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Exotic Spheres with lots of Positive Curvatures

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A basic question of global Riemannian geometry is,

> Which closed, smooth n-manifolds admit Riemannian metrics with positive (or nonnegative) sectional curvature?

This problem is almost entirely open. In the case of positive sectional curvature, the list of known examples is very sparse (see [1, 2, 3, 7, 8], and [19]); for simply connected, closed manifolds, M, there are no obstructions known to sec $M > 0$ that are not also obstructions to either positive scalar curvature or nonnegative sectional curvature.

The principle underlying the construction of all known examples of manifolds of positive curvature is that Riemannian submersions are curvature nondecreasing on horizontal planes [15]. The image of a Riemannian submersion of a positively curved manifold is thus positively curved.

Many more examples could be constructed if this process could be reversed in some cases. That is: When does the total space of a fiber bundle whose fiber and base are positively curved admit a metric with positive curvature?

"Always" is the answer to the analogous question for positive Ricci curvature and for almost nonnegative sectional curvature [14, 16], and [9]. However, it is clear that if there are any further examples of this sort with positive sectional curvature, they are very difficult to find. For example, if the fibers are totally geodesic, then a necessary condition is that the bundle be "fat" (see [20]). This imposes some pretty severe constraints on the topology.

Among the known fiber bundles whose base and fibers admit positive curvature, perhaps the ones for which this question is most intriguing are the S^3-bundles over S^4. They are easy to construct, fairly abundant, and include 16 of the 28 oriented diffeomorphism classes of exotic 7-spheres among their total spaces.

Recall that the S^3-bundles over S^4 are classified by $\mathbb{Z} \oplus \mathbb{Z}$ as follows [12, 18]. The bundle that corresponds to $(n, m) \in \mathbb{Z} \oplus \mathbb{Z}$ is obtained by gluing two copies of $\mathbb{R}^4 \times S^3$ together via the

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diffeomorphism $g_{n,m} : (\mathbb{R}^4\backslash\{0\}) \times S^3 \longrightarrow (\mathbb{R}^4\backslash\{0\}) \times S^3$ given by

$$g_{m,n}(u, v) \longrightarrow \left(\frac{u}{|u|^2}, \frac{u^m v u^n}{|u|^{n+m}} \right), \tag{0.1}$$

where we have identified $\mathbb{R}^4$ with $\mathbb{H}$ and S^3 with $\{v \in \mathbb{H} \mid |v| = 1\}$. We will call the bundle obtained from $g_{m,n}$ "the bundle of type (m, n)," and we will denote it by $E_{m,-n}$.

In this article we will prove the following.

Theorem A. *The exotic 7-spheres that are total spaces of S^3-bundles over S^4 all admit sequences of almost nonnegatively curved Riemannian metrics that have positive sectional curvature at a point and an effective, isometric $O(2) \times SO(3)$-action.*

This means that these spheres admit sequences of Riemannian metrics g_i so that

(a) $\sec(g_i) \geq -\frac{1}{i}$;
(b) $\mathrm{diam}(g_i) \leq 1$; and
(c) $\sec(g_i) > 1$ at a point.

It follows from [13] (and a Meyer-Vietoris argument) that the total space of an S^3 bundle over S^4 is homeomorphic to S^7 if and only if it is of type $(m, -(m-1))$ (or $(-m, m-1)$) for $m = 0, 1, 2, 3, \ldots$. According to [13], the residue class $(2m-1)^2$ mod 7 is a smooth invariant of the total space, and the total space has an exotic differential structure if $(2m-1)^2 \not\equiv 1$ mod 7. The complete diffeomorphism classification of these spaces can be derived from [6].

In [11], Gromoll and Meyer showed that the bundle of type $(2, -1)$, which is an exotic sphere, admits a metric with an effective, isometric $O(2) \times SO(3)$-action whose sectional curvature is nonnegative and positive at a point. As a consequence, the bundle of type $(2, -1)$ admits metrics that satisfy the conclusions of Theorem A. Thus our theorem can be thought of as extension (with a weaker conclusion) of the main result in [11]. In fact, the metric that our construction yields on the bundle of type $(2, -1)$ is, with a minor modification, the Gromoll-Meyer metric, and our construction is a natural extension of the one in [11].

The idea behind the construction of the metrics in our theorem actually works for about half of the S^3 bundles over S^4. We will also give the construction for the other bundles, and show that the metrics are almost nonnegatively with effective, isometric $0(2) \times SO(3)$ actions. However, we will not give further curvature computations for the other bundles in this article.

In [5], Davis showed that every S^3-bundle over S^4 admits an $O(2) \times SO(3)$ action by bundle maps. It is easy to see that Davis's $SO(3)$ action is the same as ours. The author thinks that the two $O(2) \times SO(3)$ actions are smoothly equivalent, but this has not been checked. In any case, our actions are constructed by different means than those of [5].

The outline of this article is as follows.

In Section 1 we give explicit formulas for the Hopf fibrations, $\tilde{h}$ and h, that come from the free S^3-actions on S^7 from the left and from the right. These formulas are probably well known, but the author does not know a reference for them. Since they are needed to rigorously check our topological computations, they are included for the sake of completeness.

In Section 2 we give some motivation for our construction. The results of this section will not be explicitly used in the sequel, but they do suggest a way to generalize our method, and obtain a procedure that constructs new positively curved manifolds from old ones. Unfortunately, this method does not work in all cases, but it seems likely that it could work on an ad hoc basis.

In Section 3 we define a sequence of principal S^3-bundles,

$$Sp(2,m) \overset{p_{1,\ldots m-1}}{\longrightarrow} Sp(2,m-1) \overset{p_{1,\ldots,m-2}}{\longrightarrow} \cdots \overset{p_{1,2}}{\longrightarrow} Sp(2) \longrightarrow S^7 \longrightarrow S^4 .$$

These are essentially obtained by iteratively pulling back various versions of h and $\tilde{h}$ over each other. We also define m-actions of the $(m-1)$-fold product $S^3 \times S^3 \times \cdots \times S^3$ on $Sp(2,m)$ and show that the quotients of these actions can naturally be viewed as the S^3-bundles over S^4 of type $(m,0), (m,-1), \ldots, (m,-(m-1))$. This topological identification is carried out by explicitly constructing bundle charts in a manner that is similar to the one used in [11]. Keeping in the true spirit of an article about examples, these topological computations are done explicitly for the bundles of types $(3,0)$, $(3,-1)$, and $(3,-2)$ in Section 3 and the bundles of types $(4,0)$, $(4,-1)$, $(4,-2)$, and $(4,-3)$ in the Appendix. Although a single computation for the general case could probably be done with less ink, the resulting exposition would probably be more cumbersome and less informative. It is hoped that the approach taken here will make these computations very accessible and thereby facilitate the study of these spaces.

In Section 4 we review the result of Fukaya and Yamaguchi that we will use to conclude that our bundles have almost nonnegative curvature. Before we can simultaneously achieve the other curvature property mentioned in the theorem, we will have to make a further study of the geometry of our spaces. This is done in Sections 5, 6, and 7, where the $O(2) \times SO(3)$ symmetries are constructed, the tangent space of $Sp(2,m)$ is found, and a metric on $Sp(2,m)$ is constructed that is invariant under all of the relevant group actions.

In Section 8 we show that our metrics on the exotic spheres have positive curvature at a point, completing the proof of our theorem.

Background and Notation: We assume that the reader is familiar with the notions of the vertical and horizontal distributions associated to a Riemannian submersion, $\pi : M \longrightarrow N$, and with O'Neill's "horizontal curvature equation" (see equation {4} p. 464 of [15]).

We denote the vertical and horizontal distributions of π by V_π and H_π and the components of a vector, w, that lie in V_π and H_π by w^{v_π} and w^{h_π}, respectively. When we are discussing only one submersion and there is no possibility of confusion we may omit the subscript $_\pi$ from the superscripts ^{v} and ^{h}.

The sphere of radius r in $\mathbb{R}^{n+1}$ is denoted $S^n(r)$. $A_{\tilde{h}}$ and A_h are the free actions of S^3 on S^7 from the left and from the right that give the Hopf fibrations.

1. An explicit formula for the Hopf fibration

In this section we write down explicit formulas for the Hopf fibrations $h : S^7 \longrightarrow S^4$ and $\tilde{h} : S^7 \longrightarrow S^4$ corresponding to the right and left actions of S^3 on S^7. These formulas are probably well known, but were not found by the author even in the basic reference [10].

We denote points on S^7 by pairs of quaternions written as column vectors. Then the multiplications by S^3 on the left or the right are the Hopf S^3-actions on S^7. The quotients of each are of course S^4. We will need the following concrete descriptions of the quotient maps.

The quotient map for action on the right is

$$h : \begin{pmatrix} a \\ c \end{pmatrix} \mapsto \left(a\bar{c}, \frac{1}{2}\left(|a|^2 - |c|^2\right)\right) ,$$

and the quotient map for action on the left is

$$\tilde{h} : \begin{pmatrix} a \\ c \end{pmatrix} \mapsto \left(\bar{a}c, \frac{1}{2} \left(|a|^2 - |c|^2 \right) \right) . \tag{1.1}$$

The image is $S^4(\frac{1}{2})$, which we are viewing as a subset of $\mathbb{H} \oplus \mathbb{R}$.

