathbb{R}^{n+1}$ is given by

$$\mathcal{H}_h(\mathbf{E}) := \lim_{\delta \to 0} \inf\left\{ \sum_{k=1}^{\infty} h(\operatorname{diam} B_k) : \; \mathbf{E} \subset \bigcup_{k=1}^{\infty} B_k, \; \operatorname{diam} B_k < \delta \right\} ,$$

and the inner Hausdorff measure by

$$\underline{\mathcal{H}}_h(\mathbf{E}) = \sup\{\mathcal{H}_h(K)\} \,,$$

where the supremum is taken over all closed sets $K \subset \mathbf{E}$.

For $h(r) = \frac{\theta_n}{2^n} r^s$ $(s > 0)$, where θ_n represents the volume of the unit ball in $\mathbb{R}^n$, we write $\mathcal{H}^s$ $(\underline{\mathcal{H}}^s)$ instead of $\mathcal{H}_h$ $(\underline{\mathcal{H}}_h)$. Note that $\mathcal{H}^{n+1}$ coincides with the Lebesgue measure $\mathcal{L}^{n+1}$ in $\mathbb{R}^{n+1}$.

Let us recall the definition of Hausdorff dimension.

Definition 2.1. Let $\mathbf{K}$ be a bounded set in $\mathbb{R}^{n+1}$. The Hausdorff dimension $\alpha_H(\mathbf{K})$ of $\mathbf{K}$ is defined by

$$\alpha_H(\mathbf{K}) := \inf\left\{s > 0 : \mathcal{H}^s(\mathbf{K}) < \infty\right\}.$$

If an n-dimensional set $\mathbf{K} \subset \mathbb{R}^{n+1}$ has Hausdorff dimension $\alpha_H(\mathbf{K}) > n$, then it is called a fractal set in the sense of Mandelbrot.

For more details concerning the Hausdorff measure and dimension we refer the reader to [15, 16, 26].

The following obvious properties of the Hausdorff dimension will be useful later: For $s < \alpha_H(\mathbf{K})$, $\mathcal{H}^s(\mathbf{K}) = \infty$ while $\mathcal{H}^s(\mathbf{K}) = 0$ when $s > \alpha_H(\mathbf{K})$.

3. Upper Minkowski dimension

Let $\mathbf{K}$ be a compact set of $\mathbb{R}^{n+1}$ and suppose R_0 denotes a grid consisting of $(n+1)$-dimensional cubes with sides of length 1 and vertices with integer coordinates. The grid R_k is obtained from R_0 by division of each of the cubes in R_0 into $2^{(n+1)k}$ different cubes with side length 2^{-k}. Denote by $m_k(\mathbf{K})$ the number of cubes of the grid R_k which intersect $\mathbf{K}$. Then the value

$$\alpha(\mathbf{K}) := \overline{\lim}_{k\to\infty} \frac{\log_2 m_k(\mathbf{K})}{k}$$

is the upper Minkowski dimension of the set $\mathbf{K}$. The quantity $\alpha(\mathbf{K})$ is also known as the fractal dimension, box dimension, cell dimension, etc.

Throughout the article we denote by Γ a compact topological surface which is the boundary of a Jordan domain Ω^+ in $\mathbb{R}^{n+1}$ (see [20, 21]) and by Ω^- the complement of $\Omega^+ \cup \Gamma$. The boundary of Ω^+ is not required to satisfy the condition $\mathcal{H}^n(\Gamma) < \infty$, when it is the case, this will be indicated.

Definition 3.1. A surface Γ is called an n-rectifiable surface if $\mathcal{H}^n(\Gamma) < \infty$ and it is the Lipschitz image of some bounded set of $\mathbb{R}^n$.

The following lemma is probably well known, but for the reader's convenience we shall consider its inclusion in our article.

Lemma 3.2. *The upper Minkowski dimension of a surface Γ has the following properties:*

(i) $n \leq \alpha_H(\Gamma) \leq \alpha(\Gamma) \leq n+1$.

