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Classes of Singular Integral Operators Along Variable Lines

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ABSTRACT. We prove estimates for classes of singular integral operators along variable lines in the plane, for which the usual assumption of nondegenerate rotational curvature may not be satisfied. The main L^p estimates are proved by interpolating L^2 bounds with suitable bounds in Hardy spaces on product domains. The L^2 bounds are derived by almost-orthogonality arguments. In an appendix we derive an estimate for the Hilbert transform along the radial vector field and prove an interpolation lemma related to restricted weak type inequalities.

1. Introduction

For a special class of non-vanishing smooth vector fields $v : \mathbb{R}^2 \to \mathbb{R}^2$ we study the Hilbert transform H along the lines $\ell_x = \{y : y = x - tv(x), t \in \mathbb{R}\}$, defined by

$$Hf(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x - tv(x)) \frac{dt}{t} . \tag{1.1}$$

We also consider the related maximal operator M defined by

$$Mf(x) = \sup_{h>0} \frac{1}{h} \int_0^h |f(x - tv(x))| dt \tag{1.2}$$

and it is our objective to prove L^p estimates for H and M.

Presently it seems to be an open problem whether for every smooth v the operators H and M are bounded in $L^p(\mathbb{R}^2)$, for any $p \in (1, \infty)$ (although the globally defined operators (1.1) and (1.2) may fail to be L^p bounded if $p \leq 2$, see the remark in Section 6). If the curvature of the integral curves of v never vanishes to infinite order (as a function defined on an integral curve), then local versions of H and M are indeed bounded in L^p, for all $p \in (1, \infty)$; see [3], [10], and [11]. We

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are concerned here with obtaining estimates in some globally defined model examples as well as in cases in which the curvature may vanish to infinite order. We shall assume that our vector field depends only on x_1,

$$v(x_1, x_2) = (1, a(x_1)) \ . \tag{1.3}$$

It is well known that in this case the L^2 boundedness of H can be derived from Hunt's extension of Carleson's Theorem [8], [16] (this was perhaps first pointed out by Coifman and El–Kohen). However, neither the L^p boundedness for $p \neq 2$, nor any result on M seems to be a corollary of the Carleson–Hunt Theorem. In this paper we restrict ourselves to vector fields of the form (1.3) where a' is monotone for $t \neq t_0$. and $\lim_{t\to t_0} a'(t) = 0$ (here we allow the cases $t_0 = \pm\infty$). It is of course possible to estimate the Hilbert transform for $x_1 > t_0$ and $x_1 < t_0$ separately, so without loss of generality we assume that $t_0 < \infty$ and consider the operators

$$\mathfrak{H}f(x) = \chi_{(t_0,\infty)}(x_1) \int_{-\infty}^{\infty} f(x_1 - s, x_2 - sa(x_1)) \frac{ds}{s} \tag{1.4}$$

$$\mathfrak{M}f(x) = \chi_{(t_0,\infty)}(x_1) \sup_{h>0} \frac{1}{h} \int_0^h |f(x_1 - s, x_2 - sa(x_1))| \, ds \ , \tag{1.5}$$

and we assume that a' is nonnegative, monotonic, and increasing in (t_0, ∞). Then the monotonicity of a' implies the sets

$$I(\tau) = \{t > t_0 : \tau/2 \le a'(t) \le 2\tau\}$$

are intervals for all $\tau > 0$ and we shall always make the following assumptions. The first hypothesis is that the length of $I(\tau)$ is not changing too fast, specifically

$$0 < \inf_{\tau>0} \frac{|I(2\tau)|}{|I(\tau)|} \le \sup_{\tau>0} \frac{|I(2\tau)|}{|I(\tau)|} < \infty \, . \tag{1.6}$$

As a second hypothesis we impose the condition

$$\sup_{\tau>0} \frac{1}{\tau} \int_0^\tau \frac{|I(\sigma)|}{|I(\tau)|} d\sigma < \infty \, , \tag{1.7}$$

see also Lemma 1.1 for an alternative hypothesis.

Theorem. *Let $a : (t_0, \infty) \to [0, \infty)$ be a C^1 function satisfying $\lim_{t\to t_0} a'(t) = 0$ and suppose that a' is increasing in (t_0, ∞). Suppose that the assumptions (1.6) and (1.7) are satisfied. Then the operators $\mathfrak{H}$ and $\mathfrak{M}$ are bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$.*

Remarks.

(i) If $t_0 = 0$ and $a(t) = t^\gamma$, then $|I(\tau)| \approx \tau^{\frac{1}{\gamma-1}}$. If $t_0 = -\infty$ and $a(t) = e^t$, then $|I(\tau)| \approx 1$. In both cases (1.6) and (1.7) are clearly satisfied. The L^p version of the theorem is new for globally defined examples such as $a(t) = e^t$.

(ii) Notational changes in our proof yield local versions of the theorem. Assume $t_0 = 0$. If we set

$$\mathcal{H}f(x) = \chi_{[0,1]}(x_1) \, \text{p.v.} \int_{-\beta}^{\beta} f(x_1 - t, x_2 - ta(x_1)) \frac{dt}{t}$$

$$\mathcal{M}f(x) = \chi_{[0,1]}(x_1) \sup_{0<h<\beta} \frac{1}{h} \int_0^h |f(x_1 - t, x_2 - ta(x_1))| \, dt$$

and if we assume that (1.6) and (1.7) hold with the modification that the supremum in τ is only extended over all $\tau < \tau_{\max}$ for suitable $\tau_{\max}$, then $\mathcal{H}$ and $\mathcal{M}$ are bounded on L^p for $1 < p < \infty$. This version applies to examples such as $a(t) = \exp(-1/t)$ or $a(t) = \exp(-\exp(1/t))$, $t > 0$.

(iii) Similarly for the global version it is not necessary to assume that a' vanishes at t_0. If $\lim_{t\to t_0} a'(t) = \tau_{\min} > 0$, then we assume that in (1.6) and (1.7) the supremum in τ is only extended to over all $\tau > 2\tau_{\min}$, and the conclusion of the theorem holds. This version applies to examples such as $a(t) = \exp(\exp(t))$.

We point out that we may always assume that $a(t_0) = 0$. To see this let $\tilde{a}(t) = a(t) - a(t_0)$ and let $\tilde{\mathfrak{H}}$ be as in (1.4) with a replaced by $\tilde{a}$. Define $Ax = (x_1, x_2 + a(t_0)x_1)$, then $\mathfrak{H}f(Ax) = \tilde{\mathfrak{H}}[f(A\cdot)](x)$ and a satisfies our assumptions if and only if $\tilde{a}$ does. Moreover, we may assume without loss of generality that $a'(t) > 0$ for $t > t_0$. For if a' vanishes in (c, d), then the Hilbert transform $\mathfrak{H}f(x)$ coincides for $x_1 \in (c, d)$ with the translation invariant Hilbert transform along a fixed line and the L^p-boundedness of this operator is of course well known. Assuming these normalizations, an alternative formulation of the theorem can be obtained from the following result (which states that the hypothesis (1.6) and (1.7) is then equivalent to the hypothesis (1.6) and (1.9) below).

Lemma 1.1. *Let $a : [t_0, \infty) \to [0, \infty)$ be a C^1 function satisfying $\lim_{t\to t_0} a(t) = 0$ and $\lim_{t\to t_0} a'(t) = 0$ and assume that a' is strictly increasing in (t_0, ∞). Suppose that condition (1.6) is satisfied. Then there is a positive constant C such that*

$$\sup_{t\in I(\tau)} \frac{a'(t)|I(\tau)|}{a(t)} \le C \tag{1.8}$$

for all $\tau > 0$. Moreover, condition (1.7) is satisfied if and only if there exists a positive constant b such that

$$\inf_{t\in I(\tau)} \frac{a'(t)|I(\tau)|}{a(t)} \ge b \tag{1.9}$$

uniformly in $\tau > 0$.

Proof. Let $t \in I(\tau)$ and choose $s \in I(\tau/16)$. Then

$$a(t) \ge a(t) - a(s) \ge \int_{I(\tau/4)} a'(\sigma)d\sigma \ge \frac{\tau}{8}|I(\tau/4)| \ge c\tau|I(\tau)|$$

where in the last inequality we have used (1.6).

Suppose now that the expression in (1.7) is D. Then for $t \in I(\tau)$

$$\begin{aligned} a(t) &\le \int_{t_0}^{t} a'(s)ds \le c_1 \sum_{l\ge 0} \left|I\left(\tau 2^{-l}\right)\right| \tau 2^{-l} \le c_2 \int_0^{2\tau} |I(\sigma)|d\sigma \\ &\le c_2 D 2\tau |I(2\tau)| \le c_3 D a'(t)|I(\tau)|\,; \end{aligned}$$

here we have used (1.6) and (1.7). Conversely, if (1.9) holds and if $t \in I(\tau)$ and T is the right endpoint of the interval $I(\tau/8)$, then

$$\begin{aligned} \int_0^\tau |I(\sigma)|d\sigma &\le c_1 \sum_{2^{-k}\le 4\tau} 2^{-k}\left|I\left(2^{-k}\right)\right| \le c_2 \sum_{2^{-k}\le \tau/8} 2^{-k}\left|I\left(2^{-k}\right)\right| \le c_3 \sum_{2^{-k}\le\tau/8} \int_{I(2^{-k})} a'(s)ds \\ &\le c_3 \int_{t_0}^{T} a'(s)ds = c_3 a(T) \le c_3 a(t) \le c_3 b^{-1} a'(t)|I(\tau)| \le c_4 b^{-1}\tau|I(\tau)|\,. \qquad \square \end{aligned}$$

We shall now give an outline of the proof of the theorem, leaving the main technical details to Sections 2 and 3. We shall assume that $\lim_{t\to t_0} a(t) = 0$ and that $a'(t) > 0$ for $t > t_0$; as pointed out above, this is no loss of generality.

