The Journal of Geometric Analysis
Volume 11, Number 2, 2001

Generalized Low Pass Filters and MRA Frame Wavelets

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ABSTRACT. *A tight frame wavelet* ψ *is an* $L^2(\mathbb{R})$ *function such that* $\{\psi_{jk}(x)\} = \{2^{\frac{j}{2}}\psi(2^j x - k), j, k \in \mathbb{Z}\}$, *is a tight frame for* $L^2(\mathbb{R})$. *We introduce a class of "generalized low pass filters" that allows us to define (and construct) the subclass of MRA tight frame wavelets. This leads us to an associated class of "generalized scaling functions" that are not necessarily obtained from a multiresolution analysis. We study several properties of these classes of "generalized" wavelets, scaling functions and filters (such as their multipliers and their connectivity). We also compare our approach with those recently obtained by other authors.*

1. Introduction

We assume the reader is familiar with the *Multiresolution Analysis (MRA)* method for constructing wavelet bases for $L^2(\mathbb{R})$. For the sake of completeness, however, and, also because we need to establish an appropriate notation that allows us to explain the contents of this article and how our results compare with those of other authors, we begin with a brief description of the MRA method.

An MRA consists of a sequence $\{V_j\}$, $j \in \mathbb{Z}$, of closed subspaces of $L^2(\mathbb{R})$ that is increasing, $V_j \subset V_{j+1}$, the members of this sequence are dyadic dilates of, say, V_0 in the sense that $f \in V_j$ if and only if $f(2^{-j}\cdot) \in V_0$, $L^2(\mathbb{R}) = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$ and, lastly, there exists an element φ in V_0 (a *scaling function*) such that the sequence of its integral translates $\varphi_n(x) \equiv \varphi(x-n)$ makes up an orthonormal basis of V_0.

An (orthonormal) *wavelet* is a function $\psi \in L^2(\mathbb{R})$ such that the system $\psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, is an orthonormal basis of $L^2(\mathbb{R})$. The construction of such a ψ from an MRA is rather simple and elegant. If we can produce a ψ in the orthogonal complement, W_0, of V_0, within V_1, such that $\{\psi_k\} = \{\psi(\cdot - k)\}$, $k \in \mathbb{Z}$, is an orthonormal basis of W_0, then the properties of the MRA $\{V_j\}$ easily imply that ψ is a wavelet. We refer the reader to [4] for these

Math Subject Classifications. 42, 42A99.
Key Words and Phrases. filters, wavelets, frames.
Acknowledgements and Notes. This research was supported by a grant from the Southwestern Bell Telephone Company and AFOSR grant F49620-96-1-0287. The first named author's research was also supported by grant 2P03A05814 from KBN.

ISSN 1050-6926

details (as well as other that are related to much of the discussion of wavelets we shall present). In this reference and here the Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx \tag{1.1}$$

for $f \in L^1(\mathbb{R})$.

One obtains the function ψ from the MRA by making use of the fact that associated with each scaling function φ there is a unique 2π periodic function $m \in L^2([0, 2\pi)) \equiv L^2(T)$ such that

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi) \tag{1.2}$$

a.e. A basic property of m is that

$$|m(\xi)|^2 + |m(\xi + \pi)|^2 = 1 \tag{1.3}$$

for a.e. $\xi \in \mathbb{R}$. One can then show that $\psi \in L^2(\mathbb{R})$ satisfying

$$\hat{\psi}(2\xi) = m_1(\xi)\hat{\varphi}(\xi)\,, \tag{1.4}$$

where $m_1(\xi) = e^{i\xi}\overline{m(\xi + \pi)}$, is a wavelet. These properties of m_1 and ψ are straightforward consequences of the fact that we are seeking an element $\psi \in V_1$ orthogonal to V_0 with the properties we described.

These are the MRA wavelets. There are wavelets that cannot be obtained in this way. The class of *all* wavelets ψ in $L^2(\mathbb{R})$ can be characterized by two equations and the property $||\psi||_2 \geq 1$:

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(2^j\xi\right)\right|^2 = 1 \quad \text{a.e.}\,, \tag{1.5}$$

and

$$t_q(\xi) = \sum_{j\geq 0} \hat{\psi}\left(2^j\xi\right)\overline{\hat{\psi}\left(2^j(\xi + 2q\pi)\right)} = 0 \quad \text{a.e.} \tag{1.6}$$

whenever q is an odd integer. We remind the reader that the book [4] presents a complete account of these facts; in Chapter 7 of this book one can find the characterization of all scaling functions as well. There it is pointed out that if the condition $||\psi||_2 \geq 1$ is not assumed, then the two equations (1.5) and (1.6) characterize the systems $\{\psi_{jk}\}$, $j, k \in \mathbb{Z}$, that are tight frames with constant 1.

A *frame* in a Hilbert space H, with inner product $< \cdot, \cdot >$, is a family $\{\varphi_n, n \in \Delta\}$ of elements in H for which there exist two positive constants, A and B, such that

$$A||f||^2 \leq \sum_{n\in\Delta} |< f, \varphi_n >|^2 \leq B||f||^2 \tag{1.7}$$

for all $f \in H$. The numbers A and B are called the *frame constants;* if $A = B$, $\{\varphi_n\}$ is called a *tight frame* and, after a renormalization, we can assume $A = B = 1$ (we shall suppose this to be the case in this article and, therefore, the term "with constant 1" will often be tacitly assumed). The indexing set Δ for the family $\{\varphi_n\}$ can be quite general; we assume it to be countable and, in particular, we often will be dealing with the case where $\{\varphi_n\}$ is the sequence of translates $\{\varphi(\cdot - n)\}$ of a function $\varphi \in H \subset L^2(\mathbb{R})$, $n \in \mathbb{Z}$, or the indexing set consists of the pairs $(j, k) \in \mathbb{Z} \times \mathbb{Z}$.

Many investigators have considered the case when the system $\{\psi_{jk}\}$, $j, k \in \mathbb{Z}$, is a frame for $L^2(\mathbb{R})$ rather than an orthonormal basis. This leads to the question of giving meaning and studying such frames that arise from a construction that extends the one we described above that produced the MRA wavelets. This is the purpose of this study: to introduce an appropriate definition of an *MRA wavelet frame* and, then, to study the properties of such frames.

Other authors have posed this problem and have obtained interesting and useful solutions of it (see [1], [3], and [6]). Our approach is different from the ones we have cited and the MRA wavelet frames we obtain belong to a class that strictly includes the ones obtained by the other authors.

The next section is devoted to the definition and construction of MRA wavelet frames. We consider, at first, only the case of MRA wavelet tight frames. In the fifth and last section, when we compare our results with those obtained by others, we show how the study of the general frame in [1] can be reduced to the case of tight frames.

The novelty of our approach is that we define and construct our MRA wavelet frames by making use of a collection of general low pass-filters. In the wavelet case, the function m introduced in (1.2) is called a *low pass filter;* the pair (m, m_1) is often referred to as a pair of *quadrature mirror filters,* and m_1 is called a *high pass filter.* The MRA method we described above begins with the spaces V_0 and V_1 and a scaling function φ and, from these, one constructs W_0, the orthogonal complement of V_0 within V_1. The wavelet is then a member ψ of W_0 such that $\{\psi(\cdot - n)\}$, $n \in \mathbb{Z}$, is an orthonormal basis for W_0. The MRA wavelet frames introduced by [1] and [2] follow this approach after introducing the notion of a "frame MRA" which is defined as we did above, except that the integral translates of the "pseudo-scaling function" φ form a frame for V_0 (instead of an orthonormal basis).

An important implementation of the MRA construction begins with a low pass filter m and, from it, a scaling function and its associated MRA are then produced in order to obtain wavelets. This method is very powerful and provides important information about the properties of the wavelet it produces. It was used very effectively, for example, by Daubechies [3] when she constructed her compactly supported wavelets. Unless a complete characterization is known of these 2π periodic functions $m \in L^2(T)$, satisfying (1.3) that are low pass filters for an MRA, constructing wavelets, by the "standard" method from the "known" low pass filters, does not produce the general MRA wavelet. Fortunately, some of us were able to find such a characterization [5]. In this article we show how one can use the information obtained from [5] in order to develop a general theory of MRA frame wavelets based on an appropriate "generalized low pass filter." This then will be the principal material included in the second section.

We want to emphasize that our approach is to obtain tight frames (or, more generally, frames) of the form $\{\psi_{jk}(x)\} = \{2^{\frac{j}{2}}\psi(2^j x - k)\}$, $j, k \in \mathbb{Z}$, from a function ψ constructed from an appropriate generalized filter m (a 2π periodic function satisfying (1.3)). Such a filter need *not* be associated with a "generalized" MRA (say, of the type introduced in [1]). In the literature, the function $m(\xi) = (1 + e^{3i\xi})/2$ is often presented as an example of a 2π-periodic function satisfying (1.3) that is *not* a low pass filter. It is then cast away as "useless." We shall show, in fact, that this m can be used for constructing an "MRA tight frame wavelet" (MRA TFW) and provides an example of such a generalized filter that *does not* arise from a "generalized MRA." Our moral is simply: Do not discard a generalized filter even though it is not a low pass filter; it may very well yield a bonafide "MRA tight frame wavelet."

In the third section we study the "multipliers" associated with the classes of MRA frame wavelets, the corresponding "pseudo-scaling functions," and generalized low pass filters. That is, we characterize those measurable functions ν such that $(\nu\hat{\psi})^\vee$ belong to the class $\mathcal{C}$ whenever

ψ belongs to $\mathcal{C}$ ($\mathcal{C}$ will be the class of frame wavelets, MRA frame wavelets, and pseudo scaling functions that we have introduced). We also characterize those measurable functions μ such that μm is a generalized low pass filter whenever m is such a filter. In the fourth section we study certain consequences of the multiplier results in the previous section and the connectivity of the classes of wavelet frames we introduced. This is very much in the spirit of [7]. The fifth section is devoted to a comparison of our approach with those used by other investigators [1], [2], and [6].

We want to thank Ziemowit Rzeszotnik for his suggestions and corrections.

2. Preliminaries and basic definitions

We introduced the notion of a tight frame immediately after the inequality (1.7). We shall study such frames when they are generated by a function in $L^2(\mathbb{R})$.

Definition 2.1. A function $\psi \in L^2(\mathbb{R})$ is a *tight frame wavelet* (for short, TFW) if the system $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$, where $\psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$, is a tight frame (with constant 1) for $L^2(\mathbb{R})$; that is, for all $f \in L^2(\mathbb{R})$,

$$f = \sum_{j,k\in\mathbb{Z}} < f, \psi_{jk} > \psi_{jk} \tag{2.1}$$

unconditionally in $L^2(\mathbb{R})$.

This is equivalent to the condition

$$||f||_2^2 = \sum_{j,k\in\mathbb{Z}} |< f, \psi_{jk} >|^2 , \tag{2.2}$$

for every $f \in L^2(\mathbb{R})$ (this corresponds to the definition of a tight frame we discussed in the first section; see [4, Chapters 7 and 8]).

We shall use the following characterization (already mentioned in the first section) of TFW's, which is essentially proved in [4], Theorem 1.6 of Chapter 7 (the frame terminology is not used in Theorem 1.6, since in [4] frames are introduced in Chapter 8).

Theorem 2.2. *A function $\psi \in L^2(\mathbb{R})$ is a TFW if and only if ψ satisfies* (1.5) *and* (1.6).

Following [5], we shall denote by $\tilde{\mathbf{F}}$ the set of *generalized filters;* i.e., $m \in \tilde{\mathbf{F}}$ if m is 2π-periodic, and satisfies (1.3).

Definition 2.3. A function $\varphi \in L^2(\mathbb{R})$ is called *a pseudo-scaling function* if there exists $m \in \tilde{\mathbf{F}}$ such that

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi) \qquad \text{for a.e.} \quad \xi \in \mathbb{R} . \tag{2.3}$$

Remark 2.4. Notice that m is not uniquely determined by the pseudo-scaling function φ. Therefore, we shall denote by $\tilde{\mathbf{F}}_\varphi$ the set of all $m \in \tilde{\mathbf{F}}$ such that m satisfies (2.3) for φ. For example, if $\varphi = 0$, then, $\tilde{\mathbf{F}}_\varphi = \tilde{\mathbf{F}}$, while, if φ is a scaling function for an MRA (see [4, Chapters 2 and 7]), then, $\tilde{\mathbf{F}}_\varphi$ is a singleton.

Note that for a pseudo-scaling function φ, the function $|\hat{\varphi}|^\vee$ is also a pseudo-scaling function, and if $m \in \tilde{\mathbf{F}}_\varphi$, then, $|m| \in \tilde{\mathbf{F}}_{|\hat{\varphi}|^\vee}$. Let us recall (see [5, Section 3]) that for every $m \in \tilde{\mathbf{F}}$, we can define $\varphi_{|m|} \in L^2(\mathbb{R})$ by letting $\hat{\varphi}_{|m|}(\xi) = \prod_{j=1}^{\infty} |m(\frac{\xi}{2^j})|$. However, in general, it is not true that $|\hat{\varphi}|$ can be obtained as the infinite product of the values of $|m|$ (take $\varphi \equiv 0$ for example). Hence, it is not necessarily true that $|\hat{\varphi}| = \hat{\varphi}_{|m|}$. In order to explore this further let us mention (see [5, Lemma 1]) that for almost every $\xi \in \mathbb{R} \setminus \{0\}$ the limit $\lim_{n\to\infty} \hat{\varphi}_{|m|}(2^{-n}\xi)$ exists and is either equal to 0 or 1. Since this limit is not going to change if we replace ξ with any dyadic dilate $2^k\xi$, $k \in \mathbb{Z}$, of ξ, it is important to consider the Lebesgue measure of the set

$$N_0(|m|) := \left\{\xi \in I,\ \lim_{n\to\infty} \hat{\varphi}_{|m|}\left(2^{-n}\xi\right) = 0\right\}$$

where $I = [-\pi, \pi) \setminus [-\frac{\pi}{2}, \frac{\pi}{2})$ (see also [5, Theorem 2 and Lemma 4] for the significance of $N_0(|m|)$).

Proposition 2.5. *Suppose that φ is a pseudo-scaling function and $m \in \tilde{\mathbf{F}}_\varphi$. If*

$$\lim_{n\to\infty} \left|\hat{\varphi}(2^{-n}\xi)\right| = 1 \qquad \text{for a.e. } \xi \in \mathbb{R} \tag{2.4}$$

then,

$$\left|\hat{\varphi}(\xi)\right| = \prod_{j=1}^{\infty} \left|m\left(\frac{\xi}{2^j}\right)\right| = \hat{\varphi}_{|m|}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R} \tag{2.5}$$

and, obviously, $|N_0(|m|)| = 0$.

Proof. By (2.3), we have, for every $n \in \mathbb{N}$,

$$\left|\hat{\varphi}(\xi)\right| = \left[\prod_{j=1}^{n} \left|m\left(\frac{\xi}{2^j}\right)\right|\right] \left|\hat{\varphi}\left(2^{-n}\xi\right)\right| .$$

Using (2.4), we obtain that $|\hat{\varphi}(\xi)| = \hat{\varphi}_{|m|}(\xi)$ and, therefore, (2.5) and $|N_0(|m|)| = 0$ are clearly satisfied. □

Following [5], we shall call a *generalized filter m* with the property that $|N_0(|m|)| = 0$, a *generalized low pass filter* (see [5, Section 3], for more details).