(ii) *If Γ is an n-rectifiable surface then $\alpha_H(\Gamma) = \alpha(\Gamma) = n$.*

Proof. The first assertion follows from the fact $m_k(\Gamma)2^{-(n+1)k} \leq m_0(\Gamma)$ and from the definition of the Hausdorff dimension.

Now let $\Gamma = \rho(G)$, where G is a bounded set of $\mathbb{R}^n$ and ρ is a Lipschitz function, i.e., there exists $c > 0$ such that $|\rho(x) - \rho(y)| \leq c|x-y|$ for all $x, y \in \mathbb{R}^n$ (c is called Lipschitz coefficient).

Let Q be an n-dimensional cube such that $G \subset Q$, let d be the diameter of Q and put $\delta_k := 2^{-k}/c$, where c is the Lipschitz coefficient of ρ. Then Q can be divided into $(\lfloor d/\delta_k \rfloor + 1)^n$

n-dimensional cubes Q^j of diameter not greater than δ_k. Here $\lfloor x \rfloor$ stands for the largest integer less than or equal to x. Therefore $\rho(Q^j \cap G)$ intersects not more than 2^{n+1} cubes of the grid R_k, whence

$$m_k(\Gamma) \le 2^{n+1}(\lfloor d/\delta_k \rfloor + 1)^n \le C\, 2^{nk} ,$$

where the constant C only depends on Γ. This completes the proof. □

Notice that property (ii) shows that the upper Minkowski dimension and Hausdorff dimension can be equal, although this is not always valid.

4. Boundary values of the Cauchy transform on Hölder spaces

If $\mathbf{E}$ is a bounded subset of $\mathbb{R}^{n+1}$, and u is a bounded $\mathbb{R}_{0,n}$-valued function defined on $\mathbf{E}$ we define the modulus of continuity of the function u as the nonnegative function $\omega(u,t)$, $t > 0$, by setting

$$\omega(u,t) = \sup_{|x-y|\le t} \{|u(x) - u(y)| : x, y \in \mathbf{E}\} .$$

Let ν be a real number with $0 < \nu \le 1$. We call a function u defined on $\mathbf{E} \subset \mathbb{R}^{n+1}$ Hölder continuous with exponent ν in $\mathbf{E}$ (Lipschitz continuous for $\nu = 1$) if

$$\sup_{0<t\le\delta} \frac{\omega(u,t)}{t^\nu} < \infty ,$$

where δ is the diameter of $\mathbf{E}$. Moreover, the set of Hölder continuous functions on $\mathbf{E}$ is denoted by $C^{0,\nu}(\mathbf{E})$ $(0 < \nu \le 1)$. With the norm

$$\|u\|_\nu := \|u\|_\infty + \sup_{0<t\le\delta} \frac{\omega(u,t)}{t^\nu} ,$$

where $\|u\|_\infty$ is the sup norm, the space $C^{0,\nu}(\mathbf{E})$ becomes a real Banach space.

Zygmund class, or quasismooth class, has long been used in one complex variable functions theory and on a natural analogue in higher dimension (several real variables) a great amount of work has been done.

Let us now present a Zygmund class version in the Clifford analysis context, which could therefore be used in our framework, see Section 5.

An $\mathbb{R}_{0,n}$-valued continuous function u belongs to the Zygmund class if is bounded and there exists a positive constant C such that

$$|u(x+y) - 2u(x) + u(x-y)| \le C\,|y| ,$$

for all $x, y \in \mathbb{R}^{n+1}$.

The boundedness of u and the above condition imply the continuity of u.

As Clifford analysis generalizes complex analysis to Euclidean spaces, we may expect that in this higher-dimensional function theory, the Zygmund class is related to the theory of sets of removable singularities just as in the case of the complex plane. The first two authors already indicated this analogy in three-dimensional spaces within the framework of quaternionic analysis (see [5]).

In what follows, Γ is an AD-regular surface, i.e., there exists a constant $C > 0$ such that for all $x \in \Gamma$ and $0 < r < \operatorname{diam}\Gamma$

$$C^{-1}r^n \leq \mathcal{H}^n(\Gamma \cap B(x,r)) \leq C\, r^n\,,$$

where $B(x, r)$ denotes the open ball with center x and radius r.