Following [21], [22] we decompose the operator, according to the size of the curvature of the integral curves. For $\ell \in \mathbb{Z}$ let

$$I_\ell = \left\{ t > t_0 : 2^{-\ell-1} < a'(t) \le 2^{-\ell} \right\} ;$$

then I_ℓ is an interval by the monotonicity assumption on a'. Let $\delta > 0$ be such that

$$\begin{aligned} 10\delta < |I_{\ell+1}| / |I_\ell| &< (10\delta)^{-1} \\ \delta &\le b/10 \end{aligned} \tag{1.10}$$

for all $\ell \in \mathbb{Z}$. Let $\chi \in C_0^\infty$ such that $\chi(t) \ge 0$ for all t, $\chi(t) > 0$ if $|t| \le 1/2$ and $\chi(t) = 0$ if $|t| > \delta + 1/2$. Let s_ℓ be the center of I_ℓ and let

$$\rho_\ell(t) = \frac{\chi\left(|I_\ell|^{-1}(t - s_\ell)\right)}{\sum_{m\in\mathbb{Z}} \chi\left(|I_m|^{-1}(t - s_m)\right)} .$$

Then the family $\{\rho_\ell\}$ forms a partition of unity of the interval (t_0, ∞). Moreover,

$$I_\ell \subset \operatorname{supp} \rho_\ell \subset I_{\ell-1} \cup I_\ell \cup I_{\ell+1} \tag{1.11}$$

and therefore

$$2^{-\ell-2} \le a'(t) \le 2^{-\ell+2} \quad \text{if } t \in \operatorname{supp} \rho_\ell ; \tag{1.12}$$

also $\operatorname{supp} \rho_\ell \cap \operatorname{supp} \rho_m = \emptyset$ if $|\ell - m| \ge 4$. Finally observe that

$$\left|\rho_\ell'(t)\right| \le C\, |I_\ell|^{-1} . \tag{1.13}$$

We choose an odd function $\psi \in C^\infty$ with support in $\{t : 1/2 \le |t| \le 2\}$, such that

$$\sum_{j\in\mathbb{Z}} 2^j \delta^{-1} \psi\left(2^j \delta^{-1} t\right) = \frac{1}{t}$$

and set

$$\psi_j(t) = 2^j \delta^{-1} \psi\left(2^j \delta^{-1} t\right) .$$

Here the factor δ is as in (1.10); this normalization is introduced for convenience and simplifies the notation later; note in particular that $\operatorname{supp} \rho_\ell + \operatorname{supp} \psi_j \subset I_{\ell-1} \cup I_\ell \cup I_{\ell+1}$ if $2^{-j} \le |I_\ell|$. We split

$$\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{H}_2$$

where

$$\mathfrak{H}_2 f(x) = \sum_\ell \rho_\ell(x_1) \sum_{2^{-j} \le |I_\ell|} \int \psi_j(t) f(x_1 - t, x_2 - t a(x_1))\, dt .$$

Lemma 1.2. *$\mathfrak{H}_2$ is bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$.*

Proof. For $\ell, m \in \mathbb{Z}$ let $R_{\ell m} = \{y \in \mathbb{R}^2 : y_1 \in I_\ell,\ (m-1)2^{-\ell}|I_\ell|^2 < y_2 \le m 2^{-\ell}|I_\ell|^2\}$ and let $f_{\ell m} = f \chi_{R_{\ell m}}$. Set

$$\mathfrak{H}_{2,\ell m} f(x) = \sum_{2^{-j} \le |I_\ell|} \int \psi_j(t) f_{\ell m}(x_1 - t, x_2 - t a(x_1))\, dt .$$

Note that $|x_1 - y_1| a(x_1) \le 2^{-j} b^{-1} \delta 2^{-\ell+2} |I_\ell| \le 2^{-\ell} |I_\ell|^2$ if $x_1 \in \cup_{j=\ell-1}^{\ell+1} I_j$, $x_1 - y_1 \in \operatorname{supp} \psi_j$ and $2^{-j} \le |I_\ell|$ [cf. (1.10)].

Therefore, $\mathfrak{H}_2 f_{\ell m}(x) = 0$ if x does not belong to the union of rectangles $R_{\lambda\mu}$ with $\ell - 2 \le \lambda \le \ell + 2$ and $m - 2 \le \mu \le m + 2$. It follows that

$$\|\mathfrak{H}_2\|_{L^p \to L^p} \le C \sup_{\ell,m} \left\|\mathfrak{H}_{2,\ell m}\right\|_{L^p \to L^p} ;$$

hence it suffices to obtain a uniform L^p bound for $\mathfrak{H}_{2,\ell m}$.

Define $A_{\ell m} x = (|I_\ell|^{-1}(x_1 - u_1^{\ell m}), 2^\ell |I_\ell|^{-2}(x_2 - u_2^{\ell m}))$ where $(u_1^{\ell m}, u_2^{\ell m})$ is the center of $R_{\ell m}$. Then the affine transformation $A_{\ell m}$ maps the rectangle $R_{\ell m}$ to the unit square Q centered at 0 and $\mathfrak{H}_{2,\ell m} f(x) = \widetilde{\mathfrak{H}}_{2,\ell m}\big[f_{\ell m}(A_{\ell m}^{-1} \cdot)\big](A_{\ell m} x)$ with

$$\widetilde{\mathfrak{H}}_{2,\ell m} g(z_1, z_2) = \sum_{2^{-j} \le |I_\ell|} \int 2^j \, |I_\ell| \, \delta^{-1} \psi\left(2^j \delta^{-1} \, |I_\ell| \, t\right) g_Q(z_1 - t, z_2 - a_{\ell m}(z_1)) \, dt$$

where $a_{\ell m}(z_1) = 2^\ell |I_\ell|^{-1} a(u_1^{\ell m} + |I_\ell| z_1)$ and $g_Q = g \chi_Q$. Note that $a'_{\ell m}$ is bounded above and below, uniformly in ℓ, m. This is essentially the case of nonvanishing rotational curvature, however standard theorems [10], [11], [15] or [20] cannot be immediately applied since we are dealing with a globally defined operator and since a is not smooth enough. Nevertheless, standard arguments can be applied and indeed the operators $\widetilde{\mathfrak{H}}_{2,\ell m}$ and therefore the operators $\mathfrak{H}_{2,\ell m}$ are uniformly bounded in $L^p(\mathbb{R}^2)$, $1 < p < \infty$. More details are carried out in Section 5. □

The nontrivial contribution comes from the operator $\mathfrak{H}_1$. We choose a non-negative C^∞ function ϕ supported in $\{\mu : 1/2 \le |\mu| \le 2\}$ with $\sum_{r \in \mathbb{Z}} \phi(2^{-r}\mu) = 1$ for $\mu \ne 0$. Then $\mathfrak{H}_1$ is a sum of operators

$$T_{j\ell}^r f(x) = \rho_\ell(x_1) \int \psi_j(x_1 - y_1) \, f(y) \int \phi\left(2^{-r}\mu\right) e^{i\mu[x_2 - y_2 - a(x_1)(x_1 - y_1)]} d\mu \, dy \qquad (1.14)$$

where $|I_\ell| < 2^{-j}$. We decompose $\mathfrak{H}_1 = \mathcal{T} + R$ where

$$\mathcal{T} = \sum_\ell \sum_{2^{-j} > |I_\ell|} \sum_{r \ge 2j + \ell} T_{j\ell}^r$$

The operator $R = \mathfrak{H}_1 - \mathcal{T}$ can be handled by standard arguments from Calderón–Zygmund theory.

Lemma 1.3. *R is bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$.*

Proof. We expand $e^{-i\mu a(x_1)(x_1 - y_1)}$ in a power series in $\mu a(x_1)(x_1 - y_1)$ and observe that the terms (1.14) which contribute to R satisfy $2^r |a(x_1)(x_1 - y_1)| \le c b^{-1} \delta 2^{r-j-\ell} |I_\ell| \le c'$. Define operators $\mathfrak{S}_{k,r}$ by

$$\mathfrak{S}_{k,r} g(x_1, x_2) = \sum_\ell \sum_{\substack{2^{-j} > |I_\ell| \\ r < 2j + \ell}} \rho_\ell(x_1) \int \psi_j(x_1 - y_1) \left[2^r a(x_1)(x_1 - y_1)\right]^k g(y_1, x_2) \, dy_1 .$$

Next define Littlewood–Paley operators L_r, $\widetilde{L}_{r,k}$ in the second variable by $\widehat{L_r f}(\xi) = \phi(2^{-r}\xi_2)\widehat{f}(\xi)$ and $\widehat{\widetilde{L}_{r,k} f}(\xi) = (2^{-r}\xi_2)^k \widetilde{\phi}(2^{-r}\xi_2)\widehat{f}(\xi)$; here $\widetilde{\phi}$ is supported in $\pm(1/4, 4)$ and equals 1 on $\operatorname{supp} \phi$. Then

$$R = \sum_{k=0}^\infty \frac{(-i)^k}{k!} \sum_{r \in \mathbb{Z}} \widetilde{L}_{r,k} \mathfrak{S}_{k,r} [L_r f] .$$

By standard Calderón–Zygmund theory

$$\Big\|\sum_r \widetilde{L}_{r,k}h_r\Big\|_p \le c_p 10^k \Big\|\Big(\sum_r |L_r h_r|^2\Big)^{1/2}\Big\|_p$$

for $1 < p < \infty$. By another application of Littlewood–Paley theory it clearly suffices to show that the vector-valued operator $F = \{f_r\}_{r\in\mathbb{Z}} \mapsto \{\mathfrak{S}_{k,r} f_r\}_{r\in\mathbb{Z}}$ maps $L^p(\ell^2)$ into itself with operator norm bounded by CB^k,[1] for some positive constant B.

Observe that $\mathfrak{S}_{0,r}$ is essentially dominated by a maximal Hilbert transform in the first variable; in fact Cotlar's inequality [24, p. 35]) holds:

$$|\mathfrak{S}_{0,r}g(x)| \le C\,(M_1[g](x) + M_1\,[H_1 g]\,(x))\ ;$$

here M_1 and H_1 denote the standard Hardy–Littlewood maximal function and the Hilbert transform in the first variable, respectively, and C does not depend on r. If $k > 0$ and r, ℓ are fixed, then for $x_1 \in I_\ell$

$$\begin{aligned} |\mathfrak{S}_{k,r}g(x)| &\le C\sum_\ell \sum_{\substack{2^{-j}>|I_\ell| \\ 2j+\ell>r}} \rho_\ell(x_1)\int \left|2^r a(x_1)(x_1-y_1)\right|^k \left|\psi_j(x_1-y_1)\right| |g(y_1,x_2)|\,dy_1 \\ &\le C\sum_\ell \rho_\ell(x_1) \sum_{2j+\ell>r} \left(2b^{-1}\delta 2^{r-\ell-2j}\right)^k M_1 g(x) \le C' B^k M_1 g(x)\,. \end{aligned}$$

By the Fefferman–Stein inequality for sequences of maximal functions [12] and a vector valued inequality for the Hilbert transform

$$\begin{aligned} \Big\|\Big(\sum_r |\mathfrak{S}_{k,r}f_r|^2\Big)^{1/2}\Big\|_p &\le CB^k\left[\Big\|\Big(\sum_r |f_r|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_r |H_1 f_r|^2\Big)^{1/2}\Big\|_p\right] \\ &\le C'B^k \Big\|\Big(\sum_r |f_r|^2\Big)^{1/2}\Big\|_p\,. \end{aligned}$$

□

Our main estimates concern the operator $\mathcal{T}$ and we shall introduce a further decomposition. For nonnegative integers s and n let

$$\mathfrak{A}_s = \left\{(j,\ell) : 2^{-j-s} > |I_\ell| \ge 2^{-j-s-1}\right\} \tag{1.15}$$

and

$$\mathcal{T}_{sn} = \sum_{(j,\ell)\in\mathfrak{A}_s} T_{j\ell}^{2j+\ell+n}\ ; \tag{1.16}$$

then $\mathcal{T} = \sum_{s,n=0}^{\infty} \mathcal{T}_{sn}$.

