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Kähler-Ricci Soliton Typed Equations on Compact Complex Manifolds with $C_1(M) > 0$

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ABSTRACT. *As a generalization of Calabi's conjecture for Kähler-Ricci forms, which was solved by Yau in 1977, we discuss the existence of Kähler-Ricci soliton typed equation on a compact Kähler manifold (M, g) with positive first Chern $C_1(M) > 0$ as well as the uniqueness. For a given positively definite (1,1)-form $\Omega \in C_1(M)$ of M and a holomorphic vector field X on M, we prove that there is a Kähler form ω in the Kähler class $[\omega_g]$ solving the Kähler-Ricci soliton typed equation if and only if, i) X is belonged to a reductive subalgebra of holomorphic vector fields and the imaginary part of X generates a compact one-parameter transformations subgroup of M; and ii) $L_X\Omega$ is a real-valued (1,1)-form. Moreover, the solution ω is unique in the class $[\omega_g]$.*

1. Introduction

Let (M, g) be an n-dimensional Kähler manifold with its Kähler form $\omega_g = \sqrt{-1}\sum g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. Then it is well known that any (1,1)-form Ω representing the first Chern class $C_1(M)$ is the Ricci form of some Kähler form ω in the Kähler class $[\omega_g]$. This result is usually called the Calabi's conjecture for Kähler-Ricci forms, which was solved by Yau in his celebrated work in 1977 [16]. Namely, ω satisfies

$$\mathrm{Ric}(\omega) - \Omega = 0\,, \tag{1.1}$$

where $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω. Moreover, such ω is unique in the class $[\omega_g]$.

The case $C_1(M) > 0$ is more subtle in many related topics in complex geometry, such as the existence of Kähler-Einstein metrics [14]. Many difficulties come from a nontrivial continuous group of holomorphic transformations, in particular, generated by a holomorphic vector field on M(if it exists). This may introduce some degenracies [6], [5]. On the other hand, by the Hodge theorem, there is a smooth complex-valued function θ_X of M for any Kähler form ω such that

$$L_X\omega = \sqrt{-1}\partial\bar{\partial}\theta_X\,,$$

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where L_X denotes the Lie derivative along X. As a generalization of Equation (1.1), we may ask if there is a Kähler form ω such that

$$\mathrm{Ric}(\omega) - \Omega = L_X\omega . \tag{1.2}$$

Equation (1.2) is here called Kähler-Ricci soliton typed [9], [14].

Another motivation to study Equation (1.2) is that the Kähler-Ricci soliton is a solution of Equation (1.2) when the (1,1)-form Ω is equal to ω. The notation of Ricci solitons was first introduced by Hamilton in his work on Ricci flow in 1993 [9]. A Ricci soliton can be considered as a good replacement, when a manifold does not admit an Einstein metric. In fact, Equation (1.2) was studied in connection with Kähler-Einstein metrics with positive scalar curvature by Tian in his paper [14]. Some examples of Kähler-Ricci solitons on certain compact Kähler manifolds were found by Koiso and Cao in [10] and [3] and [4], respectively.

In this paper, we shall discuss the existence of Equation (1.2) as well as the uniqueness. Let $\mathrm{Aut}(M)$ be a connected component containing the identity of holomorphism transformations group of M and $\eta(M)$ its Lie algebra consisting of all holomorphic vector fields on M. Then it is well known that there is a semidirect decomposition of $\mathrm{Aut}(M)$ (cf. [8]),

$$\mathrm{Aut}(M) = \dot{\mathrm{Aut}}(M) \propto R_u ,$$

where $\dot{\mathrm{Aut}}(M) \subset \mathrm{Aut}(M)$ is a reductive subgroup on M which is a complexification of a maximal compact subgroup K on M, and R_u is the unipotent radical of $\mathrm{Aut}(M)$. In particular, the Lie subalgebra $\dot{\eta}(M) \subset \eta(M)$ of $\dot{\mathrm{Aut}}(M)$ is reductive. More precisely, $\dot{\eta}(M)$ is the complexification of real compact Lie algebra of K.

Our main theorem can be stated as follows.

Main Theorem. *Let (M, ω_g) be a compact Kähler manifold with positive first Chern $C_1(M) > 0$. Let $\Omega \in C_1(M)$ be a positively definite (1,1)-form of M and X a holomorphic vector field on M. Then there is a Kähler form ω in the Kähler class $[\omega_g]$ solving Equation (1.2) if and only if*

i) X belongs to a reductive algebra $\dot{\eta}(M)$ of reductive Lie subgroup $\dot{\mathrm{Aut}}(M)$ of $\mathrm{Aut}(M)$ and the imaginary part of X generates a compact one-parameter transformations subgroup of $\mathrm{Aut}(M)$.

ii) $L_X\Omega$ is a real-valued (1,1)-form of M. Moreover, the solution ω of Equation (1.2) is unique in the class $[\omega_g]$.

As an application of the Main Theorem, we can prove that the Kähler-Ricci soliton on a compact Kähler manifold with $C_1(M) > 0$ is unique modula the holomorphic transformations group $\mathrm{Aut}(M)$ of M in our subsequent paper [15]. In case of a Kähler-Einstein metric, the uniqueness problem was solved by Bando and Mabuchi in 1985 [2].

In order to prove the Main Theorem, we reduce Equation (1.2) to solving certain Monge-Ampére equations and use the continuity method as in [16], [2], and [12] to prove the existence and uniqueness. The present Monge-Ampére equations are more complicated than one in [16] and all *a priori* estimates including C^0-estimate, C^2-estimate, and C^3-estimate need to be done again.

Since the Calabi's conjecture is true for any $\Omega \in C_1(M)$, one may believe that the assumption of positively definite on $\Omega \in C_1(M)$ can be removed.

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2. Necessity conditions

In this section, we shall verify the necessity conditions stated in the Main Theorem. Let M be an n-dimensional compact Kähler manifold with positive first Chern class $C_1(M) > 0$. Let $\Omega \in C_1(M)$ be a positively definite (1,1)-form of M and X a holomorphic vector field on M. We assume that the Kähler metric h with its Kähler form $\omega_h = \sqrt{-1}\sum h_{i\bar{j}}dz^i \wedge d\bar{z}^j$ satisfies the following Kähler-Ricci soliton typed equation,

$$\mathrm{Ric}\,(\omega_h) - \Omega = L_X\omega_h \tag{2.1}$$

where L_X denotes Lie derivative along X and the Ricci curvature has the following expression in local coordinates,

$$\begin{cases} R_{i\bar{j}} = -\partial_i\partial_{\bar{j}}\log\det\left(h_{k\bar{l}}\right) \\ \mathrm{Ric}\,(\omega_h) = \sqrt{-1}R_{i\bar{j}}dz^i \wedge d\bar{z}^j \,. \end{cases} \tag{2.2}$$

From Equation (2.1), we see that the (1, 1)-form $L_X\omega_h$ is real-valued, which implies the imaginary part of X generates a one-parameter isometric subgroup associated with ω_h. In particular, this one-parameter transformations subgroup is compact. Let Aut(M) be a connected component containing the identity of holomorphism transformations group of M. Then there is a maximal compact subgroup K of Aut(M) containing the above one-parameter isometric subgroup such that Aut(M) has the following semidirect decomposition [8],

$$\mathrm{Aut}(M) = \dot{\mathrm{A}}\mathrm{ut}(M) \propto R_u \,,$$

where $\dot{\mathrm{A}}\mathrm{ut}(M) \subset \mathrm{Aut}(M)$ is the reductive subgroup on M which is the complexification of maximal compact subgroup K, and R_u is the unipotent radical of Aut(M). Let $\eta(M)$ be the linear space of holomorphic vector fields of M. Then $\eta(M)$ is the Lie algebra of Aut(M) and the Lie subalgebra $\dot{\eta}(M)$ of $\dot{\mathrm{A}}\mathrm{ut}(M)$ is reductive. More precisely, $\dot{\eta}(M)$ is the complexification of real compact Lie subalgebra of K. In particular, $X \in \dot{\eta}(M)$.