It is worth noting that $\mathcal{S}_\Gamma$ extends to a bounded linear operator on $C^{0,\nu}(\Gamma)$, satisfying $\mathcal{S}_\Gamma{}^2 = \mathcal{I}$, where $\mathcal{I}$ is the identity operator and that $\mathcal{S}_\Gamma$ appears naturally when studying the boundary behavior of the Cauchy transform $\mathcal{C}_\Gamma$ of a function $u \in C^{0,\nu}(\Gamma)$, $0 < \nu \leq 1$.

The central formula establishing the relation between the boundary value of $\mathcal{C}_\Gamma$ and $\mathcal{S}_\Gamma$ is the so-called Plemelj-Sokhotski formula. The following theorem is concerned with this main result and we refer the reader to [1, 2] for the proof. In [3, 4, 11] are much related work on this topic.

Theorem 4.1. *Let Γ be such that $\mathcal{H}^n(\Gamma) < \infty$. If $u \in C^{0,\nu}(\Gamma)$, $0 < \nu \leq 1$, we have that*

(i) $\mathcal{C}_\Gamma u \in C^{0,\nu}(\Omega^\pm \cup \Gamma)$ *with* $\mathcal{C}_\Gamma u(\infty) = 0$.
(ii) $\mathcal{C}_\Gamma u$ *is left monogenic in* $\mathbb{R}^{n+1} \setminus \Gamma$.
(iii) (Plemelj-Sokhotski Formula). *For all* $z \in \Gamma$,

$$(\mathcal{C}_\Gamma{}^{\pm} u)(z) := \lim_{\Omega^\pm \ni x \to z} (\mathcal{C}_\Gamma u)(x) = \frac{1}{2}((\mathcal{S}_\Gamma u)(z) \pm u(z))\,.$$

Theorem 4.2. *Let Γ be such that $\mathcal{H}^n(\Gamma) < \infty$ and let $u \in C^1(\Omega^+) \cap C(\overline{\Omega^+})$. Then the Borel-Pompeiu formula holds:*

$$\int_\Gamma e(y-x)\kappa(y)u(y)\, d\mathcal{H}^n(y) - \int_{\Omega^+} e(y-x)\partial_y u(y)\, d\mathcal{L}^{n+1}(y) = \begin{cases} u(x), & x \in \Omega^+ \\ 0, & x \in \Omega^-\,. \end{cases}$$

In terms of the Theodorescu transform T_{Ω^+} where (see [19])

$$(T_{\Omega^+}u)(x) := -\int_{\Omega^+} e(y-x)u(y)\, d\mathcal{L}^{n+1}(y),\ x \in \mathbb{R}^{n+1}\,,$$

the Borel-Pompeiu formula also reads:

$$(\mathcal{C}_\Gamma u)(x) + (T_{\Omega^+}\partial_x u)(x) = \begin{cases} u(x), & x \in \Omega^+ \\ 0, & x \in \Omega^-\,. \end{cases}$$

5. Removable singularities for monogenic functions

The aim of this section is to describe shortly in the framework of the Clifford analysis an approach of the following problem.

Let Ω be a domain in $\mathbb{R}^{n+1}$ and let $\mathbf{F}$ be a subset of Ω. Given a class of functions on $\Omega \setminus \mathbf{F}$, the problem is to prescribe conditions on this class of functions and on the set $\mathbf{F}$ under which these functions can be extended monogenically across $\mathbf{F}$.