Remark 2.6. Note that even for a pseudo-scaling function φ which satisfies (2.4), the set $\tilde{\mathbf{F}}_\varphi$ is not necessarily a singleton. For example, take $\hat{\varphi}(\xi) = \chi_{[-\epsilon,\epsilon]}(\xi)$, $(0 \leq \epsilon < \frac{\pi}{2})$. Any $m \in \tilde{\mathbf{F}}$, $m \geq 0$, such that $m|_{[-\epsilon,\epsilon]} = \chi_{[-\frac{\epsilon}{2},\frac{\epsilon}{2}]}|_{[-\epsilon,\epsilon]}$ is an element of $\tilde{\mathbf{F}}_\varphi$. Since m restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}) \setminus [-\epsilon, \epsilon)$ can be any measurable function with values in $[0, 1]$ (m can then be extended to $[-\pi, \pi)$ so that it satisfies $m(\xi)^2 + m(\xi + \pi)^2 = 1$), it is clear that $\tilde{\mathbf{F}}_\varphi$ contains infinitely many elements.

Definition 2.7. A TFW ψ is an MRA TFW if there exists a pseudo-scaling function φ and $m \in \tilde{\mathbf{F}}_\varphi$ such that

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}} \overline{m\left(\frac{\xi}{2} + \pi\right)} \hat{\varphi}\left(\frac{\xi}{2}\right) \qquad \text{for a.e. } \xi \in \mathbb{R} . \tag{2.6}$$

The following lemma is an elementary result in integration theory. Since we shall apply it several times in the sequel, we provide a proof for the reader's convenience.

Lemma 2.8. *If $f \in L^1(\mathbb{R})$, then, for a.e. $\xi \in \mathbb{R}$, $\lim_{n\to+\infty} f(2^n\xi) = 0$.*

Proof. Without loss of generality, we can assume that $f \geq 0$. Assuming that $f \in L^1(\mathbb{R})$ and applying the monotone convergence theorem we obtain

$$\int_{\mathbb{R}} \sum_{n\in\mathbb{N}} f\left(2^n u\right) du = \sum_{n\in\mathbb{N}} \int_{\mathbb{R}} f\left(2^n u\right) du = \sum_{n\in\mathbb{N}} 2^{-n} \int_{\mathbb{R}} f(u)\, du = \|f\|_1 < \infty .$$

It follows that for a.e. u, $\sum_{n\in\mathbb{N}} f(2^n u)$ is finite. Therefore, for a.e. u, $\lim_{n\to+\infty} f(2^n u) = 0$. □

Proposition 2.9. *Suppose ψ is an MRA TFW and φ is a pseudo-scaling function satisfying* (2.6). *Then φ satisfies* (2.4); *thus, m is a generalized low pass filter.*

Proof. Since ψ is a TFW, we can use (1.5) and, since ψ is an MRA TFW, we can use (2.6) to obtain

$$\begin{aligned}
1 &= \sum_{j\in\mathbb{Z}} \left|\hat{\psi}(2^j\xi)\right|^2 = \sum_{j\in\mathbb{Z}} \left|m\left(2^{j-1}\xi + \pi\right)\right|^2 \left|\hat{\varphi}\left(2^{j-1}\xi\right)\right|^2 \\
&= \lim_{n\to\infty} \sum_{j=-n}^{n} \left|m\left(2^{j-1}\xi + \pi\right)\right|^2 \left|\hat{\varphi}\left(2^{j-1}\xi\right)\right|^2 \\
&= \lim_{n\to\infty} \sum_{j=-n}^{n} \left\{1 - \left|m\left(2^{j-1}\xi\right)\right|^2\right\} \left|\hat{\varphi}\left(2^{j-1}\xi\right)\right|^2 \\
&= \lim_{n\to\infty} \left\{\left|\hat{\varphi}\left(2^{-n-1}\xi\right)\right|^2 - \left|\hat{\varphi}\left(2^{n}\xi\right)\right|^2\right\} .
\end{aligned}$$

Since $\varphi \in L^2(\mathbb{R})$, Lemma 2.8 implies $\lim_{n\to\infty} |\hat{\varphi}(2^n\xi)|^2 = 0$ for a.e. ξ. This shows that for a.e. $\xi \in \mathbb{R}$, $\lim_{n\to\infty} |\hat{\varphi}(2^{-n}\xi)| = 1$. □

Remark 2.10. Observe that if ψ in Definition 2.7 happens to be a wavelet, then, ψ is an MRA wavelet in the usual sense. To see this, recall that (2.6) implies

$$|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} \left|\hat{\psi}\left(2^j\xi\right)\right|^2 . \tag{2.7}$$

A consequence of (2.7) is that $\|\hat{\varphi}\|_2 = \|\hat{\psi}\|_2$, and, since ψ is a wavelet, this implies that $\|\varphi\|_2 = 1$. Using Proposition 2.9 and Proposition 2.5, we conclude that the generalized filter m in (2.6) satisfies $|N_0(|m|)| = 0$, $\hat{\varphi}_{|m|} = |\hat{\varphi}|$ and $\|\varphi_{|m|}\|_2 = 1$. We can now apply [5, Theorem 2] to conclude that m is a low pass filter and φ is a scaling function. Therefore, ψ is an MRA wavelet.

The notions presented suggest a natural way of constructing MRA TFWs from generalized low pass filters. We will, in fact, show that this offers us a method that yields all the MRA TFWs that we just introduced.

The construction of MRA TFWs. Suppose that m is a generalized low pass filter. Using results from [7] and [5], we know that there exists a filter multiplier μ (i.e., μ is unimodular and 2π-periodic) such that $m(\xi) = \mu(\xi)|m(\xi)|$ and a scaling function multiplier ν (i.e., ν is unimodular and $\nu(2\xi)\overline{\nu(\xi)}$ is 2π-periodic) such that $\nu(2\xi)\overline{\nu(\xi)} = \mu(\xi)$ for a.e. $\xi \in \mathbb{R}$. If we define a function $\varphi \in L^2(\mathbb{R})$ by $\hat{\varphi}(\xi) := \nu(\xi)\hat{\varphi}_{|m|}(\xi)$, then φ is a pseudo-scaling function, $|\hat{\varphi}| = \hat{\varphi}_{|m|}$ and $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$ a.e. We then define a function $\psi \in L^2(\mathbb{R})$ by (2.6)

$$\hat{\psi}(\xi) := e^{i\frac{\xi}{2}}\overline{m\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right).$$

First, we show that ψ is a TFW (this would imply immediately that ψ is an MRA TFW). Secondly, we will show, as promised, that all MRA TFWs are obtained in this way.

Theorem 2.11. *The function ψ defined by the procedure we just presented using equality* (2.6) *is a TFW.*

Proof. Theorem 2.2 shows that it is enough to prove that $\hat{\psi}$ satisfies (1.5) and (1.6).

Note that since $|N_0(|m|)| = 0$ and $|\hat{\varphi}| = \hat{\varphi}_{|m|}$, the function φ satisfies (2.4) (see comments preceding (2.5)). The argument in the proof of Proposition 2.9 shows that

$$\sum_{j\in\mathbb{Z}}\left|\hat{\psi}\left(2^j\xi\right)\right|^2 = \lim_{n\to+\infty}\left[\left|\hat{\varphi}\left(2^{-n-1}\xi\right)\right|^2 - \left|\hat{\varphi}\left(2^n\xi\right)\right|^2\right].$$

Lemma 2.8 and (2.4) now immediately yield (1.5).

The following argument shows (1.6). When q is an odd integer we have

$$\begin{aligned}
&\sum_{j=0}^{\infty}\hat{\psi}\left(2^j\xi\right)\overline{\hat{\psi}\left(2^j(\xi+2\pi q)\right)} \\
&= \hat{\psi}(\xi)\overline{\hat{\psi}(\xi+2\pi q)} + \sum_{j=1}^{\infty}\hat{\psi}\left(2^j\xi\right)\overline{\hat{\psi}\left(2^j(\xi+2\pi q)\right)} \\
&= e^{i\frac{\xi}{2}}\overline{m\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right)\cdot e^{-i\frac{\xi}{2}}\cdot(-1)\cdot m\left(\frac{\xi}{2}+\pi(q+1)\right)\overline{\hat{\varphi}\left(\frac{\xi}{2}+\pi q\right)} \\
&\quad + \sum_{j=1}^{\infty}\overline{m\left(2^{j-1}\xi+\pi\right)}\hat{\varphi}\left(2^{j-1}\xi\right)\cdot m\left(2^{j-1}\xi+\pi\right)\overline{\hat{\varphi}\left(2^{j-1}\xi+2^j\pi q\right)} \\
&= -\hat{\varphi}(\xi)\overline{\hat{\varphi}(\xi+2\pi q)} + \sum_{j=1}^{\infty}\left(1-\left|m\left(2^{j-1}\xi\right)\right|^2\right)\hat{\varphi}\left(2^{j-1}\xi\right)\overline{\hat{\varphi}\left(2^{j-1}(\xi+2\pi q)\right)} \\
&= -\hat{\varphi}(\xi)\overline{\hat{\varphi}(\xi+2\pi q)} + \sum_{j=1}^{\infty}\left\{\hat{\varphi}\left(2^{j-1}\xi\right)\overline{\hat{\varphi}\left(2^{j-1}(\xi+2\pi q)\right)}\right. \\
&\quad \left. - \hat{\varphi}\left(2^j\xi\right)\overline{\hat{\varphi}\left(2^j(\xi+2\pi q)\right)}\right\} \\
&= \lim_{N\to+\infty}\left\{-\hat{\varphi}\left(2^N\xi\right)\overline{\hat{\varphi}\left(2^N(\xi+2\pi q)\right)}\right\} \\
&= 0
\end{aligned}$$

for a.e. $\xi \in \mathbb{R}$. □

Theorem 2.11 completes our construction since it shows that, indeed, we always obtain an MRA TFW by this method that starts with an arbitrary $m \in \tilde{\mathbf{F}}$ satisfying $|N_0(|m|)| = 0$.

We still have to answer the opposite question: Suppose we have an MRA TFW ψ (meaning, of course, that there exists a pseudo-scaling function φ which satisfies (2.6)), is it true that ψ can be obtained by the construction described above? The problem is that it is not a priori clear that there exists a scaling function multiplier ν such that $\hat{\varphi}(\xi) = \nu(\xi)\hat{\varphi}_{|m|}(\xi)$, where $m \in \tilde{\mathbf{F}}_\varphi$ satisfies (2.6) for ψ. The following theorem gives a positive answer to this question.

Theorem 2.12. *Suppose that ψ is an MRA TFW and φ is the associated pseudo-scaling function which satisfies* (2.6). *Then, ψ and φ can be constructed in the way described preceding Theorem 2.11.*

Proof. ψ is defined by (2.6) in terms of a pseudo-scaling function φ and an associated $m \in \tilde{\mathbf{F}}_\varphi$. Hence, we need to show that φ is obtained by our construction from $m \in \tilde{\mathbf{F}}$. More precisely, we shall prove that there exists a scaling function multiplier ν such that $\hat{\varphi}(\xi) = \nu(\xi)\hat{\varphi}_{|m|}(\xi)$. By Proposition 2.9 φ satisfies (2.4). Then, by Proposition 2.5, $|\hat{\varphi}| = \hat{\varphi}_{|m|}$ and $|N_0(|m|)| = 0$. We also know that there is a filter multiplier μ such that $m = \mu|m|$. Consider a signum function for $\hat{\varphi}$, i.e., a function t which is unimodular and $\hat{\varphi}(\xi) = t(\xi)|\hat{\varphi}(\xi)|$ (obviously, such a function exists). Hence, we only need to prove that there exists a unimodular function ν such that $\nu(2\xi)\overline{\nu(\xi)} = \mu(\xi)$ a.e. and $\nu(\xi) = t(\xi)$ when $\hat{\varphi}(\xi) \neq 0$. If we do this, then our construction based on this m gives us precisely this φ when we choose μ and ν as indicated.

Take $\xi \in \mathbb{R}$ such that $\hat{\varphi}(2\xi) \neq 0$. Therefore, $\hat{\varphi}(\xi) \neq 0$ and $m(\xi) \neq 0$. The following simple computation

$$\mu(\xi)|m(\xi)|\hat{\varphi}(\xi) = m(\xi)\hat{\varphi}(\xi) = \hat{\varphi}(2\xi) = t(2\xi)\left|\hat{\varphi}(2\xi)\right| = t(2\xi)\overline{t(\xi)}|m(\xi)|\hat{\varphi}(\xi)$$

shows that in this case $t(2\xi)\overline{t(\xi)} = \mu(\xi)$.

Consider an arbitrary $\xi \in \mathbb{R}$ for which (2.4) is true (as we observed above, a.e. $\xi \in \mathbb{R}$ has this property). If $\hat{\varphi}(2^n\xi) \neq 0$ for every $n \in \mathbb{Z}$, we define $\nu(2^n\xi) := t(2^n\xi)$ for every $n \in \mathbb{Z}$. Otherwise, by (2.4) and (2.3) we must have an $n_0 \in \mathbb{Z}$ such that $\hat{\varphi}(2^{n_0}\xi) \neq 0$ and $\hat{\varphi}(2^n\xi) = 0$ for $n \geq n_0 + 1$, while $\hat{\varphi}(2^n\xi) \neq 0$ for $n \leq n_0$. In this case, we define ν as follows: for $n \leq n_0$, $\nu(2^n\xi) := t(2^n\xi)$, while, for $n \geq n_0 + 1$, $\nu(2^n\xi) := \nu(2^{n-1}\xi)\mu(2^{n-1}\xi)$. This clearly completes the proof. □

We shall complete this section by addressing yet another question that appears naturally in our construction of MRA TFWs. As we have shown above, given $m \in \tilde{\mathbf{F}}$ such that $|N_0(|m|)| = 0$ we can construct an MRA TFW ψ from m. It is important to emphasize that we can construct infinitely many different ψ's, since our choice of the corresponding pseudo-scaling function φ depends on our choice of a scaling function multiplier ν. Recall that the only requirement for ν was to be unimodular and to satisfy that $\nu(2\xi)\overline{\nu(\xi)} = \mu(\xi)$, where μ is a filter multiplier such that $m(\xi) = \mu(\xi)|m(\xi)|$. It was shown in [7] that there are infinitely many ν's with those properties, since we can define ν to be an arbitrary unimodular function on I (defined immediately before Proposition 2.5) and then extend it inductively by using $\nu(2^n\xi) = \nu(2^{n-1}\xi)\mu(2^{n-1}\xi)$ for $n \geq 1$ and using $\nu(2^{n-1}\xi) = \nu(2^n\xi)\overline{\mu(2^{n-1}\xi)}$ for $n \leq -1$. Observe that under our assumption it is not necessarily true that the product $\prod_{j=1}^{\infty} m(\frac{\xi}{2^j})$ converges a.e.[1] Moreover, even in the case when the

[1]For example, if $\tilde{m}(\xi) = -|m(\xi)|$, the product $\prod_{j=1}^{\infty} \tilde{m}(2^{-j}\xi)$ is *not* convergent. The reader might amuse him/herself by producing a $\tilde{\varphi}$ such that $\tilde{m} \in \tilde{\mathbf{F}}_{\tilde{\varphi}}$.

product $\prod_{j=1}^{\infty} m(\frac{\xi}{2^j})$ converges a.e. to a function $\hat{\tilde{\varphi}} \in L^2(\mathbb{R})$ (obviously, $\tilde{\varphi}$ is then a pseudo-scaling function and $m \in \tilde{\mathbf{F}}_{\tilde{\varphi}}$), by choosing an arbitrary ν as above we may not pick exactly $\tilde{\varphi}$ (although, by Theorem 2.12, we know that there is a proper ν among those available). The question now becomes: Can we prescribe a particular choice of ν (and, therefore, of φ as well) in the general situation so that in the case when the product $\prod_{j=1}^{\infty} m(\frac{\xi}{2^j})$ converges a.e. our φ happens to be exactly $\tilde{\varphi}$? The answer to this question is yes. Let us prove that.