Proposition 2.1. *Let X be a holomophic vector field on M and Ω a positively definite (1,1)-form of M as above. Assume that there is a Kähler form ω_h solving Equation* (2.1). *Then X belongs to the reductive Lie subalgebra $\dot{\eta}(M) \subset \eta(M)$ and $L_X\Omega$ is a real-valued (1,1)-form of M.*

Proof. It remains to prove that $L_X\Omega$ is a real-valued (1,1)-form of M. Since the interior product $i_X(\omega)$ is a closed (0,1)-form, then by the Hodge theorem and the fact that $L_X\omega_h$ is real-valued (1,1)-form, there is a smooth real-valued function θ of M such that

$$L_X\omega_h = di_X\,(\omega_h) = \sqrt{-1}\partial\bar{\partial}\theta \,. \tag{2.3}$$

On the other hand, we can choose a local coordinate system so that $\omega_h = \sqrt{-1}\partial\bar{\partial}\phi = \sqrt{-1}\phi_{i\bar{j}}dz^i \wedge d\bar{z}^j$ for some potential function ϕ. Then

$$L_X\omega_h = \sqrt{-1}L_X\left(\partial\bar{\partial}\phi\right) = \sqrt{-1}\partial\bar{\partial}(X(\phi)) \,.$$

Hence by (2.3), it follows

$$\Delta X(\phi) = \Delta\theta \,, \tag{2.4}$$

where Δ denotes the Lapalacian operator associated with Kähler metric h.

Since

$$
\begin{aligned}
L_X\mathrm{Ric}(\omega_h) &= -\sqrt{-1}\partial\bar\partial\left(X\left(\log\det\left(h_{k\bar l}\right)\right)\right)\\
&= -\sqrt{-1}\partial\bar\partial\left(h^{k\bar l}X^i\left(h_{k\bar l}\right)_i\right)\\
&= -\sqrt{-1}\partial\bar\partial\left(h^{k\bar l}X^i\left(\phi_{k\bar l}\right)_i\right)\\
&= -\sqrt{-1}\partial\bar\partial\left(h^{k\bar l}\left(X^i\phi_i\right)_{k\bar l}-h^{k\bar l}X^i_{,k}\phi_{i\bar l}\right)\\
&= -\sqrt{-1}\partial\bar\partial\left(h^{k\bar l}(X(\phi))_{k\bar l}-X^k_{,k}\right)\\
&= -\sqrt{-1}\partial\bar\partial(\Delta(X(\phi)))\,,
\end{aligned}
\tag{2.5}
$$

where $(h^{k\bar l})$ is the inverse of matrix $(h_{k\bar l})$, then inserting (2.4) into (2.5), we get

$$
L_X\mathrm{Ric}(\omega_h) = -\sqrt{-1}\partial\bar\partial(\Delta\theta)\,. \tag{2.6}
$$

On the other hand, we have

$$
\begin{aligned}
L_X\left(L_X\omega_h\right) &= \sqrt{-1}L_X\left(\partial\bar\partial\theta\right)=\sqrt{-1}\partial\bar\partial(X(\theta))\\
&= \sqrt{-1}\partial\bar\partial\left(h^{k\bar l}\theta_{\bar l}\theta_k\right)\\
&= \sqrt{-1}\partial\bar\partial\left(\|\theta\|_h^2\right)\,.
\end{aligned}
\tag{2.7}
$$

Hence, combining (2.6) and (2.7), we prove

$$
\begin{aligned}
L_X\Omega &= L_X\mathrm{Ric}(\omega_h) - L_X(\omega_h)\\
&= -\sqrt{-1}\partial\bar\partial\left(\Delta\theta+\|\theta\|_h^2\right)\,,
\end{aligned}
$$

which is a real-valued (1,1)-form. □

3. Reduction to certain complex Monge-Ampére equations

Keep the notation in Section 2. We assume in this section that a holomorphic vector field X on M is belonged to a reductive subalgebra $\dot\eta(M)$ of $\eta(M)$ such that the imaginary part of X generates a compact one-parameter transformations subgroup on M, and $L_X\Omega$ is a real-valued (1,1)-form of M. Let K be the maximal subgroup of $\mathrm{Aut}(M)$ generated by $\dot\eta(M)$. Then one can choose a K-invariant Kähler metric g of M with its Kähler form $\omega_g=\sqrt{-1}\sum g_{i\bar j}dz^i\wedge d\bar z^j$. In particular, $L_X\omega_g$ is a real-valued (1,1) form of M. Hence, by the Hodge theorem, there is a smooth real-valued function θ_X of M such that

$$
L_X\omega_g = di_X\left(\omega_g\right)=\sqrt{-1}\partial\bar\partial\theta_X\,. \tag{3.1}
$$

Since Ricci curvature form $\mathrm{Ric}(\omega_g)$ of ω_g represents $C_1(M)$, there is a unique smooth real-valued function f of M such that

$$
\begin{cases}
\mathrm{Ric}\left(\omega_g\right)-\Omega=\sqrt{-1}\partial\bar\partial f\\
\int_M e^f\omega_g^n=\int_M\omega_g^n\,,
\end{cases}
\tag{3.2}
$$

where $\omega_g^n = \omega_g \wedge \ldots \wedge \omega_g$. Moreover, from the proof of Proposition 2.1, we see $L_X \mathrm{Ric}(\omega_g)$ is a real-valued (1,1)-form of M. Hence, by the assumption of Ω,

$$\sqrt{-1}L_X\left(\partial\bar{\partial}f\right) = \sqrt{-1}\partial\bar{\partial}(X(f))$$

is a real-valued (1,1)-form of M. This shows $X(f)$ is a real-valued function.

Let $\omega = \omega_g + \sqrt{-1}\partial\bar{\partial}\phi$ be a solution of the Kähler-Ricci soliton typed equation, i.e., ω_ϕ satisfies

$$\mathrm{Ric}\left(\omega_\phi\right) - \Omega = L_X\omega_\phi . \tag{3.3}$$

Then by combining (3.1) through (3.3), it is easy to see that Equation (3.3) is equivalent to the following complex Monge-Ampére equation:

$$\begin{cases} \det\left(g_{i\bar{j}} + \phi_{i\bar{j}}\right) = \det\left(g_{i\bar{j}}\right)\exp\{f - (\theta_X + X(\phi)) + c\} \\ \left(g_{i\bar{j}} + \phi_{i\bar{j}}\right) > 0 \end{cases} \tag{3.4}$$

for some constant c.

From Equation (3.4) we see that $X(\phi)$ is real-valued function of M. For this reason, we introduce the two functions spaces as follows:

$$\mathcal{M}_X = \left\{\phi \in C^\infty(M) \mid \quad \omega_\phi = \omega_g + \sqrt{-1}\partial\bar{\partial}\phi \quad \text{is a Kähler form} \right.$$
$$\left. \text{and} \quad X(\phi) \quad \text{is a real-valued function}\right\} .$$

and

$$\mathcal{W}_X = \left\{\phi \in C^\infty(M) \mid \quad X(\phi) \quad \text{is a real-valued function}\right\} .$$

Clearly, $\mathcal{M}_X \subseteq \mathcal{W}_X$.