In the case of domains in $\mathbb{R}^2$, removable singularities for holomorphic functions related to continuous complex-valued functions is stated as follows:

Let Δ be an open connected subset of the complex plane $\mathbb{C}$. Let $f : \Delta \longrightarrow \mathbb{C}$ be a continuous function which is holomorphic in $\Delta \setminus \mathbf{K}$, where $\mathbf{K} \subset \Delta$. Then Dolzhenko's theorem (see [14]) tells us that if **K** has zero inner Hausdorff measure with respect to the measure function $h(r) = r\omega(f, r)$, where $\omega(f, r)$ is the usual modulus of continuity of a function f, then f is holomorphic throughout Δ. Moreover, Dolzhenko also obtained the following result (see [14]): **K** is a removable set of singularities for holomorphic functions satisfying a Hölder condition of order ν if and only if $\underline{\mathcal{H}}^{1+\nu}(\mathbf{K}) = 0$ $(0 < \nu < 1)$. In [27] Nguyen proved that this result is also true for the case $\nu = 1$.

One can find important applications of a generalization of this result in the case of higher dimension to guarantee the uniqueness of the solution to the jump problem for monogenic functions.

Theorem 5.1 (Dolzhenko type theorem). *Let u be a continuous function with modulus of continuity $\omega(u, r)$ in the domain Ω and monogenic in $\Omega \setminus \mathbf{F}$, where $\mathbf{F} \subset \Omega$. If $\underline{\mathcal{H}}_h(\mathbf{F}) = 0$ for $h(r) = r^n\omega(u, r)$, then u is monogenic throughout Ω.*

Proof. Let K be the set of points belonging to the domain Ω where the function u is not monogenic. Then obviously K is closed, $K \subset \mathbf{F}$, and $\mathcal{H}_h(K) = 0$.

We can assume that $u(x)$ is a nonconstant function and that $\omega(u, r) \geq cr$, where c is a positive constant that does not depend on r. Then $r^{n+1} \leq c\,h(r)$ and $\mathcal{L}^{n+1}(K) = 0$, whence it follows that K is no where dense in Ω.

Let B be a fixed open ball which together with its boundary ∂B lies in Ω. Let us take a sufficiently small number $\epsilon > 0$ in such a way that $B \setminus K_{2\epsilon} \neq \emptyset$, where $K_{2\epsilon} = \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, K) \leq 2\epsilon\}$.

Let $\eta > 0$. Since $\mathcal{H}_h(K) = 0$, there is a cover $\{B_1, B_2, \dots\}$ of $K \cap \overline{B}$ by open balls B_k with center a_k and radius $r_k \leq \epsilon/2$ such that

$$\sum_{k=1}^{\infty} r_k^n \omega(u, r_k) < \frac{\epsilon^n}{2^{n/2}}\,\eta\,.$$

Since $K \cap \overline{B}$ is compact, a finite number $B_1, \dots, B_m$ of the B_k also cover $K \cap \overline{B}$ and

$$\sum_{k=1}^{m} r_k^n \omega(u, r_k) < \frac{\epsilon^n}{2^{n/2}}\,\eta\,.$$

We may assume at the outset that the set of balls $\{B_1, B_2, \dots\}$ has been enumerated in decreasing order of their radii and that none of them is covered by the union of the others.

Let $\Omega_1 = B \cap B_1$, $\Omega_k = B \cap (B_k \setminus \bigcup_{j=1}^{k-1} B_j)$ $(2 \leq k \leq m)$. Every set Ω_k decomposes into a finite number of nonintersecting simply connected domains $\Omega_{k,i}$, $i = 1, \dots, s_k$ with boundary $\Gamma_{k,i}$. Moreover,

$$\sum_{i=1}^{s_k} \mathcal{H}^n(\Gamma_{k,i}) \leq \mathcal{H}^n(\partial B_k) = \sigma_{n+1} r_k^n\,,$$

where ∂B_k denotes the boundary of the ball B_k.