[1] Here and in the sequel C will denote some absolute "constant" which may depend on p and whose value may change from line to line.

Proposition 1.4. *Let* $1 < p \leq 2, \gamma < 1 - 1/p$. *Then for all* $f \in L^p$

$$\|\mathcal{T}_{sn} f\|_p \leq C_{\gamma,\beta,p} 2^{-n\gamma} \min\left\{1, 2^{(n-s)\beta}\right\} \|f\|_p \qquad \text{if } \beta < 1/2$$

and

$$\left\|\mathcal{T}_{sn}^* f\right\|_p \leq C_{\gamma,\beta,p} 2^{-n\gamma} \min\left\{1, 2^{(n-s)\beta}\right\} \|f\|_p \qquad \text{if } \beta < 1 - 1/p\,.$$

Clearly the theorem follows from Lemmas 1.2, 1.3, and Proposition 1.4. The appropriate L^2 estimates for Proposition 1.4 will be derived in Section 2. The difficulty in obtaining L^p estimates is the absence of a Calderón–Zygmund theory on a suitable space of homogeneous type. Fortunately in our present analysis we can interpolate the L^2 estimates with somewhat weaker estimates on multiparameter Hardy spaces. These are derived in Section 3. In Section 4 we shall discuss the modifications needed to estimate the maximal operator $\mathfrak{M}$. Section 5 contains the estimates needed to complete the proof of Lemma 1.2 above. The final section is an appendix where we study the Hilbert transform along the radial vector field, including a general interpolation lemma related to restricted weak type estimates.

2. L^2-estimates for oscillatory integral operators

The following result is a straightforward consequence of the almost-orthogonality lemma by Cotlar and Stein (see [24, p. 280]); in our application below we will be able to choose $\epsilon = 1/2$.

Lemma 2.1. *Suppose that* $0 < \epsilon \leq 1$, $0 < C_1 \leq \sqrt{C_2}$. *Let* $\{T_j\}$ *be a collection of bounded operators on a Hilbert space* H *such that*

$$\|T_j\| \leq C_1$$

and

$$\max\left\{\left\|T_j (T_k)^*\right\|, \left\|(T_j)^* T_k\right\|\right\} \leq C_2 2^{-\epsilon|j-k|}$$

for all $j, k \in \mathbb{Z}$. *Then the partial sums* $\sum_{j=-N}^{N} T_j$ *converge in the strong operator topology to a bounded operator* T *as* $N \to \infty$ *and* T *satisfies the bound*

$$\|T\| \leq 10\epsilon^{-1} C_1 \log_2\left(1 + \sqrt{C_2}/C_1\right)\,.$$

Proof. By the Cotlar–Stein lemma

$$\|T\| \leq \sum_{n=0}^{\infty} \sup_{|j-k|=n} \max\left\{\left\|T_j (T_k)^*\right\|^{1/2}, \left\|(T_j)^* T_k\right\|^{1/2}\right\}\,.$$

Let $N = 2\epsilon^{-1}\log_2(1 + \sqrt{C_2}/C_1)$. We dominate the n^{th} term in the series by C_1 if $n < N$ and by $\sqrt{C}_2 2^{-\epsilon n/2}$ if $n \geq N$. Hence,

$$\|T\| \leq C_1 \log_2\left(1 + \sqrt{C_2}/C_1\right)\left(2\epsilon^{-1} + \left(1 - 2^{-\epsilon/2}\right)^{-1}\right)\,.$$

This implies the asserted inequality. □

In what follows we consider oscillatory integral operators acting on functions $g \in L^2(\mathbb{R})$. Suppose that $\Psi_j \in C^2(\mathbb{R} \times \mathbb{R})$ and that

$$\Psi_j(x, y) = 0 \text{ if } |x - y| \geq \delta 2^{-j+2} \text{ or } |x - y| \leq \delta 2^{-j-2}\,, \tag{2.1}$$

where δ is as in (1.10). Suppose also that

$$\left|\partial_y^\kappa \Psi_j(x,y)\right| \le A 2^j 2^{j\kappa}, \qquad \kappa = 0,1,2\,. \tag{2.2}$$

Lemma 2.2. *For given $n \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$ let $j \mapsto \ell(j)$ denote a function defined on a subset $\mathfrak{z}$ of $\mathbb{Z}$ satisfying $|\lambda|/2 \le 2^{\ell(j)+2j+n} \le 2|\lambda|$ and $(j,\ell(j)) \in \mathfrak{A}_s$ for all $j \in \mathfrak{z}$ [here $\mathfrak{A}_s$ is as in (1.15)]. Define an operator P_j acting on Schwartz functions of one variable by*

$$P_j g(x) = \rho_{\ell(j)}(x) \int e^{i\lambda a(x)(x-y)} \Psi_j(x,y) g(y)\,dy\,; \tag{2.3}$$

here Ψ_j is as in (2.1), (2.2). Then P_j is bounded on L^2 and for all $g \in L^2(\mathbb{R})$

$$\left\|P_j g\right\|_2 \le CA \min\left\{2^{-s/2}, 2^{-n/2}\right\} \|g\|_2 \tag{2.4}$$

where C does not depend on j and the particular function ℓ. Moreover, $(P_j)^* P_k = 0$ for $|j-k| \ge 10$ and the L^2 operator norm of $P_j P_k^*$ satisfies

$$\left\|P_j P_k^*\right\|_{L^2 \to L^2} \le CA^2 2^{-|j-k|/2}\,. \tag{2.5}$$

Finally if $\mathcal{P} = \sum_{j\in\mathfrak{z}} P_j$, then $\mathcal{P}$ is bounded on $L^2(\mathbb{R})$ with norm $\le CA(1+s+n)\min\{2^{-s/2}, 2^{-n/2}\}$.

Proof. The asserted L^2 bound for $\mathcal{P}$ follows from (2.4), (2.5), and Lemma 2.1. The modulus of the kernel K_{jk} of $P_j P_k^*$ is given by

$$\left|K_{jk}(x,z)\right| = \left|\rho_{\ell(j)}(x)\rho_{\ell(k)}(z) \int \Psi_j(x,y)\overline{\Psi_k(z,y)} e^{-i\lambda y[a(x)-a(z)]} dy\right|\,. \tag{2.6}$$

A crude estimate yields $|K_{jk}(x,z)| \le CA^2 \min\{2^j, 2^k\}$ and in turn

$$\int \left|K_{jk}(x,z)\right| dx + \int \left|K_{jk}(x,z)\right| dz \le CA^2 2^{-s}\,. \tag{2.7}$$

If $j = k$, then $|a(x) - a(z)| \approx 2^{-\ell(j)}|x-z|$ and if $|x-z| > 2^{j+\ell(j)}\lambda^{-1}$ we may improve the previous estimate by integrating by parts twice. This yields

$$\left|K_{jj}(x,z)\right| \le CA^2 \min\left\{2^j, 2^{3j+2\ell(j)}\lambda^{-2}|x-z|^{-2}\right\}$$

and therefore

$$\int \left|K_{jj}(x,z)\right| dx + \int \left|K_{jj}(x,z)\right| dz \le CA^2 2^{2j+\ell(j)}\lambda^{-1} \le CA^2 2^{-n}\,.$$

This together with (2.7) implies (2.4).

Now assume that $|j-k| \ge 10$; then also $|\ell(j) - \ell(k)| \ge 10$. By taking adjoints we may without loss of generality assume that $k < j$. There is an interval I_l between $I_{\ell(j)}$ and $I_{\ell(k)}$ which does not intersect either $I_{\ell(k)}$ or $I_{\ell(j)}$ but satisfies $|l - \ell(j)| \le 5$. Then by assumption (1.6) we obtain

$$|a(x) - a(z)| \ge 2^{-l-1}|I_l| \ge c2^{-\ell(j)}\left|I_{\ell(j)}\right|$$

if $x \in \operatorname{supp} \rho_{\ell(j)}$ and $z \in \operatorname{supp} \rho_{\ell(k)}$. Integrating by parts once in (2.6) yields the pointwise bound

$$|K_{jk}(x,z)| \le CA^2 \frac{2^{j+k}|\lambda|^{-1}}{|I_{\ell(j)}| 2^{-\ell(j)}} \le A^2 2^{-n} 2^{k-j} |I_{\ell(j)}|^{-1} .$$

For fixed z we integrate over $x \in \operatorname{supp} \rho_{\ell(j)}$ and obtain

$$\int |K_{jk}(x,z)| \, dx \le CA^2 2^{-n} 2^{k-j} .$$

If we also use (2.7) we obtain by the continuous version of Schur's lemma the asserted estimate (2.5), where A is actually replaced by the smaller value $A2^{-(s+n)/4}$. □

The usefulness of the following lemma has been demonstrated for example in [19]. It follows by a twofold application of Plancherel's theorem.

Lemma 2.3. *Let $m \in L^\infty(\mathbb{R})$, let $\{P_\lambda\}$ be a family of bounded linear operators on $L^2(\mathbb{R})$. Suppose that for every f in the Schwarz space $\mathcal{S}(\mathbb{R}^2)$ the function $(x_1, y_2, \lambda) \mapsto P_\lambda[f(\cdot, y_2)](x_1)$ is continuous and suppose that the L^2 operator norms of P_λ are uniformly bounded by B. For Schwartz functions $f \in \mathcal{S}(\mathbb{R}^2)$ define T by*

$$Tf(x) = \iint m(\lambda) e^{i\lambda(x_2 - y_2)} P_\lambda [f(\cdot, y_2)](x_1) \, d\lambda dy_2 .$$

Then T extends to a bounded operator on $L^2(\mathbb{R}^2)$ with operator norm bounded by cB.

Corollary 2.4. *The operator $\mathcal{T}_{sn}$ defined in (1.16) is bounded on $L^2(\mathbb{R}^2)$ with operator norm $\le C(1+s+n)\min\{2^{-n/2}, 2^{-s/2}\}$.*

Proof. We write $\mathcal{T}_{sn} = \sum_{i=0}^{4} \mathcal{T}_{sn,i}$ where $\mathcal{T}_{sn,i}$ is as in (1.16), with the additional specification that only values of ℓ with $\ell = i \bmod 5$ occur in the sum. As an immediate consequence of Lemma 2.2 and Lemma 2.3 we obtain the L^2 boundedness of $\mathcal{T}_{sn,i}$, with the required bounds. □

The following variant of Lemma 2.3 will be used when f has some cancellation property with respect to the y_2 variable.

Lemma 2.5. *Let $\{P_\lambda\}$ be a family of bounded linear operators on $L^2(\mathbb{R})$ satisfying the assumptions of Lemma 2.3. For Schwartz functions $f \in \mathcal{S}(\mathbb{R}^2)$ and fixed u_2 define S_r by*

$$S_r f(x) = \iint \phi\left(2^{-r}\lambda\right) e^{i\lambda x_2} \left(e^{-i\lambda y_2} - e^{-i\lambda u_2}\right) P_\lambda [f(\cdot, y_2)](x_1) \, d\lambda dy_2 .$$

Then

$$\|S_r f\|_2 \le CB2^r \left(\int |y_2 - u_2|^2 |f(y)|^2 dy\right)^{1/2}$$

where C does not depend on u_2.