It is well known that the metrics induced on S^4 by h and $\tilde{h}$ are isometric to $S^4(1/2)$. Not surprisingly, the actual submersion metrics on S^4 induced by h and $\tilde{h}$ are each the canonical metric on $S^4(1/2)$. (Recall that the family of metrics isometric to a given one is parameterized by the diffeomorphism group of the underlying manifold. So this is perhaps a pedantic point, but there is some actual substance.)

Proposition 1.1. *The metrics induced on $S^4(1/2)$ by h and $\tilde{h}$ are both the canonical metric.*

Proof. We give the proof only for h since the proof for $\tilde{h}$ is the same with different notation.

Set $e_5 = (0, 1) \in \mathbb{H} \oplus \mathbb{R}$.

Note that the S^3 actions on S^7 given by left multiplication with

$$\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \tag{1.2}$$

are by symmetries of h, as is the $SO(2)$ action given by left multiplication with

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{1.3}$$

The maps induced on $S^4(1/2)$ by (1.2) fix $\frac{e_5}{2}$ and are the standard S^3 and $SO(3)$-actions on the quaternion plane. The action induced on $S^4(1/2)$ by (1.3) is the direct sum of the standard $\mathbb{Z}_2$-ineffective S^1-action on the "purely real" circle in $S^4 \subset \mathbb{H} \oplus \mathbb{R}$ and the trivial action on the purely imaginary quaternions.

Let g_s denote the submersion metric on $S^4(1/2)$ induced by h and g_{can} the canonical metric. Let $S^{can}(p, \rho)$ and $S^s(p, \rho)$ denote the metric spheres about p of radius ρ with respect to g_{can} and g_s. Let r denote the distance from $\frac{e_5}{2}$ with respect to g_{can}, and $\frac{\partial}{\partial r}$ denote the unit radial field emanating from $\frac{e_5}{2}$ with respect to g_{can}. Using the S^3-action above we can conclude that

$$g_S \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \text{ depends only on } r . \tag{1.4}$$

Next we claim that $\frac{\partial}{\partial r}$ is also the radial field from $\frac{e_5}{2}$ with respect to g_s. To see this observe that the orbit, $\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$, of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under (1.3) is a geodesic in $S^7(1)$ that is horizontal with respect to h. Since $h\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e_5}{2}$, it follows that the orbit of $\frac{e_5}{2}$ with respect to (1.3) is a geodesic with respect to g_s. The tangent field to this orbit coincides with part of $\frac{\partial}{\partial r}$, and its image under (1.2) is all of $\frac{\partial}{\partial r}$. It follows that $\frac{\partial}{\partial r}$ is also the radial field from $\frac{e_5}{2}$ with respect to g_s.

Therefore, $g_S(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = g_{can}(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) = 1$ and

$$S^{can}\left(\frac{e_5}{2}, r\right) = S^s\left(\frac{e_5}{2}, r\right) \tag{1.5}$$

for all r.

It remains to check that $g_S|_{S_{can}(\frac{e_5}{2},r)} = g_{can}|_{S_{can}(\frac{e_5}{2},r)}$ for all r. The actions (1.2) show that the matrices for the two metrics (with respect to a fixed basis) differ by no more than a multiplicative constant for each r, and (1.5) and the fact that (S^4, g_s) is isometric $S^4(1/2)$ implies that this constant is 1. □

2. Motivation for the construction

A few of the S^3-bundles over S^4 are already known to admit nonnegative sectional curvature. Those of types $(1, 0)$, $(1, 1)$, and $(2, -1)$ are, respectively, the Hopf fibration, the unit tangent bundle of S^4, and the exotic sphere of Gromoll and Meyer [4, 11, 18]. In addition, Rigas has observed that types $(2, 0)$, $(1, -1)$, and $(2, -2)$ admit nonnegative curvature [17].

The construction of the metrics on all of these bundles is intimately related to the biinvariant metric on $Sp(2)$. In fact, the metrics on the type $(1, 0)$, $(1, 1)$, $(2, 0)$, and $(2, -1)$ bundles are S^3-quotients of the biinvariant metric on $Sp(2)$. Therefore it is rather intriguing that $Sp(2)$ itself can be constructed from the Hopf fibration.

Proposition 2.1. *$Sp(2)$ is diffeomorphic to the total space of the pullback of the Hopf fibration $S^7 \xrightarrow{h} S^4$ via $S^7 \xrightarrow{a\circ h} S^4$, where $a : S^4 \longrightarrow S^4$ is the antipodal map.*

In fact, the biinvariant metric on $Sp(2)$ is isometric (up to rescaling) to the subspace metric on the pullback $(a \circ h)^*(S^7) \subset S^7(1) \times S^7(1)$, where $S^7(1)$ is the unit 7-sphere and $S^7(1) \times S^7(1)$ has the product metric.

Sketch of Proof. View $Sp(2)$ as

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{H}, |a|^2 + |c|^2 = |b|^2 + |d|^2 = |a|^2 + |b|^2 = |c|^2 + |d|^2 = 1 \right.$$
$$\left. \text{and } a\bar{c} + b\bar{d} = 0 \right\}.$$

The first two equations show that the columns of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are each in S^7, so $Sp(2)$ can be identified with a subset of $S^7 \times S^7$. Using the other equations, it can be shown that this subset is $(a \circ h)^*(S^7)$.

The direct sum of the standard $Sp(2)$ action on $\mathbb{R}^8$ with itself restricts to an action on $S^7 \times S^7$. The restriction of this action to $(a \circ h)^*(S^7) = Sp(2)$ is just the standard left action of $Sp(2)$ on itself. Thus it is a tautology that $(a \circ h)^*(S^7) = Sp(2)$ is invariant under this action. A similar argument shows that $(a \circ h)^*(S^7) = Sp(2)$ is invariant under the right action of $Sp(2)$. Since these actions are by isometries on $S^7 \times S^7$, they are also by isometries on $(a \circ h)^*(S^7) = Sp(2)$. Hence the subspace metric on $(a \circ h)^*(S^7) = Sp(2)$ is biinvariant. □

It follows that we can think of the metrics on the bundles of type $(2, 0)$, $(2, -1)$, and $(1, 1)$ as being "created" from the Hopf fibration, h, via the following recipe: pull back h via $a \circ h$ and mod out the total space of the pullback by an appropriate free S^3 action. Thus it is possible to "change the type" of an S^3 bundle over S^4 by pulling it back over another S^3 bundle over S^4 and modding out by a free S^3 action. This leads us to ask:

Question 1: Can we obtain metrics on different S^3 bundles over S^4 by replacing one copy of S^7 in the recipe above with one of our other known examples, i.e., with the bundle of type $(1, 1)$, $(1, -1)$, $(2, 0)$, $(2, -1)$, or $(2, -2)$?

Question 2: Will some of the metrics obtained in this way be nonnegatively (or positively) curved?

General Question: Suppose M and N are compact, positively curved manifolds admitting Riemannian submersions, π_M and π_N, onto the same base B. Can we find a free isometric group action on the pull back $\pi_M^*(N)$ whose quotient admits positive curvature?

The answer to Question 1 at least, is yes. In fact, by starting with the bundle of type $(2, 0)$ and iteratively applying our recipe (pull back some bundle over $a \circ h$ and mod out by a free S^3 action), we can obtain every bundle of type $(m, -n)$, where $m > n \geq 0$. In particular, we can get all bundles whose total spaces are exotic spheres, since these are of type $(n + 1, -n)$, with $n > 0$.

We will not carry out this program explicitly here. We have described it only to motivate the construction of Section 3, which will accomplish the same goal in a manner that is easier to execute.

The author does not think that the answer to the general question is always yes, but it seems reasonable that the answer is occasionally yes, and that this offers a new (ad hoc) method for constructing manifolds of positive curvature.

3. $Sp(2, m)$–A space which fibers over the bundles of type $(m, -k)$, $(m > k \geq 0)$

In this section we define a sequence of spaces and submersions,

$$Sp(2, m) \overset{p_{1,\ldots,m-1}}{\longrightarrow} Sp(2, m-1) \overset{p_{1,\ldots,m-2}}{\longrightarrow} \cdots \overset{p_{1,2}}{\longrightarrow} Sp(2) \longrightarrow S^7 \longrightarrow S^4 ,$$

and show that there are free actions of the $(m-1)$-fold product $S^3 \times S^3 \times \cdots \times S^3$ on $Sp(2, m)$ whose quotients are the S^3-bundles over S^4 of type $(m, -k)$ for all $k \in \mathbb{N}$ with $m > k \geq 0$.

$Sp(2, m)$ is the subset of the m-fold product $S^7 \times \cdots \times S^7$,

$$Sp(2, m) = \Big\{(u_1, u_2, u_3, \ldots , u_m) \in S^7 \times \cdots \times S^7 \;\Big|\; h(u_1) = a \circ h(u_2), \tilde{h}(u_2) = a \circ h(u_3) ,$$
$$\tilde{h}(u_3) = a \circ h(u_4), \ldots , \tilde{h}(u_{m-1}) = a \circ h(u_m)\Big\} .$$

The submersion $p_{1,\ldots,m-1} : Sp(2, m) \longrightarrow Sp(2, m-1)$ is simply the restriction of the projection map of the m-fold product $S^7 \times S^7 \times \cdots \times S^7$ onto its first $m - 1$ factors.