Let

$$H(x) = u(x) - \int_{\partial B} e(y - x)\kappa(y)u(y)\,d\mathcal{H}^n(y), \quad x \in B\,.$$

Since $u(x)$ is monogenic in $B \setminus K$ and $\bigcup_{k,i} \Omega_{k,i} \subset K_\epsilon$ ($K_\epsilon = \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, K) \leq \epsilon\}$), then for $x \in B \setminus K_{2\epsilon}$, by virtue of the Cliffordian Cauchy formula, we obtain

$$H(x) = -\sum_{k=1}^{m} \sum_{i=1}^{s_k} \int_{\Gamma_{k,i}} e(y-x)\kappa(y)u(y)\, d\mathcal{H}^n(y) .$$

Therefore

$$\begin{aligned} |H(x)| &\leq \frac{2^{n/2}}{\sigma_{n+1}} \sum_{k=1}^{m} \sum_{i=1}^{s_k} \int_{\Gamma_{k,i}} \frac{1}{|y-x|^n} |u(y) - u(a_k)|\, d\mathcal{H}^n(y) \\ &\leq \frac{2^{n/2}}{\sigma_{n+1}\epsilon^n} \sum_{k=1}^{m} \omega(u, r_k) \sum_{i=1}^{s_k} \mathcal{H}^n(\Gamma_{k,i}) < \eta . \end{aligned}$$

In view of the arbitrary choice of η, we have $H(x) \equiv 0$ for $x \in B \setminus K_{2\epsilon}$, and as we also chose ϵ to be arbitrary, we obtain $H(x) \equiv 0$ in $B \setminus K$.

Taking into account that $B \setminus K$ is dense in B, and that the function $H(x)$ is continuous in B, we obtain the equality $H(x) \equiv 0$, for $x \in B$. Hence, u is monogenic in B and the theorem is proved. □

Combined with Liouville's theorem, see [12], Theorem 5.1 enables us to prove the following corollary.

Corollary 5.2. *Let $\mathbf{F} \subset \mathbb{R}^{n+1}$ be a bounded closed set. Then, given a nonnegative and nondecreasing $\mathbb{R}$-valued function $\omega(r)$ for $r \geq 0$ such that $\mathcal{H}_h(\mathbf{F}) = 0$ for the measure function $h(r) = r^n\omega(r)$, then the class of functions monogenic in $\mathbb{R}^{n+1} \setminus \mathbf{F}$ and continuous in $\overline{\mathbb{R}^{n+1}}$ for which $\omega(u, r) \leq \omega(r)$ consists of constants only.*

We now establish other consequences of Theorem 5.1.

Corollary 5.3 (Painlevé Theorem). *Let the set $\mathbf{F} \subset \Omega \subset \mathbb{R}^{n+1}$ be such that $\mathcal{H}^n(\mathbf{F}) < \infty$. If the function u is monogenic in $\Omega \setminus \mathbf{F}$ and continuous in Ω, then u is monogenic in Ω.*

Proof. Let $h(r) = r^n\omega(u, r)$. Then we have

$$\inf\left\{\sum_{k=1}^{\infty} h(\operatorname{diam} B_k)\right\} \leq \omega(u, \delta) \inf\left\{\sum_{k=1}^{\infty} (\operatorname{diam} B_k)^n\right\} ,$$

where the infimum is taken over all countable δ-coverings $\{B_k\}$ of $\mathbf{F}$.

Since $\mathcal{H}^n(\mathbf{F}) < \infty$ and $\omega(u, \delta) \to 0$ as $\delta \to 0$, then by letting δ tend to zero in the above inequality, we may conclude that $\mathcal{H}_h(\mathbf{F}) = 0$, which in view of Theorem 5.1 implies the desired result. □

Corollary 5.4. *Let $\underline{\mathcal{H}}^{n+\nu}(\mathbf{F}) = 0$ $(0 < \nu \leq 1)$. Then a function $u \in C^{0,\nu}(\Omega)$ which is monogenic in $\Omega \setminus \mathbf{F}$ is monogenic in Ω.*

Proof. The proof follows by making use of the elementary fact that for any function $u \in C^{0,\nu}(\Omega)$ we have that $\underline{\mathcal{H}}_h(\mathbf{F}) \leq c\, \underline{\mathcal{H}}^{n+\nu}(\mathbf{F})$, with $h(r) = r^n\omega(u, r)$ and c is a positive constant. □

Remark. As an immediate consequence of the definition of the Hausdorff dimension it follows that if $\alpha_H(\mathbf{F}) < n + \nu$ $(0 < \nu \leq 1)$, then the condition of our corollary is satisfied.