Proof. We write the difference of exponentials as an integral over a derivative and see that $S_r = \int_0^1 S_{r,\sigma} d\sigma$ where

$$S_{r,\sigma} f(x) = -i \iint \lambda\phi\left(2^{-r}\lambda\right) e^{i\lambda(x_2 - (1-\sigma)u_2 - \sigma y_2)} (y_2 - u_2) P_\lambda [f(\cdot, y_2)](x_1) \, dy_2 \, d\lambda .$$

Set $G_\mu(y_1) = \int e^{-i\mu y_2}(y_2-u_2)f(y_1,y_2)\,dy_2 = \mathcal{F}_2\big[(\cdot - u_2)f(y_1,\cdot)\big](\mu)$ where $\mathcal{F}_2$ denotes the Fourier transform in the y_2 variable. Then

$$S_{r,\sigma}f(x) = i\int \lambda\phi\left(2^{-r}\lambda\right)e^{i\lambda(x_2-(1-\sigma)u_2)}P_\lambda\left[G_{\lambda\sigma}\right](x_1)\,d\lambda\,.$$

From applications of Plancherel's theorem and Fubini's theorem it follows that

$$\begin{aligned}\left\|S_{r,\sigma}f\right\|_2 &= \sqrt{2\pi}\left(\iint\left|\lambda\phi\left(2^{-r}\lambda\right)\right|^2\left|P_\lambda\left[G_{\lambda\sigma}\right](x_1)\right|^2dx_1d\lambda\right)^{1/2}\\ &\leq B\sqrt{2\pi}\left(\iint\left|\lambda\phi\left(2^{-r}\lambda\right)\right|^2\left|G_{\lambda\sigma}(y_1)\right|^2d\lambda\,dy_1\right)^{1/2}\\ &\leq C\sigma^{-1/2}B2^r\left(\int\left|y_2-u_2\right|^2\left|f(y)\right|^2dy\right)^{1/2}\end{aligned}$$

and the desired estimate is obtained by integrating in σ. □

3. Estimates for rectangle atoms

The L^p estimates for $\mathcal{T}_{sn}$ and their adjoints are derived by interpolation of the L^2 estimates in the previous section with appropriate estimates on the Hardy space $H^1_{\text{prod}}(\mathbb{R}\times\mathbb{R})$ with the multiparameter dilation structure. The interpolation theorem can be found in [9]. In order to prove the H^1 estimates we use the version of Calderón–Zygmund theory as developed by Journé [17]. A particularly elegant variant of it which is valid in two parameters was proved by Fefferman [12]. In this setting it suffices to check the behavior of the singular integral operator on rectangle atoms.

Let $R = J_1\times J_2$ be a rectangle with edges parallel to the coordinate axes and center (u_1,u_2). Then f is called a *rectangle atom associated to* R if f is supported in R, if

$$\|f\|_2 \leq |R|^{-1/2}$$

and if

$$\begin{aligned}\int f(x_1,x_2)\,dx_1 &= 0 \text{ for almost every } x_2\in J_2\,,\\ \int f(x_1,x_2)\,dx_2 &= 0 \text{ for almost every } x_1\in J_1\,.\end{aligned}$$

Let $w_{R,\epsilon}(x) = \prod_{i=1}^2(1+|x_i-u_i|/|J_i|)^\epsilon$. Suppose that the operator T is bounded on L^2 and suppose that there is $\epsilon>0$ such that for all R and all rectangle atoms f_R associated to R

$$\int |Tf_R(x)|\,w_{R,\epsilon}(x)dx \leq B \tag{3.1}$$

where B does not depend on R. Then according to Fefferman's theorem, the operator T maps $H^1_{\text{prod}}(\mathbb{R}\times\mathbb{R})$ to $L^1(\mathbb{R}^2)$ and there is the estimate

$$\|T\|_{H^1\to L^1} \leq c\|T\|_{L^2\to L^2} + C_\epsilon B\,.$$

In what follows we fix a rectangle atom f associated to a rectangle R and estimate $\mathcal{T}_{sn}f$ in rectangular regions in the complement of R. Given $m=(m_1,m_2)$ with nonnegative integers m_1, m_2 and given a rectangle $R = J_1\times J_2$ as above we define $\mathcal{J}_1(m_1)$, $\mathcal{J}_2(m_2)$, and $\mathcal{R}(m)$ by

$$\mathcal{J}_i(m_i) = \begin{cases}\{x_i : |x_i-u_i|\leq 8\,|J_i|\} & \text{if } m_i=0\\ \left\{x_i : 2^{m_i+3}\,|J_i| < |x_i-u_i| \leq 2^{m_i+4}\,|J_i|\right\} & \text{if } m_i>0\end{cases} \tag{3.2}$$

and

$$\mathcal{R}(m) = \mathcal{J}_1(m_1) \times \mathcal{J}_2(m_2) \ . \tag{3.3}$$

It is our objective to prove the following proposition which together with Corollary 2.4 implies Proposition 1.4.

Proposition 3.1. *Let f be a rectangle atom associated to the rectangle $R = J_1 \times J_2$ with center (u_1, u_2) and let $\mathcal{R}(m)$ be as in (3.3). Then for $0 < \epsilon < 1/2$*

$$\int_{\mathcal{R}(m)} |\mathcal{T}_{sn} f(x)|\, dx \leq C_\epsilon 2^{2(s+n)\epsilon} 2^{-\epsilon(m_1+m_2)} \min\left\{1, 2^{(n-s)/2}\right\} \tag{3.4}$$

$$\int_{\mathcal{R}(m)} \left|\mathcal{T}_{sn}^* f(x)\right| dx \leq C_\epsilon 2^{2(s+n)\epsilon} 2^{-\epsilon(m_1+m_2)} \ . \tag{3.5}$$

Consequently $\mathcal{T}_{sn}$ and $\mathcal{T}_{sn}^*$ map $H^1_{\text{prod}}(\mathbb{R} \times \mathbb{R})$ boundedly into L^1 and, for every $\alpha > 0$, the operator norms are bounded by $C_\alpha 2^{\alpha n}$ and $C_\alpha 2^{\alpha(s+n)}$, respectively.

We now decompose $\mathcal{T}_{sn} = \sum_r \mathcal{T}_{sn}^r$ where

$$\mathcal{T}_{sn}^r = \sum_{\substack{(j,\ell)\in\mathfrak{A}_s \\ \ell+2j=r-n}} T_{j\ell}^r \ .$$

Lemma 3.2. *Let f be a rectangle atom associated to the rectangle $R = J_1 \times J_2$ with center (u_1, u_2) and let $\mathcal{R}(m)$ be as in (3.3). Then*

$$\begin{aligned} &\|\mathcal{T}_{sn} f\|_{L^1(\mathcal{R}(m))} + \left(2^r |J_2|\right)^{-1} \left\|\mathcal{T}_{sn}^r f\right\|_{L^1(\mathcal{R}(m))} \\ &\quad \leq C(1+s+n) 2^{(m_1+m_2)/2} \min\left\{2^{-s/2}, 2^{-n/2}\right\} \ . \end{aligned} \tag{3.6}$$

and the same estimates hold if $\mathcal{T}_{sn}$ and $\mathcal{T}_{sn}^r$ are replaced by their adjoints.

Proof. We have already proved the L^2 bounds for $\mathcal{T}_{sn}$ in Section 2 (see Corollary 2.4), and the asserted estimate for $\mathcal{T}_{sn}$ follows by the Cauchy–Schwarz inequality and the size estimate for the atom. Similarly, in view of the y_2 cancellation of f we can use Lemma 2.5 instead of Lemma 2.3 to obtain also the estimate for $\mathcal{T}_{sn}^r$. □

Lemma 3.3. *Let f be a rectangle atom associated to the rectangle $R = J_1 \times J_2$ with center (u_1, u_2) and let $\mathcal{J}_2(m_2)$ be as in (3.2). Then for $M = 0, 1, 2, \dots$*

$$\begin{aligned} &\int_{\mathcal{J}_2(m_2)} \left(\int_{\mathcal{J}_1(0)} \left|\mathcal{T}_{sn}^r f(x)\right|^2 dx_1\right)^{1/2} dx_2 \\ &\quad \leq C_M 2^{n/2}(1+s+n) \left(\frac{2^{n-m_2}}{2^r |J_2|}\right)^M \min\left\{1, 2^{(n-s)/2}\right\} \\ &\quad\quad \times \min\left\{1;\ 2^r |J_2| + 2^{-m_2}\right\} \|f\|_{L^1(L^2)} \end{aligned} \tag{3.7}$$

where $\|f\|_{L^1(L^2)} = \int \left(\int |f(x_1, x_2)|^2 dx_1\right)^{1/2} dx_2$. The same estimates remain true when $\mathcal{T}_{sn}^r$ is replaced by its adjoint.

Proof. Denote by $K^r_{j\ell}$ the kernel of the operator $T^r_{j\ell}$. By an integration by parts with respect to the frequency variable λ and the Leibniz rule we express $K^r_{j\ell} = \sum_{\nu=0}^{M+1} K^r_{j\ell\nu}$, where

$$K^r_{j\ell\nu}(x,y) = \rho_\ell(x_1)\,\widetilde{\rho}_\ell(x_1)\int \Gamma_{\lambda,M+1}(x_2-y_2)\,\Psi_{j,\nu,\lambda}(x_1,y_1)\,e^{i\lambda a(x_1)(x_1-y_1)}d\lambda\,, \tag{3.8}$$

where $\Gamma_{\lambda,M+1}(u) = e^{i\lambda u}u^{-M-1}$ and

$$\Psi_{j,\nu,\lambda}(x_1,y_1) = c_\nu\widetilde{\rho}_\ell(x_1)\,(a(x_1)(x_1-y_1))^\nu\,2^{-r(M+1-\nu)}\phi^{(M+1-\nu)}\left(\lambda 2^{-r}\right)\psi_j(x_1-y_1)\;;$$

here $\widetilde{\rho}_\ell(x_1)$ is supported in $\cup_{i=-2}^{2} I_{\ell+i}$ and equal to 1 on the support of ρ_ℓ. If $\ell = r-n-2j$ the functions $\Psi_{j,\nu,\lambda}$ satisfy (2.2) with $A = A_\nu$ where

$$A_\nu \le C\left(|I_\ell|2^{-\ell-j}\right)^\nu 2^{-r(M+1-\nu)} \le C'2^{-s\nu}2^{-(2j+\ell)\nu}2^{-r(M+1-\nu)} \le C''2^{-s\nu}2^{(n-r)(M+1)}$$

and C may depend on M. We fix ν and $\lambda \in \operatorname{supp}\phi(2^{-r}\cdot)$ and define an oscillatory integral operator by

$$\mathcal{P}_{\lambda,\nu}g(u) = \sum_{\substack{(j,\ell)\in\mathfrak{A}_s\\ \ell=r-n-2j}} \rho_\ell(u)\int\Psi_{j,\nu,\lambda}(u,w)e^{-i\lambda a(u)(u-w)}g(w)dw\,.$$