Suppose, as above, that $m \in \tilde{\mathbf{F}}$ and $|N_0(|m|)| = 0$. Consider a filter multiplier μ such that $m(\xi) = \mu(\xi)|m(\xi)|$. For $\xi \in I$ consider a sequence $\{\prod_{j=1}^{N} \mu(2^{-j}\xi)\}_{N\in\mathbb{N}}$. Since this is a sequence of points in the torus we can pick (measurably with respect to ξ) an accumulation point on the torus of this sequence. Let $\nu(\xi)$ be this accumulation point, i.e., we have defined ν properly on I. As before, we extend ν to $\mathbb{R}$ and define φ by $\hat{\varphi}(\xi) := \nu(\xi)\hat{\varphi}_{|m|}(\xi)$, $\xi \in \mathbb{R}$. We claim that this is the proper choice of φ.

Suppose that $\hat{\tilde{\varphi}}(\xi) = \prod_{j=1}^{\infty} m(2^{-j}\xi)$ exists. We have to prove that in this case $\varphi = \tilde{\varphi}$ (as elements in $L^2(\mathbb{R})$, of course). By our assumption $\hat{\tilde{\varphi}}(\xi) = \lim_{N\to+\infty} \left[\prod_{j=1}^{N} \mu(2^{-j}\xi) \cdot \prod_{j=1}^{N} |m(2^{-j}\xi)|\right]$. Since $|N_0(|m|)| = 0$, for almost every $\xi \in \mathbb{R}$, there exists $n = n(\xi) \in \mathbb{N}$ such that $\hat{\varphi}_{|m|}(2^{-n}\xi) \neq 0$ and $\hat{\tilde{\varphi}}(2^{-n}\xi) \neq 0$. This implies that for $u = 2^{-n}\xi$ the limit $\lim_{N\to+\infty} \prod_{j=1}^{N} \mu(2^{-j}u)$ exists. It follows that the limit $\lim_{N\to+\infty} \prod_{j=1}^{N} \mu(2^{-j}\xi)$ exists for a.e. $\xi \in \mathbb{R}$. By our choice of ν above it is clear that, for a.e. $\xi \in I$, $\nu(\xi) = \prod_{j=1}^{\infty} \mu(2^{-j}\xi)$. Since the product $\prod_{j=1}^{\infty} \mu(2^{-j}\xi)$ exists for almost every $\xi \in \mathbb{R}$ and $\nu(2\xi)\overline{\nu(\xi)} = \mu(\xi)$, we conclude by induction that, for a.e. $\xi \in \mathbb{R}$, $\nu(\xi) = \prod_{j=1}^{\infty} \mu(2^{-j}\xi)$. It then follows that, for a.e. $\xi \in \mathbb{R}$,

$$\hat{\tilde{\varphi}}(\xi) = \prod_{j=1}^{\infty} \mu\left(2^{-j}\xi\right) \prod_{j=1}^{\infty} \left|m\left(2^{-j}\xi\right)\right| = \nu(\xi)\hat{\varphi}_{|m|}(\xi) = \hat{\varphi}(\xi)\,,$$

which completes the proof and concludes this section.

3. Multiplier results

We will now describe the multiplier classes associated with TFWs.

Definition 3.1. (1) *A TFW multiplier* is a function ν such that $\tilde{\psi} = (\hat{\psi}\nu)^\vee$ is a TFW whenever ψ is a TFW.

(2) *An MRA TFW multiplier* is a function ν such that $\tilde{\psi} = (\nu\hat{\psi})^{\vee}$ is an MRA TFW whenever ψ is an MRA TFW.

(3) *A pseudo-scaling function multiplier* is a function ν such that $\tilde{\varphi} = (\hat{\varphi}\nu)^{\vee}$ is a pseudo-scaling function associated with an MRA TFW whenever φ has the same property.

(4) Finally, *a generalized low pass filter multiplier* is a function μ such that $\tilde{m} = m\mu$ is a generalized low pass filter whenever m is a generalized low pass filter.

The next theorem shows that the class of TFW multipliers is identical to the class of wavelet multipliers (see [7, Theorem II]).

Theorem 3.2. *A measurable function ν is a TFW multiplier if and only if ν is unimodular and $\nu(2\xi)\overline{\nu(\xi)}$ is 2π-periodic.*

Proof. (*if*) TFW's are characterized as elements of $L^2(\mathbb{R})$ satisfying (1.5) and (1.6). Being an element of $L^2(\mathbb{R})$ and (1.5) are properties invariant under multiplication by a unimodular function on the Fourier transform side. Thus, let us consider (1.6). Let ψ be a TFW, so that (1.6) holds. Let us assume ν is unimodular and $\nu(2\xi)\overline{\nu(\xi)} = \mu(\xi)$ is a 2π-periodic function, necessarily unimodular. Let $\hat{\tilde{\psi}}(\xi) = \hat{\psi}(\xi)\nu(\xi)$. Let q be an odd integer, and let $j \geq 0$. Then

$$\hat{\tilde{\psi}}\left(2^j\xi\right)\overline{\hat{\tilde{\psi}}\left(2^j(\xi+2q\pi)\right)} = \hat{\psi}\left(2^j\xi\right)\overline{\hat{\psi}\left(2^j(\xi+2q\pi)\right)}\nu\left(2^j\xi\right)\overline{\nu\left(2^j(\xi+2q\pi)\right)}. \tag{3.1}$$

If $j \geq 1$ then,

$$\begin{aligned}
&\nu\left(2^j\xi\right)\overline{\nu\left(2^j(\xi+2q\pi)\right)} \\
&= \mu\left(2^{j-1}\xi\right)\nu\left(2^{j-1}\xi\right)\overline{\mu\left(2^{j-1}(\xi+2q\pi)\right)\nu\left(2^{j-1}(\xi+2q\pi)\right)} \\
&= \mu\left(2^{j-1}\xi\right)\overline{\mu\left(2^{j-1}\xi\right)}\nu\left(2^{j-1}\xi\right)\overline{\nu\left(2^{j-1}(\xi+2q\pi)\right)} \\
&= \nu\left(2^{j-1}\xi\right)\overline{\nu\left(2^{j-1}(\xi+2q\pi)\right)}
\end{aligned}$$

by 2π periodicity and unimodularity of μ. If $j-1 \geq 1$, then we can repeat the above argument until we obtain

$$\nu\left(2^j\xi\right)\overline{\nu\left(2^j(\xi+2q\pi)\right)} = \nu(\xi)\overline{\nu(\xi+2q\pi)} \qquad \text{for} \quad j \geq 1\,.$$

Using this equality in (3.1), and summing over $j \geq 0$ we obtain

$$\sum_{j=0}^{\infty}\hat{\tilde{\psi}}\left(2^j\xi\right)\overline{\hat{\tilde{\psi}}\left(2^j(\xi+2q\pi)\right)} = \nu(\xi)\overline{\nu(\xi+2q\pi)}\sum_{j=0}^{\infty}\hat{\psi}\left(2^j\xi\right)\overline{\hat{\psi}\left(2^j(\xi+2q\pi)\right)}\,.$$

Since, by (1.6) the right-hand side is 0, we conclude that $\tilde{\psi}$ also satisfies (1.6). Hence, $\tilde{\psi}$ is a TFW and, thus, ν is a TFW multiplier.

(*only if*) Let ν be a TFW multiplier. We will first show the unimodularity. Let ψ be the Haar wavelet:

$$\psi(x) = \chi_{[0,\frac{1}{2})}(x) - \chi_{[\frac{1}{2},1)}(x)\,.$$

It follows that $|\hat{\psi}(\xi)| > 0$ for a.e. ξ. By assumption, for every $n \geq 1$, $(\hat{\psi}\,\nu^n)^{\vee}$ is a TFW and, thus, satisfies (1.5):

$$\sum_{j\in\mathbb{Z}}\left|\nu\left(2^j\xi\right)\right|^{2n}\left|\hat{\psi}\left(2^j\xi\right)\right|^2 = 1 \quad \text{a.e.} \quad \xi\,.$$

In particular, for a.e. ξ and every $n \in \mathbb{N}$,

$$|\nu(\xi)|^n \left|\hat{\psi}(\xi)\right|^2 \leq 1 .$$

This is only possible if $|\nu(\xi)| \leq 1$ a.e. since $\hat{\psi}$ almost never vanishes. Using (1.5) for ψ and $(\nu\hat{\psi})^{\vee}$, and subtracting, we obtain

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(2^j\xi\right)\right|^2 \left(1 - \left|\nu\left(2^j\xi\right)\right|^2\right) = 0 \quad \text{a.e.} \quad \xi$$

which is only possible if all terms vanish. Thus, $|\nu(\xi)| = 1$ a.e.

Having established this unimodularity, we see that the application of the multiplier ν does not effect the $L^2(\mathbb{R})$ norm of ψ ($||\tilde{\psi}||_2 = ||\psi||_2$). It follows then that ν is a wavelet multiplier (see (1.5), (1.6), and the discussion immediately preceding and following these two equations). We can, therefore, use [7, Theorem II] and conclude that $\nu(2\xi)\overline{\nu(\xi)}$ is a 2π-periodic function. □

The next result shows that the class of TFW multipliers coincides with the class of MRA TFW multipliers.

Theorem 3.3. *A measurable function ν is an MRA TFW multiplier if and only if it is unimodular and $\nu(2\xi)\overline{\nu(\xi)}$ is 2π-periodic.*

Proof. (*if*) Let ν be unimodular, and let

$$s(\xi) = \nu(2\xi)\overline{\nu(\xi)} \tag{3.2}$$

be 2π-periodic, necessarily unimodular. We now use [7, Lemma 2.1] to obtain a unimodular, 2π-periodic function t such that

$$s(\xi) = \overline{t(\xi)}t\left(\frac{\xi}{2}\right)t\left(\frac{\xi}{2} + \pi\right) . \tag{3.3}$$

Let $\mu(\xi) = \nu(\xi)t(\frac{\xi}{2})t(\frac{\xi}{2} + \pi)$; then μ is unimodular, and

$$\begin{aligned} \mu(2\xi)\overline{\mu(\xi)} &= \nu(2\xi)\overline{\nu(\xi)}t(\xi)t(\xi + \pi)\overline{t\left(\tfrac{\xi}{2}\right)t\left(\tfrac{\xi}{2} + \pi\right)} \\ &= s(\xi)t(\xi)t(\xi + \pi)\overline{t\left(\tfrac{\xi}{2}\right)t\left(\tfrac{\xi}{2} + \pi\right)} \\ &= t(\xi + \pi) \end{aligned} \tag{3.4}$$

is a 2π-periodic function. In the above computation, we have used (3.2) and (3.3) as well as the unimodularity and the periodicity of t. Let ψ be an MRA TFW, φ an associated pseudo-scaling function with $m \in \tilde{\mathbf{F}}_{\varphi}$ such that (2.6) holds. Let

$$\tilde{m}(\xi) = m(\xi)t(\xi + \pi)$$

and

$$\hat{\tilde{\varphi}}(\xi) = \hat{\varphi}(\xi)\mu(\xi) .$$

Then, $\tilde{m} \in \tilde{\mathbf{F}}_{\tilde{\varphi}}$:

$$\hat{\tilde{\varphi}}(2\xi) = \mu(2\xi)\hat{\varphi}(2\xi) = t(\xi + \pi)\mu(\xi)m(\xi)\hat{\varphi}(\xi) = \tilde{m}(\xi)\hat{\tilde{\varphi}}(\xi) . \tag{3.5}$$

Let

$$\hat{\tilde{\psi}}(\xi) = \nu(\xi)\hat{\psi}(\xi) .$$

Since ψ is a TFW, we can apply Theorem 3.2 to deduce that $\tilde{\psi}$ is a TFW. It follows, that $\tilde{\psi}$ is an MRA TFW, since

$$\begin{aligned}
e^{i\frac{\xi}{2}}\overline{\tilde{m}\left(\frac{\xi}{2}+\pi\right)}\hat{\tilde{\varphi}}\left(\frac{\xi}{2}\right) &= e^{i\frac{\xi}{2}}\overline{t\left(\frac{\xi}{2}+2\pi\right)m\left(\frac{\xi}{2}+\pi\right)}\mu\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right) \\
&= \hat{\psi}(\xi)\overline{t\left(\frac{\xi}{2}\right)}\mu\left(\frac{\xi}{2}\right) = \hat{\psi}(\xi)\overline{t\left(\frac{\xi}{2}\right)}\mu(\xi)\overline{t\left(\frac{\xi}{2}+\pi\right)} \\
&= \hat{\psi}(\xi)\nu(\xi) \\
&= \hat{\tilde{\psi}}(\xi) ,
\end{aligned}$$

where we have used (3.4), and the definition of μ.

(*only if*) The Haar wavelet is, in particular, an MRA TFW; we can then proceed as in the proof of Theorem 3.2 to show the unimodularity of ν. Using Remark 2.10 we conclude, that ν is an MRA wavelet multiplier. The theorem then follows from [7, Theorem II]. □

The next theorem characterizes the class of generalized low pass filter multipliers. Generalized low pass filters are functions $m \in \tilde{\mathbf{F}}_\varphi$ for some pseudo-scaling function φ satisfying (2.4). That is, $m \in \tilde{\mathbf{F}}$ and $|N_0(|m|)| = 0$. We will use this fact in the following result.

Theorem 3.4. *A measurable function ν is a generalized low pass filter multiplier if and only if ν is unimodular and 2π-periodic.*

Proof. (*if*) If we let $\tilde{m}(\xi) = \nu(\xi)m(\xi)$, and ν is unimodular and 2π periodic, then, clearly, if $m \in \tilde{\mathbf{F}}$, $|N_0(|m|)| = 0$ then these properties are also true for $\tilde{m}$.

(*only if*) We proceed as in the proof of the two previous theorems.

Let ψ be the Haar wavelet, and m the corresponding low pass filter, since $|\hat{\psi}| > 0$ a.e., it follows, that $|m| > 0$ a.e.. Let

$$\tilde{m}(\xi) = \nu(\xi)m(\xi) .$$

By assumption $\tilde{m}(\xi) \in \tilde{\mathbf{F}}$; in particular, $\tilde{m}$ is 2π-periodic, and so ν is also 2π-periodic. Applying ν repetitively, we obtain

$$|\nu(\xi)|^n|m(\xi)| \le 1 \quad \text{a.e.} \quad \xi \qquad n \ge 1 .$$

This implies, that $|\nu(\xi)| \le 1$ a.e.. Unimodularity follows, since both m and $\tilde{m}$ are in $\tilde{\mathbf{F}}$ and, thus, satisfy $|\tilde{m}(\xi)|^2 + |\tilde{m}(\xi+\pi)|^2 = 1 = |m(\xi)|^2 + |m(\xi+\pi)|^2$. □

The above three results provide description of TFW, MRA TFW, and generalized low pass filter multipliers. These classes are identical with the respective multiplier classes of wavelets. This fact is basically a consequence of the fact that all of these multiplier operations necessarily preserve the $L^2(\mathbb{R})$ norm of the TFW ψ.

The following result is somewhat surprising, since it shows that the situation for pseudo-scaling function multipliers is completely different. A pseudo-scaling function (psf) multiplier is a function ν which transforms, via the associated multiplier transformation, pseudo-scaling functions, satisfying (2.4) into pseudo-scaling functions satisfying the same requirement. Let us observe, that pseudo-scaling functions satisfying (2.4) are exactly those, which appear in (2.6). This follows from Propositions 2.9 and 2.5, and Theorem 2.11. Let us introduce some notation for the next theorem.

For a measurable function ν, let $E = \{\xi : \nu(\xi) \neq 0\}$ and $\mu(\xi) = \dfrac{\nu(2\xi)}{\nu(\xi)}$ on E.

Theorem 3.5. *ν is a psf multiplier if and only if*

(1) $|\nu(2\xi)| \le |\nu(\xi)|$ a.e. and $\lim_{j\to\infty} |\nu(2^{-j}\xi)| = 1$ a.e.

(2) $\mu(\xi)$ extends to a 2π-periodic function.

(3) If $\xi, \eta \in E$, and $\xi - \eta$ is an odd multiple of π, then $|\mu(\xi)| = |\mu(\eta)| = 1$.