Now define actions of the $(m-1)$-fold product $S^3 \times S^3 \times \cdots \times S^3$ on the m-fold product $S^7 \times \cdots \times S^7$ by

$$(q_1, q_2, \ldots , q_{m-1})(u_1, u_2, u_3, \ldots , u_m)$$
$$= (q_1 u_1 \bar{q}_{m-n}, \; q_1 u_2 \bar{q}_2, \; q_2 u_3 \bar{q}_3, \; \ldots , q_{m-2} u_{m-1} \bar{q}_{m-1}, \; q_{m-1} u_m) . \quad (3.1)$$

and

$$(q_1, q_2, \ldots , q_{m-1})(u_1, u_2, u_3, \ldots , u_m)$$
$$= (q_1 u_1, \; q_1 u_2 \bar{q}_2, \; q_2 u_3 \bar{q}_3, \; \ldots , q_{m-2} u_{m-1} \bar{q}_{m-1}, \; q_{m-1} u_m) , \quad (3.2)$$

where the n in (3.1) satisfies $1 \leq n \leq m-1$.

We will call the actions in (3.1) and (3.2), $A_{m,-n}$ and $A_{m,0}$, respectively.

Theorem 3.1.

(i) *$A_{m,-n}$ and $A_{m,0}$ are free and leave $Sp(2,m)$ invariant.*

(ii) *The quotient space of $A_{m,-n}$ is diffeomorphic to the total space of the S^3-bundle over S^4 of type $(m,-n)$.*

(iii) *The quotient space of $A_{m,0}$ is diffeomorphic to the total space of the S^3-bundle over S^4 of type $(m,0)$.*

Proof of Theorem 3.1 (i). Suppose

$$(q_1 u_1 \bar{q}_{m-n},\ q_1 u_2 \bar{q}_2,\ q_2 u_3 \bar{q}_3,\ \dots,\ q_{m-2} u_{m-1} \bar{q}_{m-1},\ q_{m-1} u_m) = (u_1, u_2, \dots, u_m) \ . \quad (3.3)$$

The equality of the last factors of (3.3) yields $q_{m-1} = 1$. Equality of the next factor then implies $q_{m-2} = 1$. Proceeding in this manner it follows from a reverse induction argument that $(q_1, q_2, \dots, q_{m-1}) = (1, 1, \dots, 1)$. So $A_{m,-n}$ is free. A similar argument shows that $A_{m,0}$ is free.

To see that these actions leave $Sp(2,m)$ invariant, note that both the left and the right actions of S^3 on S^7 are by symmetries of h and $\tilde{h}$ [10]. When viewed as symmetries of $\tilde{h}$ the left action induces the identity on S^4 and the right action induces the standard $\mathbb{Z}_2$-ineffective S^3-action, which conjugates the purely imaginary 2-sphere in S^4 and fixes its complementary circle. When viewed as symmetries of h the right action induces the identity on S^4 and the left action induces the standard $\mathbb{Z}_2$-ineffective S^3-action, which conjugates the purely imaginary 2-sphere in S^4 and fixes its complementary circle. Having made these observations, it can now be seen that the defining equations of $Sp(2,m)$ are preserved by the actions in (3.1) and (3.2). □

The proofs of Theorem 3.1(ii, iii) are by explicit computation of bundle charts and their overlap maps. As mentioned in the introduction, these computations will not be done for the general case. Instead we will cover the cases when $m = 3$ in the following subsections, and the cases when $m = 4$ in the appendix. By going through the proofs of these cases the reader will be able to concretely see how the computation changes when we increase m and how the computation changes when we increase n.

3.1. $E_{3,0}$

Consider the $S^3 \times S^3$-action on $Sp(2,3)$ that is given by

$$(q_1, q_2)\,(u, v, w) = (q_1 u, q_1 v \bar{q}_2, q_2 w) \ .$$

The quotient, $E_{3,0}$, is an S^3 bundle over S^4. Indeed the map

$$p_{3,0} : \text{orbit}\,(u, v, w) \mapsto \tilde{h}(w)$$

is a bundle map. This is because it satisfies $p_{3,0} \circ q_{3,0} = \tilde{h} \circ p_3^3$, where $q_{3,0} : Sp(2,3) \longrightarrow E_{3,0}$ is the quotient map and $p_3^3 : Sp(2,3) \longrightarrow S^7$ is the projection of $Sp(2,3)$ onto its last factor. Since each of $q_{3,0}$, $\tilde{h}$, and p_3^3 are smooth submersions it follows that $p_{3,0}$ is a smooth submersion and hence (since $Sp(2,3)$ is compact) a bundle map.

Define $\phi : \mathbb{R}^4 \longrightarrow \mathbb{R}$ by

$$\phi(u) = \frac{1}{\sqrt{1+|u|^2}}\,. \tag{3.4}$$

Proposition 3.2. *$(E_{3,0}, p_{3,0})$ is the S^3-bundle over S^4 of type $(3,0)$.*

Proof. We prove this by constructing explicit bundle charts and computing the overlap map. Our computations closely resemble those of [11].

The charts $h_1, h_2 : \mathbb{R}^4 \times S^3 \longrightarrow E_{3,0}$ are defined by

$$h_1(u,q) = \text{ orbit }\left(\begin{pmatrix} uq \\ q \end{pmatrix}\phi(u), \begin{pmatrix} 1 \\ -\bar{u} \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u)\right). \tag{3.5}$$

and

$$h_2(v,r) = \text{ orbit }\left(\begin{pmatrix} r \\ \bar{v}r \end{pmatrix}\phi(v), \begin{pmatrix} v \\ -1 \end{pmatrix}\phi(v), \begin{pmatrix} 1 \\ v \end{pmatrix}\phi(v)\right). \tag{3.6}$$

h_1 and h_2 are embeddings onto the open dense sets

$$\begin{aligned} U_1 &= \left\{\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}\right) \,\middle|\, \gamma \neq 0\right\} \quad \text{and} \\ U_2 &= \left\{\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}\right) \,\middle|\, \alpha \neq 0\right\}, \end{aligned} \tag{3.7}$$

respectively.

In fact, the formulas for the inverses are given by

$$h_1^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}\right)\right) = \left(\frac{\bar{\alpha}\gamma}{|\gamma|^2}, \frac{\bar{\gamma}\bar{b}c}{|\gamma||b||c|}\right)$$

and

$$h_2^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}\right)\right) = \left(\frac{\bar{\alpha}\gamma}{|\alpha|^2}, -\frac{\bar{\alpha}\bar{d}a}{|\alpha||d||a|}\right),$$

as can easily be verified.

It follows that

$$\begin{aligned} h_2^{-1} \circ h_1(u,q) &= h_2^{-1}\left[\text{orbit}\left(\begin{pmatrix} uq \\ q \end{pmatrix}\phi(u), \begin{pmatrix} 1 \\ -\bar{u} \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u)\right)\right] \\ &= \left(\frac{u}{|u|^2}, \frac{uuuq}{|u|^3}\right) = \left(\frac{u}{|u|^2}, \frac{u^3 q}{|u|^3}\right), \end{aligned}$$

and hence by (0.1) that $E_{3,0}$ is the bundle of type $(3,0)$. □

3.2. $E_{3,-1}$

Next consider the $S^3 \times S^3$-action on $Sp(2,3)$ that is given by

$$(q_1, q_2)(u, v, w) = (q_1 u \bar{q}_2, q_1 v \bar{q}_2, q_2 w) \ .$$

The quotient, $E_{3,-1}$, is an S^3 bundle over S^4. Indeed the map

$$p_{3,-1} : \text{orbit}\,(u, v, w) \mapsto \tilde{h}(w)$$

is a bundle map. This can be seen via the argument that we gave for $E_{3,0}$.

Proposition 3.3. *$(E_{3,-1}, p_{3,-1})$ is the S^3-bundle over S^4 that is of type $(3,-1)$.*

Proof. As in Proposition 3.2 we define charts $h_1, h_2 : \mathbb{R}^4 \times S^3 \longrightarrow E$ for $p_{3,-1}$ by the formulas (3.5) and (3.6). As before h_1 and h_2 are embeddings onto the open dense sets, U_1 and U_2 in (3.7).