The left-hand side of (3.7) is bounded by a linear combination of terms of type

$$\int_{\mathcal{J}_2(m_2)}|x_2-y_2|^{-M-1}\int\left\|\mathcal{P}_{\lambda,\nu}\left[f(\cdot,y_2)\right]\right\|_{L^2(\mathbb{R})}d\lambda dy_2dx_2\,;$$

note also that $\mathcal{P}_{\lambda,\nu}[f(\cdot,y_2)] = 0$ if $2^{-r}\lambda \notin \operatorname{supp}\phi$. The operator norm of $\mathcal{P}_{\lambda,\nu}$ is bounded by $\min\{2^{-s/2},2^{-n/2}\}(s+n+1)A_\nu$; this follows from Lemma 2.2. Therefore, we obtain

$$\begin{aligned}&\int_{\mathcal{J}_2(m_2)}\left(\int|T^r_{sn}f(x)|^2dx_1\right)^{1/2}dx_2\\&\quad\le C_M 2^{n/2}(n+s+1)\left(\frac{2^{n-m_2}}{2^r|J_2|}\right)^M\min\left\{1,2^{(n-s)/2}\right\}\int\left(\int|f(x_1,x_2)|^2dx_1\right)^{1/2}dx_2\,.\end{aligned}$$

This proves one of the estimates claimed in (3.7). If we also use the cancellation of the atom in the y_2 variable we may replace the term $\Gamma_{\lambda,M+1}(x_2-y_2)$ in (3.8) by

$$\Gamma_{\lambda,M+1}(x_2-y_2)-\Gamma_{\lambda,M+1}(x_2-u_2) = O\left(|J_2|\,|x_2-y_2|^{-M-1}\left[|x_2-y_2|^{-1}+|\lambda|\right]\right)$$

and the previous argument yields the second estimate in (3.7), with the factor $2^r|J_2|+2^{-m_2}$. The same argument applies to the adjoint operator. □

Lemma 3.4. *Let f be a rectangle atom associated to the rectangle $R = J_1\times J_2$ with center (u_1,u_2). Let $M_1 > 0$ and let $\mathcal{R}(m)$ be as in (3.3). Assume $|I_\ell| \le 2^{-j}$. Then $T^r_{j\ell}f(x) = (T^r_{j\ell})^*f(x) = 0$ if $x\in\mathcal{R}(m)$ and $2^j|J_1| > 2^{-m_1}$.*

If $r = \ell+2j+n$ and $(j,\ell)\in\mathfrak{A}_s$ then for $0\le\theta_1,\theta_2\le 1$

$$\int_{\mathcal{R}(m)}\left|T^r_{j\ell}f(x)\right|dx \;\le\; C2^{-s}\left(1+2^{n-s}\right)^{\theta_1}\left(2^j|J_1|\right)^{\theta_1}\left(2^r|J_2|\right)^{\theta_2} \tag{3.9}$$

$$\int_{\mathcal{R}(m)}\left|\left(T^r_{j\ell}\right)^*f(x)\right|dx \;\le\; C\left(1+2^n+2^s\right)^{\theta_1}\left(2^j|J_1|\right)^{\theta_1}\left(2^r|J_2|\right)^{\theta_2}\;; \tag{3.10}$$

moreover if also $2^r|J_2| \geq 10b^{-1}2^{n-s}2^{-m_2}$ *then*

$$\int_{\mathcal{R}(m)} \left|T^r_{j\ell} f(x)\right| dx \leq C2^{-s}\left(1+2^{n-s}\right)^{\theta_1}\left(2^j|J_1|\right)^{\theta_1}\left(2^{m_2}2^r|J_2|\right)^{-1} \tag{3.11}$$

$$\int_{\mathcal{R}(m)} \left|\left(T^r_{j\ell}\right)^* f(x)\right| dx \leq C\left(1+2^n+2^s\right)^{\theta_1}\left(2^j|J_1|\right)^{\theta_1}\left(2^{m_2}2^r|J_2|\right)^{-1}. \tag{3.12}$$

Proof. The first statements are obvious and we give the proof for (3.9) through (3.12). It suffices to prove these inequalities for $\theta_1, \theta_2 \in \{0, 1\}$; the general case then follows by taking geometric means. Denote by $K^r_{j\ell}$ and $\widetilde{K}^r_{j\ell}$ the kernels of $T^r_{j\ell}$ and $(T^r_{j\ell})^*$, respectively. Then $\widetilde{K}^r_{j\ell}(x,y) = \overline{K^r_{j\ell}(y,x)}$ and

$$K^r_{j\ell}(x,y) = \rho_\ell(x_1)\,\psi_j(x_1-y_1)\,2^r\mathcal{F}^{-1}\phi\left(2^r\left(x_2-y_2-a(x_1)(x_1-y_1)\right)\right)$$

where $\mathcal{F}^{-1}\phi$ is the inverse Fourier transform of ϕ. Let $\omega_{r,M}(x,y) = 2^r(1+2^r|x_2-y_2-a(x_1)(x_1-y_1)|)^{-M}$. Then it is straightforward to check from (1.11) through (1.13) that for $\theta_1, \theta_2 \in \{0, 1\}$

$$\left|\partial^{\theta_1}_{y_1}\partial^{\theta_2}_{y_2} K^r_{j\ell}(x,y)\right| \leq C2^j\left(2^j+2^{r-\ell}|I_\ell|\right)^{\theta_1}2^{r\theta_2}\omega_{r,M}(x,y)$$

$$\left|\partial^{\theta_1}_{y_1}\partial^{\theta_2}_{y_2} \widetilde{K}^r_{j\ell}(x,y)\right| \leq C2^j\left(2^j+2^{r-\ell-j}+|I_\ell|^{-1}\right)^{\theta_1}2^{r\theta_2}\omega_{r,M}(y,x).$$

Since $K^r_{j\ell}(x,y) = 0$ if $|x_1-y_1| \geq C2^{-j}$ or $x_1 \notin \operatorname{supp}\rho_\ell$ we use the cancellation properties of the atom to obtain

$$\int \left|T^r_{j\ell} f\right| dx \leq C2^{-s}2^j|I_\ell|\left(1+2^{r-\ell-2j}\right)^{\theta_1}\left(2^j|J_1|\right)^{\theta_1}\left(2^r|J_2|\right)^{\theta_2}$$

$$\int \left|\left(T^r_{j\ell}\right)^* f\right| dx \leq C\left(1+2^{r-\ell-2j}+2^{-j}|I_\ell|^{-1}\right)^{\theta_1}\left(2^j|J_1|\right)^{\theta_1}\left(2^r|J_2|\right)^{\theta_2}$$

which implies (3.9) and (3.10).

Note that if also $2^r|J_2| \geq 10b^{-1}2^{n-s}2^{-m_2}$, then $2^{m_2}|J_2| \geq b^{-1}2^{-\ell-j}|I_\ell|$ and therefore

$$\omega_{r,M}(x,y)+\omega_{r,M}(y,x) \leq C_M 2^r\left(1+2^r|x_2-y_2|\right)^{-M}$$

for $x \in \mathcal{R}(m)$, $y \in R$. Now the previous argument also yields (3.11) and (3.12). □

We now decompose $\mathcal{T}_{sn} = \sum_j \mathcal{T}_{j,s,n}$ where

$$\mathcal{T}_{j,s,n} = \sum_{\ell:(j,\ell)\in\mathfrak{A}_s} T^{\ell+2j+n}_{j\ell}.$$

The proof of the following lemma is similar to the proof of Lemma 3.4.

Lemma 3.5. *Let f be a rectangle atom associated to the rectangle $R = J_1 \times J_2$ with center (u_1, u_2) and let $\mathcal{J}_1(m_1)$ be as in (3.2). Assume $|I_\ell| \leq 2^{-j}$. Then $\mathcal{T}_{j,s,n}f(x) = 0$ if $x \in \mathcal{R}(m)$ and $2^j|J_1| > 2^{-m_1}$; moreover for $0 \leq \theta \leq 1$*

$$\int_{\mathcal{J}_1(m_1)}\left(\int\left|\mathcal{T}_{j,s,n}f(x_1,x_2)\right|^2dx_2\right)^{1/2}dx_1$$

$$\leq C2^{-s}(1+2^{n-s})^\theta\left(2^j|J_1|\right)^\theta\int\left(\int|f(y)|^2dy_2\right)^{1/2}dy_1 \tag{3.13}$$

$$\int_{\mathcal{J}_1(m_1)} \left(\int \left| \mathcal{T}^*_{j,s,n} f(x_1, x_2) \right|^2 dx_2 \right)^{1/2} dx_1$$
$$\leq C \left(1 + 2^n + 2^s\right)^\theta \left(2^j |J_1|\right)^\theta \int \left(\int |f(y)|^2 dy_2 \right)^{1/2} dy_1 \tag{3.14}$$

Proof. The first statement is obvious. Let $\mathcal{E}(x_1, y_1, \lambda) = \rho_\ell(x_1)\psi_j(x_1 - y_1)e^{i\lambda a(x_1)(x_1 - y_1)}$, then

$$\begin{aligned} |\mathcal{E}(x_1, y_1, \lambda) - \mathcal{E}(x_1, u_1, \lambda)| &\leq C2^j \left(2^j + 2^{-\ell} |I_\ell| |\lambda|\right) |J_1| \\ |\mathcal{E}(x_1, y_1, \lambda) - \mathcal{E}(u_1, y_1, \lambda)| &\leq C2^j \left(|I_\ell|^{-1} + 2^j + 2^{-\ell-j} |\lambda|\right) |J_1| \end{aligned}$$

Note that in the present case, if $|\lambda| \approx 2^r$ then $2^{-\ell}|I_\ell||\lambda| \leq C2^{j+n-s}$, $2^{-\ell-j}|\lambda| \leq 2^{j+n}$ and $|I_\ell|^{-1} \leq 2^{j+s}$.