Proof. (*if*) Let φ be a pseudo-scaling function satisfying (2.4), and suppose ν satisfies (1)-(3). Let

$$\hat{\tilde{\varphi}}(\xi) = \nu(\xi)\hat{\varphi}(\xi) .$$

Using condition (1), we see, that $\tilde{\varphi}$ satisfies (2.4):

$$\lim_{j\to\infty}\left|\hat{\tilde{\varphi}}\left(2^{-j}\xi\right)\right| = \lim_{j\to\infty}\left|\nu\left(2^{-j}\xi\right)\right| \cdot \lim_{j\to\infty}\left|\hat{\varphi}\left(2^{-j}\xi\right)\right| = 1 .$$

Let us now examine the 2-scale equation

$$\hat{\tilde{\varphi}}(2\xi) = \tilde{m}(\xi)\hat{\tilde{\varphi}}(\xi) . \tag{3.6}$$

We claim that there exists $\tilde{m} \in \tilde{\mathbf{F}}$ such that

$$\nu(2\xi)m(\xi) = \nu(\xi)\tilde{m}(\xi) . \tag{3.7}$$

If $\xi \in E$, then (3.7) is equivalent to

$$\tilde{m}(\xi) = \frac{\nu(2\xi)}{\nu(\xi)} m(\xi) . \tag{3.8}$$

If $\xi \notin E$, then, by (1), $2\xi \notin E$ and, thus, (3.7) is satisfied automatically. Requirement (2) implies that $\tilde{m}$ defined on E by (3.8) is 2π-periodic on E; that is, if $\xi, \eta \in E, \xi \equiv \eta (\mathrm{mod}\, 2\pi)$, then $\tilde{m}(\xi) = \tilde{m}(\eta)$. We will now define a 2π-periodic extension of $\tilde{m}$ to $\mathbb{R}$ satisfying

$$|\tilde{m}(\xi)|^2 + |\tilde{m}(\xi + \pi)|^2 = 1 . \tag{3.9}$$

Let us consider $[-\pi, \pi]$. Define $\tilde{m}(\xi)$ on (part of)$[-\pi, \pi]$ by the following: If $\xi + 2k\pi \in E$ for some $k \in \mathbb{Z}$, then

$$\tilde{m}(\xi) = \tilde{m}(\xi + 2k\pi) .$$

The definition is consistent with the 2π periodicity of m on E. Let $\xi, \eta \in [-\pi, \pi]$ and $|\xi - \eta| = \pi$. Then one of the following conditions must hold:

(a) $\exists k, \ell \in \mathbb{Z}$, such that $\xi + 2k\pi \in E$ and $\eta + 2\ell\pi \in E$;

(b) $\exists k \in \mathbb{Z}$, such that $\xi + 2k\pi \in E$ and for any $\ell \in \mathbb{Z}$, $\eta + 2\ell\pi \notin E$; and

(c) for any $k \in \mathbb{Z}$, $\xi + 2k\pi \notin E$ and $\eta + 2k\pi \notin E$.

In the (a) case, $(\xi + 2k\pi) - (\eta + 2\ell\pi)$ is an odd multiple of π, so by (3), $1 = |\mu(\xi + 2k\pi)| = |\mu(\eta + 2\ell\pi)|$. Thus, $|\tilde{m}(\xi)| = |\tilde{m}(\xi + 2k\pi)| = |m(\xi + 2k\pi)| = |m(\xi)|$ and, similarly, $|\tilde{m}(\eta)| = |m(\eta)|$.

Since $m \in \tilde{\mathbf{F}}$ this implies

$$|\tilde{m}(\xi)|^2 + |\tilde{m}(\eta)|^2 = 1 .$$

We have either $\xi + \pi = \eta$ or $\eta + \pi = \xi$, so (3.9) holds.

In the (b) case we extend the definition of $\tilde{m}$ to the set of all such $\eta \in [-\pi, \pi)$ by

$$\tilde{m}(\eta) = \sqrt{1 - |\tilde{m}(\xi)|^2} .$$

In the (c) case, we let

$$\tilde{m}(\zeta) = 1$$

if $\zeta \in [-\frac{\pi}{2}, \frac{\pi}{2})$, and

$$\tilde{m}(\zeta) = 0$$

if $\zeta \in [-\pi, \pi) \setminus [-\frac{\pi}{2}, \frac{\pi}{2})$, whenever $\zeta = \xi$ or η.

We have thus extended $\tilde{m}$ to the entire interval $[-\pi, \pi)$ so that (3.9) holds if ξ and $\xi + \pi$ are in $[-\pi, \pi)$. We now extend $\tilde{m}$ to $\mathbb{R}$ by 2π-periodicity. Clearly, $\tilde{m} \in \tilde{\mathbf{F}}$, and (3.6) is satisfied.

(*only if*) Let φ be the Shannon scaling function; that is, $\hat{\varphi}(\xi) = \chi_{[-\pi,\pi)}(\xi)$. Then $\hat{\tilde{\varphi}}(\xi) = \nu(\xi)\hat{\varphi}(\xi) = \nu(\xi)$ on $[-\pi, \pi)$. By (2.4) for $\tilde{\varphi}$ we have

$$\lim_{j\to\infty} \left|\nu\left(2^{-j}\xi\right)\right| = 1 .$$

Now, let φ be the Haar scaling function, so that $|\hat{\varphi}| > 0$ a.e. Take the unique $m \in \tilde{\mathbf{F}}_\varphi$, and select $\tilde{m} \in \tilde{\mathbf{F}}_{\tilde{\varphi}}$; then

$$\nu(2\xi)m(\xi)\hat{\varphi}(\xi) = \nu(2\xi)\hat{\varphi}(2\xi) = \hat{\tilde{\varphi}}(2\xi) = \tilde{m}(\xi)\hat{\tilde{\varphi}}(\xi) = \tilde{m}(\xi)\nu(\xi)\hat{\varphi}(\xi) .$$

So,

$$\nu(2\xi)m(\xi) = \tilde{m}(\xi)\nu(\xi) \quad \text{a.e.} \quad \xi \in \mathbb{R} . \tag{3.10}$$

For $\xi \in E$,

$$\frac{\nu(2\xi)}{\nu(\xi)} = \frac{\tilde{m}(\xi)}{m(\xi)} . \tag{3.11}$$

This establishes (2), since the right-hand side is 2π-periodic. Since $m(\xi) \neq 0$ a.e., then it follows from (3.10) that if $\xi \notin E$, then $2\xi \notin E$. To conclude (1), we need to establish

$$|\nu(2\xi)| \leq |\nu(\xi)| \qquad \text{for} \quad \xi \in E .$$

We do this by using specific filters m. Given any $\xi_0 \in E$ that is not an odd multiple of π, we can produce a generalized low pass filter m_1, such that $m_1(\xi_0) = 1$, m_1 is smooth and $m_1(\xi) \neq 0$ a.e. It follows that the associated pseudo-scaling function φ_1 satisfies $\hat{\varphi}_1 \neq 0$ a.e. and, thus, for this m_1, instead of m, (3.10) holds. So,

$$\frac{|\nu(2\xi_0)|}{|\nu(\xi_0)|} = \frac{|\tilde{m}(\xi_0)|}{|m_1(\xi_0)|} \leq 1 .$$

Hence, (1) is established. Let us return to the Haar low pass filter and consider (3.11). Let $\xi, \eta \in E$, and $\xi - \eta$ be an odd multiple of π,

$$\tilde{m}(\xi) = \mu(\xi)m(\xi), \quad \tilde{m}(\eta) = \mu(\eta)m(\eta) .$$

Then,

$$\begin{aligned} 1 &= |\tilde{m}(\xi)|^2 + |\tilde{m}(\eta)|^2 = |\mu(\xi)|^2|m(\xi)|^2 + |\mu(\eta)|^2|m(\eta)|^2 \\ &\leq |m(\xi)|^2 + |m(\eta)|^2 = 1 \end{aligned}$$

by (1). Hence, the inequality is actually an equality and we obtain (3). □

Remark 3.6. Let $\nu(\xi) = \chi_{[-\frac{\pi}{4},\frac{\pi}{4})}(\xi)$. It is easy to see, that (1)-(3) in Theorem 3.5 are satisfied. Thus ν is a pseudo-scaling function multiplier, which is not unimodular. It cannot be a scaling function multiplier.

4. Connectivity question for MRA TFWs

As we have seen in Sections 2 and 3, MRA TFWs have several common traits with MRA wavelets. More precisely, as in the case for MRA wavelets, MRA TFWs satisfy equations (2.6) and (2.7), and the class of MRA TFW multipliers coincides with the class of MRA wavelet multipliers (Theorem 3.3). This suggests that the question of connectivity of the set of all MRA TFWs can be treated by the argument applied in [7]; where the connectivity of the set of MRA wavelets was proved.

We shall prove in this section that this suggestion is only partially true. It turns out that the proof (given in [7]) of the connectivity of the MRA wavelets ψ with given absolute value $|\hat{\psi}|$ translates word for word to our situation. On the other hand, the path we constructed in [7] connecting two MRA wavelets having Fourier transforms with different absolute values fails to do so in the case of MRA TFWs.

We begin by proving the analog of Theorem 3 from [7]. Suppose that ψ_0 is an MRA TFW and φ_0 is a pseudo-scaling function associated with ψ_0 (in the sense that φ_0 and ψ_0 satisfy (2.6)). Although such φ_0 is not uniquely determined by ψ_0, we know, by (2.7), that $|\hat{\varphi}_0|$ is unique. Following [7, p. 578–579], it makes sense to define the three classes of MRA TFW's, ψ, determined by ψ_0 (notice that in the following definitions, φ denotes a pseudo-scaling function associated with an MRA TFW ψ by (2.6), while ν denotes an MRA TFW multiplier):

$$\mathcal{W}_{\psi_0}^{TF} := \left\{\psi : \left|\hat{\psi}(\xi)\right| = \left|\hat{\psi}_0(\xi)\right| \quad \text{a.e.}\right\} , \tag{4.1}$$

$$\mathcal{S}_{\psi_0}^{TF} := \left\{\psi : \left|\hat{\varphi}(\xi)\right| = \left|\hat{\varphi}_0(\xi)\right| \quad \text{a.e.}\right\} , \tag{4.2}$$

$$\mathcal{M}_{\psi_0}^{TF} := \left\{\psi : \exists\, \nu \text{ such that } \hat{\psi}(\xi) = \nu(\xi)\hat{\psi}_0(\xi) \quad \text{a.e.}\right\} . \tag{4.3}$$

Theorem 4.1. *If ψ_0 is an MRA TFW, then*

$$\mathcal{W}_{\psi_0}^{TF} = \mathcal{S}_{\psi_0}^{TF} = \mathcal{M}_{\psi_0}^{TF} .$$

Proof. The proof of this theorem is essentially the same as the proof of MRA wavelet case given in [7]. Let us only highlight the main steps of the proof.

Notice that (2.7) immediately implies that $\mathcal{W}_{\psi_0}^{TF} \subseteq \mathcal{S}_{\psi_0}^{TF}$ and that

$$\left|\hat{\psi}(\xi)\right|^2 = \left|\hat{\varphi}\left(\frac{\xi}{2}\right)\right|^2 - \left|\hat{\varphi}(\xi)\right|^2 . \tag{4.4}$$

Obviously, (4.4) implies that $\mathcal{S}_{\psi_0}^{TF} \subseteq \mathcal{W}_{\psi_0}^{TF}$.

By Theorem 3.3 we know that an MRA TFW multiplier is unimodular; thus, $\mathcal{M}_{\psi_0}^{TF} \subseteq \mathcal{W}_{\psi_0}^{TF}$. It remains to prove that $\mathcal{S}_{\psi_0}^{TF} \subseteq \mathcal{M}_{\psi_0}^{TF}$. This part of the proof requires some subtle modifications of the proof in [7]; we shall emphasize these details.

Suppose $\psi_1 \in \mathcal{S}_{\psi_0}^{TF}$. By (2.6), there exist pseudo-scaling functions φ_0 and φ_1, and generalized filters m_0 and m_1 such that $|\hat{\varphi}_0(\xi)| = |\hat{\varphi}_1(\xi)|$ a.e., $m_j \in \tilde{\mathbf{F}}_{\varphi_j}$ and $\hat{\psi}_j(\xi) = e^{i\frac{\xi}{2}}\overline{m_j(\frac{\xi}{2}+\pi)}\hat{\varphi}_j(\frac{\xi}{2})$, $j = 0, 1$. In particular, since $\mathcal{S}_{\psi_0}^{TF} = \mathcal{W}_{\psi_0}^{TF}$, $|\hat{\psi}_0(\xi)| = |\hat{\psi}_1(\xi)|$ a.e. Therefore, it makes sense to define $\tilde{\psi} \in L^2(\mathbb{R})$ by

$$\hat{\tilde{\psi}}(\xi) := e^{i\frac{\xi}{2}}\left|\hat{\psi}_j(\xi)\right|; \qquad j = 0,\ 1\ . \tag{4.5}$$

Notice that $\tilde{\psi}$ is well defined by (4.5), despite the fact that it is not necessarily true that $|m_0|$ and $|m_1|$ are equal (see Remark 2.6). $|m_i|$ and $|\hat{\varphi}_i|^{\vee}$, $i = 0, 1$, are a generalized low pass filter and pseudo scaling function satisfying (1.2); thus, it is clear that (4.5) provides a function $\tilde{\psi}$ that is an MRA TFW. As in the proof of [7, Theorem 3] it is enough to show that there exists MRA TFW multipliers ν_j such that $\hat{\psi}_j = \nu_j\hat{\tilde{\psi}}$ $(j = 0, 1)$ (by Theorem 3.3, these multipliers satisfy exactly the same properties as the MRA wavelet multipliers constructed in [7]). Without loss of generality we shall consider the case $j = 1$.

Toward this end, let

$$F := \left\{\xi \in \mathbb{R} : \hat{\varphi}_1\left(2^{\ell+1}\xi\right) = m_1\left(2^{\ell}\xi\right)\hat{\varphi}_1\left(2^{\ell}\xi\right) \quad \text{for all} \quad \ell \in \mathbb{Z}\right\}\ , \tag{4.6}$$

and it is clear that

$$|\mathbb{R} \setminus F| = 0\ . \tag{4.7}$$

Also let $E = \{\xi \in F : \hat{\varphi}_1(\xi) \neq 0\}$. We then have $E \subset 2E$, and, consequently,

$$2^nE \subset 2^{n+1}E \quad \text{for} \quad n = 0, 1, \cdots\ . \tag{4.8}$$

It follows that if we define $\Delta_0 = E$, $\Delta_n = 2^nE \setminus 2^{n-1}E$ for $n \geq 1$, then $\Delta_m \bigcap \Delta_n = \emptyset$ for $m \neq n$. We claim that

$$\left|F \setminus \bigcup_{n\geq 0} 2^nE\right| = 0\ . \tag{4.9}$$

If we accept (4.9) the rest of the proof follows verbatim the proof in [7]. Indeed, since

$$\mathbb{R} \setminus \bigcup_{n\geq 0} 2^nE = \left\{(\mathbb{R} \setminus F) \setminus \bigcup_{n\geq 0} 2^nE\right\} \bigcup \left\{F \setminus \bigcup_{n\geq 0} 2^nE\right\}\ ,$$

(4.7) and (4.9) imply $|\mathbb{R} \setminus \bigcup_{n\geq 0} 2^nE| = 0$. Thus, it suffices to define ν_1 on the disjoint union $\bigcup_{n\geq 0} \Delta_n = \bigcup_{n\geq 0} 2^nE$ which has full measure. We accomplish this by defining first the function μ

such that $\mu(\xi)|m_1(\xi)| = m_1(\xi)$ if $m_1(\xi) \neq 0$, and $\mu(\xi) = 1$ if $m_1(\xi) = 0$ (notice that μ is unimodular and 2π-periodic), and then, we define the function $t(\xi)$ inductively on $\bigcup_{n \geq 0} \Delta_n$ so that

$$t(\xi) = \frac{|\hat{\varphi}_1(\xi)|}{\hat{\varphi}_1(\xi)} \qquad \text{for} \quad \xi \in \Delta_0 ,$$

and

$$t(\xi) = \overline{\mu\left(\frac{\xi}{2}\right) t\left(\frac{\xi}{2}\right)} \qquad \text{for} \quad \xi \in \Delta_n .$$

We define ν_1 by

$$\nu_1(\xi) := \overline{\mu\left(\frac{\xi}{2} + \pi\right)} \cdot \overline{t\left(\frac{\xi}{2}\right)}, \tag{4.10}$$

and it follows that ν_1 is unimodular and $\nu_1(2\xi)\overline{\nu_1(\xi)} = \overline{\mu(\xi + \pi)}\mu(\frac{\xi}{2} + \pi)\mu(\frac{\xi}{2})$ is 2π-periodic, i.e., ν_1 is an MRA TFW multiplier and $\hat{\psi}_1 = \nu_1 \cdot \hat{\tilde{\psi}}$ a.e.