The limiting case $\nu = 1$ is exceptionally interesting and is treated in [27] within the framework of Complex Analysis. Here, we can see, that in the complex case, where $n = 1$, the condition $\mathcal{L}^2(\mathbf{F}) = 0$ is sufficient for the removability of a compact set $\mathbf{F}$ for continuous holomorphic functions satisfying a Lipschitz condition. This implication is well known (see, e.g., [18], Chapter 3, Section 2). And in fact, Corollary 5.4 can be seen as a generalization of the Garnett result to the higher-dimensional Euclidean spaces setting.

We do not know whether or not the converse to the Corollary 5.4, for $0 < \nu < 1$, is true in general. If the answer is affirmative a complete generalization would thus be obtained of the result proved by Dolzhenko (see [14], Theorem 3).

Corollary 5.5. *Let the set $\mathbf{F} \subset \Omega \subset \mathbb{R}^{n+1}$ has zero inner Hausdorff measure with respect to the measure function $h_1(r) = r^{n+1} \log 1/r$. If u is of the Zygmund class in Ω and is monogenic in $\Omega \setminus \mathbf{F}$, then u is monogenic in Ω.*

Proof. By an analogous argument as in the classical complex case for a function u satisfying the Zygmund condition, one has $\omega(u, r) \leq C r \log 1/r$. Therefore, $\underline{\mathcal{H}}_h(\mathbf{F}) = 0$ for the measure function $h(r) = r^n \omega(u, r)$. Theorem 5.1 then implies the desired result. □

6. Jump problem for monogenic functions

In this section heavy use of the Theorem 5.1 will be made in order to obtain sufficient conditions for the uniqueness of the solution for the so-called jump problem for monogenic functions in Euclidean spaces. The jump problem for monogenic functions consists in finding a function Φ, monogenic in $\mathbb{R}^{n+1} \setminus \Gamma$, such that Φ satisfies

$$\Phi^+(x) - \Phi^-(x) = g(x), \; x \in \Gamma; \; \Phi^-(\infty) = 0 \,. \tag{6.1}$$

Hereby g is a given continuous function on Γ, and $\Phi^+(x)$ and $\Phi^-(x)$ represent the limit values of the desired function Φ at a point $x \in \Gamma$ as this point is approached from inside Ω^+ and Ω^-, respectively. We also shall write $\Phi^\pm(x)$, by abuse of notation, for the respective restrictions of Φ to $\Omega^\pm$.

The role of the Cauchy transform $\mathcal{C}_\Gamma$ in solving this problem is well known for smooth surfaces (see e.g., [8, 28, 29]). A more general context was considered by the authors in [1, 2, 9, 10].

If Γ is an AD-regular surface and $u \in C^{0,\nu}(\Gamma)$, then the Cauchy transform $\mathcal{C}_\Gamma u$ gives a solution of the jump problem (6.1) and by Painlevé's Theorem, it is unique (see e.g., [2]).

If Γ is such that $\mathcal{H}^n(\Gamma) = \infty$, then, although the Cauchy transform loses its meaning, the jump problem remains meaningful.

In the case $n = 1$, in [22, 23], Kats presented a new method for solving the jump problem, which does not use contour integration and can thus be used on nonrectifiable and fractal curves. Moreover, Harrison and Norton [20, 21] defined integration along nonsmooth boundaries in $\mathbb{R}^{n+1}$, for $n \geq 1$. Similar integral methods for $n = 1$ were developed independently in [24, 25].

In the articles [6, 7], such method was adapted immediately within Quaternionic Analysis. Seemingly, a possible generalization for Euclidean space of higher dimensions with Clifford Analysis could also be envisaged.

In order to develop further investigation on the existence of solutions to problem (6.1) for domains bounded by nonsmooth surfaces, such as $\mathcal{H}^n(\Gamma) = \infty$, we start with some auxiliary definitions and remarks.