Let $\mathcal{F}_2 f$ denote the Fourier transform of f in the second variable. If $2^j|J_1| \leq 1$, we use the cancellation of f in the y_1 variable and we obtain the estimate

$$\left(\int \left| \mathcal{T}_{j,n,s} f(x_1, x_2) \right|^2 dx_2 \right)^{1/2} \leq$$
$$C \min\left\{1, \left(1 + 2^{n-s}\right) 2^j |J_1|\right\} \int \left(\int \left| \sum_r \phi\left(2^{-r}\lambda\right) \mathcal{F}_2 f(y_1, \lambda) \right|^2 d\lambda \right)^{1/2} dy_1$$

where the sum is extended over all r that can be written as $r = \ell + 2j + n$ with $(j, \ell) \in \mathfrak{A}_s$. Also note that the expression on the left-hand side is supported on I_ℓ. We apply Plancherel's theorem and perform the x_1 integration to arrive at (3.13), with $\theta = 1$. The general case follows by taking geometric means. A similar argument also yields (3.14). □

Proof of Proposition 3.1. Since (3.4) implies (3.1) we only have to prove the estimate for rectangle atoms by Fefferman's theorem. This in turn follows from the above lemmas by applications of the Cauchy–Schwarz inequality and by summing geometric series. Specifically we use Lemma 3.2 for $\mathcal{T}_{sn}$ if $m_1 + m_2 \leq 10 + (n+1)(1+\epsilon)$. For $m_1 \leq 10$ and $m_2 \geq (n+1)(1+\epsilon)$ we estimate the operators $\mathcal{T}^r_{sn}$ and their adjoints and then sum in r. Here we use Lemma 3.2 if $2^r|J_2| \leq 2^{-m_2}$, Lemma 3.3 with $M = 0$ if $2^{-m_2} \leq 2^r|J_2| \leq 2^{-2m_2\epsilon}$, and Lemma 3.3 with $M = 10/\epsilon$ if $2^r|J_2| \geq 2^{-2m_2\epsilon}$.

For $m_2 \leq 10$ and $m_1 \geq (n+1)(1+\epsilon)$ we estimate the operators $\mathcal{T}_{j,s,n}$ and $\mathcal{T}^*_{j,s,n}$ and then sum in j. Only terms with $2^j|J_1| \leq C2^{-m_1}$ will occur and the desired estimate follows from Lemma 3.5, with $\theta = \epsilon$.

For $m_2 \geq 10$ and $m_1 \geq (n+1)(1+\epsilon)$, we estimate $T^r_{j\ell}$ with $\ell = r - 2j - n$, $(j, \ell) \in \mathfrak{A}_s$ using Lemma 3.4 with $\theta_1 = \epsilon$ and sum in r,j; again only terms with $2^j|J_1| \leq C2^{-m_1}$ will occur. We consider $\mathcal{T}_{sn}$ and distinguish two cases, depending on whether $2^{n-m_2/2}10b^{-1}$ is large or small. In the first case where $2^{n-m_2/2}10b^{-1} \geq 1$ we also have $2^{m_2} \leq C2^{2n}$ and we use (3.9) with $\theta_2 = 1$ if $2^r|J_2| \leq 2^{-n}$, (3.9) with $\theta_2 = 0$ if $2^{-n} < 2^r|J_2| \leq 10b^{-1}2^n$, and (3.11) if $2^r|J_2| < 10b^{-1}2^n$. In the second case where $2^{n-m_2/2}10b^{-1} < 1$ we use (3.9) with $\theta_2 = 1$ if $2^r|J_2| \leq 2^{-n-m_2/2}10b^{-1}$, (3.11) with $\theta_2 = 1$ if $2^r|J_2| > 2^{-n-m_2/2}10b^{-1}$. Finally, this analysis also applies to the operator $(T^r_{j\ell})^*$ if in the previous argument we replace (3.9) by (3.10) and (3.11) by (3.12). □

Remarks.

(i) It should be possible to extend our result to cover similar classes of vector fields in $\mathbb{R}^n$. Instead of Fefferman's theorem one would have to use the version of Calderón–Zygmund theory in [5]. In our two-dimensional setting we used Fefferman's theorem for convenience, but we verified in effect the hypotheses of Theorem 1 in [5].

(ii) There is the open problem of L^p boundedness for the Hilbert transform associated to an arbitrary C^∞ vector field. As a first step one might try to find a version of our theorem for vector fields v which do not necessarily depend on only one variable.

(iii) It would be interesting if there is an underlying Calderón–Zygmund theory for our operators that is different from the product theory. In a different context such variants have been considered in [6].

4. The maximal operator

The arguments in the previous sections apply equally well to prove the L^p boundedness for the maximal operator $\mathfrak{M}$; in fact, some of those arguments simplify. Let Ψ be a nonnegative C^∞ function with support in $(1/2, 2)$ and assume that $\Psi(t) = 1$ for $t \in (1/\sqrt{2}, \sqrt{2})$. Let $\Psi_j(t) = 2^j\delta^{-1}\Psi(2^j\delta^{-1}t)$. Then it is straightforward to see that

$$\mathfrak{M}f(x) \le C \sup_j \sum_\ell \rho_\ell(x_1) \int \Psi_j(t) \, |f(x_1 - t, x_2 - ta(x_1))| \, dt$$

and we may clearly assume that f is nonnegative. Then the estimate

$$\left(\int \left| \sum_\ell \rho_\ell(x_1) \sup_{2^{-j} \le |I_\ell|} \int \Psi_j(t) f(x_1 - t, x_2 - ta(x_1)) \, dt \right|^p dx \right)^{1/p} \le C\|f\|_p \tag{4.1}$$

follows by the rescaling argument in Lemma 1.2 and known estimates for maximal operators in the case of nonvanishing rotational curvature.

Let $S^r_{j\ell}$ be defined as $T^r_{j\ell}$ in (1.14), but with ψ_j replaced by Ψ_j. For $k = 0, 1, \ldots$ define

$$S^r_{j\ell k}f(x) = \rho_\ell(x_1)\int \left(2^r a(x_1)(x_1 - y_1)\right)^k \Psi_j(x_1 - y_1) \, f(y) \int \left(2^{-r}\mu\right)^k \phi\left(2^{-r}\mu\right) e^{i\mu[x_2 - y_2]} d\mu \, dy$$

so that $S^r_{j\ell} = \sum_{k=0}^\infty (-i)^k (k!)^{-1} S^r_{j\ell k}$. In order to complete the proof we have to show that

$$\left\| \left(\sum_j \left| \sum_{\ell: 2^{-j} > |I_\ell|} \rho_\ell \sum_{r \ge 2j+\ell} S^r_{j\ell} f \right|^2 \right)^{1/2} \right\|_p \le C_p\|f\|_p \tag{4.2}$$

$$\left\| \sum_\ell \rho_\ell \sup_{j: 2^{-j} > |I_\ell|} \left| \sum_{r \le 2j+\ell} S^r_{j\ell k} f \right| \right\|_p \le C_p B^k \|f\|_p \tag{4.3}$$

Note that the cancellation of ψ was not used in the estimates for $\mathcal{T}$ and in fact straightforward modifications of the arguments in Sections 2 and 3 also yield (4.2). In order to see (4.3) we argue as in the proof of Lemma 1.3. Let M_1, M_2 be the Hardy–Littlewood maximal operators acting in the first and the second variable, respectively, and let

$$\Gamma_k f(x) = \sup_m \left| \sum_{r<m} \widetilde{L}_{r,k} L_r f(x) \right|$$

where L_r, $\widetilde{L}_{r,k}$ are as in the proof of Lemma 1.3. Then Cotlar's inequality [24, p. 35] applies:

$$|\Gamma_k f(x)| \leq C M_2 f(x) + C M_2 \left[\sum_{r=-\infty}^{\infty} \widetilde{L}_{r,k} L_r f \right](x) ;$$

moreover,

$$\sum_\ell \rho_\ell(x_1) \sup_{2^{-j} > |I_\ell|} \left| \sum_{r \leq 2j+\ell} S^r_{j\ell k} f(x) \right| \leq C 10^k M_1 [\Gamma_k f](x) .$$

Since the operator $\sum_{r=-\infty}^{\infty} \widetilde{L}_{r,k} L_r$ is bounded on L^p with norm $O(c_p B^k)$ and suitable B the two previous inequalities imply (4.3). The asserted estimate for the maximal operator $\mathfrak{M}$ follows from (4.1), (4.2), and (4.3).

5. The case of nonvanishing rotational curvature, revisited

We consider the operator defined for smooth functions by

$$Tf(x) = \chi(x_1) \sum_{j \geq 0} \int \Psi_j(x_1, y_1) f(y_1, x_2 + S(x_1, y_1)) \, \chi(y_1) \, dy_1 . \tag{5.1}$$

Here χ and Ψ_j are C^2 functions; χ is supported in the interval $[-1, 1]$, and $\Psi_j(x_1, y_1) = 0$ unless $2^{-j-3} \leq |x_1 - y_1| \leq 2^{-j+3}$. We assume that (2.2) holds and that Ψ_j has the additional cancellation property

$$\int \Psi_j(x, y) dy = \int \Psi_j(x, y) dx = 0 . \tag{5.2}$$

As a model case for S we consider the example $S(x_1, y_1) = -a(x_1)(x_1 - y_1)$, and with the appropriate choice of Ψ_j we recover a local version of the Hilbert transform in (1.1). The assumption of rotational curvature is that the mixed derivative $S_{x_1 y_1}$ does not vanish from below.

Proposition 5.1. *Suppose that S is a C^1 function on $[-1, 1]^2$ and assume that the partial derivatives $S_{x_1 y_1}$, $S_{x_1 y_1 y_1}$, $S_{x_1 y_1 y_1 y_1}$ exist and are continuous in $[-1, 1]$. Assume that $S_{x_1 y_1}$ does not vanish in $[-1, 1]$. Then T extends to a bounded operator on L^p, $1 < p < \infty$.*

As previously mentioned the proof is quite standard, and we shall be sketchy. If ϕ is as in (1.14), then we define $\zeta_r(x, y) = 2^r \mathcal{F}^{-1}[\phi](2^r(x_2 - y_2 + S(x_1, y_1)))$ and $\Theta_k = 1 - \sum_{r>k} \zeta_r$. Then $T = \sum_{n=1}^{\infty} T_{1,n} + T_2$ where

$$T_{1,n} f(x) = \sum_{j \geq 0} \int \chi(x_1) \chi(y_1) \Psi_j(x_1, y_1) \zeta_{2j+n}(x, y) f(y) dy$$

$$T_2 f(x) = \sum_{j \geq 0} \int \chi(x_1) \chi(y_1) \Psi_j(x_1, y_1) \Theta_{2j}(x, y) f(y) dy .$$

It turns out that for $1 < p \le 2$

$$\left\|T_{1,n}f\right\|_p \le C_p n^{-1+2/p} 2^{-n(1-1/p)} \|f\|_p \tag{5.3}$$

$$\|T_2 f\|_p \le C_p \|f\|_p \tag{5.4}$$

and that the same estimates hold for the adjoint operators. This of course proves Proposition 5.1. □

By Lemma 2.3 the case $p = 2$ can be reduced to estimates for certain oscillatory integral operators in one dimension. Let λ be fixed, $|\lambda| \ge 1/2$, and define the operator

$$P_j g(u) = \chi(u) \int e^{i\lambda S(u,w)} \Psi_j(u, w) \chi(w) g(w)\, dw\,.$$

For the first result we assume that Ψ_j is as above, but we do not actually need the cancellation condition (5.2).