Therefore, it remains to prove (4.9); however, we have to modify the argument presented in [7] to do so.

Suppose K is a measurable subset of $F \setminus \bigcup_{n \geq 0} 2^n E = \bigcap_{n \geq 0} (F \setminus 2^n E)$. If $\xi \in K$, then, $\xi \in F = 2^n F$ for all $n \in \mathbb{Z}$. Hence, $2^{-n}\xi \in F$ for all $n \in \mathbb{Z}$; moreover, $2^{-n}\xi \notin E$ for all $n \geq 0$. It follows that $\hat{\varphi}_1(2^{-n}\xi) = 0$, for all $n \geq 0$. We conclude that for all $n \geq 0$,

$$\chi_K(\xi)\hat{\varphi}_1\left(2^{-n}\xi\right) = 0 . \tag{4.11}$$

Apply (4.4) for ψ_1 and φ_1 to conclude that for all $j \leq 0$, $j \in \mathbb{Z}$,

$$\left|\hat{\psi}_1\left(2^j\xi\right)\right| \chi_K(\xi) = 0 . \tag{4.12}$$

Since ψ_1 is a TFW, it satisfies (1.5). Thus, by (2.7), we obtain

$$1 = \sum_{j \in \mathbb{Z}} \left|\hat{\psi}_1\left(2^j\xi\right)\right|^2 = |\hat{\varphi}_1(\xi)|^2 + \sum_{j \leq 0} \left|\hat{\psi}_1\left(2^j\xi\right)\right|^2 \quad \text{a.e.} \tag{4.13}$$

Observe that (4.11) for $n = 0$, (4.12), and (4.13) imply that $\chi_K(\xi) = 0$ a.e., i.e., $|K| = 0$. This proves (4.9) and completes the proof of Theorem 4.1. $\square$

Let us now consider the proof of [7, Theorem 4]; in particular, the part of the proof which establishes the connectivity of the set $\mathcal{M}_{\psi_0}$ (see [7, p. 587–588]). Notice that this part of the proof uses only the properties of wavelet multipliers, which are, by Theorem 3.2, exactly the same as the properties of TFW multipliers. Observe also that the class $\mathcal{M}_{\psi_0}^{TF}$, defined by (4.3), is well defined for an arbitrary TFW ψ_0. Hence, the first part of the proof of [7, Theorem 4] translates verbatim to this situation and shows that the following theorem is valid.

Theorem 4.2. *If ψ_0 is a TFW, then $\mathcal{M}_{\psi_0}^{TF}$ is arcwise connected in $L^2(\mathbb{R})$.*

Using Theorem 4.1 we obtain the following corollary in the MRA case.

Corollary 4.3. *If ψ_0 is an MRA TFW, then $\mathcal{W}_{\psi_0}^{TF}$ is arcwise connected in $L^2(\mathbb{R})$.*

Remark 4.4. With Corollary 4.3 we have reduced the question of connectivity of MRA TFWs significantly. What remains to be answered is the following question.

Suppose that ψ_0 and ψ_1 are MRA TFWs that are obtained from pseudo-scaling functions whose Fourier transforms $\hat{\varphi}_0$ and $\hat{\varphi}_1$ are nonnegative. Can we connect ψ_0 and ψ_1 with a continuous path in $L^2(\mathbb{R})$, within the class of MRA TFWs?

Observe that without loss of generality we can assume that, say, ψ_0 is a particular MRA TFW; for example ψ_0 can be the Shannon wavelet. As is well known (see [7, Theorem 4]), this question has been answered positively for MRA wavelets ψ_0 and ψ_1. However, as we shall see below, the analogy with methods [7] breaks down in the MRA TFW case.

In order to see that, let us recall (see [5]) that a generalized nonnegative filter m which is Hölder continuous at zero and bounded below a.e. by a positive constant on $[-\frac{\pi}{3}, \frac{\pi}{3}]$ has to be a filter of an MRA wavelet.

Choose ψ_0 to be the Shannon wavelet and ψ_1 to be an MRA TFW whose associated pseudo-scaling function is φ such that $\hat{\varphi}(\xi) = \chi_{[-\epsilon,\epsilon]}(\xi)$, where $0 < \epsilon < \frac{\pi}{2}$ (see Remark 2.6). Then, obviously, the norm of ψ_1 satisfies

$$\|\psi_1\|_2 < 1 ,$$

and it is impossible to connect ψ_0 and ψ_1 with a path which has the property that ψ_t is an MRA wavelet for every $t \in [0, 1)$. Unfortunately, if we define our path as it was done in [7, p. 588], then, for every $t \in [0, 1)$, m_t is going to be a Hölder continuous at zero, generalized nonnegative filter with the property

$$m_t(\xi) \geq 1 - t \qquad \text{for} \quad \xi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] .$$

Hence, ψ_t has to be an MRA wavelet for $t \in [0, 1)$. The same thing happens if we choose to define m_t by the following formula

$$m_t = \sqrt{(1-t)m_0^2 + tm_1^2} .$$

Therefore, the above question remains open and would require a different method.

5. A comparison with other constructions of MRA wavelet frames and the relation to the case of tight frames

In [1] and [2] a very natural construction of MRA wavelet frames is presented based on an extension of the notion of an MRA we described at the beginning of Section 1. As we explained there, their extension differs from the "classical" one only in the last assumption: the existence of an element $\varphi \in V_0$ such that the sequence $\{\varphi_n\}$, $n \in \mathbb{Z}$, of its integral translates is an orthonormal basis of V_0 is replaced by the existence of such a φ such that $\{\varphi_n\}$ is a frame in V_0; that is, they assume that inequalities (1.7) are satisfied by $\{\varphi_n\}$ for each $f \in V_0$. These authors then find conditions that guarantee the existence of a function $\psi \in V_1$ that belongs to the orthogonal complement of V_0 whose integral translates form a frame for $W_0 \equiv V_1 \ominus V_0 = V_1 \cap V_0^{\perp}$. If this is achieved, it follows immediately from the other properties of an MRA that the system $\{\psi_{jk}\}$, $j, k \in \mathbb{Z}$, is then a frame for $L^2(\mathbb{R})$. We shall show that these systems form a proper subset of those constructed by our method in Section 2. In particular, the systems obtained by Benedetto and his collaborators, in [1], [2], are *semiorthogonal;* that is, $\psi_{jk} \perp \psi_{mn}$ if $j \neq m$ (i.e., the spaces W_j and W_m are orthogonal when they are the 2^j- and 2^m- dilates of W_0). This is not always the case for the frames we construct. Before showing these things it is useful to consider some properties of the frames we are considering. We shall often refer to such a generator ψ as being semiorthogonal if $\{\psi_{jk}\}$ is a semiorthogonal system.

Suppose $\varphi \in L^2(\mathbb{R})$ has the property that its integral translates $\varphi_n(x) = \varphi(x-n)$ form a frame for the closed subspace V_0 that they span. Thus, the inequalities (1.7) are satisfied for all $f \in V_0$. From these inequalities we can easily deduce several properties related to the *frame operator* $S : V_0 \to \ell^2 \equiv \ell^2(\mathbb{Z})$ defined by $Sf = \{< f, \varphi_n >\}$, $n \in \mathbb{Z}$. Most of these results and the consequences we derive from them are well known and we shall give appropriate references later in this section; we need to state them in the notation that is consistent with this article and we find it useful to include some of the arguments that establish them.

It follows from (1.7) that S is a bounded operator on V_0 with norm $||S|| \le \sqrt{B}$, and the range of S

$$\mathfrak{R} = \mathfrak{R}(S) = \left\{a \in \ell^2(\mathbb{Z}) : a = Sf \text{ for some } f \in V_0\right\}$$

is a closed subspace of $\ell^2(\mathbb{Z})$; moreover, S maps V_0 one-to-one, onto $\mathfrak{R}$. The inverse of S is also a one-to-one, onto map, $S^{-1} : \mathfrak{R} \to V_0$ and $||S^{-1}|| \le \frac{1}{\sqrt{A}}$. We consider the adjoint S^* of S to be a map from ℓ^2 onto V_0 or as a map from $\mathfrak{R}$ onto V_0. In the latter case S^* is one-to-one and has an inverse $(S^*)^{-1} = (S^{-1})^*$. We also have $||S^*|| \le \sqrt{B}$ and $||(S^*)^{-1}|| \le \frac{1}{\sqrt{A}}$.

A simple calculation shows that if $a = \{a_n\} \in \ell^2$, then

$$S^*a = \sum_{n\in\mathbb{Z}} a_n\varphi_n \,, \tag{5.1}$$

where the (say) symmetric partial sums of this series converge in the norm of $L^2(\mathbb{R})$. Taking Fourier transforms of both sides of (5.1) and writing $f = S^*a$ we obtain

$$\hat{f}(\xi) = \mu(\xi)\hat{\varphi}(\xi) \,, \tag{5.2}$$

where $\mu(\xi) = \sum_{n\in\mathbb{Z}} a_n e^{-in\xi}$. Since $a = \{a_n\} \in \ell^2$ this last series converges in the $L^2(T)$-norm, where T denotes the torus (which we identify with the interval $[0, 2\pi)$ together with addition modulo 2π). The norm on $L^2(T)$ we shall use is given by

$$||\mu||^2_{L^2(T)} = \int_0^{2\pi} |\mu(\xi)|^2 \frac{d\xi}{2\pi} = ||a||^2_{\ell^2} \tag{5.3}$$

and we consider μ to be a 2π periodic function on $\mathbb{R}$ having Fourier coefficients $a = \{a_n\}$. We can (and shall) consider the correspondence $\mu \longleftrightarrow a$ to be an identification of the space ℓ^2 with the space $L^2(T)$. Thus, we may regard S^* to be a mapping of $L^2(T)$ onto V_0 with norm not exceeding $\sqrt{B}$: $||S^*\mu||_2^2 \le B||\mu||^2_{L^2(T)}$. Moreover, since S^* is onto, every $f \in V_0$ satisfies equality (5.2).

If $\mu(\xi) = \sum_{n\in\mathbb{Z}} a_n e^{-in\xi} \in L^2(T)$ corresponds to $a = \{a_n\} \in \ell^2$, the last inequality gives us

$$\left\|S^*\mu\right\|_2^2 = \left\|S^*a\right\|_2^2 \le B||a||^2_{\ell^2} = B||\mu||^2_{L^2(T)} \tag{5.4}$$

In particular, we have shown the following.

Theorem 5.1. *If $\varphi \in L^2(\mathbb{R})$ is such that the sequence $\{\varphi_n\} = \{\varphi(\cdot - n)\}, n \in \mathbb{Z}$, is a frame for $V_0 = \overline{\operatorname{span}\{\varphi_n, n \in \mathbb{Z}\}}$, then $\mu\hat{\varphi} \in L^2(\mathbb{R})$ whenever μ is a 2π-periodic function in $L^2(T)$; moreover,*

$$V_0 = \left\{ f \in L^2 : \hat{f} = \mu\hat{\varphi},\ \mu\ 2\pi\text{-periodic in } L^2(T) \right\} .$$

The function μ for which (5.2) is satisfied for an $f \in V_0$ is not, in general, unique among the elements of $L^2(T)$. Since S^*, as an operator on $\mathfrak{R}$, is one-to-one and onto V_0, it follows from (5.1) that the unique element $a = \{a_n\} \in \mathfrak{R}$ such that $f = S^*a$ produces a unique $\lambda(\xi) = \sum_{n\in\mathbb{Z}} a_n e^{-in\xi}$ such that $\hat{f}(\xi) = \lambda(\xi)\hat{\varphi}(\xi)$. As we shall show, there is a simple characterization of these functions $\lambda \in L^2(T)$ for which the sequence of their Fourier coefficients is a member of $\mathfrak{R}$.

In order to obtain this characterization we introduce the function

$$\sigma_\varphi(\xi) = \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(\xi + 2k\pi)\right|^2 \tag{5.5}$$

which is immediately seen to belong to $L^1(T)$. Consider the 2π-periodic subset of $\mathbb{R}$

$$U = \left\{\xi \in \mathbb{R} : \sigma_\varphi(\xi) \neq 0\right\} .$$

The general $\lambda \in L^2(T)$ having Fourier coefficients that make up a sequence in $\mathfrak{R}$ has the form

$$\lambda(\xi) = \sum_{n\in\mathbb{Z}} < f, \varphi_n > e^{-in\xi}, \qquad f \in V_0 . \tag{5.6}$$

It is not hard to see that

$$\lambda(\xi) = \sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi)\overline{\hat{\varphi}(\xi + 2k\pi)} . \tag{5.7}$$

The expression on the right is clearly a 2π-periodic function in $L^1(T)$. Its Fourier coefficients are

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} e^{-in\xi} \sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi)\overline{\hat{\varphi}(\xi + 2k\pi)}\, d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\overline{\hat{\varphi}(\xi)} e^{-in\xi}\, d\xi \\ &= < f, \varphi_{-n} > \ , \end{aligned}$$

$n = 0, \pm 1, \pm 2, \cdots$. This, together with (5.6), shows (5.7) as well as the fact that $\lambda \in L^2(T)$.

Now suppose $\xi \in U^c = \{\eta : \sigma_\varphi(\eta) = 0\}$ and, thus, $\hat{\varphi}(\xi + 2k\pi) = 0$ for all $k \in \mathbb{Z}$. It follows that $\lambda(\xi) = 0$ and , consequently, $U^c \subset \{\xi : \lambda(\xi) = 0\}$. If $\lambda_1\hat{f} = \lambda_2\hat{f}$ for $\lambda_1, \lambda_2 \in L^2(T)$ and $\lambda_1(\xi) = 0 = \lambda_2(\xi)$ if $\xi \in U^c$, we claim that $\lambda_1(\xi) = \lambda_2(\xi)$. Indeed, λ_1 and λ_2 agree on U^c while, if $\xi \in U$ then there exists $k \in \mathbb{Z}$ such that $\hat{\varphi}(\xi + 2k\pi) \neq 0$. Since λ_1 and λ_2 are 2π-periodic,

$$\lambda_1(\xi)\hat{\varphi}(\xi + 2k\pi) = \hat{f}(\xi + 2k\pi) = \lambda_2(\xi)\hat{\varphi}(\xi + 2k\pi) ;$$

consequently, $\lambda_1(\xi) = \lambda_2(\xi)$ and the claim is established.

We have established the following.