For any $\phi \in \mathcal{M}_X$, we define a family of functionals,

$$I_t(\phi) = \int_0^1 \int_M \dot{\phi}_\tau e^{t(\theta_X + X(\phi_\tau))}\omega_{\phi_\tau}^n \wedge d\tau , \tag{3.5}$$

where ϕ_τ is a path in $\mathcal{M}_X$ from 0 to ϕ and $\dot{\phi}_\tau = \frac{d}{d\tau}\phi_\tau$. $I_t(\phi)$ are modifications of one functional used in [1] and [12].

Lemma 3.1. *$I_t(\phi)$ are all independent of path. So I_t is a family of functionals on $\mathcal{M}_X$. In particular,*

$$I_t(\phi) = \int_0^1 \int_M \phi e^{t(\theta_X + \tau X(\phi))}\omega_{\tau\phi}^n \wedge d\tau , \tag{3.6}$$

Proof. Assume that ϕ_τ is a path in $\mathcal{M}_X$ so that $\phi_0 = \phi_1 = \phi = 0$. Then the lemma is equivalent to prove $I_t(\phi) = 0$. Let $\phi_{\tau,\delta} = (1-\delta)\phi_\tau = \phi'$. Then

$$\begin{aligned} I_t(\phi) &= \int_0^1\int_0^1\int_M d_{\tau,\delta}\phi' \wedge d_{\tau,\delta}\left(e^{t(\theta_X + X(\phi'))}\omega_{\phi'}^n\right) \\ &= n\sqrt{-1}\int_0^1\int_0^1\int_M \left(\frac{\partial\phi'}{\partial\tau}\partial\bar{\partial}\frac{\partial\phi'}{\partial\delta} - \frac{\partial\phi'}{\partial\delta}\partial\bar{\partial}\frac{\partial\phi'}{\partial\tau}\right)e^{t(\theta_X + X(\phi'))}\omega_{\phi'}^{n-1} \wedge d\tau \wedge d\delta \\ &\quad + \sqrt{-1}t\int_0^1\int_0^1\int_M \left[\frac{\partial\phi'}{\partial\tau}X\left(\frac{\partial\phi'}{\partial\delta}\right) - \frac{\partial\phi'}{\partial\delta}X\left(\frac{\partial\phi'}{\partial\tau}\right)\right]e^{t(\theta_X + X(\phi'))}\omega_{\phi'}^n \wedge d\tau \wedge d\delta . \end{aligned} \tag{3.7}$$

By using integral by part, the first part of second equality becomes

$$n\sqrt{-1}t\int_0^1\int_0^1\int_M\left[\frac{\partial\phi'}{\partial\tau}\bar\partial\frac{\partial\phi'}{\partial\delta}\wedge\partial\left(\theta_X+X\left(\phi'\right)+\frac{\partial\phi'}{\partial\delta}\partial\frac{\partial\phi'}{\partial\tau}\wedge\bar\partial\left(\theta_X+X\left(\phi'\right)\right)\right]\right.$$

$$\times e^{t(\theta_X+X(\phi'))}\omega_{\phi'}^{n-1}\wedge d\tau\wedge d\delta$$

$$-\sqrt{-1}\int_0^1\int_0^1\int_M\left(\partial\frac{\partial\phi'}{\partial\tau}\wedge\bar\partial\frac{\partial\phi'}{\partial\delta}+\bar\partial\frac{\partial\phi'}{\partial\delta}\wedge\partial\frac{\partial\phi'}{\partial\tau}\right)e^{t(\theta_X+X(\phi'))}\omega_{\phi'}^{n-1}\wedge d\tau\wedge d\delta$$

$$=-\sqrt{-1}t\int_0^1\int_0^1\int_M\left[\frac{\partial\phi'}{\partial\tau}\overline{X\left(\frac{\partial\phi'}{\partial\delta}\right)}-\frac{\partial\phi'}{\partial\delta}X\left(\frac{\partial\phi'}{\partial\tau}\right)\right]e^{t(\theta_X+X(\phi'))}\omega_{\phi'}^{n}\wedge d\tau\wedge d\delta$$

$$=-n\sqrt{-1}t\int_0^1\int_0^1\int_M\left[\frac{\partial\phi'}{\partial\tau}X\left(\frac{\partial\phi'}{\partial\delta}\right)-\frac{\partial\phi'}{\partial\delta}X\left(\frac{\partial\phi'}{\partial\tau}\right)\right]e^{t(\theta_X+X(\phi'))}\omega_{\phi'}^{n}\wedge d\tau\wedge d\delta \quad (3.8)$$

Inserting (3.8) into (3.7), we prove $I_t(\phi)=0$. □

In order to prove the existence of a solution of Equation (3.4), we use the continuity method like the one in [16] and [12] and consider the following normalized equations with parameter $t\in[0,1]$:

$$\begin{cases}\det\left(g_{i\bar j}+\phi_{i\bar j}\right)=\det\left(g_{i\bar j}\right)\exp\{f-t\left(\theta_X+X(\phi)\right)+I_t(\phi)\}\\ \left(g_{i\bar j}+\phi_{i\bar j}\right)>0\,.\end{cases}\tag{3.9}$$

Since θ_X is a smooth real-valued function of M, $X(\phi_t)$ are all real-valued smooth functions if ϕ_t are smooth solutions of Equation (3.9) at t. Moreover, by differentiating log of Equation (3.9), ω_{ϕ_t} satisfies the following Ricce equations:

$$\mathrm{Ric}\left(\omega_{\phi_t}\right)-\Omega=\sqrt{-1}\partial\bar\partial t\left(\theta_X+X\left(\phi_t\right)\right)=tL_X\omega_{\phi_t}\,.\tag{3.10}$$

4. Openness

Let F be a functional on $\mathcal{M}_X\times[0,1]$ defined by

$$F(\phi,t)=\log\det\left(g_{i\bar j}+\phi_{i\bar j}\right)-\log\det\left(g_{i\bar j}\right)-f+t\left(\theta_X+X(\phi)\right)-I_t(\phi)\,.\tag{4.1}$$

Since by Lemma 3.1 the linearized functional of $I_t(\phi)$ at ϕ is

$$I'(\psi)=\int_M\psi e^{t(\theta_X+X(\phi))}\omega_\phi^n\,,$$

the Fréchet derivative $L_{(\phi,t)}$ of F at (ϕ,t) with respect to the first factor is given by

$$L_{(\phi,t)}\psi=\Delta'\psi+tX(\psi)-\int_M\psi e^{t(\theta_X+X(\phi))}\omega_\phi^n\,,\tag{4.2}$$

where Δ' denotes the Lapalacian operator associated with Kähler form ω_ϕ.

Lemma 4.1. *Let $\phi\in\mathcal{M}_X$ and $\psi\in\mathcal{W}_X$. Then $L_{(\phi,t)}\psi\in\mathcal{W}_X$ and $F(\phi,t)\in\mathcal{W}_X$.*

Proof. For each $p\in M$, choose a local coordinate system $(x_1,\dots,x_n)$ so that $\omega_\phi=\sqrt{-1}\omega^i\wedge\overline{\omega}^i$ and $\psi_{i\bar j}(p)=\delta_{ij}\psi_{i\bar i}(p)$. Then

$$L_X(\omega_\phi)=\sqrt{-1}\left(\theta_X+X(\phi)\right)_{i\bar i}\omega^i\wedge\overline{\omega}^i=\sqrt{-1}X_{\bar i;i}\omega^i\wedge\overline{\omega}^i\,.$$

In particular, $X_{i\bar{i}}$ are all real-valued. Thus

$$\begin{aligned} X\left(\Delta'\psi\right) &= X_{\bar{k}}\psi_{i\bar{i}k} = X_{\bar{k}}\psi_{k\bar{i}i} \\ &= \left(X_{\bar{k}}\phi_k\right)_{i\bar{i}} - X_{\bar{k}i}\psi_{ki} = (X(\psi))_{i\bar{i}} - X_{i\bar{i}}\psi_{i\bar{i}} \end{aligned} \tag{4.3}$$

is real-valued. It follows $\Delta'\psi \in \mathcal{W}_X$. On the other hand, $\overline{X}(X(\psi)) = \overline{X}_{\bar{i}}X_{\bar{i}}\psi_{i\bar{i}}$ and $X(\psi)$ are both real-valued, then $X(X(\psi)) = \overline{X}(X(\psi))$ is real-valued and consequently $X(X(\psi)) \in \mathcal{W}_X$. Thus $L_{(\phi,t)}\psi \in \mathcal{W}_X$.