Lemma 5.2. *Suppose that S is a C^1 function on $[-1, 1]^2$ and assume that the partial derivatives S_{uw}, S_{uww}, S_{uwww} exist and are continuous in $[-1, 1]^2$. Assume that S_{uw} does not vanish in $[-1, 1]^2$. Then for $2^{2j} \le |\lambda|$ the $L^2 \to L^2$ operator norm of P_j is bounded by $CA2^j|\lambda|^{-1/2}$.*

Proof. This is a version of the argument in Lemma 2.2. One writes out the kernel $K_j(u, z)$ of the operator $P_j P_j^*$, and integrates by parts twice if $|u-z| \ge 2^j|\lambda|^{-1}$. If $\Phi(u, w, z) = S(u, w) - S(z, w)$, then our assumptions guarantee that $|\Phi_w(u, w, z)|$ is bounded below by $c|u - z|$ and that Φ_{ww} and Φ_{www} are $O(|u - z|)$. Therefore, a consequence of the integration by parts is the pointwise estimate

$$\left|K_j(u, z)\right| \le 2^j \left(1 + \left|\lambda 2^{-j}(u - z)\right|^2\right)^{-1}$$

and the desired estimate follows by Schur's Lemma. □

In the next lemma we use the cancellation of the Ψ_j but not the assumption of rotational curvature.

Lemma 5.3.

Suppose that Ψ_j is as above and satisfies the additional cancellation property (5.2). Suppose that S is a C^1 function on $[-1, 1]^2$ and assume that the partial derivative S_{uw} exists and is continuous in $[-1, 1]$. Then the operator $\sum_{2^{2j} \ge \lambda} P_j$ is bounded on L^2.

Proof. We verify that $\|P_j^* P_k\| + \|P_j P_k^*\| \le 2^{-|j-k|}$, provided that $2^{2j} \ge |\lambda|$, $2^{2k} \ge |\lambda|$. We may assume $j \ge k$. The kernel of $P_j^* P_k$ is given by

$$K_{jk}(u, z) = \overline{\chi(u)}\chi(z) \int q_k(u, z, w)\Psi_j(u, w)dw$$

where $q_k(u, z, w) = e^{i\lambda[S(u,w)-S(z,w)]}|\chi(w)|^2\overline{\Psi_k(z, w)}$. Observe that for $u, z \in \operatorname{supp}\rho$, $|w - u| \le 2^{-j}$, $|w - z| \le 2^{-k}$ we have $|S_y(u, w) - S_y(z, w)| \le C2^{-k}$ and, since $\lambda 2^{-2k} \le 1$,

$$|q_k(u, z, w) - q_k(u, z, u)| \le C2^k + \left|\lambda\left(S_y(u, w) - S_y(z, w)\right)\right| \le C'2^k\,.$$

Now using the cancellation of Ψ_j in the second variable we see that $\int |K(u, z)|dz \le 2^{-j+k}$ and $\int |K(u, z)|du \le 2^{-j+k}$ and the desired estimate for $P_j^* P_j$ follows.

Next, the kernel of $P_j P_k^*$ is given by

$$L_{jk}(u,z) = \chi(u)\overline{\chi(z)} \int r_k(u,z,w)\overline{\Psi_j(w,u)}dw$$

where $r_k(u,z,w) = |\chi(w)|^2 e^{i\lambda(S(w,u)-S(w,z))}\Psi_k(w,z)$. The desired estimate follows from the cancellation of Ψ_j in the first variable since $|\partial_w r_k| = O(|\lambda|2^{-k}+1) = O(2^k)$. □

The L^2 estimates for $T_{1,n}$ and T_2 immediately follow from the two previous lemmas and Lemma 2.3. In order to show the L^p estimates, one shows that T_2 and its adjoint are of weak type (1, 1), moreover $T_{1,n}$ and its adjoint satisfy a weak-type inequality with constant $O(n)$. From this the L^p estimates follow by the Marcinkiewicz interpolation theorem.

The weak-type estimates rely on Calderón–Zygmund theory in $[-1,1]\times\mathbb{R}$ which is made into a suitable space of homogeneous type (cf. [24, Ch. I]). The underlying distance function is $d(x,y) = |x_1-y_1| + |x_2-y_2+S(x_1,y_1)|^{1/2}$, with the balls $B(y,\delta) = \{x : d(x,y) < \delta\}$. Our assumption is that $S \in C^1$ and the mixed derivative $S_{x_1y_1}$ exists and is continuous. The standard properties of this metric were derived in [14], in a more general context; see also [19]. In particular d is essentially symmetric, $d(x,y) \approx d(y,x)$. Let $\mathcal{K}_{j,n}(x,y) = \chi(x_1)\chi(y_1)\Psi_j(x_1,y_1)\zeta_{2j+n}(x,y)$ and $\mathcal{L}_j(x,y) = \chi(x_1)\chi(y_1)\Psi_j(x_1,y_1)\Theta_{2j}(x,y)$. It is a straightforward exercise to verify that for suitable large D and for $y' \in B(y,\delta)$

$$\begin{aligned}\int_{\mathbb{R}^2\setminus B(y,D\delta)} \left|\mathcal{K}_{j,n}(x,y') - \mathcal{K}_{j,n}(x,y)\right| dx &\le C\min\left\{1, 2^n 2^j\delta, 2^n 2^{-j}\delta^{-1}\right\}\\ \int_{\mathbb{R}^2\setminus B(y,D\delta)} \left|\mathcal{L}_j(x,y') - \mathcal{L}_j(x,y)\right| dx &\le C\min\left\{2^j\delta, 2^{-j}\delta^{-1}\right\} ;\end{aligned}$$

we omit the details. This implies the asserted weak-type estimates for $T_{1,n}$, T_2 and by the symmetry of the situation the estimates for the adjoints follow in the same way.

Similar considerations can be applied to the analogous maximal operator, defined by

$$Mf(x) = \sup_j \left|A_j f(x)\right| \tag{5.5}$$

where

$$A_j f(x) = \chi(x_1) \int \Phi_j(x_1,y_1)\, f(y_1, x_2 + S(x_1,y_1))\, \chi(y_1)\, dy_1 ;$$

here S satisfies the assumptions of Proposition 5.1, and Φ_j is as Ψ_j above, but does not necessarily have any cancellation property. Let δ_0 be an even Schwartz function on the real line such that $\widehat{\delta_0}(\lambda) = 1$ for $|\lambda| \le 1$. Let

$$B_j f(x) = \chi(x_1) \iint \Phi_j(x_1,y_1)\, 2^{2j}\delta_0\left(2^{2j}y_2\right) f(y_1, x_2 - y_2 + S(x_1,y_1))\, \chi(y_1)\, dy_1 dy_2 ;$$

then

$$Mf(x) \le \sup_j \left|B_j f(x)\right| + \left(\sum_{j\ge 0}\left|A_j f(x) - B_j f(x)\right|^2\right)^{1/2} . \tag{5.6}$$

The maximal function $\sup|B_j f|$ is pointwise controlled by the Hardy–Littlewood maximal function with respect to the nonisotropic balls $B(y,\delta)$ defined above; it is bounded on L^p for $1 < p \le \infty$. The square-function in (5.6) can be considered as the ℓ^2 norm of a vector valued singular integral and the L^p boundedness follows as above.

6. Appendix

6.1. The Hilbert transform in the radial direction

We now study the operators H and M for the radial vector field $v(x) = x/|x|$, in d dimensions, $d \geq 2$, i.e.,

$$Hf(x) = \text{p.v.} \int_{-\infty}^{\infty} f(x + tx/|x|) \frac{dt}{t} \tag{6.1}$$

and the maximal operator M defined by

$$Mf(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x + tx/|x|)|\, dt\ . \tag{6.2}$$

For this example the critical exponent for L^p boundedness turns out to be the dimension d, and for $p = d$ we prove a restricted weak type inequality (for a similar result on the Kakeya maximal operator acting on radial functions see [4]). In what follows let $L^{p,q}$ denote the Lorentz space.

Proposition. *Let H and M be as in (6.1), (6.2), respectively. Then H is bounded on $L^p(\mathbb{R}^d)$ if and only if $d < p < \infty$. M is bounded on $L^p(\mathbb{R}^d)$ if and only if $d < p \leq \infty$.*

Moreover, H and M map $L^{d,q}(\mathbb{R}^d)$ to $L^{d,r}(\mathbb{R}^d)$ if and only if $q = 1$ and $r = \infty$.

Proof. The proof of these results is elementary. One introduces polar coordinates to reduce matters to standard estimates for Hilbert transforms, maximal operators, and Hardy operators in one dimension. We shall give only the proof for the operator H. The proof for the maximal operator M is similar.

We split

$$H = H_1 + H_2 + H_3$$

where

$$\begin{aligned} H_1 f(x) &= \text{p.v.} \int_{|t|\geq 4|x|} f(x + tx/|x|) \frac{dt}{t} \\ H_2 f(x) &= \text{p.v.} \int_{-|x|/4}^{4|x|} f(x + tx/|x|) \frac{dt}{t} \\ H_3 f(x) &= \int_{-4|x|\leq t \leq -|x|/4} f(x + tx/|x|) \frac{dt}{t}\ . \end{aligned}$$

We first show that H_1 is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. For $l = 0, 1, 2, \ldots$ set

$$H_{1,l} f(x) = \int_{2^{l+2}|x|\leq |t| \leq 2^{l+3}|x|} f(x + tx/|x|) \frac{dt}{t}\ ,$$

then $H_1 = \sum_{l=0} H_{1,l}$. Let $F_p(s, \theta) = f(s\theta)s^{(d-1)/p}$ and let $\mathcal{M}_1$ denote the Hardy–Littlewood maximal operator in the s-variable. Then

$$\begin{aligned} \|H_{1,l} f\|_q &\leq \left(\iint_{S^{d-1}\times\mathbb{R}^+} \left[\int_{2^{l+2}r\leq|t|\leq 2^{l+3}r} |f((r+t)\theta)| \frac{dt}{t} \right]^p r^{d-1}\, dr\, d\theta \right)^{1/p} \\ &\leq C 2^{-l(d-1)/p} \left(\iint_{S^{d-1}\times\mathbb{R}^+} \left[\int_{2^{l+2}r\leq|t|\leq 2^{l+3}r} \right.\right. \end{aligned}$$

$$\left|f((r+t)\theta)(r+t)^{(d-1)/p}\right| \frac{dt}{|t|}\Bigg]^p dr\,d\theta\Bigg)^{1/p}$$

$$\leq C2^{-l(d-1)/p}\left(\iint_{S^{d-1}\times\mathbb{R}^+}\left[\mathcal{M}_1\left[F_p(\cdot,\theta)\right](r)\right]^p dr\,d\theta\right)^{1/p}$$

$$\leq C2^{-l(d-1)/p}\left(\iint_{S^{d-1}\times\mathbb{R}^+}\left|F_p(r,\theta)\right|^p dr\,d\theta\right)^{1/p} \leq C'2^{-l(d-1)/p}\|f\|_p$$

and the L^p boundedness of H_1 follows.