Theorem 5.2. *Given $f \in V_0$ there exists precisely one function $\lambda = \lambda_f \in L^2(T)$ satisfying $\hat{f}(\xi) = \lambda(\xi)\hat{\varphi}(\xi)$ such that the sequence of Fourier coefficients of λ belongs to $\mathfrak{R}$. This is the unique function in $L^2(T)$ satisfying (5.2) that vanishes outside U. This establishes one-to-one correspondences between the pairs $(V_0, \mathfrak{R})$ and $(\mathfrak{R}, L^2(U))$, where $L^2(U) = \{\lambda \in L^2(T) : \lambda(\xi) = 0 \text{ if } \xi \in U^c\}$. $f \longleftrightarrow (S^*)^{-1} f = a$ and $a \longleftrightarrow \lambda_f$. Furthermore, $\lambda \in L^2(U)\,(\subset L^2(T))$ is characterized by the minimality property $||\lambda||_{L^2(T)} \leq ||\mu||_{L^2(T)}$ for all μ satisfying (5.2).*

The various norms of elements in $L^2(U)$, ℓ^2, V_0 we have been considering are related as follows:

$$\begin{aligned} A^2 \left\|\lambda_f\right\|^2_{L^2(U)} &= A^2||a||^2_{\ell^2} \leq A||f||^2_2 \leq \|\{< f, \varphi_n >\}\|^2_{\ell^2} \\ &\leq B||f||^2_2 \leq B^2||a||^2_2 = B^2 \left\|\lambda_f\right\|^2_{L^2(U)} . \end{aligned} \tag{5.8}$$

It is also of interest to characterize the functions $\varphi \in L^2(\mathbb{R})$ that generate such frames by their integral translates $\{\varphi_n\}$:

Theorem 5.3. *Suppose $\varphi \in L^2(\mathbb{R})$, then $\{\varphi_n\}$ is a frame for the space $V_0 = \overline{\text{span}\{\varphi_n : n \in \mathbb{Z}\}}$ if and only if there exists a 2π-periodic measurable subset $U \subset \mathbb{R}$ such that*

$$A\chi_U(\xi) \leq \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(\xi + 2k\pi)\right|^2 = \sigma_\varphi(\xi) \leq B\chi_U(\xi) \tag{5.9}$$

for a.e. $\xi \in \mathbb{R}$.

Proof. Suppose $\{\varphi_n\}$ satisfies (1.7) and suppose $F = \{\xi : \sigma_\varphi(\xi) > B\}$ has positive measure. Since χ_F is 2π-periodic it follows from Theorem 5.1 that the function f defined by $\hat{f} = \chi_F\hat{\varphi}$ belongs to V_0. But, using Theorem 5.2 and (5.8),

$$\begin{aligned} \frac{1}{2\pi}\,|F \cap [0, 2\pi]|\, B &< \int_0^{2\pi} \chi_F(\xi)\sigma_\varphi(\xi)\frac{d\xi}{2\pi} = \frac{1}{2\pi}\int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2 d\xi \\ &= ||f||^2_2 \leq B\,\|\chi_F\|^2_{L^2(T)} = \frac{B}{2\pi}|F \cap [0, 2\pi]| . \end{aligned}$$

This is clearly impossible and we conclude that $|F| = 0$. This establishes the right side of (5.9). A completely similar argument, in which the role of F is replaced by $G = \{\xi \in U : \sigma_\varphi(\xi) < A\}$, shows that this last set has measure 0. This shows that the inequalities (5.9) are satisfied when $\{\varphi_n\}$, $n \in \mathbb{Z}$, is a frame for V_0 satisfying (1.7).

Now suppose φ satisfies (5.9) and $f \in L^2(\mathbb{R})$. The argument we used to establish (5.6) and (5.7) shows that

$$\lambda(\xi) = \sum_{n\in\mathbb{Z}} < f, \varphi_n > e^{-in\xi} = \sum_{k\in\mathbb{Z}} \hat{f}(\xi + 2k\pi)\overline{\hat{\varphi}(\xi + 2k\pi)} .$$

Thus,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |< f, \varphi_n >|^2 &= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k\in\mathbb{Z}} \hat{f}(\xi+2k\pi)\overline{\hat{\varphi}(\xi+2k\pi)}\right|^2 d\xi \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \left\{\sum_{k\in\mathbb{Z}} \left|\hat{f}(\xi+2k\pi)\right|^2\right\}\left\{\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi+2k\pi)|^2\right\} d\xi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\{\sum_{k\in\mathbb{Z}} \left|\hat{f}(\xi+2k\pi)\right|^2\right\} \sigma_\varphi(\xi)\, d\xi\,.\end{aligned} \tag{5.10}$$

Applying the right-hand inequality in (5.9) to $\sigma_\varphi(\xi)$ (taking into account the 2π-periodicity of χ_U) we obtain

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |< f, \varphi_n >|^2 &\le \frac{B}{2\pi}\sum_{k\in\mathbb{Z}}\int_0^{2\pi} \left|\hat{f}(\xi+2k\pi)\right|^2 \chi_U(\xi+2k\pi)\, d\xi \\ &= \frac{B}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 \chi_U(\xi)\, d\xi \le \frac{B}{2\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\xi)\right|^2 d\xi = B\|f\|_2^2\,.\end{aligned}$$

Thus, the second inequality in (1.7) is true whenever $f \in L^2(\mathbb{R})$ (not just for $f \in V_0$). When $f \in V_0$ it follows from Theorem 5.2 that $\hat{f} = \hat{f}\chi_U$. Thus, applying the left-hand inequality in (5.9), we have

$$\begin{aligned}A\|f\|_2^2 &= \frac{A}{2\pi}\int_{-\infty}^{\infty} \chi_U(\xi)\left|\hat{f}(\xi)\right|^2 d\xi \le \frac{1}{2\pi}\int_{-\infty}^{\infty} \sigma_\varphi(\xi)\left|\hat{f}(\xi)\right|^2 d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{\sum_{k\in\mathbb{Z}} \left|\hat{f}(\xi+2k\pi)\right|^2\right\} \sigma_\varphi(\xi)\, d\xi = \sum_{n\in\mathbb{Z}} |< f, \varphi_n >|^2\,,\end{aligned}$$

where the last equality is a consequence of (5.10). This shows that the first inequality in (1.7) is valid as well when $f \in V_0$. □

The following result shows that one can replace the assumption that the translates $\{\varphi_n\}$ produce a general frame to the case where $\{\varphi_n\}$ forms a tight frame with constant 1.

Theorem 5.4. *Suppose $\varphi \in L^2(\mathbb{R})$ is such that $\{\varphi_n\} = \{\varphi(\cdot - n)\}, n \in \mathbb{Z}$, is a frame for the space $V_0 = \overline{\operatorname{span}\{\varphi_n, n \in \mathbb{Z}\}}$, then there exists $\tilde{\varphi} \in V_0$ such that $\{\tilde{\varphi}(\cdot - n)\}$ is a tight frame with constant* 1.

Proof. We have shown that under these hypotheses φ satisfies (5.9). Let $\tilde{\varphi}$ be defined by letting $(\tilde{\varphi})^\wedge = \dfrac{\hat{\varphi}}{\sqrt{\sigma_\varphi}}$. $\tilde{\varphi}$ is, then, well defined since $\hat{\varphi}(\xi) = 0$ when $\sigma_\varphi(\xi) = 0$ (we let $(\tilde{\varphi})^\wedge(\xi) = 0$ for such ξ). Moreover, $\lambda(\xi) = \dfrac{\chi_U(\xi)}{\sigma_\varphi(\xi)}$ is a bounded 2π-periodic function. It follows from Theorem 5.1 that $\tilde{\varphi} \in V_0$. Since $\sigma_{\tilde{\varphi}}(\xi) = \chi_U(\xi)$ (5.9) is transformed into an equality with $A = B = 1$, and the sequence $\{\tilde{\varphi}(\cdot - n)\}, n \in \mathbb{Z}$, forms a tight frame with constant 1. □

Let us now return to the construction of [1] and [2]. The following lemma is a version of Proposition (4.3) in the second of these citations.

Lemma 5.5. *Suppose $(\{V_j\}, \varphi)$ is an MRA of the type we described at the beginning of this section with $\{\varphi(\cdot - n)\}$, $n \in \mathbb{Z}$, a tight frame with constant 1 for V_0, then there exists a 2π-periodic function m, whose restriction to U is unique, satisfying*

(i) $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$ *for a.e.* ξ,

and

(ii) $|m(\xi)|^2 + |m(\xi+\pi)|^2$ *is either* 0 *or* 1 *when* $\xi \in U \cap (U+\pi)$,

(iii) $|m(\xi)|$ *is either* 0 *or* 1 *when* $\xi \in U \setminus (U+\pi)$.

Proof. Since $\hat{\varphi}(2\cdot) \in \hat{V}_{-1} \subset \hat{V}_0$ Theorem 5.2 tells us that there exists a unique $m \in L^2(T)$ that vanishes on $\mathbb{R} \setminus U = U^c$ satisfying (i). The values m assumes on U^c are irrelevant to the validity of (i); we shall see that these are appropriate non-zero choices for us later on. If $\xi \in U$, then there exists $k_0 \in \mathbb{Z}$ such that $\hat{\varphi}(\xi + 2k_0\pi) \neq 0$ and we have

$$m(\xi) = m(\xi + 2k_0\pi) = \frac{\hat{\varphi}(2\xi + 4k_0\pi)}{\hat{\varphi}(\xi + 2k_0\pi)}$$

and, in particular, we see that m is completely determined on U.

Since $\sigma_\varphi(\xi) = \chi_U(\xi)$ (see Theorem 5.4 and its proof) we have, summing separately over k even and k odd,

$$\begin{aligned}
\chi_U(2\xi) &= \sum_{k\in\mathbb{Z}} \left|\hat{\varphi}(2\xi + 2k\pi)\right|^2 \\
&= \sum_{\ell\in\mathbb{Z}} \left|\hat{\varphi}(2(\xi + 2\ell\pi))\right|^2 + \sum_{\ell\in\mathbb{Z}} \left|\hat{\varphi}(2(\xi + (2\ell+1)\pi))\right|^2 \\
&= \sum_{\ell\in\mathbb{Z}} \left|\hat{\varphi}(\xi + 2\ell\pi)\right|^2 |m(\xi + 2\ell\pi)|^2 \\
&\quad + \sum_{\ell\in\mathbb{Z}} \left|\hat{\varphi}(\xi + \pi + 2\ell\pi)\right|^2 |m(\xi + \pi + 2\ell\pi)|^2 \\
&= |m(\xi)|^2 \chi_U(\xi) + |m(\xi+\pi)|^2 \chi_U(\xi+\pi)\,.
\end{aligned}$$

It is clear that (ii) and (iii) are an immediate consequence of the last equalities. □

Given this result we shall describe the method used in [2] and [1] to obtain the function ψ that generates the wavelet tight frame associated with this MRA. We believe that we are clarifying the ideas if we express them in terms of the notion of a generalized low pass filter we have introduced in this article and the construction of a wavelet ψ by means of equality (2.6). Lemma 5.5 provides us with the function m, defined on U, that is the candidate for the generalized low pass filter we seek. In order to have equality (1.3) satisfied on $U \cap (U+\pi)$ we shall assume that $|m(\xi)|^2 + |m(\xi+\pi)|^2 > 0$ a.e. on this intersection. Lemma 5.5 (ii) then assures us that (1.3) is true for these ξ. If $\xi \in U \cap (U+\pi)^c$, Lemma 5.5 (iii) tells us that $|m(\xi)|$ is either 0 or 1. In this case, $\eta = \xi + \pi$ lies outside U and this allows us to define $m(\xi+\pi) = m(\eta)$ so that $|m(\xi+\pi)|$ is 0 if $|m(\xi)| = 1$ and $|m(\xi+\pi)| = 1$ if $|m(\xi)|$ is 0. In fact, this extends m to $U^c \cap (U+\pi)$. On the remaining set $U^c \cap (U+\pi)^c$ we can extend m so that the resulting function is measurable, 2π-periodic and satisfies (1.3) (since $\xi \in U^c \cap (U+\pi)^c$ if and only if $\xi + \pi \in U^c \cap (U+\pi)^c$); however, these values play no role in our discussion.

In order to obtain the generator, ψ, of the MRA TFW, we have to produce such a function $\psi \in W_0 = V_1 \cap V_0^\perp$ such that $\{\psi_n\} = \{\psi(\cdot - n)\}$, $n \in \mathbb{Z}$, is a tight frame for the space W_0. We do this by defining ψ to be the function satisfying

$$\hat{\psi}(2\xi) = e^{i\xi}\overline{m(\xi+\pi)}\hat{\varphi}(\xi)\,; \tag{5.11}$$

that is, we use equation (2.2) with m defined as we described in the last paragraph. We shall show that $\{\psi_{jk}\}$, $j, k \in \mathbb{Z}$, is a tight frame for $L^2(\mathbb{R})$. In particular, this also shows that the systems obtained in [2] and [1] are of the type we introduced in this work. We shall also show that if $\psi \in W_0$ does generate a tight frame, $\{\psi(\cdot - n)\}$, $n \in \mathbb{Z}$, for W_0, then it is of the form (5.11). In addition we shall present an example of an MRA TFW of the type we constructed in Section 2 that is not one that we have just produced; thus, the class of MRA TFW we obtain is strictly larger than the class produced in [2] and [1].

Assuming, then, that ψ satisfies (5.11), with m described in the paragraph that follows the proof of Lemma 5.5, we have, since $\sigma_\varphi(\xi) = \chi_U(\xi)$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\left|\hat{\psi}(2\xi+2n\pi)\right|^2 &= \sum_{n\in\mathbb{Z}}|m(\xi+n\pi+\pi)|^2\left|\hat{\varphi}(\xi+n\pi)\right|^2\\ &= \sum_{\ell\in\mathbb{Z}}|m(\xi+2\ell\pi+\pi)|^2\left|\hat{\varphi}(\xi+2\ell\pi)\right|^2\\ &\quad+\sum_{\ell\in\mathbb{Z}}|m(\xi+2(\ell+1)\pi)|^2\left|\hat{\varphi}(\xi+\pi+2\ell\pi)\right|\\ &= |m(\xi+\pi)|^2\chi_U(\xi)+|m(\xi)|^2\chi_U(\xi+\pi)\,.\end{aligned}$$

Since $|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1$ if $\xi \in U \cap (U+\pi)$ and $|m(\xi)|$ is either 0 or 1 when $\xi \in U \cap (U+\pi)^c$ or $\xi \in U^c \cap (U+\pi)$ we see that

$$\sigma_\psi(2\xi) \equiv \sum_{n\in\mathbb{Z}}\left|\hat{\psi}(2\xi+2n\pi)\right|^2 \tag{5.12}$$

assumes only the values 0 or 1 a.e. (observe that this is independent of how m is defined on $U^c \cap (U+\pi)^c$). That is,

$$\sigma_\psi(\eta) = \chi_E(\eta) \tag{5.13}$$

a.e., where E is a 2π-periodic measurable set $E \subset \mathbb{R}$. Moreover, since m is a 2π-periodic function in $L^2(T)$, Theorem 5.1 and equality (5.11) imply that $\psi(2^{-1}\cdot) \in V_0$. In this case this is equivalent to $\psi \in V_1$. We want to show that $\psi \in W_0$ (equivalently, $\psi \perp V_0$) and, finally that $\{\psi(\cdot - n)\}$, $n \in \mathbb{Z}$, is a tight frame for W_0.