Let $\phi_s = s\phi$. Then we have

$$\begin{aligned} X\left(\Delta'_{\phi_s}\phi\right) &= X\left(\frac{d}{ds}\left(\log\det\left(g_{i\bar{j}} + s\phi_{i\bar{j}}\right) - \log\det\left(g_{i\bar{j}}\right)\right)\right) \\ &= \frac{d}{ds}\left(X\left(\log\det\left(g_{i\bar{j}} + s\phi_{i\bar{j}}\right) - \log\det\left(g_{i\bar{j}}\right)\right)\right) , \end{aligned} \tag{4.4}$$

and

$$X\left(\log\det\left(g_{i\bar{j}} + \phi_{i\bar{j}}\right) - \log\det\left(g_{i\bar{j}}\right)\right) = \int_0^1 X\left(\Delta'_{\phi_s}\phi\right) ds . \tag{4.5}$$

As similar as the proof of (4.3), it is easy to see $X(\Delta'_{\phi_s}\phi)$ are all real-valued. So

$$X\left(\log\det\left(g_{i\bar{j}} + \phi_{i\bar{j}}\right) - \log\det\left(g_{i\bar{j}}\right)\right)$$

is real-valued. On the other hand, $X(\theta_X + X(\phi)) = X_{\bar{i}}\overline{X}_{\bar{i}}$ is real-valued. It follows $X(F(\phi, t))$ is real-valued and consequently $F(\phi, t) \in \mathcal{W}_X$. □

Let $H_{k+2}(M) = C^{k+2}(M) \cap \mathcal{W}_X$. We define a family of inner products $<, >$ on $H_{k+2}(M)$ by

$$< f, g >= \int_M fge^{t(\theta_X + X(\phi))}\omega_\phi^n$$

for any $f, g \in H_{k+2}(M)$, where $\omega_\phi \in \mathcal{M}_X$. Then one can extend $H_{k+2}(M)$ to a family of Hilbert L^2-spaces $H^2_{k+2}(M)$ with the family of products $<, >$.

Lemma 4.2. *i). Let $\phi \in \mathcal{M}_X$. Then $L_{(\phi,t)}$ are all self-adjoint with respect to the products $<, >$, i.e., we have*

$$\int_M gL_{(\phi,t)}fe^{t(\theta_X + X(\phi))}\omega_\phi^n = \int_M fL_{(\phi,t)}ge^{t(\theta_X + X(\phi))}\omega_\phi^n \tag{4.6}$$

for any $f, g \in \mathcal{W}_X$.

ii). Suppose Ω is a positive form in $C_1(M)$ and $\phi = \phi_t$ is a smooth solution of Equation (3.9) *at t. Then the first eigenvalue of $L_{(\phi,t)}$ is positively definite.*

Proof. i). Let $\widetilde{L}_{(\phi,t)}f = \Delta' f + tX(f)$. Then by (4.2), it suffices to prove $< \widetilde{L}_{(\phi,t)}f, g >=< \widetilde{L}_{(\phi,t)}g, f >$. For each $p \in M$, choose a local coordinate system $(x_1, \ldots, x_n)$ so that $\omega_\phi =$

$\sqrt{-1}\omega^i \wedge \overline{\omega}^i$. Then $(\theta_X + X(\phi))_{\bar{i}} = X_{\bar{i}}$. By using integral by part, we get

$$\begin{aligned}\int_M g\widetilde{L}_{(\phi,t)} f e^{t(\theta_X+X(\phi))}\omega_\phi^n &= \int_M \left(gf_{i\bar{i}} + tgX_{\bar{i}}f_i\right) e^{t(\theta_X+X(\phi))}\omega_\phi^n \\ &= \int_M \left(fg_{i\bar{i}} + tfg_{\bar{i}}\overline{X}_{\bar{i}} + tfg_iX_{\bar{i}} + tgfX_{i\bar{i}} + t^2 fgX_{\bar{i}}\overline{X}_{\bar{i}}\right) e^{t(\theta_X+X(\phi))}\omega_\phi^n \\ &\quad - t\int_M \left(fg_iX_{\bar{i}} + gfX_{i\bar{i}} + tfgX_{\bar{i}}\overline{X}_{\bar{i}}\right) e^{t(\theta_X+X(\phi))}\omega_\phi^n \\ &= \int_M \left(fg_{i\bar{i}} + tf\overline{X_{\bar{i}}g_i}\right) e^{t(\theta_X+X(\phi))}\omega_\phi^n \\ &= \int_M \left(fg_{i\bar{i}} + tfX(g)\right) e^{t(\theta_X+X(\phi))}\omega_\phi^n = \int_M f\widetilde{L}_{(\phi,t)} g e^{t(\theta_X+X(\phi))}\omega_\phi^n .\end{aligned} \tag{4.7}$$

ii). Let λ be the first eigenvalue of $L_{(\phi,t)}$ and ψ an eigenfunction of λ, i.e.,

$$\Delta'\psi + tX(\psi) - \int_M \psi e^{t(\theta_X+X(\phi))}\omega_\phi^n = -\lambda\psi .$$

Clearly, $\lambda = \int_M e^{t(\theta_X+X(\phi))}\omega_\phi^n > 0$ if $\psi \equiv \text{const} \neq 0$.

By using integral by part together with Ricci formula and identities (3.10), we have

$$\begin{aligned}\lambda\int_M \psi_i\psi_{\bar{i}} e^{t(\theta_X+X(\phi))}\omega_\phi^n &= -\int_M \left(\Delta'\psi + tX(\psi)\right)_i \psi_{\bar{i}} e^{t(\theta_X+X(\phi))}\omega_\phi^n \\ &= -\int_M \psi_{j\bar{j}i}\psi_{\bar{i}} e^{t(\theta_X+X(\phi))}\omega_\phi^n - t\int_M \left(X_{\bar{j}}\psi_{\bar{i}}\psi_{ij} + X_{\bar{j}i}\psi_{\bar{i}}\psi_j\right) e^{t\theta_X+X(\phi))}\omega_\phi^n \\ &= \int_M \left(R_{i\bar{j}} - tX_{\bar{j}i}\right)\psi_{\bar{i}}\psi_j e^{t(\theta_X+X(\phi))}\omega_\phi^n + \int_M \psi_{ij}\psi_{\bar{i}\bar{j}} e^{t(\theta_X+X(\phi))}\omega_\phi^n \\ &\geq \int_M \Omega_{i\bar{j}}\psi_{\bar{i}}\psi_j e^{t(\theta_X+X(\phi))}\omega_\phi^n .\end{aligned} \tag{4.8}$$

Since Ω is positively definite, we prove $\lambda > 0$. □

Set $I = \{t \in [0,1] |$ there is a smooth solution ϕ_t of Equation (3.9) at t $\}$. Then we have the following:

Proposition 4.3. *The set I is nonempty and open.*

Proof. By Yau's solution for Calabi's conjecture, there is a unique smooth solution ϕ_0 of Equation (3.9) at $t = 0$ with $I_0(\phi_0) = \int_0^1\int_M \phi_0\omega_{\tau\phi_0}^n \wedge d\tau = 0$. So I is nonempty. Now we suppose ϕ_t is a smooth solution of Equation (3.9) at t. Consider the map $F(\phi,t) : H_{k+2}(M) \times [0,1] \to H_k(M)$ defined by (4.1). By Lemma 4.1 and the standard regularity theorem of elliptic equations, the linearized operator $L_{(\phi_t,t)} : H_{k+2}(M) \to H_k(M)$ of $F(\phi,t)$ with respect to the first factor at (ϕ_t,t) is invertible. Then applying the implicit function theorem to the map $F(\phi,t)$, there is a small number $\delta > 0$ such that there are C^{k+2}-functions ϕ_s of Equation (3.9) at any $s \in [t, t+\delta)$. By the regularity theorem of Monge-Ampére equations [16], ϕ_s are in fact smooth. This proves I is an open set. □

5. C^0-estimates

Let (S^2, ω_{g_0}) be a unit two-sphere in R^3 with the standard metric g_0. Since S^2 is conformal to $C^1 \cup \{+\infty\}$, a holomorphic vector field X of S^2 can be denoted by $X = (a+\sqrt{-1}b)(r\frac{\partial}{\partial r} + \sqrt{-1}\frac{\partial}{\partial\theta})$, where (r, θ) is the polar coordinate system on C^1 and a, b are two real numbers.

Lemma 5.1. *Let X be the above nontrivial holomorphic vector field and ω_g a Kähler form of S^2 with $\omega_g \le A\omega_{g_0}$ for some positive constant A. Suppose that $\omega_g + \sqrt{-1}\partial\bar{\partial}\phi$ is a Kähler form such that $X(\phi)$ is a real-valued function. Then*

$$|X(\phi)| \le 2|a|A .$$

Proof. Since $X(\phi)$ is real-valued, we have

$$br\frac{\partial\phi}{\partial r} + a\frac{\partial\phi}{\partial\theta} = 0 . \tag{5.1}$$

First assume $a = 0$. Then $\frac{\partial\phi}{\partial r} = 0$. It follows

$$X(\phi) = ar\frac{\partial\phi}{\partial r} - b\frac{\partial\phi}{\partial\theta} = -b\frac{\partial\phi}{\partial\theta} \equiv 0 ,$$

since $X(\phi)|_{r=0} = 0$. So we may assume $a \ne 0$. By (5.1), we have

$$X(\phi) = \left(a + \frac{b^2}{a}\right)^n r\frac{\partial\phi}{\partial r} \tag{5.2}$$

and

$$\frac{\partial^2\phi}{\partial\theta^2} = \left(\frac{b}{a}\right)^2 r\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) . \tag{5.3}$$

On the other hand, since

$$\omega_{g_0} = \sqrt{-1}\frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} ,$$

$\omega_g + \sqrt{-1}\partial\bar{\partial}\phi > 0$ implies

$$\begin{aligned} \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + r^{-2}\frac{\partial^2\phi}{\partial\theta^2} &= \Delta\phi \\ &\ge \frac{-4}{(1+r^2)^2}\frac{\omega_g}{\omega_{g_0}} \ge \frac{-4A}{(1+r^2)^2} . \end{aligned} \tag{5.4}$$

It follows by (5.3),

$$r\frac{\partial\phi}{\partial r} \ge \frac{-4A}{1 + \left(\frac{b}{a}\right)^2}\int_0^r \frac{s}{(1+s^2)^2}ds \ge \frac{-2A}{1 + \left(\frac{b}{a}\right)^2} \tag{5.5}$$

and

$$r\frac{\partial\phi}{\partial r} \le \frac{4A}{1 + \left(\frac{b}{a}\right)^2}\int_r^\infty \frac{s}{(1+s^2)^2}ds \le \frac{2A}{1 + \left(\frac{b}{a}\right)^2} . \tag{5.6}$$

Combining (5.2), (5.5), and (5.6), we obtain

$$|X(\phi)| \le 2|a|A . \qquad \square \tag{5.7}$$

***Definition* 5.2.** Let K be a closed set of complex manifold M and S a connected compact manifold. A family $\mathcal{F}$ of complex curves Γ_s(possibly singular) with base points $O_s(s \in S)$ is said to be a smooth S^2-fiber covering of M with K if there exist

i) a differentiable manifold $\mathcal{Y}$ and a smooth submersion $\theta : \mathcal{Y} \to S$ whose fibers are S^2-complex curves and

ii) two smooth maps $\sigma : S \to \mathcal{Y}$ and $\tau : \mathcal{Y} \to M$ such that for each $s \in S$

(a) $\theta^{-1}(s) \cong S^2$ is the normalization of Γ_s under the map τ

(b) $\tau(\sigma(s)) = O_s$ and $M = \mathcal{F} \cup K$.

Corollary 5.3. *Let (M, ω_g) be a Kähler manifold with a nontrivial holomorphic vector field X. Suppose that ϕ is a smooth function of M such that $\omega_g + \sqrt{-1}\partial\bar{\partial}\phi$ is a Kähler form and $X(\phi)$ is a real-valued function. Then there is a uniform constant C independent of ϕ such that $|X(\phi)| < C$.*

Proof. Let $K = \{p \in M | \quad X(p) = 0\}$. Then span{Real($X$), Im($X$)} defines a two-dimensional distribution on $M \setminus K$ and the closures of its integral orbits through a point $p \in M \setminus K$ are all complex curves with possible singularities in M. Since there is no nontrivial holomorphic vector field on any compact Riemannian surfaces with genus ≥ 1, we see that all normalizations of orbits are holomorphically isomorphic to S^2. So one can construct a connected compact manifold S and a family $\mathcal{F}$ of integral orbits Γ_s with base points $O_s(s \in S)$ such that $\mathcal{F}$ is a smooth S^2-fiber covering of M with K. Applying Lemma 5.1 to each normalization fiber S^2, we see there is a uniform constant C independent of ϕ such that $|X(\phi)| < C$. □

Proposition 5.4. *Let ϕ_t be solutions of Equation* (3.9) *at t. Then there is a uniform constant C such that $|\phi_t| < C$.*

Proof. From Equation (3.9), we have

$$e^{I_t(\phi_t)} \int_M e^{f-t(\theta_X + X(\phi_t))} \omega_g^n = \int_M \omega_{\phi_t}^n = \int_M \omega_g^n . \tag{5.8}$$

It follows by Corollary 5.3, there is a uniform constant C_1 such that $|I_t(\phi_t)| \leq C_1$. Let $\widetilde{f}_t = I_t(\phi_t) + f - t(\theta_X + X(\phi_t))$. Then $|\widetilde{f}_t| \leq C_2$ for some uniform constant C_2 and Equation (3.9) becomes

$$\det\left(g_{i\bar{j}} + \widetilde{\phi}_{t i\bar{j}}\right) = \det\left(g_{i\bar{j}}\right) e^{\widetilde{f}_t} , \tag{5.9}$$

where $\widetilde{\phi}_t = \phi_t - c_t$ and c_t are constants chosen so that $\sup_M \widetilde{\phi}_t = -1$. By an argument of C^0-estimate in [13, p. 157–159], we see that there is a uniform constant C_3 such that $|\widetilde{\phi}_t| \leq C_3$. On the other hand, by (3.6) in Lemma 3.1, we have

$$\begin{aligned} I_t(\phi_t) &= \int_0^1 \int_M \phi_t e^{t(\theta_X + sX(\phi_t))} \omega_{s\phi_t}^n ds \\ &= c_t \int_0^1 \int_M e^{t(\theta_X + sX(\phi_t))} \omega_{s\phi_t}^n \wedge ds + \int_0^1 \int_M \widetilde{\phi}_t e^{t(\theta_X + sX(\phi_t))} \omega_{s\phi_t}^n ds . \end{aligned} \tag{5.10}$$