Next, we show that H_2 is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. For a function of two variables denote by H_* the maximal Hilbert transform in the first variable. Let χ_k be the characteristic function of the interval $[2^{k-3}, 2^{k+4}]$. Let $F_{k,p}(s,\theta) = 2^{k(n-1)/p} f(s\theta)\chi_k(s)$. Then

$$\begin{aligned}\|H_2 f\|_p &\leq \left(\sum_k \int_{S^{d-1}}\int_{2^k}^{2^{k+1}}\left|\text{p.v.}\int_{-|x|/4}^{4|x|} f(x+t\theta)\frac{dt}{t}\right|^p r^{d-1}dr\,d\theta\right)^{1/p}\\ &\leq C\left(\int_{S^{d-1}}\sum_k\int\left|H_*F_{k,p}(r,\theta)+\mathcal{M}_1\left(F_{k,p}\right)(r,\theta)\right|^p dr\,d\theta\right)^{1/p}\\ &\leq C\left(\int_{S^{d-1}}\sum_k\int\left|F_{k,p}(s,\theta)\right|^p ds\,d\theta\right)^{1/p} \leq C'\|f\|_p\,.\end{aligned}$$

Finally we estimate H_3 where the restriction $p > d$ is needed. Observe that

$$\begin{aligned}\|H_3 f\|_q &\leq \left(\iint_{S^{d-1}\times\mathbb{R}^+}\left[\frac{4}{r}\int_{-4r}^{-r/4}|f((r+t)\theta)|\,dt\right]^p r^{d-1}\,dr\,d\theta\right)^{1/p}\\ &\leq 2\left(\iint_{S^{d-1}\times\mathbb{R}^+}\left[\frac{4}{r}\int_0^{4r}|f(s\theta)|\,ds\right]^p r^{d-1}\,dr\,d\theta\right)^{1/p}.\end{aligned}$$

Let for $j = 0, 1, \ldots$

$$S_j g(r) = \frac{1}{r}\int_{2^{-j+1}r\leq|s|\leq 2^{-j+2}r} g(s)\,ds.$$

Then

$$\begin{aligned}\left(\int_0^\infty \left|S_j g(r)\right|^p r^{d-1}dr\right)^{1/p} &\leq C2^{-j(1-1/p)}\left(\int_0^\infty r^{d-2}\int_{2^{-j+1}r}^{2^{-j+2}r}|g(s)|^p\,ds dr\right)^{1/p}\\ &\leq C2^{-j(1-d/p)}\left(\int_0^\infty |g(s)|^p s^{d-1}ds\right)^{1/p}.\end{aligned}$$

Now for $f \in L^p(\mathbb{R}^d)$ define $H_{3,j}$ by $H_{3,j}f(r\theta) = S_j[f(\cdot\theta)](r)$. Then $|H_3 f(r\theta)| \leq \sum_{j=0}^\infty |H_{3,j}[|f|](r\theta)|$ and $H_{3,j}$ is bounded on $L^p(\mathbb{R}^d)$ with operator norm $\leq C2^{j(-1+d/p)}$. This implies the asserted L^p estimate for $p > d$. It also implies that H_3 is of restricted weak type (d, d), that is T_3 maps $L^{d,1}$ into $L^{d,\infty}$, see Section 6.2 below.

We now turn to the necessary conditions. It is easy to see that H does not map L^∞ to L^∞. In order to check the sharpness of the L^p estimates, we test H on the characteristic function χ of the ball of radius 1, centered at the origin. Then $\|f\|_p \le C$ and $|Hf(x)| \ge c|x|$ for $|x| \ge 2$. This implies that L^p boundedness only holds for $p > d$; moreover, if H maps $L^{d,q}$ to $L^{d,r}$ then necessarily $r = \infty$. We still have to show that $L^{d,q} \to L^{d,\infty}$ boundedness can hold only for $q = 1$. Since by interpolation the above estimates show that H_1 and H_2 are bounded on all $L^{p,q}$ spaces for $1 < p < \infty$, it suffices to consider H_3. For large N define $f_N(x) = 1/|x|$ if $1 \le |x| \le N$ and $f_N(x) = 0$ otherwise. Then $\|f_N\|_{L^{d,q}} \approx [\log N]^{1/q}$ and for $10 \le |x| \le N/2$ we have $|H_3 f_N(x)| \ge c|x|^{-1}\log N$. This shows that $\|H_3 f_N\|_{L^{d,\infty}}/\|f_N\|_{L^{d,q}} \ge C[\log N]^{1-1/q}$. Now if H is bounded from $L^{d,q}$ to $L^{d,\infty}$, then H_3 is bounded from $L^{d,q}$ to $L^{d,\infty}$ and this can only happen if $q = 1$. □

Remark. One may construct a C^∞ vector field which coincides with $v(x) = x/|x|$ if $|x| \ge 1$ and $|x_d| \ge |x|/2$. There are the same obstructions to L^p boundedness as for the radial vector field and in fact L^p boundedness for the Hilbert transform (1.1) will fail if $p \le d$. The same remark applies to the maximal function (1.2). These obstructions are not present if one considers local versions of the Hilbert transform or the maximal operator.

6.2. An interpolation lemma

Suppose $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ are two couples of normed vector spaces, compatible in the sense of interpolation theory. Suppose that we are given a sequence of operators T_j mapping $A_0 + A_1$ to $B_0 + B_1$ such that

$$\|T_j a\|_{B_s} \le M_s 2^{j\alpha_s} \|a\|_{A_s}, \qquad s = 0, 1 \tag{6.3}$$

where $\alpha_0 < 0 < \alpha_1$. Then it is easy to see that $T = \sum T_j$ maps $A_0 \cap A_1$ to $B_0 + B_1$. In fact if $a \in A_0 \cap A_1$, we obtain

$$\begin{aligned} \Big\|\sum_{j>m} T_j a\Big\|_{B_0} + t\Big\|\sum_{j\le m} T_j a\Big\|_{B_1} &\le \sum_{j>m} M_0 2^{j\alpha_0}\|a\|_{A_0} + t\sum_{j\le m} M_1 2^{j\alpha_1}\|a\|_{A_1} \\ &\le C\left[M_0 2^{m\alpha_0}\|a\|_{A_0} + tM_1 2^{m\alpha_1}\|a\|_{A_1}\right]. \end{aligned} \tag{6.4}$$

Recall the definition of the Peetre K-functional

$$K\left(t, a, \overline{A}\right) = \inf\left\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} : a = a_0 + a_1,\ a_0 \in A_0,\ a_1 \in A_1\right\}$$

and the definition of the real interpolation space $\overline{A}_{\theta,q} = K_{\theta,q}(\overline{A})$ with norm

$$\|a\|_{\overline{A}_{\theta,q}} = \left(\int \left[t^{-\theta} K\left(t, a, \overline{A}\right)\right]^q \frac{dt}{t}\right)^{1/q},$$

with the natural modification in the case $q = \infty$.

If for fixed t we choose m in (6.4) such that $2^{m(\alpha_1-\alpha_0)} \approx M_0\|a\|_{A_0}/(tM_1\|a\|_{A_1})$, we see that for $\theta = \alpha_0/(\alpha_0 - \alpha_1) \in (0, 1)$ and $a \in A_0 \cap A_1$

$$\|Ta\|_{\overline{B}_{\theta,\infty}} = \sup_{t>0} t^{-\theta} K\left(t, Ta, \overline{B}\right) \le C M_0^{1-\theta} M_1^{\theta} \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta}. \tag{6.5}$$

This inequality is an extension of an inequality implicitly in [2], for L^p spaces. For the concrete case $A_s = B_s = L^{p_s}$, $s = 0, 1$ we may apply (6.5) for a being the characteristic function of a measurable set and then (6.5) becomes a restricted weak type inequality. This implies [26, Ch. V] that T maps the Lorentz space $L^{p,1}$ into $L^{p,\infty}$ if $(1-\theta)/p_0 + \theta/p_1 = 1/p$ and $\theta = \alpha_0/(\alpha_0 - \alpha_1)$.

The following lemma is an abstract extension of this interpolation result. It implies (6.5), since $K_{\theta,1}$ is an interpolation functor of exponent θ (see [1, p. 40].

Lemma. *Let $\{T_j\}$ be a sequence of operators mapping $A_0 + A_1$ to $B_0 + B_1$ and satisfying (6.3), with $\alpha_0 < 0 < \alpha_1$. Let $\theta = \alpha_0/(\alpha_0 - \alpha_1)$. Then $T = \sum T_j$ extends to a bounded operator mapping $\overline{A}_{\theta,1}$ to $\overline{B}_{\theta,\infty}$, with operator norm bounded by $CM_0^{1-\theta}M_1^{\theta}$; here $C = O((\alpha_1 - \alpha_0)2^{(\alpha_1-\alpha_0)\theta})$.*

Proof. Since $A_0 \cap A_1$ is dense in $\overline{A}_{\theta,1}$ (see [1, p. 47]) it suffices to prove the required inequality for $a \in A_0 \cap A_1$. Fix t and for every $j \in \mathbb{Z}$ split $a = a_0^j + a_1^j$ such that

$$\left\|a_0^j\right\|_{A_0} + 2^{j(\alpha_1-\alpha_0)}tM_1M_0^{-1}\left\|a_1^j\right\|_{A_1} \leq 2K\left(2^{j(\alpha_1-\alpha_0)}tM_1M_0^{-1}, a, \overline{A}\right). \tag{6.6}$$

Then

$$\begin{aligned} t^{-\theta}K\left(t, Ta, \overline{B}\right) &\leq t^{-\theta}\left[\left\|\sum_j T_j a_0^j\right\|_{B_0} + t\left\|\sum_j T_j a_1^j\right\|_{B_1}\right] \\ &\leq t^{-\theta}\left[\sum_j M_0 2^{j\alpha_0}\left\|a_0^j\right\|_{A_0} + t\sum_j M_1 2^{j\alpha_1}\left\|a_1^j\right\|_{A_1}\right] \\ &\leq M_0\sum_j\left(2^{j(\alpha_1-\alpha_0}t\right)^{-\theta}\left[\left\|a_0^j\right\|_{A_0} + 2^{j(\alpha_1-\alpha_0)}tM_1M_0^{-1}\left\|a_1^j\right\|_{A_1}\right]. \end{aligned}$$

By (6.6) and the monotonicity of the K functional one easily obtains

$$\begin{aligned} &(2^{j(\alpha_1-\alpha_0)}t)^{-\theta}\left[\|a_0^j\|_{A_0} + 2^{j(\alpha_1-\alpha_0)}tM_1/M_0\|a_1^j\|_{A_1}\right] \\ &\leq \tfrac{2}{\alpha_1-\alpha_0}\tfrac{\alpha_0}{2^{\alpha_0}-1}\int_{2^{j(\alpha_1-\alpha_0)}t}^{2^{(j+1)(\alpha_1-\alpha_0)}t} s^{-\theta}K(sM_1/M_0, a, \overline{A})\tfrac{ds}{s} \end{aligned}$$

and therefore

$$\|Ta\|_{\overline{B}_{\theta,\infty}} \leq CM_0\int_0^{\infty} s^{-\theta}K\left(sM_1/M_0, a, \overline{A}\right)\frac{ds}{s} = CM_0^{1-\theta}M_1^{\theta}\|a\|_{\overline{A}_{\theta,1}}. \qquad \square$$

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