The formulas for the inverses are now given by

$$h_1^{-1}\left(\text{orbit}\left(\left(\begin{array}{c} a \\ c \end{array}\right), \left(\begin{array}{c} b \\ d \end{array}\right), \left(\begin{array}{c} \alpha \\ \gamma \end{array}\right)\right)\right) = \left(\frac{\bar{\alpha}\gamma}{|\gamma|^2}, \frac{\bar{\gamma}\bar{b}c\gamma}{|\gamma|^2|b||c|}\right)$$

and

$$h_2^{-1}\left(\text{orbit}\left(\left(\begin{array}{c} a \\ c \end{array}\right), \left(\begin{array}{c} b \\ d \end{array}\right), \left(\begin{array}{c} \alpha \\ \gamma \end{array}\right)\right)\right) = \left(\frac{\bar{\alpha}\gamma}{|\alpha|^2}, -\frac{\bar{\alpha}\bar{d}a\alpha}{|\alpha|^2|d||a|}\right) .$$

Thus

$$\begin{aligned} h_2^{-1} \circ h_1(u, q) &= h_2^{-1}\left[\text{orbit}\left(\left(\begin{array}{c} uq \\ q \end{array}\right)\phi(u), \left(\begin{array}{c} 1 \\ -\bar{u} \end{array}\right)\phi(u), \left(\begin{array}{c} \bar{u} \\ 1 \end{array}\right)\phi(u)\right)\right] \\ &= \left(\frac{u}{|u|^2}, \frac{uuuq\bar{u}}{|u|^4}\right) = \left(\frac{u}{|u|^2}, \frac{u^3 q\bar{u}}{|u|^4}\right) . \end{aligned}$$

So by (0.1) $E_{3,-1}$ is the bundle of type $(3,-1)$. □

3.3. $E_{3,-2}$

The $S^3 \times S^3$ action on $Sp(2,3)$ that gives $E_{3,-2}$ is

$$(q_1, q_2)(u, v, w) = (q_1 u \bar{q}_1, q_1 v \bar{q}_2, q_2 w) \ . \tag{3.8}$$

The quotient, $E_{3,-2}$, is an S^3-bundle over S^4. Indeed, arguing as in the beginning of Subsection 3.1 we see that the map

$$p_{3,-2} : \text{orbit}\,(u, v, w) \mapsto \tilde{h}(w)$$

is a bundle map.

Proposition 3.4. *The quotient of the action in (3.8) is the total space of the S^3-bundle over S^4 of type $(3,-2)$. It is therefore an exotic 7-sphere that is different from the space of nonnegative curvature discovered by Gromoll and Meyer in [11].*

Proof. We define charts $h_1, h_2 : \mathbb{R}^4 \times S^3 \longrightarrow E$ for $p_{3,-2}$ by the formulas (3.5) and (3.6).

As before, h_1 and h_2 are embeddings onto the open dense sets, U_1 and U_2 from (3.7), but now their inverses are given by

$$h_1^{-1}\left(\text{orbit}\left(\left(\begin{array}{c} a \\ c \end{array}\right), \left(\begin{array}{c} b \\ d \end{array}\right), \left(\begin{array}{c} \alpha \\ \gamma \end{array}\right)\right)\right) = \left(\frac{\bar{\alpha}\gamma}{|\gamma|^2}, \frac{\bar{\gamma}\bar{b}cb\gamma}{|\gamma|^2|b|^2|c|}\right)$$

and

$$h_2^{-1}\left(\text{orbit}\left(\left(\begin{array}{c} a \\ c \end{array}\right), \left(\begin{array}{c} b \\ d \end{array}\right), \left(\begin{array}{c} \alpha \\ \gamma \end{array}\right)\right)\right) = \left(\frac{\bar{\alpha}\gamma}{|\alpha|^2}, \frac{\bar{\alpha}\bar{d}ad\alpha}{|\alpha|^2|d|^2|a|}\right).$$

Thus

$$\begin{aligned} h_2^{-1} \circ h_1(u, q) &= h_2^{-1}\left[\text{orbit}\left(\left(\begin{array}{c} uq \\ q \end{array}\right)\phi(u), \left(\begin{array}{c} 1 \\ -\bar{u} \end{array}\right)\phi(u), \left(\begin{array}{c} \bar{u} \\ 1 \end{array}\right)\phi(u)\right)\right] \\ &= \left(\frac{u}{|u|^2}, \frac{uuuq\bar{u}\bar{u}}{|u|^5}\right) = \left(\frac{u}{|u|^2}, \frac{u^3q\bar{u}^2}{|u|^5}\right). \end{aligned}$$

So by (0.1) $E_{3,-2}$ is the bundle of type $(3, -2)$. □

4. Almost nonnegative curvature

In [9], Fukaya and Yamaguchi proved the following theorem.

Theorem 4.1 (Fukaya-Yamaguchi). *Let a compact manifold M admit the structure of a fiber bundle*

$$F \hookrightarrow M \longrightarrow N$$

with a compact Lie Group G as the structure group. Suppose that the fiber F admits a G-invariant metric of nonnegative sectional curvature and that for every $\varepsilon > 0$, there exists a metric h_ε on N such that $\sec_{h_\varepsilon} \operatorname{diam}(h_\varepsilon)^2 > -\varepsilon$. Then M also admits a metric g_ε satisfying the same curvature–diameter inequality.

The submersion $Sp(2, k) \overset{p_{1,\dots k-1}}{\longrightarrow} Sp(2, k-1)$ is actually a principal S^3-bundle. The S^3 action is given by

$$q\,(u_1, \dots, u_k) = (u_1, u_2, \dots, u_{k-1}, u_k\bar{q}).$$

Thus, applying Theorem 4.1 successively to our sequence of submersions

$$Sp(2, m) \overset{p_{1,\dots m-1}}{\longrightarrow} Sp(2, m-1) \overset{p_{1,\dots,m-2}}{\longrightarrow} \cdots \overset{p_{1,2}}{\longrightarrow} Sp(2) \overset{p_1}{\longrightarrow} S^7 \overset{h}{\longrightarrow} S^4$$

yields metrics of almost nonnegative curvature on $Sp(2, m)$.

It also follows from Theorem 4.1 that all S^3-bundles over S^4 admit almost nonnegative sectional curvature.

To obtain metrics with the additional curvature property asserted in Theorem A we will need to define more specific metrics on these bundles. Our metrics will be obtained from Riemannian submersions of almost nonnegatively curved metrics on the $Sp(2, m)$'s.

Thus we will need to find metrics on $Sp(2, m)$ that simultaneously satisfy the construction of Fukaya-Yamaguchi and are invariant under the actions $A_{m,-n}$ and $A_{m,0}$.

Before defining these metrics we discuss the symmetries of $E_{(m,-n)}$ and the tangent space of $Sp(2, m)$ in the next two sections.

5. The symmetries of $E_{(m,-n)}$

As was the case in [11], our bundles admit either an $O(2) \times SO(3)$ or an $O(2) \times S^3$ symmetry group. These groups also act by isometries with respect to the metric that we will define in Section 7.

The $O(2)$-action is from the left. On the level of $Sp(2, m)$, it is given by

$$A(u_1, u_2, \ldots, u_m) = (Au_1, Au_2, \ldots, Au_m) , \tag{5.1}$$

where we are viewing $u_i \in S^7$ as a (2×1)-column vector with quaternion entries, and we are exploiting the natural embedding $O(2) \hookrightarrow Sp(2)$ induced from the embedding $\mathbb{R} \hookrightarrow \mathbb{H}$.

The fact that (5.1) leaves $Sp(2, m)$ invariant follows from the fact that the $O(2)$-action on S^7 that is given by

$$(A, u) \mapsto Au \tag{5.2}$$

is by symmetries of both h and $\tilde{h}$. It is by symmetries of h, because the S^3-action that gives h is on the right and hence commutes with (5.2). (5.2) is by symmetries of $\tilde{h}$ because the S^3-action that gives $\tilde{h}$ can be described via the left multiplication

$$\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} u .$$

Since this commutes with (5.2), (5.2) is by symmetries of $\tilde{h}$.

Since the action in (5.1) commutes with $A_{m,-n}$ and $A_{m,0}$, it induces an $O(2)$ action on each of the $E_{(m,-n)}$'s.

The $SO(3)$-action is induced by the S^3-action

$$q(u_1, u_2, \ldots, u_m) = (u_1, u_2, \ldots, u_m\bar{q}) . \tag{5.3}$$

This clearly commutes with $A_{m,-n}$, $A_{m,0}$, and with (5.1). Hence, (5.1) and (5.3) together induce an $O(2) \times S^3$-action on $E_{(m,-n)}$. However, on the level of $E_{(m,-n)}$, the S^3-action is $\mathbb{Z}_2$-ineffective if $n \neq 0$. To see this, simply note that

$$(u_1, u_2, \ldots, u_m(-1)) = (-1, -1, \ldots, -1)(u_1, u_2, \ldots, u_m) , \tag{5.4}$$

where the action on the righthand side is one of the $A_{m,-n}$'s with $n \neq 0$. On the other hand, (5.4) is false if the action on the right is $A_{m,0}$, and a similar argument shows that $-1 \in S^3$ acts freely on $E_{m,0}$.

To find the full kernel of our actions we prove the following.

Proposition 5.1.