It follows by Corollary 5.3,

$$\left| c_t \int_0^1 \int_M e^{t(\theta_X + sX(\phi_t))} \omega_{s\phi_t}^n \wedge ds - I_t(\phi_t) \right| \leq C_4$$

and consequently $|c_t| \leq C_5$ for some uniform constants C_4 and C_5. Therefore, we prove

$$|\phi_t| \leq \left|\tilde{\phi}_t\right| + |c_t| \leq C_3 + C_5 . \qquad \square$$

6. C^2-estimates and C^3-estimates

Proposition 6.1. *Let ϕ_t be solutions of Equation (3.9) at t. Then there are two uniform constants C and c such that*

$$n + \Delta\phi_t \leq C\exp\{c(\phi_t - \inf_M \phi_t)\} .$$

Proof. For simplicity, let $\phi = \phi_t$. Given each $p \in M$, choose a local coordinate system $(x_1, \dots, x_n)$ so that $g_{i\bar{j}}(p) = \delta_{ij}$ and $\phi_{i\bar{j}}(p) = \delta_{ij}\phi_{i\bar{i}}(p)$. Then by using Yau's C^2-estimate [16], [11], one can obtain

$$\begin{aligned}\Delta'((n + \Delta\phi)\exp\{-c\phi\}) &= \exp\{-c\phi\}\left(\Delta(f - t(\theta_X + X(\phi))) - n^2\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\\ &- c\exp\{c\phi\}n\,(n + \Delta\phi) + \left(c + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp\{-c\phi\}(n + \Delta\phi)\left(\sum_i \frac{1}{1 + \phi_{i\bar{i}}}\right) , \qquad (6.1)\end{aligned}$$

where Δ' denotes the Lapalacian operator associated with Kähler form ω_ϕ.

Let p be the maximal point of function $\Delta'((n + \Delta\phi)\exp\{-c\phi\})$. Then at this point, we have $\phi_{l\bar{l}i} = c(n + \Delta\phi)\phi_i$. It follows

$$\begin{aligned}\phi_{i\bar{l}l}X_{\bar{i}} &= \phi_{l\bar{l}i}X_{\bar{i}} = c(n + \Delta\phi)\phi_i X_{\bar{i}}\\ &= c(n + \Delta\phi)X(\phi) \leq c(n + \Delta\phi)\sup_M(X(\phi)) . \qquad (6.2)\end{aligned}$$

Thus by Corollary 5.3, we have

$$\begin{aligned}\Delta(-f + t(\theta_X + X(\phi))) &= -\Delta f + t\,(\theta_X + X(\phi))_{i\bar{i}}\\ &= -\Delta f\,\left(X_{\bar{k}}\left(g_{k\bar{i}} + \phi_{k\bar{i}}\right)\right)_i\\ &= -\Delta f + tX_{\bar{k}}g_{k\bar{i}i} + t\phi_{k\bar{i}i}X_{\bar{k}} + t\left(X_{\bar{k}i}\left(g_{k\bar{i}} + \phi_{k\bar{i}}\right)\right)\\ &\leq C_1 + t(n + \Delta\phi)\sup_k X_{k\bar{k}} + ct(n + \Delta\phi)\sup_M X(\phi)\\ &\leq (n + \Delta\phi)\,(C_2 + cC_3) + C_1 \qquad (6.3)\end{aligned}$$

for some uniform constants C_1, C_2, and C_3.

Inserting (6.3) into (6.1), we have

$$\begin{aligned}&\Delta'((n + \Delta\phi)\exp\{-c\phi\})\\ &\geq \exp\{-c\phi\}\left(-C_1 - n^2\inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right) - \exp\{-c\phi\}(n + \Delta\phi)\,(cn + C_2 + cC_3)\\ &\quad + \left(c + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp\{-c\phi\}(n + \Delta\phi)\left(\sum_i \frac{1}{1 + \phi_{i\bar{i}}}\right)\\ &\geq -C_4\exp\{-c\phi\} - cC_5\exp\{-c\phi\}(n + \Delta\phi)\\ &\quad + \left(c + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}\right)\exp\{-c\phi\}(n + \Delta\phi)\left(\sum_i \frac{1}{1 + \phi_{i\bar{i}}}\right) . \qquad (6.4)\end{aligned}$$

Choose c sufficiently large so that $c + \inf_{i \neq l} R_{i\bar{i}l\bar{l}} \geq \frac{c}{2}$. Then by using Equation (3.9), one can get

$$\begin{aligned}\Delta'((n+\Delta\phi)\exp\{-c\phi\}) &\geq -\exp\{-c\phi\}\,(C_4 + cC_5(n+\Delta\phi)) \\ &\quad + C_6\exp\{-c\phi\}(n+\Delta\phi)^{1+\frac{1}{m-1}}\,. \end{aligned} \tag{6.5}$$

Now applying the maximal principle theory to the function $\exp\{-c\phi\}(n+\Delta\phi)$ at the point p the same as in [16], we see there is a uniform constant C such that

$$n + \Delta\phi_t \leq C\exp\{c\,(\phi_t - \inf_M \phi_t)\}\,. \qquad \square$$

Combining Proposition 5.4 and Proposition 6.1, we have the following.

Corollary 6.2. *Let ϕ_t be solutions of Equation (3.9) at t. Then there is a uniform constant C such that $n + \Delta\phi_t \leq C$.*

Keep the notation in Proposition 6.1. Let $g'_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}}$ and

$$S = \sum g'^{i\bar{r}} g'^{\bar{j}s} g'^{k\bar{t}} \phi_{i\bar{j}k} \phi_{\bar{r}s\bar{t}}\,,$$

where $(g'^{i\bar{j}})$ is the inverse of matrix $(g'_{i\bar{j}})$. Then we have the following.

Proposition 6.3. *Let ϕ_t be solutions of Equation (3.9) at t. Then there is a uniform constant C such that $S \leq C$.*

Proof. For convenience, we use a notation as in [16]. We say that $A \cong B$ if $|A - B| \leq a(S + \sqrt{S}) + b$ for some constants a and b which can be estimated. By using Corollary 5.3, one can compute (c.f. [16])

$$\begin{aligned}\Delta S &\cong g'^{i\bar{r}} g'^{\bar{j}s} g'^{k\bar{t}} F_{i\bar{j}k}\phi_{\bar{r}s\bar{t}} + g'^{i\bar{r}} g'^{\bar{j}s} g'^{k\bar{t}} \phi_{i\bar{j}k} F_{\bar{r}s\bar{t}} \\ &\quad - \left(g'^{i\bar{q}} g'^{p\bar{r}} g'^{\bar{j}s} g'^{k\bar{t}} + g'^{i\bar{r}} g'^{p\bar{j}} g'^{s\bar{q}} g'^{k\bar{t}} + g'^{i\bar{r}} g'^{s\bar{j}} g'^{p\bar{t}} g'^{k\bar{q}}\right) \\ &\quad \times F_{p\bar{q}}\phi_{i\bar{j}k}\phi_{\bar{r}s\bar{t}} + S_0\,,\end{aligned} \tag{6.6}$$