We shall show that $\psi(\cdot - n) \perp \varphi$ for all $n \in \mathbb{Z}$ and, thus, $\operatorname{span}\{\psi(\cdot - n) : n \in \mathbb{Z}\} \subset W_0$. By the Plancherel theorem this orthogonality is equivalent to

$$\int_{\mathbb{R}}\hat{\psi}(\eta)e^{-in\eta}\overline{\hat{\varphi}(\eta)}\,d\eta = 0$$

for all $n \in \mathbb{Z}$. Changing variables by letting $\eta = 2\xi$ this equality is equivalent to

$$0 = \int_0^{2\pi} e^{-in2\xi}\sum_{k\in\mathbb{Z}}\hat{\psi}(2\xi+2k\pi)\overline{\hat{\varphi}(2\xi+2k\pi)}\,d\xi$$

for all $n \in \mathbb{Z}$. Since this last sum is a π-periodic function in $L^1([0,\pi))$ this last equality is equivalent to

$$\sum_{k\in\mathbb{Z}}\hat{\psi}(2\xi+2k\pi)\overline{\hat{\varphi}(2\xi+2k\pi)} = 0 \tag{5.14}$$

for a.e. $\xi \in \mathbb{R}$. Using 5.11 and Lemma 5.5 (i) the left-hand side of (5.14) equals

$$\begin{aligned}\sum_{k\in\mathbb{Z}} & e^{i\xi}\overline{m(\xi+k\pi+\pi)\hat{\varphi}(\xi+k\pi)}\overline{m(\xi+k\pi)\hat{\varphi}(\xi+k\pi)} \\ &= e^{i\xi}\overline{m(\xi+\pi)m(\xi)}\left[\sigma_\varphi(\xi)-\sigma_\varphi(\xi+\pi)\right] \\ &= e^{i\xi}\overline{m(\xi+\pi)m(\xi)}\left[\chi_U(\xi)-\chi_U(\xi+\pi)\right] .\end{aligned}$$

But the difference in the bracket is 0 if $\xi \in U \cap (U+\pi)$ or $\xi \in U^c \cap (U+\pi)^c$. Furthermore, our extension of m to $[U^c \cap (U+\pi)] \cup [U \cap (U+\pi)^c]$ is such that the product $\overline{m(\xi+\pi)}\,\overline{m(\xi)}$ is 0 on this union. This establishes (5.14) and, consequently, $\widetilde{W_0} \equiv \overline{\operatorname{span}\{\psi(\cdot - n) : n \in \mathbb{Z}\}} \subset W_0 = V_1 \cap V_0^{\perp}$.

Finally, we show that $\widetilde{W_0} = W_0$. Let P be the orthogonal projection of $L^2(\mathbb{R})$ onto $V_0 \oplus \widetilde{W_0}$. Since $V_1 = V_0 \oplus W_0$ the desired equality is true if the image of P is V_1. The general $f \in V_1$ satisfies

$$\hat{f}(2\xi) = \mu(\xi)\hat{\varphi}(\xi) , \tag{5.15}$$

where μ is a 2π periodic function in $L^2(T)$ (this follows from Theorem 5.1 and the fact that $\{V_j\}$, $j \in \mathbb{Z}$, is a tight frame MRA (TF-MRA), as described at the beginning of this section, with $\{\varphi_n\}$ a tight frame for V_0). Since m satisfies (1.3) we have

$$\begin{aligned}\mu(\xi)\hat{\varphi}(\xi) &= \mu(\xi)\left\{m(\xi)\overline{m(\xi)} + e^{i\xi}\overline{m(\xi+\pi)}e^{-i\xi}m(\xi+\pi)\right\}\hat{\varphi}(\xi) \\ &= \mu(\xi)\overline{m(\xi)}\hat{\varphi}(2\xi) + \mu(\xi)e^{-i\xi}m(\xi+\pi)\hat{\psi}(2\xi) .\end{aligned}$$

Thus,

$$\hat{f}(2\xi) = \mu(\xi)\overline{m(\xi)}\hat{\varphi}(2\xi) + \mu(\xi)e^{-i\xi}m(\xi+\pi)\hat{\psi}(2\xi) . \tag{5.16}$$

The equality (B) in Theorem 1.7 of Chapter 3 in [4] can easily be seen to be valid if we apply it to the projections onto the subspaces $\overline{\operatorname{span}\{\varphi(\cdot - n) : n \in \mathbb{Z}\}}$ and $\widetilde{W_0}$. That is,

$$(Pf)^\wedge(2\xi) = \sum_{n\in\mathbb{Z}} \hat{f}(2\xi+2n\pi)\left\{\overline{\hat{\varphi}(2\xi+2n\pi)}\hat{\varphi}(2\xi) + \overline{\hat{\psi}(2\xi+2n\pi)}\hat{\psi}(2\xi)\right\} .$$

The last expression, using (5.15) and the usual summation over the odd and even n, then gives us

$$\begin{aligned}&(Pf)^\wedge(2\xi) \\ &= \left\{\mu(\xi)\overline{m(\xi)}\chi_U(\xi) + \mu(\xi+\pi)\overline{m(\xi+\pi)}\chi_U(\xi+\pi)\right\}\hat{\varphi}(2\xi) \\ &\quad + \left\{\mu(\xi)e^{-i\xi}m(\xi+\pi)\chi_U(\xi)\right. \\ &\quad \left. - \mu(\xi+\pi)e^{-i\xi}m(\xi)\chi_U(\xi+\pi)\right\}\hat{\psi}(2\xi) .\end{aligned} \tag{5.17}$$

We now claim.

Lemma 5.6. *$(Pf)^\wedge = \hat{f}$ whenever $f \in V_1$.*

Remark: As pointed out above this implies the desired result $\widetilde{W_0} = W_0$.

Proof of Lemma 5.6. If $\xi \in U \cap (U+\pi)^c$ then $\xi + \pi \notin U$ (recall that U is a 2π-periodic set) and (5.17) reduces to

$$(Pf)^\wedge(2\xi) = \mu(\xi)\overline{m(\xi)}\hat{\varphi}(2\xi) + \mu(\xi)e^{-i\xi}m(\xi+\pi)\hat{\psi}(2\xi) = \hat{f}(2\xi)$$

(the last equality is (5.16)).

If $\xi \notin U$ then $(Pf)^\wedge(2\xi) = 0$ since $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi) = 0 = e^{i\xi}\overline{m(\xi+\pi)}\cdot\hat{\varphi}(\xi) = \hat{\psi}(2\xi)$ (recall that $\operatorname{supp}\hat{\varphi} \subset U$). Also, by (5.15), $\hat{f}(2\xi) = \mu(\xi)\hat{\varphi}(\xi) = 0$. Again we obtain $(Pf)^\wedge(2\xi) = \hat{f}(2\xi)$.

Finally, if $\xi \in U \cap (U+\pi)$ we have, by (5.16) and (5.17),

$$\begin{aligned}
&\hat{f}(2\xi) - (Pf)^\wedge(2\xi) \\
&= \mu(\xi)\overline{m(\xi)}\hat{\varphi}(2\xi) + \mu(\xi)e^{-i\xi}m(\xi+\pi)\hat{\psi}(2\xi) \\
&\quad - \left\{\mu(\xi)\overline{m(\xi)} + \mu(\xi+\pi)\overline{m(\xi+\pi)}\right\} m(\xi)\hat{\varphi}(\xi) \\
&\quad - \left\{\mu(\xi)e^{-i\xi}m(\xi+\pi) - \mu(\xi+\pi)e^{-i\xi}m(\xi)\right\} e^{i\xi}\overline{m(\xi+\pi)}\hat{\varphi}(\xi) \\
&= 0\,.
\end{aligned}$$

□

Let us summarize what we have established with the arguments we just presented.

Theorem 5.7. *Suppose $\{V_j\}$, $j \in \mathbb{Z}$, is a tight frame MRA with $\varphi \in V_0$ such that $\{\varphi(\cdot - n)\}$, $n \in \mathbb{Z}$, is a tight frame for V_0. Suppose the function m, defined on U, by equality (i) in Lemma 5.5 satisfies $|m(\xi)|^2 + |m(\xi+\pi)|^2 > 0$ for $\xi \in U \cap (U+\pi)$. Then m can be extended to all of $\mathbb{R}$ so that this extension, also denoted by m, is a generalized low pass filter such that the function $\psi \in W_0 = V_1 \cap V_0^\perp$, defined by* (5.11), *generates a tight frame, $\{\psi(\cdot - n)\}$, $n \in \mathbb{Z}$, for W_0. As a consequence,*

$$\{\psi_{jk}(x)\} = \left\{2^{\frac{j}{2}}\psi\left(2^j x - k\right)\right\}, \; j, k \in \mathbb{Z}\,,$$

is an MRA wavelet tight frame for $L^2(\mathbb{R})$.

This is, essentially, the construction of [2] and [1]. We presented it in the terms we introduced in this article and, by doing so, we did not need the smoothness or decrease at ∞ conditions imposed in the two papers cited. We are also restricting our attention to one dimension; [1] and [2] consider these frames in $\mathbb{R}^n$ as is the case in other treatments of this subject, (see [6]). We shall comment on these other approaches. Before doing so, however, we establish a "converse" of Theorem 5.7.

Theorem 5.8. *Suppose $(\{V_j\}, \varphi)$ is a tight frame MRA (as in the last theorem) and there exists a $\psi \in W_0 = V_1 \cap V_0^\perp$ such that $\{\psi(\cdot - n)\}$ is a tight frame for W_0, then ψ satisfies* (5.11),

$$\hat{\psi}(2\xi) = e^{i\xi}\overline{m_0(\xi+\pi)}\hat{\varphi}_0(\xi)\,, \tag{5.18}$$

where m_0 is a generalized low pass filter in $\widetilde{\mathbf{F}}_{\varphi_0}$ and φ_0 is also a generator of a tight frame $\{\varphi_0(\cdot - n)\}$, $n \in \mathbb{Z}$, for V_0.

Proof. By Theorem 5.1 there exists an $m_1 \in L^2(T)$, 2π periodic and completely determined on U such that

$$\hat{\psi}(2\xi) = m_1(\xi)\hat{\varphi}(\xi)\,. \tag{5.19}$$

We also have a function $m \in L^2(T)$, 2π periodic and completely determined on U such that Lemma 5.5 (i) is satisfied: $\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)$. We claim that

$$|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1 \text{ for } \xi \in U \cap (U+\pi) . \tag{5.20}$$

By Lemma 5.5 (ii) it suffices to show that the sum in (5.20) is positive for almost all such ξ. Suppose this were not the case, then the set

$$E = \{\xi \in U \cap (U+\pi) : m(\xi) = 0 = m(\xi+\pi)\}$$

has positive measure. Let $\lambda_1(\xi) = \chi_E(\xi)$ and

$$\lambda_2(\xi) = \begin{cases} \chi_E(\xi) & \text{if } \xi \in [2k\pi, (2k+1)\pi) \\ -\chi_E(\xi) & \text{if } \xi \in [(2k+1)\pi, 2(k+1)\pi) \end{cases}$$

for $k \in \mathbb{Z}$. Since E is π-periodic, both of these functions are 2π periodic (in fact, λ_1 is π-periodic). We then define f_1 and f_2 by letting $\hat{f}_j(2\xi) = \lambda_j(\xi)\hat{\varphi}(\xi)$, $j = 1, 2$. It is clear that f_1, f_2 belong to V_1. We claim that $f_1, f_2 \in W_0$. This follows from the fact that

$$\sum_{k\in\mathbb{Z}} \hat{f}_j(2\xi + 2k\pi)\overline{\hat{\varphi}(2\xi + 2k\pi)}$$
$$= \lambda_j(\xi)\overline{m(\xi)}\chi_U(\xi) + \lambda_j(\xi+\pi)\overline{m(\xi+\pi)}\chi_U(\xi+\pi) = 0$$

(since $\operatorname{supp} \lambda_j \subset E$ and $m(\xi) = 0 = m(\xi+\pi)$ for $\xi \in E$) and, thus, $f_j \perp V_0$, $j = 1, 2$. Now, since we determined that these two functions are in W_0, we can find (again, using Theorem 5.1) 2π periodic $\alpha_j \in L^2(T)$, $j = 1, 2$, such that

$$\hat{f}_j(\xi) = \alpha_j(\xi)\hat{\psi}(\xi) = \alpha_j(\xi)m_1\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right) .$$

From these equalities and the definition of f_j (after multiplying by $\overline{\hat{\varphi}(\xi)}$) we obtain

$$\lambda_j(\xi)\left|\hat{\varphi}(\xi)\right|^2 = \alpha_j(2\xi)m_1(\xi)\left|\hat{\varphi}(\xi)\right|^2 .$$

Applying the usual periodization argument, we obtain

$$\lambda_j(\xi)\chi_U(\xi) = \alpha_j(2\xi)m_1(\xi)\chi_U(\xi),\ j = 1, 2 . \tag{5.21}$$

Since the left side of (5.21) equals $\pm\chi_E(\xi)$ it follows that $\alpha_j(2\xi) \neq 0$, for $j = 1, 2$, when $\xi \in E$.

Since E is π-periodic it follows that if $\xi \in E \cap [0, \pi)$, then $\xi + \pi \in E \cap [\pi, 2\pi)$. When $\xi \in E \cap [0, \pi)$ then $\lambda_1(\xi) = \lambda_2(\xi)$ and, using (5.21), we have

$$\alpha_1(2\xi)m_1(\xi) = \alpha_2(2\xi)m_1(\xi) .$$

For this same ξ we also have $\lambda_1(\xi+\pi) = -\lambda_2(\xi+\pi)$; using the 2π periodicity of α_1 and α_2 we then have

$$\alpha_1(2\xi)m_1(\xi+\pi) = -\alpha_2(2\xi)m_1(\xi+\pi) .$$

Suppose neither $m_1(\xi)$ nor $m_1(\xi+\pi)$ is 0, then the last two equalities imply $\alpha_2(2\xi) = \alpha_1(2\xi) = -\alpha_2(2\xi)$ which is impossible since $\alpha_j(2\xi) \neq 0$, $j = 1, 2$, when $\xi \in E$. It follows that either $m_1(\xi) = 0$ or $m_1(\xi+\pi) = 0$ when $\xi \in E \cap [0, \pi)$. Since we are assuming that $|E| > 0$ and, since E is π-periodic, $|E \cap [0, \pi)| > 0$, at least one of the two sets in the union $\{\xi \in E \cap [0, \pi) : m_1(\xi) = 0\} \cup \{\xi \in E \cap [0, \pi) : m_1(\xi+\pi) = 0\} = E \cap [0, \pi)$ must have positive measure.

Since E is π-periodic it follows that $\tilde{E} \equiv \{\eta : m_1(\eta) = 0\} \cap E$ has positive measure. Let f be the nonzero function defined by the equality $\hat{f}(2\xi) = \chi_{\tilde{E}}(\xi)\hat{\varphi}(\xi)$. The argument that showed that $f_j \in W_0$, $j = 1, 2$, applies to f. Moreover, $\chi_{\tilde{E}}(\xi) m_1(\xi) = 0$ a.e. and this implies that $f \perp \overline{\text{span}\{\psi(\cdot - n) : n \in \mathbb{Z}\}}$ and it follows that $\{\psi(\cdot - n)\}$ cannot be a tight frame for W_0. This contradiction establishes the claim that m satisfies (5.20).