(i) *For any $n = 0, 1, 2, \ldots, m-1$, the $O(2) \times S^3$-action on $E_{m,-n}$ induced by (5.1) and (5.3) is by symmetries of $p_{m,-n} : E_{m,-n} \longrightarrow S^4$. The induced action on S^4 is the restriction of the action on $\mathbb{R}^5$ obtained by taking the direct sum of the standard $\mathbb{Z}_2$-ineffective $O(2)$-action on $\mathbb{R}^2$ and the standard $SO(3)$-action on $\mathbb{R}^3$. Equivalently, it is the join of the standard $\mathbb{Z}_2$-ineffective $O(2)$-action on S^1 with the standard $SO(3)$-action on S^2.*

(ii) *For any $n = 1, 2, \ldots, m-1$, the kernel of (5.1) $\times$ (5.3) on $E_{m,-n}$ is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In the former case it is generated by $\{(0, -id)\} \subset O(2) \times S^3$. In the latter it is generated by $\{(-id, 0), (0, -id)\} \subset O(2) \times S^3$.*

(iii) *The kernel of (5.1) × (5.3) on $E_{m,0}$ is isomorphic to* 0 *when m is odd and to* $\mathbb{Z}_2$ *when m is even; in the latter case, the kernel is generated by* $\{(-id, 0)\} \subset O(2) \times S^3$.

Proof. Let $p_m^m : Sp(2,m) \longrightarrow S^7$ denote the projection onto the last factor, and let $q_{m,-n} : Sp(2,m) \longrightarrow E_{m,-n}$ be the quotient map. Observe that $p_{m,-n} \circ q_{m,-n} = \tilde{h} \circ p_m^m$. The $O(2) \times S^3$-action on $Sp(2,m)$ is clearly by symmetries of p_m^m. The induced action on S^7 is the product of the action in (5.2) with the action

$$(q,u) \mapsto u\bar{q} \; . \tag{5.5}$$

We showed on page 171 that (5.2) is by symmetries of $\tilde{h}$, and it is clear that (5.5) is by symmetries of $\tilde{h}$. Therefore our $O(2) \times SO(3)$-action on $E_{m,-n}$ is by symmetries of $p_{m,-n}$.

By direct computation we see that the actions induced on S^4 by (5.2) and (5.5) via $\tilde{h}$ are as described in the statement of (i). In fact we have already given part of the computation for (5.2) in the proof of (1.1).

The circle in $S^4(1/2)$ that is fixed by $SO(3)$ is the intersection of $S^4(1/2)$ and the "purely real" copy of $\mathbb{R}^2$ in $\mathbb{H} \oplus \mathbb{R} = i\mathbb{R}^3 \oplus \mathbb{R} \oplus \mathbb{R}$, where $i\mathbb{R}^3 \oplus \mathbb{R}$ is the decomposition of $\mathbb{H}$ into purely imaginary and purely real numbers.

The S^2 that is fixed by $O(2)$ is the intersection of $S^4(1/2)$ and $i\mathbb{R}^3$. This completes our outline of the proof of (i).

Given (i) it follows that the subgroups listed in (ii) are the only possibilities for the kernel of (5.1) × (5.3) on $E_{m,-n}$. The exact kernel is therefore determined by the action of $(-id, 0) \subset O(2) \times SO(3)$. By direct computation it is easy to see that this element either acts freely or trivially on $E_{m,-n}$ and that both possibilities can occur. For example it acts freely on $E_{2,-1}$ and trivially on $E_{3,-1}$.

Similarly, it follows from (i) and the exposition before the statement of the proposition that the group listed in (iii) is the only possible kernel for the action on $A_{m,0}$. Direct computation completes the proof of (iii). □

Notation: Throughout this article the symbols $S^1_{\mathbb{R}}$ and S^2_{im} will stand for the circle and 2-sphere in S^4 that were described in the preceding proof.

Since $O(2) \times SO(3)$ will ultimately act by isometries on $Sp(2,m)$, we may reduce the problem of studying curvature at an arbitrary point of $Sp(2,m)$ to the problem of studying it at certain special points.

Proposition 5.2.

(i) *If $n > 0$, then every point in $E_{m,-n}$ has a point in its orbit under $O(2) \times SO(3)$ that can be represented in $Sp(2,m)$ by a point of the form*

$$\left(\begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix} p \, , \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} \cos t \\ -\alpha \sin t \end{pmatrix}, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix} \cdots \right), \tag{5.6}$$

with $t \in [0, \frac{\pi}{4}]$, $re(\alpha) = 0$, $|\alpha| = 1$, and p an arbitrary member of S^3.

(ii) *Every point in $E_{m,0}$ has a point in its orbit under $O(2) \times S^3$ that can be represented in $Sp(2,m)$ by a point of the form*

$$\left(\begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix}, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} \cos t \\ -\alpha \sin t \end{pmatrix}, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix} \cdots \right), \tag{5.7}$$

with $t \in [0, \frac{\pi}{4}]$, $re(\alpha) = 0$, *and* $|\alpha| = 1$.

Proof. Suppose our point is represented by

$$\left(\begin{pmatrix} a_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ c_2 \end{pmatrix}, \cdots \begin{pmatrix} a_m \\ c_m \end{pmatrix} \right) \in Sp(2, m) .$$

Replace this point by a point in the same 0(2)-orbit for which $|a_1|$ is maximal. At such a point $\mathrm{Re}(a_1 \bar{c}_1) = 0$. Indeed,

$$\frac{d}{dt} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a_1 \\ c_1 \end{pmatrix} \Bigg|_{t=0} = \begin{pmatrix} c_1 \\ -a_1 \end{pmatrix} .$$

Thus if a_1 is not perpendicular to c_1, then by moving $\begin{pmatrix} a_1 \\ c_1 \end{pmatrix}$ slightly (forward or backward) in its $SO(2)$ orbit we can increase the size of its $(1, 1)$ entry, contradicting our hypothesis. Therefore a_1 is necessarily perpendicular to c_1.

It follows from the defining equations for $Sp(2, m)$ and the definitions of h and $\tilde{h}$ that $|a_1| = |a_o| = |c_e|$ for each odd number o and each even number e in $\{1, 2, \ldots, m\}$. Hence if $|a_1|$ is maximal for the $O(2)$-orbit, then $|a_o|$ and $|c_e|$ are too. By the argument above $\langle a_0, c_0 \rangle = \langle a_e, c_e \rangle = 0$.

In case (i), we now choose $q_{m-1}, q_{m-2}, \ldots, q_2, q_1$ so as to arrange that $im(a_o) = im(c_e) = 0$ for all odd numbers o and all even numbers e in $\{2, \ldots, m\}$.

In case (ii) we insure that $im(a_o) = im(c_e) = 0$ for all odd numbers o and all even numbers e in $\{1, 2, \ldots, m\}$ by choosing $q_1, q_2, \ldots q_{m-1}$ appropriately, and then choosing the appropriate $q \in S^3$ for the action in (5.3).

Our point now has the form of (5.6) or (5.7). □

6. The tangent space of $Sp(2, m)$

Since the $O(2) \times SO(3)$ action will ultimately be by isometries, to study the curvature of $E_{m,-n}$ we may assume that we are at a point in $Sp(2, m)$ of the form

$$\left(\begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix} p, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} \cos t \\ -\alpha \sin t \end{pmatrix}, \ldots, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix} \right) \tag{6.1}$$

if m is even, or

$$\left(\begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix} p, \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} \cos t \\ -\alpha \sin t \end{pmatrix}, \ldots, \begin{pmatrix} \cos t \\ -\alpha \sin t \end{pmatrix} \right) \tag{6.2}$$

if m is odd.

Convention 6.1. *To avoid repetition, whenever we need to write tangent vectors or other objects explicitly we will do so as though we were only studying the case when m is even. The only difference when m is odd is the substitution of the last column in (6.2) for the last column in (6.1).*

Set $N_1^p \equiv \begin{pmatrix} \cos t \\ \alpha \sin t \end{pmatrix} p$, $N_e \equiv \begin{pmatrix} \alpha \sin t \\ \cos t \end{pmatrix}$, and $N_o \equiv \begin{pmatrix} \cos t \\ -\alpha \sin t \end{pmatrix}$. N_k stands for N_e if k is even and N_o if k is odd. If a statement about N_1^p is valid for all p we may omit the p. Notice that $N_1^1 p = N_1^p$, and that N_1^p, N_e, N_o, and N_k are all points in S^7.

It will usually be possible to make statements that are valid for both N_e and N_o simultaneously. There will also be times when a statement for N_e will have an obvious (but not identical) analog for N_o. Whenever either of these situations occurs we will write N instead of N_e or N_o, and only make one statement. When the proof of such a statement requires us to explicitly write down N_e or N_o, we will prove one of the cases and leave it to the reader to consider the other case.

The vectors

$$\mathfrak{v}^1 \equiv N_1^1 \alpha p, \ \ \mathfrak{v}^e \equiv N_e \alpha, \ \text{ and } \ \mathfrak{v}^o \equiv N_o \alpha$$

at the points N_1^p, N_e, and N_o in S^7 are vertical with respect to both h and $\tilde{h}$. Most often the location of the foot point will be clear so we will omit the superscripts and write simply $\mathfrak{v}$ for $\mathfrak{v}^1$, $\mathfrak{v}^e$, or $\mathfrak{v}^o$.