where $F = f - t(\theta_X + X(\phi)) + I_t(\phi)$ and

$$\begin{aligned}S_0 &= \sum (1+\phi_{i\bar{i}})^{-1}\left(1+\phi_{j\bar{j}}\right)^{-1}(1+\phi_{k\bar{k}})^{-1}(1+\phi_{l\bar{l}})^{-1} \\ &\quad \times \left\{ \left|\phi_{\bar{i}j\bar{k}l} - \sum \phi_{\bar{i}p\bar{k}}\phi_{\bar{p}jl}\,(1+\phi_{p\bar{p}})^{-1}\right|^2 \right. \\ &\quad \left. + \left|\phi_{i\bar{j}kl} - \sum_p \left(\phi_{\bar{p}il}\phi_{p\bar{j}k} + \phi_{\bar{p}ik}\phi_{p\bar{j}l}\right)(1+\phi_{p\bar{p}})^{-1}\right|^2 \right\} > 0\,.\end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
F_{i\bar{j}k} &= \left(f - t\left(\theta_X + X(\phi)\right)\right)_{i\bar{j}k} \\
&= (f - t\theta_X)_{i\bar{j}k} - t\left(\phi_{l\bar{j}i}X^l_k + \phi_{l\bar{j}k}X^l_i + \phi_{l\bar{j}}X^l_{ik}\right) - t\phi_{l\bar{j}ik}X^l \,, \qquad (6.7)\\
F_{\bar{r}s\bar{t}} &= (f - t\theta_X)_{\bar{r}s\bar{t}} - t\left(\phi_{l\bar{r}\bar{t}}X^l_s + \phi_{l\bar{r}}X^l_{s\bar{t}}\right) - t\phi_{l\bar{r}s\bar{t}}X^l \\
&= (f - t\theta_X)_{\bar{r}s\bar{t}} - t\left(\phi_{l\bar{r}\bar{t}}X^l_s + \phi_{l\bar{r}}X^l_{s\bar{t}}\right) - t\phi_{\bar{r}s\bar{t}l}X^l \\
&\quad - t\left(\phi_{\bar{r}p}R_{s\bar{p}l\bar{t}} - \phi_{\bar{p}s}R_{p\bar{r}l\bar{t}}\right) \,, \qquad (6.8)
\end{aligned}
$$

and

$$
F_{p\bar{q}} = (f - t\theta_X)_{p\bar{q}} - t\left(\phi_{l\bar{q}}X^l_p + \phi_{l\bar{q}p}X^l\right) \,, \qquad (6.9)
$$

where $R_{i\bar{j}k\bar{l}}$ are sectional curvatures associated with the Kähler metric g.

Inserting (6.7) through (6.9) into (6.6) and using Corollary 6.2, it follows

$$
\begin{aligned}
\Delta S &\cong -tX^l\Big\{g'^{i\bar{r}}g'^{\bar{j}s}g'^{k\bar{t}}\phi_{i\bar{j}kl}\phi_{\bar{r}s\bar{t}} + g'^{i\bar{r}}g'^{\bar{j}s}g'^{k\bar{t}}\phi_{i\bar{j}k}\phi_{\bar{r}s\bar{t}l} \\
&\quad - \left(g'^{i\bar{q}}g'^{p\bar{r}}g'^{\bar{j}s}g'^{k\bar{t}} + g'^{i\bar{r}}g'^{p\bar{j}}g'^{s\bar{q}}g'^{k\bar{t}} + g'^{i\bar{r}}g'^{s\bar{j}}g'^{p\bar{t}}g'^{k\bar{q}}\right) \\
&\quad \times \phi_{p\bar{q}l}\phi_{i\bar{j}k}\phi_{\bar{r}s\bar{t}}\Big\} + S_0 \\
&= -tX^l\Big\{g'^{i\bar{r}}g'^{\bar{j}s}g'^{k\bar{t}}\left(\phi_{i\bar{j}k}\phi_{\bar{r}s\bar{t}}\right)_l - \phi_{p\bar{q}l}\phi_{i\bar{j}k}\phi_{\bar{r}s\bar{t}} \\
&\quad \times \left(g'^{i\bar{q}}g'^{p\bar{r}}g'^{\bar{j}s}g'^{k\bar{t}} + g'^{i\bar{r}}g'^{p\bar{j}}g'^{s\bar{q}}g'^{k\bar{t}} + g'^{i\bar{r}}g'^{s\bar{j}}g'^{p\bar{t}}g'^{k\bar{q}}\right)\Big\} + S_0 \\
&= -tX^l\left(S^1_l - S^2_l\right) + S_0 \,, \qquad (6.10)
\end{aligned}
$$

where

$$
S^1_l = g'^{i\bar{r}}g'^{\bar{j}s}g'^{k\bar{t}}\left(\phi_{i\bar{j}k}\phi_{\bar{r}s\bar{t}}\right)_l \qquad (6.11)
$$

and

$$
\begin{aligned}
S^2_l &= \phi_{p\bar{q}l}\phi_{i\bar{j}k}\phi_{\bar{r}s\bar{t}} \\
&\quad \times \left(g'^{i\bar{q}}g'^{p\bar{r}}g'^{\bar{j}s}g'^{k\bar{t}} + g'^{i\bar{r}}g'^{p\bar{j}}g'^{s\bar{q}}g'^{k\bar{t}} + g'^{i\bar{r}}g'^{s\bar{j}}g'^{p\bar{t}}g'^{k\bar{q}}\right) \,. \qquad (6.12)
\end{aligned}
$$

On the other hand, one can prove (c.f. [16]),

$$
\begin{aligned}
\Delta'\Delta\phi &\geq \Delta F + C_1 S - C_2 \\
&= (f - t\theta_X)_{p\bar{p}} - t\left(\phi_{l\bar{p}}X^l_p + \phi_{l\bar{p}p}X^l\right) \\
&\quad + C_1 S - C_2 C \geq C_3 S - C_4 \,, \qquad (6.13)
\end{aligned}
$$

for some positive uniform constants C_1, C_2, C_3, and C_4. Thus combining (6.10) and (6.13), we have

$$
\Delta'(S + c\Delta\phi) \geq cC_5 S - tX^l\left(S^1_l - S^2_l\right) - C_6 \qquad (6.14)
$$

as c sufficiently large.

Let p be the maximal point of function $S + c\Delta\phi$. Then we have $(S + c\Delta\phi)_l = 0$. It follows

$$S_l^1 - S_l^2 = -c\phi_{p\bar{p}l} \,. \tag{6.15}$$

Inserting (6.15) into (6.14) and then applying the maximal principle theory to the function $S + c\Delta\phi$, we obtain

$$\begin{aligned} 0 &\geq \Delta'(S + c\Delta\phi)(p) \\ &\geq \left(cC_5 S + ctX^l\phi_{p\bar{p}l} - C_6\right)(p) \\ &\geq cC_7 S(p) - C_8 \,, \end{aligned}$$

and consequently $S(p) \leq C_9$ for some uniform constant C_9. By Corollary 6.2, it follows

$$\begin{aligned} S &= S + c\Delta\phi - c\Delta\phi \leq \max_M(S + c\Delta\phi) - c\inf_M \Delta\phi \\ &\leq (S + c\Delta\phi)(p) + nc \\ &\leq S(p) + c\max_M \Delta\phi + nc \leq C_{10} \end{aligned} \tag{6.16}$$

for some uniform constant C_{10}. □

7. Proof of Main Theorem

Proof of Main Theorem. By Proposition 2.1, it remains to prove the existence and uniqueness of Equation (3.4). i) Existence. We shall prove that there is a smooth solution of Equation (3.9) at $t = 1$. Since we have proved I is an open set in Proposition 4.3, it suffices to prove I is closed. This is immediately followed from Proposition 5.4 and Proposition 6.3, and the regularity theory of Monge-Ampére equations (cf. [16]). In fact, we prove there are smooth solutions ϕ_t of Equation (3.9) for any $t \in [0, 1]$.