Let $\mathcal{X}$ be the characteristic function of the set $\overline{\Omega^+}$. For $g \in C(\Gamma)$, put $g^w(x) := \mathcal{X}(x)(\mathcal{E}_0 g)(x)$, where $\mathcal{E}_0$ is the Whitney extension operator (see [30]).

If $g \in C^{0,\nu}(\Gamma)$ then $g^w \in C^{0,\nu}(\overline{\Omega^+})$ and the function g^w is differentiable in Ω^+ with

$$\left|\partial_x g^w(x)\right| \leq c\,(\operatorname{dist}(x,\Gamma))^{\nu-1}\,. \tag{6.2}$$

Lemma 6.1. *Suppose $\alpha(\Gamma) < n+1$ and let $g \in C^{0,\nu}(\Gamma)$ $(0 < \nu < 1)$. Then $\partial_x g^w \in L_p(\Omega^+)$ for $p < (n+1-\alpha(\Gamma))/(1-\nu)$.*

Proof. Before starting the proof, let us recall the notion of Whitney partition. Introduce the layers

$$\Omega_k := \left\{x \in \mathbb{R}^{n+1} : \; 2\sqrt{n+1}\,2^{-k} \leq \operatorname{dist}(x,\Gamma) \leq 4\sqrt{n+1}\,2^{-k}\right\},$$

and consider the collection of cubes V_k of the grid R_k intersecting the layer Ω_k. After removing from the set $V' := \bigcup_{k\geq 0} V_k$ those cubes which are contained in larger cubes of V' the Whitney partition V is obtained.

Denote by v_k the number of cubes of the grid R_k appearing in V. Then $v_k \leq m_k(\Omega_k)$. Let Q be a cube of R_k intersecting Ω_k. Then there exits a cube Q' of the same grid intersecting Γ such that Q lies inside the ball with radius $11/2\sqrt{n+1}\,2^{-k}$, the center of which coincides with the center of Q'. Hence, $v_k \leq m_k(\Omega_k) \leq (2\lfloor(11/4)n+5\rfloor+3)^{n+1} m_k(\Gamma)$.

Let $\nu' \in (\alpha(\Gamma), n+1)$. Then $m_k(\Gamma) \leq c\;2^{k\nu'}$ for some positive constant c. To prove the lemma it is sufficient to establish the convergence of the series

$$\sum_{Q\in V}\int_Q \left|\partial_x g^w(x)\right|^p d\mathcal{L}^{n+1}(x)\,.$$

From (6.2) and taking into account that for all $Q \in V$

$$\operatorname{dist}(x,\Gamma) \leq 5\operatorname{diam} Q,\; x \in Q\,,$$

we get

$$\int_Q \left|\partial_x g^w(x)\right|^p d\mathcal{L}^{n+1}(x) \leq c\;2^{pk(1-\nu)}\int_Q d\mathcal{L}^{n+1}(x) = c\;2^{k(p(1-\nu)-(n+1))},\;\; Q \in V_k\,.$$

Hence,

$$\begin{aligned}\sum_{Q\in V}\int_Q \left|\partial_x g^w(x)\right|^p d\mathcal{L}^{n+1}(x) &\leq c\sum_{k=0}^{\infty} v_k 2^{k(p(1-\nu)-(n+1))}\\ &\leq c\sum_{k=0}^{\infty} 2^{k(p(1-\nu)-(n+1)+\nu')}\,.\end{aligned}$$

For $p < (n+1-\nu')/(1-\nu)$ this series converges. In view of the arbitrary choice of ν' the lemma is proved. □

Definition 6.2. The function $\Phi_0 \in C^1(\mathbb{R}^{n+1}\setminus\Gamma)$ is called a quasi-solution of the jump problem if the limit values of Φ_0 exist and satisfy (6.1).