Let $\gamma_1, \gamma_2 \in \mathbb{H}$ satisfy $\mathrm{Re}(\gamma_i) = 0$, $|\gamma_i| = 1$, $\alpha \perp \gamma_1 \perp \gamma_2 \perp \alpha$, and $\gamma_1\gamma_2 = \alpha$, and set

$$\vartheta_1 = N\gamma_1, \ \ \vartheta_2 = N\gamma_2, \ \ \tilde{\vartheta}_1 = \gamma_1 N, \ \text{ and } \ \tilde{\vartheta}_2 = \gamma_2 N \ .$$

Similarly at N_1^p we set

$$\vartheta_1 = N_1^1 \gamma_1 p, \ \ \vartheta_2 = N_1^1 \gamma_2 p, \ \ \tilde{\vartheta}_1 = \gamma_1 N_1^p, \ \text{ and } \ \tilde{\vartheta}_2 = \gamma_2 N_1^p \ .$$

As with $\mathfrak{v}$, we will usually make no notational distinction between the versions of the ϑ_i's and $\tilde{\vartheta}_i$'s with different foot points.

It follows that $\{\mathfrak{v}, \vartheta_1, \vartheta_2\}$ and $\{\mathfrak{v}, \tilde{\vartheta}_1, \tilde{\vartheta}_2\}$ are orthonormal bases for V_h and $V_{\tilde{h}}$, respectively, and that

$$\left\langle \vartheta_i, \tilde{\vartheta}_i \right\rangle = \cos 2t \ .$$

In particular,

$$\dim\left(V_h \cap V_{\tilde{h}}\right) = \begin{cases} 1 & \text{if } t \neq 0 \\ 3 & \text{if } t = 0 \,, \end{cases}$$

and therefore

$$\dim\left(H_h \cap H_{\tilde{h}}\right) = \begin{cases} 2 & \text{if } t \neq 0 \\ 4 & \text{if } t = 0 \,. \end{cases}$$

We denote by $\{x, y\}$ a pair of orthonormal vectors such that $\{x, y\} \subset H_h \cap H_{\tilde{h}}$ at p. More specifically, we require that x is the vector whose projection to S^4 is radial for $\mathrm{dist}(S^1_{\mathbb{R}}, \cdot)$ in $S^4 = S^1_{\mathbb{R}} * S^2_{im}$. Thus (under Convention 6.1)

$$(x, x, \ldots, x) = \left(\begin{pmatrix} -\sin t \\ \alpha \cos t \end{pmatrix} p, \ \begin{pmatrix} \alpha \cos t \\ -\sin t \end{pmatrix}, \ \begin{pmatrix} -\sin t \\ -\alpha \cos t \end{pmatrix}, \ \ldots, \ \begin{pmatrix} \alpha \cos t \\ -\sin t \end{pmatrix} \right) .$$

At N_e and N_o we get an orthonormal basis for the rest of H_h by adjoining x to

$$\{y = x\alpha, \ \ \eta_1 = x\gamma_1, \ \text{ and } \ \eta_2 = x\gamma_2\} \ .$$

Similarly, we get an orthonormal basis for $H_{\tilde{h}}$ by adjoining x to

$$\{y = \alpha x, \ \ \tilde{\eta}_1 = \gamma_1 x, \ \text{ and } \ \tilde{\eta}_2 = \gamma_2 x\} \ .$$

At N_1^p we complete our bases for H_h as follows.

$$\begin{aligned} H_h &= \operatorname{span}\left\{ \begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} p,\ \begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} \alpha p,\ \begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} \gamma_1 p,\ \begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} \gamma_2 p \right\} \\ &\equiv \{x, y, \eta_1, \eta_2\} \end{aligned}$$

and

$$\begin{aligned} H_{\tilde h} &= \operatorname{span}\left\{ \begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} p,\ \begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} \alpha p,\ \gamma_1 \begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} p,\ \gamma_2 \begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} p \right\} \\ &\equiv \{x, y, \tilde\eta_1, \tilde\eta_2\}\ . \end{aligned}$$

We will call the resulting bases Base(H_h) and Base$(H_{\tilde h})$ respectively.

Convention 6.2 **(η,ϑ–convention).** *We will use the symbol η to denote both η_1 and η_2, the symbol $\tilde\eta$ to denote both $\tilde\eta_1$ and $\tilde\eta_2$, the symbol ϑ to denote both ϑ_1 and ϑ_2, and the symbol $\tilde\vartheta$ to denote both $\tilde\vartheta_1$ and $\tilde\vartheta_2$. To facilitate this convention we let γ denote either γ_1 or γ_2. Whenever a statement is made about more than one of these vectors it is to be assumed that the index is the same unless otherwise indicated.*

We will need to know the inner products among the η's, ϑ's, $\tilde\eta$'s, and $\tilde\vartheta$ so we record them here.

Proposition 6.3. *Let $\langle \cdot\ \cdot \rangle$ denote the round metric on S^7.*

$$\left\langle \vartheta, \tilde\vartheta \right\rangle = \cos 2t\ . \tag{6.3}$$

$$\langle \eta, \tilde\eta \rangle = -\cos 2t\ . \tag{6.4}$$

$$\left\langle \eta, \tilde\vartheta \right\rangle = -\sin 2t\ . \tag{6.5}$$

$$\langle \tilde\eta, \vartheta \rangle = -\sin 2t\ . \tag{6.6}$$

Proof.

$$\begin{aligned} \left\langle \vartheta, \tilde\vartheta \right\rangle &= \left\langle \begin{pmatrix} \alpha\sin t \\ \cos t \end{pmatrix} \gamma,\ \gamma \begin{pmatrix} \alpha\sin t \\ \cos t \end{pmatrix} \right\rangle \\ & \quad -\sin^2 t + \cos^2 t = \cos 2t\ . \\ \langle \eta, \tilde\eta \rangle &= \left\langle \begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix} \gamma,\ \gamma \begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix} \right\rangle \\ &= -\cos^2 t + \sin^2 t = -\cos 2t\ . \\ \left\langle \eta, \tilde\vartheta \right\rangle &= \left\langle \begin{pmatrix} -\sin t \\ -\alpha\cos t \end{pmatrix} \gamma,\ \gamma \begin{pmatrix} \cos t \\ -\alpha\sin t \end{pmatrix} \right\rangle \\ & \quad -2\sin t\cos t = -\sin 2t\ . \\ \langle \tilde\eta, \vartheta \rangle &= \left\langle \gamma \begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix},\ \begin{pmatrix} \alpha\sin t \\ \cos t \end{pmatrix} \gamma \right\rangle \\ &= -2\sin t\cos t = -\sin(2t)\ . \end{aligned}$$

□

Using the definition of $Sp(2, m)$ it is easy to see

Proposition 6.4. *A vector* $(u_1, u_2, \dots, u_m) \in T(S^7 \times \cdots \times S^7)$ *is tangent to* $Sp(2, m)$ *if and only if its foot point is in* $Sp(2, m)$ *and it satisfies*

$$\begin{aligned} dh\,(u_1) &= d(a \circ h)\,(u_2)\,,\ d\tilde{h}\,(u_2) = d(a \circ h)\,(u_3)\,,\ d\tilde{h}\,(u_3) = d(a \circ h)\,(u_4)\,,\ \dots \\ d\tilde{h}\,(u_{m-1}) &= d(a \circ h)\,(u_m)\ . \end{aligned} \tag{6.7}$$

We will refer to the equations in (6.7) as the "tangency equations."

Because of the tangency equations we will have a recurring need to know the images of the η's, ϑ's, $\tilde{\eta}$'s, and $\tilde{\vartheta}$'s under dh and $d\tilde{h}$ and how they relate to each other. We record these relations here.

Proposition 6.5. *For all* $i = 2, \dots, m$,

(i) $d\tilde{h}_{N_i}(\eta) = \cos(2t)\, d(a \circ h)_{N_{i+1}}(\eta)$
(ii) $d\tilde{h}_{N_i}(\tilde{\eta}) = -d(a \circ h)_{N_{i+1}}(\eta)$
(iii) $d\tilde{h}_{N_i}(\vartheta) = \sin(2t)\, d(a \circ h)_{N_{i+1}}(\eta)$
(iv) $dh_{N_1^p}(\tilde{\vartheta}) = -\sin(2t)\, d(a \circ h)_{N_2}(\eta)$
(v) $dh_{N_1^p}(\tilde{\eta}) = -\cos 2t\, d(a \circ h)_{N_2}(\eta)$.

Proof. Let $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in S^7$ be an arbitrary point and $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in T_pS^7$ an arbitrary vector tangent to S^7 at q.