ii) Uniqueness. Let ω be a Kähler form solving Equation (1.2). Then there is a smooth function $\phi \in \mathcal{M}_X$ and a constant c such that $\omega = \omega_\phi = \omega_g + \sqrt{-1}\partial\bar{\partial}\phi$ and

$$\begin{cases} \det\left(g_{i\bar{j}} + \phi_{i\bar{j}}\right) = \det\left(g_{i\bar{j}}\right)\exp\{f - (\theta_X + X(\phi)) + c\} \\ \left(g_{i\bar{j}} + \phi_{i\bar{j}}\right) > 0 \,, \end{cases} \tag{7.1}$$

where f and θ_X are both real-valued functions of M determined by (3.2) and (3.1), respectivly. Choose c' such that $\tilde{\phi} = \phi + c'$ and

$$\begin{aligned} I\left(\tilde{\phi}\right) &= \int_0^1 \int_M \tilde{\phi} e^{\theta_X + sX(\tilde{\phi})} \omega_{s\tilde{\phi}}^n \wedge ds \\ &= \int_0^1 \int_M \left(\phi + c'\right) e^{\theta_X + sX(\phi)} \omega_{s\phi}^n \wedge ds = 0 \,. \end{aligned} \tag{7.2}$$

Consider the following normalized equations with the parameter $t \in [0, 1]$,

$$\begin{cases} \det\left(g_{i\bar{j}} + \tilde{\phi}_{i\bar{j}}\right) = \det\left(g_{i\bar{j}}\right)\exp\left\{f - t\left(\theta_X + X\left(\tilde{\phi}\right)\right) + c + I_t\left(\tilde{\phi}\right)\right\} \\ \left(g_{i\bar{j}} + \tilde{\phi}_{i\bar{j}}\right) > 0 \,, \end{cases} \tag{7.3}$$

where $I_t(\widetilde{\phi})$ is defined by (3.6). Clearly, $\widetilde{\phi}$ is a solution of Equation (7.3) at $t = 1$. Then by the similar arguments as in Part i), one can prove there are smooth solutions $\widetilde{\phi}_t$ of Equation (7.3) for any $t \in [0, 1]$ with $\widetilde{\phi}_1 = \widetilde{\phi} = \phi + c'$. Choose constants c_t so that $c + I_t(\widetilde{\phi}_t) = I_t(\widetilde{\phi}_t + c_t)$, i.e.,

$$c_t = c\left(\int_0^1 \int_M e^{t(\theta_X + sX(\tilde{\phi}_t))} \omega^n_{s\tilde{\phi}_t} \wedge ds\right)^{-1} . \tag{7.4}$$

Then $\widetilde{\phi}_t + c_t$ are smooth solutions of Equation (3.9) at t. On the other hand, ϕ_0 is a unique solution of Equation (3.9) at $t = 0$ (cf. the proof of Proposition 4.3). Thus, we have $\widetilde{\phi}_0 + c_0 = \phi_0$ and consequently $\widetilde{\phi}_t + c_t = \phi_t$ for any $t \in [0, 1]$, where ϕ_t are solutions of Equation (3.9) proved in Part i). In particular, by (7.2) and (7.4), we have

$$\begin{aligned} \phi &= \widetilde{\phi}_1 - c' = \phi_1 - c' - c_1 \\ &= \phi_1 - c\left(\int_0^1 \int_M e^{\theta_X + sX(\phi_1)} \omega^n_{s\phi_1} \wedge ds\right)^{-1} \\ &\quad - \left(\int_0^1 \int_M e^{\theta_X + sX(\phi_1)} \omega^n_{s\phi_1} \wedge ds\right)^{-1} \int_0^1 \int_M \phi e^{\theta_X + sX(\phi_1)} \omega^n_{s\phi_1} \wedge ds . \end{aligned} \tag{7.5}$$

This shows $\omega = \omega_{\phi_1}$ and ω is the unique solution of Equation (1.2) in the class $[\omega_g]$. □

Remark. Since Calabi's conjecture is true for any $\Omega \in C_1(M)$, one may believe that the assumption that Ω is positively definite can be removed. From the above discussions, we see that all results are still held for any $\Omega \in C_1(M)$ except in Section 4 as long as X and Ω satisfy Conditions i) and ii) in the Main Theorem, if one uses the continuity method to prove the existence and uniqueness. The only difficulty is how to prove the linearized operators of Equation (3.9) for the variable ϕ are invertiable without the assumption that Ω is positively definite.

References

[1] Aubin, T. Réduction du cas positif de l'équation de Monge-Ampére sur les variétés Kählerinnes compactes à la démonstration d'un intégalité, *J. Funct. Anal.*, **57**, 143–153, (1984).

[2] Bando, S. and Mabuchi, T. Uniqueness of Kähler-Einstein metrics modula connected group actions, Algebraic Geometry, *Adv. Studies in Pure Math.*, **10**, Sendai, (1987).

[3] Cao, H.D. Existence of gradient Kähler-Ricci solitons, Elliptic and parabolic methods in geometry, Peters, A.K., Chow, B., Gulliver, R., Levy, S., and Sullivan, J., Eds., 1–16, 1994.

[4] Cao, H.D. Limits of solutions of the Kähler-Ricci flow, *J. Differ. Geom.*, **45**, 257–272, (1997).

[5] Ding, W. and Tian, G. Kähler-Einstein metrics and the generalized Futaki invariants, *Invent. Math.*, **110**, 315–335, (1992).

[6] Futaki, A. An obstruction to the existence of Kähler-Einstein metrics, *Invent. Math.*, **73**, 437–443, (1983).

[7] Futaki, A. Kähler-Einstein metrics and integral invariants, *Lect. Notes in Math.*, **1314**, Springer-Verlag, Berlin, (1988).

[8] Futaki, A. and Mabuchi, T. Bilinear forms and extremal Kähler vector fields associated with Kähler classes, *Math. Ann.*, **301**, 199–210, (1995).

[9] Hamilton, R.S. Eternal solutions to the Ricci-flow, *J. Differ. Geom.*, **38**, 1–11, (1993).

[10] Koiso, N. On rationally symmetric Hamilton's equation for Kähler-Einstein metrics, Algebraic Geometry, *Adv. Studies in Pure Math.*, **18-1**, Sendai, (1990).

[11] Siu, Y.T. The existence of Kähler-Einstein metrics on manifolds with positive anticanonnical line bundle and a suitable symmetry group, *Ann. Math.*, **127**, 585–627, (1988).

[12] Tian, G. On Calabi's conjecture for complex surfaces with positive Chern class, *Invent. Math.*, **101**, 101–172, (1990).

[13] Tian, G. Kähler-Einstein metrics on algebraic manifolds, *Lect. Notes in Math.*, **1646**, Springer-Verlag, Berlin, (1996).

[14] Tian, G. Kähler-Einstein metrics with positive scalar curvature, *Invent. Math.*, **130**, 1–39, (1997).

[15] Tian, G. and Zhu, X.H. Uniqueness of Kähler-Ricci solitons on compact complex manifolds with $C_1(M) > 0$, to appear in *Acta Math.*

[16] Yau, S.T. On the Ricci curvature of a compact Kähler manifold and the Monge-Ampére equation, I^*, *Comm. Pure Appl. Math.*, **31**, 339–441, (1978).

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