Theorem 6.3. *The jump problem is solvable if and only if there exists a quasi-solution Φ_0 such that $\partial_x \Phi_0 \in L_p(\Omega^+)$ for some $p > n+1$.*

Proof. The if part of the theorem is obvious. Let now Φ_0 be a quasi-solution satisfying the above requirement. We shall show that the following function is a solution of the jump problem (6.1):

$$\Phi(x) := \Phi_0(x) - (T_{\Omega^+}\partial_x \Phi_0)(x) .$$

Taking into account the properties of T_{Ω^+} we get $\partial_x \Phi = 0$. Furthermore, the operator T_{Ω^+} maps functions of the class $L_p(\Omega^+)$, $p > n+1$, into continuous functions in $\mathbb{R}^{n+1}$ vanishing at ∞. Hence, it follows that Φ satisfies (6.1). □

By applying Theorem 6.3 and Lemma 6.1 we get the following.

Theorem 6.4. *Let $\alpha(\Gamma) < n+1$, and let $g \in C^{0,\nu}(\Gamma)$. If $1 \geq \nu > \alpha(\Gamma)/(n+1)$, then (6.1) is solvable and one of its solutions can be obtained from the formula*

$$\Phi(x) = g^w(x) - \big(T_{\Omega^+}\partial_x g^w\big)(x) . \tag{6.3}$$

Remark. On basis of the above results, we can describe the picture of uniqueness of the jump problem (6.1).

Note that the difference $\Psi = \Phi_1 - \Phi_2$ of two solutions of the jump problem (6.1)is monogenic in $\mathbb{R}^{n+1} \setminus \Gamma$ and continuous in $\overline{\mathbb{R}^{n+1}}$. If it is possible to deduce monogenicity of $\Psi(x)$ for $x \in \Gamma$, then the condition $\Phi_1^-(\infty) = \Phi_2^-(\infty) = 0$ implies $\Psi \equiv 0$. Thus, uniqueness of the solution of (6.1) follows from the removability of the surface Γ for the class of functions monogenic in a neighborhood of Γ and continuous on Γ. In particular, a solution of (6.1) is unique if Γ is an n-rectifiable surface.

Theorem 6.5. *Let Γ be an n-rectifiable surface and let $g \in C^{0,\nu}(\Gamma)$. Then for $1 \geq \nu > n/(n+1)$ the jump problem (6.1) has a unique solution given by $(\mathcal{C}_\Gamma g)(x)$.*

Proof. By virtue of Painlevé's Theorem the proof follows directly by using Theorem 4.2, Theorem 6.4, and Lemma 3.2. □

Under the conditions of Theorem 6.5, when Γ is an n-rectifiable surface, the solution of (6.1) is unique. Otherwise, to ensure uniqueness in the general statement we need to introduce an additional requirement: The function Φ, monogenic in $\overline{\mathbb{R}^{n+1}} \setminus \Gamma$, must satisfy a Hölder condition with exponent μ $(0 < \mu \leq 1)$ on each of the sets $\overline{\Omega^\pm}$, i.e., the functions $\Phi^\pm$ should belong to $C^{0,\mu}(\overline{\Omega^\pm})$. Solutions of the jump problem (6.1) with this additional condition are said to be solutions of class $C_{0,\mu}$.

By the remark of Corollary 5.4, the solution of the jump problem in the class $C_{0,\mu}$ will be unique if $\mu > \alpha_H(\Gamma) - n$. At the same time, if $\varphi \in L_p(\Omega^+)$, $p > n+1$, has a compact support, then $T_{\Omega^+}\varphi$ satisfies a Hölder condition with exponent $1 - \dfrac{n+1}{p}$. Therefore

$$g^w(x) - \big(T_{\Omega^+}\partial_x g^w\big)(x) \in C_{0,\mu} ,$$

for $\mu < ((n+1)\nu - \alpha(\Gamma))/((n+1) - \alpha(\Gamma))$. Thus, the following theorem is proved.

Theorem 6.6. *Let $\alpha(\Gamma) < n+1$, let $g \in C^{0,\nu}(\Gamma)$, $1 \geq \nu > \alpha(\Gamma)/(n+1)$ and let $\alpha_H(\Gamma) - n < \mu < ((n+1)\nu - \alpha(\Gamma))/((n+1) - \alpha(\Gamma))$. Then the function* (6.3) *is the unique solution of the jump problem which belongs to the class $C_{0,\mu}$.*

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