Then

$$\begin{aligned} dh_q(z) &= \frac{d}{d\theta} h(q \cos\theta + z \sin\theta)|_{\theta=0} \\ &= \left(q_1\bar{z}_2 + z_1\bar{q}_2, \frac{1}{2}\,(q_1\bar{z}_1 + z_1\bar{q}_1 - q_2\bar{z}_2 - z_2\bar{q}_2)\right) \\ &= (q_1\bar{z}_2 + z_1\bar{q}_2, \operatorname{Re}(q_1\bar{z}_1) - \operatorname{Re}(q_2\bar{z}_2))\ , \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} d\tilde{h}_q(z) &= \frac{d}{d\theta} \tilde{h}(q \cos\theta + z \sin\theta)|_{\theta=0} \\ &= \left(\bar{q}_1 z_2 + \bar{z}_1 q_2, \frac{1}{2}\,(q_1\bar{z}_1 + z_1\bar{q}_1 - q_2\bar{z}_2 - z_2\bar{q}_2)\right) \\ &= (\bar{q}_1 z_2 + \bar{z}_1 q_2, \operatorname{Re}(q_1\bar{z}_1) - \operatorname{Re}(q_2\bar{z}_2))\ . \end{aligned} \tag{6.9}$$

From here the proof is just to compute all of the differentials in the statement and compare the answers. The computations are a bit different for the cases i even, $i \geq 3$ and odd, and $i = 1$. The answers are still as given in the statement, and the computations for each of the cases are about equally difficult. We will do them explicitly for some of the cases and leave the other cases to the reader.

$$
\begin{aligned}
d\tilde{h}_{N_0}(\eta) &= d\tilde{h}_{N_0}\left(\begin{pmatrix} -\sin t \\ -\alpha\cos t \end{pmatrix}\gamma\right) = \left(-\alpha\gamma\cos^2 t + \gamma\bar{\alpha}\sin^2 t, 0\right) \\
&= \left(\gamma\alpha\cos^2 t - \gamma\alpha\sin^2 t, 0\right) = (\gamma\alpha\cos 2t, 0)\,. && (6.10) \\
d\tilde{h}_{N_0}(\vartheta) &= d\tilde{h}_{N_0}\left(\begin{pmatrix} \cos t \\ -\alpha\sin t \end{pmatrix}\gamma\right) = (-\alpha\gamma\cos t\sin t + \bar{\gamma}\bar{\alpha}\cos t\sin t, 0) \\
&= (\gamma\alpha\sin 2t, 0)\,. && (6.11) \\
d\tilde{h}_{N_0}(\tilde{\eta}) &= d\tilde{h}_{N_0}\left(\gamma\begin{pmatrix} -\sin t \\ -\alpha\cos t \end{pmatrix}\right) = \left(\gamma\bar{\alpha}\cos^2 t + \gamma\bar{\alpha}\sin^2 t, 0\right) \\
&= (-\gamma\alpha, 0)\,. && (6.12) \\
dh_{N_1^p}(\tilde{\eta}) &= dh_{\begin{pmatrix} \cos t \\ \alpha\sin t \end{pmatrix}p}\left(\gamma\begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix}p\right) = \left(p\bar{p}\bar{\alpha}\bar{\gamma}\cos^2 t + \bar{\gamma}p\bar{p}\bar{\alpha}\sin^2 t, 0\right) \\
&= \left(\alpha\gamma\cos^2 t + \gamma\alpha\sin^2 t, 0\right) = (\alpha\gamma\cos 2t, 0)\,, \\
dh_{N_1^p}\left(\tilde{\vartheta}\right) &= dh_{\begin{pmatrix} \cos t \\ \alpha\sin t \end{pmatrix}p}\left(\gamma\begin{pmatrix} \cos t \\ \alpha\sin t \end{pmatrix}p\right) = (p\bar{p}\bar{\alpha}\bar{\gamma}\cos t\sin t + \gamma p\bar{p}\bar{\alpha}\cos t\sin t, 0) \\
&= (\alpha\gamma\sin(2t), 0)\,. \\
dh_{N_e}(\eta) &= dh_{N_e}\left(\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix}\gamma\right) = \left(\alpha\gamma\sin^2 t + \alpha\gamma\cos^2 t, 0\right) \\
&= (\alpha\gamma, 0)\,. && \square
\end{aligned}
$$

We will need a more concrete description of $TSp(2,m)$. To get this we prove the following.

Proposition 6.6. *The vectors*

$$
\begin{gathered}
(\,x,\ x,\ x,\ \ldots,\ x\,) \\
(\,-y,\ y,\ y,\ \ldots,\ y\,)\ \textit{and} \\
\left(\,\eta,\ \eta,\ \cos(2t)\eta,\ \ldots,\ \cos^{m-3}(2t)\eta,\ \cos^{m-2}(2t)\eta\,\right)
\end{gathered}
$$

are all tangent to $Sp(2,m)$.

For the reader who wishes to have an even more concrete description, see Table 6.1 with these vectors for the case $m = 4$.

Sketch of Proof. One can gain a pretty good sense that the statement is correct by observing the relationship between the basis $\{x, y, \eta_1, \eta_2\}$ and our join decomposition, $S^1_{\mathbb{R}} * S^2_{im}$, for $S^4(1/2)$. Indeed, under h, x projects to the vector that is radial for $\mathrm{dist}(S^1_{\mathbb{R}}, \cdot)$; y projects to a vector that is tangent to the S^1's and η_1; and η_2 project to vectors tangent to the S^2's. A similar statement holds for the basis $\{x, y, \tilde{\eta}_1, \tilde{\eta}_2\}$ with respect to $\tilde{h}$. This observation determines the entries in the first two rows of Table 6.1 up to sign, and the entries in the second two rows up to a convex combination of $\{\gamma_1, \gamma_2\}$. Unfortunately, a precise proof cannot be obtained without direct computation. It is straightforward to do this using (6.8) and (6.9).

TABLE 6.1

vector type	1^{st} col	2^{nd} col	3^{rd} col	4^{th} col
x	$\begin{pmatrix} -sint \\ \alpha\cos t \end{pmatrix} p$	$\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix}$	$\begin{pmatrix} -sint \\ -\alpha\cos t \end{pmatrix}$	$\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix}$
y	$-\begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} \alpha p$	$\alpha\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix}$	$\begin{pmatrix} -\sin t \\ -\alpha\cos t \end{pmatrix} \alpha$	$\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix} \alpha$
η_1	$\begin{pmatrix} -sint \\ \alpha\cos t \end{pmatrix} \gamma_1 p$	$\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix} \gamma_1$	$\begin{pmatrix} -\sin t \\ -\alpha\cos t \end{pmatrix} \gamma_1 \cos(2t)$	$\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix} \gamma_1 \cos^2(2t)$
η_2	$\begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} \gamma_2 p$	$\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix} \gamma_2$	$\begin{pmatrix} -\sin t \\ -\alpha\cos t \end{pmatrix} \gamma_2 \cos(2t)$	$\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix} \gamma_2 \cos^2(2t)$

For example,

$$\begin{aligned} dh_{N_1}\left(\begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} p\right) &= \left(\bar{\alpha}\cos^2 t - \bar{\alpha}\sin^2 t, -2\sin t\cos t\right) \\ &= (-\alpha\cos 2t, -\sin 2t) \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} dh_{N_2}\left(\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix}\right) &= \left(\bar{\alpha}\sin^2 t + \alpha\cos^2 t, 2\sin t\cos t\right) \\ &= (\alpha\cos 2t, \sin 2t)\ . \end{aligned} \tag{6.14}$$

(6.13) and (6.14) explain the first two entries in the first row of Table 6.1. The first two entries of the second row are explained by the computations

$$dh_{N_1}\left(-\begin{pmatrix} -\sin t \\ \alpha\cos t \end{pmatrix} \alpha p\right) = -\left(-\cos^2 t + \bar{\alpha}\bar{\alpha}\sin^2 t, 0\right) = (1, 0)\ , \tag{6.15}$$

and

$$dh_{N_2}\left(\begin{pmatrix} \alpha\cos t \\ -\sin t \end{pmatrix} \alpha\right) = \left(\alpha\alpha\sin^2 t - \cos^2 t, 0\right) = (-1, 0)\ . \tag{6.16}$$

The first two entries of the second and third row are explained by similar computations and the last two entries of the last two rows are explained by Proposition 6.5(i).

We leave the rest of the details of these and the other necessary computations to the reader. □

7. The metric on $Sp(2, m)$

Let g_1 denote the canonical unit metric on S^7.

Our choice of metrics on $Sp(2, m)$ are the restrictions of certain product metrics

$$S^7(\tilde{\nu}_1) \times S^7(\nu_2, c_2) \times S^7(\nu_3, c_3) \times \cdots \times S^7(\nu_{m-1}, c_{m-1}) \times S^7(\nu_m, c_m)\ . \tag{7.1}$$

The metric on the first factor is obtained from g_1 by scaling $V_{\tilde{h}}$ by $\tilde{\nu}_1$ while keeping $H_{\tilde{h}} \perp V_{\tilde{h}}$. We denote the resulting Riemannian metric on S^7 by either $g_{\tilde{\nu}_1}$ or $\langle\ ,\ \rangle_{\tilde{\nu}_1}$.

The notation $S^7(\nu_i, c_i)$ for the Riemannian manifold that is the i^{th} factor (for $2 \leq i \leq m$) stands for the metric on S^7 that is obtained from g_1 by scaling V_h by ν_i and H_h by c_i and keeping $H_h \perp V_h$. We denote the resulting Riemannian metric on S^7 by either g_{ν_i, c_i} or $\langle\ ,\ \rangle_{\nu_i, c_i}$.