We can now use Theorem 5.7 and deduce that the function $\tilde{\psi}$ satisfying $(\tilde{\psi})^\wedge(2\xi) = e^{i\xi}\overline{m(\xi + \pi)}\hat{\varphi}(\xi)$ generates a tight frame $\{\tilde{\psi}(\cdot - n)\}, n \in \mathbb{Z}$, for W_0. By Theorem 5.1 we see that $\hat{\psi}(\xi) = \mu(\xi)(\tilde{\psi})^\wedge(\xi)$, where μ is a 2π periodic function in $L^2(T)$. Hence, $\sigma_\psi(\xi) = |\mu(\xi)|^2 \sigma_{\tilde{\psi}}(\xi)$ a.e.; moreover, it is clear that σ_ψ and $\sigma_{\tilde{\psi}}$ must be equal a.e. to the characteristic function of the same set $S \subset \mathbb{R}$. Thus, $|\mu(\xi)| = 1$ a.e. on this set and we might as well assume that this equality is true a.e. on $\mathbb{R}$. Now let α be a unimodular 2π periodic solution to the functional equation

$$\mu(2\xi) = \overline{\alpha(2\xi)}\alpha(\xi + \pi)\alpha(\xi) . \tag{5.22}$$

(See [7, Lemma 2.1], where this equation is discussed in detail. Also recall that we used this functional equation to obtain Theorem 3.3). We claim that $\hat{\varphi}_0(\xi) = \alpha(\xi)\hat{\varphi}(\xi)$ generates a tight frame $\{\varphi_0(\cdot - n)\}, n \in \mathbb{Z}$, for V_0 such that

$$\hat{\varphi}_0(2\xi) = m_0(\xi)\hat{\varphi}_0(\xi) , \tag{5.23}$$

where m_0 is a generalized low pass filter for which (5.18) is true. This claim establishes Theorem 5.8 and its proof is simple: first it is clear that $m_0(\xi) = \alpha(2\xi)\overline{\alpha(\xi)}m(\xi)$ satisfies (5.23). Moreover,

$$\begin{aligned} \hat{\psi}(2\xi) &= \mu(2\xi)\left(\tilde{\psi}\right)^\wedge(2\xi) = \mu(2\xi)e^{i\xi}\overline{m(\xi+\pi)}\hat{\varphi}(\xi) \\ &= e^{i\xi}\overline{\alpha(2\xi)}\alpha(\xi+\pi)\alpha(\xi)\overline{m(\xi+\pi)}\hat{\varphi}(\xi) \\ &= e^{i\xi}\overline{m_0(\xi+\pi)}\hat{\varphi}_0(\xi) . \end{aligned}$$

□

We have described the MRA wavelet frames that were introduced in [1] and [2]. As we have already stated we did this in terms of the construction we developed in Section 2; we also claimed that the class of frames we obtained is more general. Let us be specific about this. Theorems 5.7 and 5.8 can also be used in order to characterize the class of *all* MRA (tight) frame wavelets $\psi \in L^2(\mathbb{R})$ that are semiorthogonal. Recall that this term means that the spaces $W_j = \overline{\text{span}\{\psi(2^j \cdot - k) : k \in \mathbb{Z}\}}$ are orthogonal to each other as j ranges through $\mathbb{Z}$. The construction of ψ as a function in W_0, a space orthogonal to V_0, makes it clear that $\{\psi_{jk}\}$, $j, k \in \mathbb{Z}$, is a semiorthogonal tight frame for $L^2(\mathbb{R})$; that is, ψ is a MRA tight frame wavelet. Conversely, given a semiorthogonal MRA tight frame wavelet, as defined by Definition 2.7 with φ a pseudo-scaling function and $m \in \tilde{F}_\varphi$, it can be shown that it is constructed, as in Theorem 5.7. More precisely, we shall show that we have the following characterization of the semiorthogonal MRA tight frame wavelets.

Theorem 5.9. *ψ is a semiorthogonal MRA tight frame wavelet(MRA TFW) if and only if ψ is a tight frame MRA wavelet.*[2]

[2] In order to avoid confusion we remind the reader that an "MRA tight frame wavelet" is defined by Definition 2.7 and does not involve a "tight frame MRA." A "tight frame MRA wavelet," on the other hand, is a function $\psi \in W_0 = V_1 \cap V_0^\perp$ whose translates form a tight frame for W_0.

Proof. We only need to show the "only if" direction since the converse, as we just explained, is already established. Thus, we assume ψ is an MRA TFW which is semiorthogonal. We claim that the sequence of subspaces $V_j = \bigoplus_{\ell=-\infty}^{j-1} W_\ell$, together with the function φ in equality (2.6), form a tight frame MRA. That is, the translates $\varphi(\cdot - n)$, $n \in \mathbb{Z}$, form the desired tight frame for V_0. From equations (2.6) and (1.3) we obtain

$$|\hat{\varphi}(\xi)|^2 = \sum_{j=1}^{\infty} \left|\hat{\psi}\left(2^j \xi\right)\right|^2 \tag{5.24}$$

(see equality (2.16) on p. 61 of [4]). We claim that

$$\sigma_\varphi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)|^2 = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} \left|\hat{\psi}\left(2^\ell(\xi + 2k\pi)\right)\right|^2 = \chi_U(\xi) \tag{5.25}$$

for some 2π-periodic set U. By Theorem 5.3 this would then tell us that $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a tight frame for $S \equiv \overline{\text{span}\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}}$. We also claim that

$$S = V_0 = \bigoplus_{j=-\infty}^{-1} W_j \,. \tag{5.26}$$

Once these claims are established, since the semiorthogonality of the system $\{\psi_{jk}\}$ implies that $\{\psi_{0k}\} = \{\psi(\cdot - k)\}$, $k \in \mathbb{Z}$, is a tight frame for the subspace W_0, we can invoke Theorem 5.8 to obtain the desired conclusion that ψ is an MRA tight frame wavelet. This would establish Theorem 5.9.

In order to establish (5.25) we fix ξ and let $v_\ell = \{v_\ell(n) : n \in \mathbb{Z}\} \equiv \{\hat{\psi}(2^\ell(\xi + 2n\pi)) : n \in \mathbb{Z}\}$ (for a.e. $\xi \in \mathbb{R}$, $v_\ell \in \ell^2$). Equality (3.3) in Chapter 7 of [4] is easily seen to be valid in our case:

$$\hat{\psi}\left(2^\ell \xi\right) = \sum_{\ell=1}^{\infty}\sum_{k\in\mathbb{Z}} \hat{\psi}\left(2^j(\xi + 2k\pi)\right) \overline{\hat{\psi}\left(2^\ell(\xi + 2k\pi)\right)\hat{\psi}}\left(2^\ell \xi\right) .$$

Replacing ξ by $\xi + 2n\pi$, $n \in \mathbb{Z}$, in this last equality we obtain

$$v_j = \sum_{\ell=1}^{\infty} < v_j, v_\ell > v_\ell \,. \tag{5.27}$$

We now apply (3.7) in Chapter 7 of [4] in order to have

$$\sum_{\ell=1}^{\infty} \|v_\ell\|^2 = \dim \left\{\overline{\text{span}\{v_\ell : \ell \ge 1\}}\right\} .$$

That is,

$$\begin{aligned} D_\psi(\xi) &= \sum_{\ell=1}^{\infty}\sum_{k\in\mathbb{Z}} \left|\hat{\psi}\left(2^\ell(\xi + 2k\pi)\right)\right|^2 \\ &= \dim \overline{\left\{\text{span}\left\{\hat{\psi}\left(2^\ell(\xi + 2k\pi)\right)\right\}_{n\in\mathbb{Z}} : \ell \ge 1\right\}} . \end{aligned}$$

If we apply (2.6), $\hat{\psi}(2\eta) = e^{i\eta}\overline{m(\eta+\pi)}\hat{\varphi}(\eta)$, when $\eta = 2^\ell(\xi + 2n\pi)$, $n \in \mathbb{Z}$, we obtain

$$v_1 = \left\{\hat{\psi}(2(\xi + 2n\pi))\right\}_{n\in\mathbb{Z}} = e^{i\xi}\overline{m(\xi+\pi)}\left\{\hat{\varphi}(\xi + 2n\pi)\right\}_{n\in\mathbb{Z}}$$

and, for $\ell \geq 2$,

$$\begin{aligned} v_\ell &= \left\{\hat{\psi}\left(2^\ell(\xi + 2k\pi)\right)\right\}_{n\in\mathbb{Z}} \\ &= \left\{e^{i2^{\ell-1}\xi}\overline{m\left(2^{\ell-1}\xi+\pi\right)\prod_{k=0}^{\ell-2} m\left(2^k\xi\right)}\right\}\left\{\hat{\varphi}(\xi + 2k\pi)\right\}_{n\in\mathbb{Z}} . \end{aligned}$$

This shows that v_ℓ is a multiple of the same vector $\{\hat{\varphi}(\xi + 2k\pi)\}_{n\in\mathbb{Z}}$ for all $\ell \geq 1$. Consequently, the dimension of $\overline{\operatorname{span}\{v_\ell : \ell \geq 1\}}$ is either 1 or 0. Hence, $D_\psi(\xi) = \chi_U(\xi)$, where U is a 2π-periodic set. This proves (5.25).

We only need to show

$$V_0 = \bigoplus_{j=-\infty}^{-1} W_j = \overline{\operatorname{span}\{\varphi_n\}_{n\in\mathbb{Z}}} \equiv S . \tag{5.28}$$

It follows immediately from (2.6) that

$$\widehat{(\psi_{j0})}(\xi) = 2^{-\frac{j}{2}}e^{i2^{-j-1}\xi}\overline{m\left(2^{-j-1}\xi+\pi\right)\prod_{k=0}^{-j-2} m\left(2^k\xi\right)}\hat{\varphi}(\xi)$$

for $j < 0$ ($\prod_{k=0}^{-1} m(2^k\xi)$ is to be interpreted to be 1). This means that $\widehat{(\psi_{j0})}(\xi) = \mu(\xi)\hat{\varphi}(\xi)$ for some 2π-periodic $\mu \in L^2(T)$. Thus, ψ_{j0} and, consequently, $\psi_{jk} \in S$ for all $k \in \mathbb{Z}$ and $j < 0$. This shows that $V_0 = \bigoplus_{j=-\infty}^{-1} W_j \subset S$. If we show that $S \perp W_j$ for all $j \geq 0$ we would then have the desired equality (5.28). Toward this end we observe that it suffices to show $S \perp W_0$ since, for $j > 0$,

$$< \psi_{jk}, \varphi_\ell > = \frac{2^{\frac{j}{2}}}{2\pi} < \hat{\psi}, \mu\hat{\varphi} >, \text{ where } \mu(\eta) = e^{ik\eta}e^{-i\ell 2^j\eta}\prod_{n=0}^{j-1} m\left(2^n\eta\right) .$$

It follows that $\mu\hat{\varphi} \in \hat{S}$ and, then, the assumed orthogonality $S \perp W_0$ shows that the last inner product is 0 and, consequently, $< \psi_{jk}, \varphi_\ell > = 0$ for all $k, \ell \in \mathbb{Z}$, $j \geq 0$.

Our proof then is finished if we show $S \perp W_0$. But, using (5.11) and Lemma 5.5 (i), we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \hat{\psi}(2\xi + 2k\pi)\overline{\hat{\varphi}(2\xi + 2k\pi)} &= e^{i\xi}\overline{m(\xi+\pi)m(\xi)}\left\{\sigma_\varphi(\xi) - \sigma_\varphi(\xi+\pi)\right\} \\ &= e^{i\xi}\overline{m(\xi+\pi)m(\xi)}\left\{\chi_U(\xi) - \chi_U(\xi+\pi)\right\} . \end{aligned}$$

If $\xi \in [U \cap (U+\pi)] \cup [U^c \cap (U+\pi)^c]$, $\chi_U(\xi) - \chi_U(\xi+\pi) = 0$. When $\xi \in [U \cap (U+\pi)^c] \cup [U^c \cap (U+\pi)]$, Lemma 5.5) (iii) and (1.3) imply that either $m(\xi) = 0$ or $m(\xi+\pi) = 0$. Thus,

$$\sum_{k\in\mathbb{Z}} \hat{\psi}(2\xi + 2k\pi)\overline{\hat{\varphi}(2\xi + 2k\pi)} = 0$$

and, consequently,

$$
\begin{aligned}
2\pi < \psi, \varphi(\cdot - n) > &= \int_{-\infty}^{\infty} \hat{\psi}(\xi)\overline{\hat{\varphi}(\xi)} e^{in\xi}\, d\xi \\
&= 2\int_{-\infty}^{\infty} \hat{\psi}(2\xi)\overline{\hat{\varphi}(2\xi)} e^{2in\xi}\, d\xi = 2\sum_{k\in\mathbb{Z}} \int_{k\pi}^{(k+1)\pi} \hat{\psi}(2\xi)\overline{\hat{\varphi}(2\xi)} e^{2in\xi}\, d\xi \\
&= 2\int_0^{\pi} \left(\sum_{k\in\mathbb{Z}} \hat{\psi}(2\xi+2k\pi)\overline{\hat{\varphi}(2\xi+2k\pi)}\right) e^{2in\xi}\, d\xi = 0\,.
\end{aligned}
$$

This implies $S \perp W_0 = 0$ and the proof of Theorem 5.9 is completed. □

This shows that the systems studied in [1] and [2] are those we have introduced in Section 2 that are semiorthogonal. We present an example of an MRA tight frame wavelet that is not semiorthogonal. Let $m(\xi) = \frac{1}{2}(1 + e^{3i\xi})$ and $\hat{\varphi}(\xi) = \frac{i}{3\xi}(1 - e^{3i\xi})$. It is easy to check that m is a generalized low pass filter belonging to $\tilde{F}_{\varphi}$. Thus, the function ψ satisfying

$$
\hat{\psi}(\xi) = e^{i\frac{\xi}{2}} \frac{\left(1 - e^{-\frac{3}{2}i\xi}\right)\left(1 - e^{\frac{3}{2}i\xi}\right)}{-3i\xi} \tag{5.29}
$$

is an MRA TFW, by Theorem 2.11. We claim that ψ is *not* semiorthogonal. It is immediate that $\psi = \chi_{[-2,-\frac{1}{2})} - \chi_{[-\frac{1}{2},1)}$. But

$$
< \psi_{00}, \psi_{10} > = \sqrt{2}\int_{\mathbb{R}} \left(\chi_{[-2,-\frac{1}{2})} - \chi_{[-\frac{1}{2},1)}\right)\left(\chi_{[-1,-\frac{1}{4})} - \chi_{[-\frac{1}{4},\frac{1}{2})}\right) = \frac{5\sqrt{2}}{4} \neq 0\,,
$$

showing that W_0 and W_1 are not orthogonal.

Not every tight frame MRA gives rise to an MRA tight frame wavelet. Consider, for example, the function $\varphi \in L^2(\mathbb{R})$ such that $\hat{\varphi} = \chi_{[-\frac{3\pi}{4},\frac{3\pi}{4})}$. Then it is easy to check that $\sigma_{\varphi} = \chi_U$ where $U = \bigcup_{n\in\mathbb{Z}} [-\frac{\pi}{4}(8n+3), -\frac{\pi}{4}(8n-3))$. Thus, by Theorem 5.3 we know that $\{\varphi(\cdot - n)\}$ is a tight frame for the closed subspace, V_0, these translates generate. The dyadic dilates $V_j = \{f : f(2^{-j}\cdot) \in V_0\}$, $j \in \mathbb{Z}$, then, clearly, form a tight frame MRA and φ satisfies (1.2) with $m(\xi) = \sum_{k\in\mathbb{Z}} \chi_{[-\frac{3\pi}{8},\frac{3\pi}{8})}(\xi + 2k\pi)$. Theorem 5.8 and its proof (see, in particular, (5.20)) showed that, if there exists $\psi \in W_0 = V_1 \cap V_0^{\perp}$ such that $\{\psi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a tight frame for W_0, then $|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1$ for $\xi \in U \cap (U + \pi)$. In the case we are considering

$$
U \cap (U+\pi) = \bigcup_{n\in\mathbb{Z}} \left[\frac{\pi}{4}(4n+1), \frac{\pi}{4}(4n+3)\right)\,,
$$

which is a π-periodic set that contains the interval $I = [-\frac{5\pi}{8}, -\frac{3\pi}{8})$ and, a fortiori, the interval $I + \pi = [\frac{3\pi}{8}, \frac{5\pi}{8})$. But $m(\xi) = 0 = m(\xi+\pi)$ when $\xi \in I$ and, consequently, $|m(\xi)|^2 + |m(\xi + \pi)|^2 = 0 < 1$ for all $\xi \in I \subset U \cap (U+\pi)$. This shows that this tight frame MRA does not have an associated MRA tight frame wavelet.

We have shown, in considerable detail, how our construction compares with that of [1] and [2]. As we mentioned earlier, we do not need the regularity and decrease at ∞ assumptions stated in [1] and [2]. This, however, is not an essential point; perhaps the most important difference lies in the direct use of the generalized low pass filters we employ in Section 2. We would also like to point out that in [6] yet another construction of wavelet tight frames is presented that does not have, as its base, a frame MRA. Again certain regularity and behavior at ∞ assumptions are made that could be avoided. It is also important to observe that the treatments in these other works apply to $\mathbb{R}^n$ and are not restricted to $\mathbb{R}^1 = \mathbb{R}$. One of the reasons for the restriction to 1-dimension we have made is that we wanted to develop some of the material involving multipliers and connectivity presented in the earlier sections of this article.

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Received July 9, 1999
Revision received October 20, 1999

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