The metrics we will use will satisfy

$$1 >> \tilde{\nu}_1 >> \nu_2 >> \nu_3 >> \cdots >> \nu_{m-1} >> \nu_m >>$$
$$c_2 >> c_3 >> \cdots >> c_m > 0\,, \tag{7.2}$$

where the symbol ">>" means "much greater than." Its meaning is only implicit—that any assertions we make are valid provided the gaps between the numbers in (7.2) are sufficiently large.

η is not perpendicular to ϑ with respect to $g_{\tilde{\nu}_1}$. The vector, $\eta_n(\tilde{\nu}_1)$, that is normal to ϑ and satisfies

$$dh\,(\eta_n\,(\tilde{\nu}_1)) = dh(\eta) \tag{7.3}$$

is

$$\eta_n\,(\tilde{\nu}_1) = \frac{\tilde{\nu}_1^2\eta^h(1) + \eta^v(1)}{\tilde{\nu}_1^2\cos^2(2t) + \sin^2(2t)}\,, \tag{7.4}$$

where $\eta^h(1)$ and $\eta^v(1)$ denote the components of η that are $\tilde{h}$-horizontal and $\tilde{h}$-vertical with respect to $\langle\cdot,\cdot\rangle_1$. Use Proposition 6.5[(iv), (v)], (6.4), and (6.5) to verify that $\eta_n(\tilde{\nu}_1)$ satisfies (7.3).

The normalization in (7.4) is not the one that makes $\eta_n(\tilde{\nu}_1)$ a unit vector, rather it is the one that keeps $dh(\eta_n(\tilde{\nu}_1))$ constant with respect to $\tilde{\nu}_1$.

Remark on Notation–fonts for v: There are three notions we have introduced that one might naturally denote by the symbol "v." The vertical component of a vector, ^{v}, (p. 163), the specific vertical vector $\mathfrak{v}$ (p. 174), and the scale, ν or $\tilde{\nu}$, on the vertical space of the Hopf fibrations, h or $\tilde{h}$. Our hope is that these symbols look sufficiently like the letter v that the meaning will be easy to remember, and simultaneously that the different fonts are still sufficiently distinct that these notions will not become confused.

Identify $Sp(2,k)$ with the first k columns of $Sp(2,m)$. Under this convention, $Sp(2,1)$ is the first column of $Sp(2,m)$, i.e., $Sp(2,1) = S^7(\tilde{\nu}_1)$.

We will get a hold of the curvature tensor of $Sp(2,m)$ by studying the sequence of submersions

$$Sp(2,m) \overset{p_{m,m-1}}{\longrightarrow} Sp(2,m-1) \overset{p_{m-1,m-2}}{\longrightarrow} \cdots \overset{p_{3,2}}{\longrightarrow} Sp(2) \overset{p_{2,1}}{\longrightarrow} S^7\,(\tilde{\nu}_1) \overset{\tilde{h}}{\longrightarrow} S^4\,, \tag{7.5}$$

where $p_{k,k-1} : Sp(2,k) \longrightarrow Sp(2,k-1)$ is the projection of $Sp(2,k)$ onto its first $k-1$ entries.

Proposition 7.1. *Let g_k^r be the Riemannian metric on $Sp(2,k)$ obtained by restricting the metric on $Sp(2,m)$. There is a Riemannian metric g_{k-1}^s on $Sp(2,k-1)$ so that*

$$\left(\,Sp(2,k),\; g_k^r\,\right) \overset{p_{k,k-1}}{\longrightarrow} \left(\,Sp(2,k-1), g_{k-1}^s\,\right) \tag{7.6}$$

is a Riemannian submersion.

Moreover, if we fix $\tilde{\nu}_1$, c_i, and ν_i for all $i \le k-1$, and let $c_k \to 0$, then g_{k-1}^s converges to g_{k-1}^r in the C^∞–topology.

Sketch of Proof. The existence of g_{k-1}^s is an immediate consequence of the fact that $p_{k,k-1}$ is the quotient map of the free S^3–action on $Sp(2,k)$ given by

$$(q, u_1, \ldots, u_k) = (u_1, \ldots, u_k\bar{q})\;. \tag{7.7}$$

It remains to compare g^s_{k-1} with g^r_{k-1}. Let p^{k-1}_{k-1} denote the projection of $Sp(2, k-1)$ onto its last factor, and let $W = \ker\ d(\tilde{h} \circ p^{k-1}_{k-1}) \subset TSp(2, k-1)$. Let

$$W \times 0 = \{\ (\omega, 0)\ |\ \omega \in W\} \subset TSp(2, k)\ .$$

The vertical space for $p_{k,k-1}$ is spanned by

$$\{\ (0, 0, \dots, 0, \mathfrak{v}),\ (0, 0, \dots, 0, \vartheta_1),\ (0, 0, \dots, 0, \vartheta_2)\ \}\ .$$

So with respect to g^r_k, $W \times 0$ is horizontal for $p_{k,k-1}$.

To get a decomposition for the full horizontal space, we set

$$\begin{aligned} \text{Basis}\left(\hat{H}_k\right) = \Big\{ &(x,\ x,\ x,\ \dots,\ x)\ ,\\ &(-y,\ y,\ y,\ \dots,\ y)\ ,\\ &\left(\eta_{1,n}(\tilde{\nu}_1),\ \eta_1,\ \cos(2t)\eta_1,\ \dots,\ \cos^{k-4}(2t)\eta_1,\ \cos^{k-3}(2t)\eta_1,\ \cos^{k-2}(2t)\eta_1\right),\\ &\left(\eta_{2,n}(\tilde{\nu}_1),\ \eta_2,\ \cos(2t)\eta_2,\ \dots,\ \cos^{k-4}(2t)\eta_2,\ \cos^{k-3}(2t)\eta_2,\ \cos^{k-2}(2t)\eta_2\right) \Big\}, \end{aligned}$$

and

$$\hat{H}_k = \text{Span}\left\{ \text{Basis}\left(\hat{H}_k\right)\right\}\ .$$

Note that the image of $\hat{H}_k$ under $dp_{k,k-1}$ is $\hat{H}_{k-1}$.

To compare the metrics $g^r_k|_{\text{span}\{\ \hat{H}_{k-1}, W\times 0\ \}}$ and g^r_{k-1} we point out that

(a) the four vectors in Basis($\hat{H}_k$) are mutually perpendicular with respect to g^r_k;

(b) the four vectors in Basis($\hat{H}_{k-1}$) are mutually perpendicular with respect to g^r_{k-1}; and

(c) If $\omega \in W$ and $z \in H_{p_{k,k-1}} \equiv \text{Span}\{\ W \times 0,\ \hat{H}_k\}$, then

$$g^r_k(z, (\omega, 0)) = g^r_{k-1}\left(dp_{k,k-1}(z), dp_{k,k-1}(\omega, 0)\right)\ .$$

Given (a)–(c) only the possible distortion of lengths of vectors in Basis ($\hat{H}_k$) could prevent the submersion $(Sp(2,k), g^r_k) \overset{p_{k,k-1}}{\longrightarrow} (Sp(2, k-1), g^r_{k-1})$ from being Riemannian. It is easy to see that there is such a distortion and it is by an additive term whose size is no more than c_k. It follows that g^s_{k-1} converges to g^r_{k-1} in the C^∞–topology as $c_k \to 0$. □

The upshot of (7.1) is that rather than actually having a sequence of Riemannian submersions in (7.5) we have a stack

$$
\begin{array}{ccccccccc}
Sp(2,m) & \overset{p_{m,m-1}}{\longrightarrow} & \left(Sp(2,m-1), g^s_{m-1}\right) & & & & & & \\
 & & \wr & & & & & & \\
 & & \left(Sp(2,m-1), g^r_{m-1}\right) & \overset{p_{m-1,m-2}}{\longrightarrow} & \left(Sp(2,m-2), g^s_{m-2}\right) & & & & \\
 & & & & \wr & & & & \\
 & & & & \left(Sp(2,m-2), g^r_{m-2}\right) & \overset{p_{m-2,m-3}}{\longrightarrow} & \cdots & & \\
 & & & \vdots & & & & & \\
\left(Sp(2,4), g^r_4\right) & \overset{p_{4,3}}{\longrightarrow} & \left(Sp(2,3), g^s_3\right) & & & & & & \\
 & & \wr & & & & & & \\
 & & \left(Sp(2,3), g^r_3\right) & \overset{p_{3,2}}{\longrightarrow} & \left(Sp(2), g^s_2\right) & & & & \\
 & & & & \wr & & & & \\
 & & & & \left(Sp(2), g^r_2\right) & \overset{p_{2,1}}{\longrightarrow} & S^7\left(\tilde{\nu}_1\right) . & &
\end{array}
\tag{7.8}
$$

Since the change in the metric when we "go down one of the $\wr$'s" is small in the C^∞ topology, we may (and will) think of (7.8) as a sequence of Riemannian submersions, provided we remember that the curvature computations we make are subject to an error that is very small compared to ν_m.

Of course there really is a sequence of *Riemannian* submersions (7.5), but due to the errors introduced at each stage in (7.8), the precise description of the metrics that make the submersions in (7.5) Riemannian is fairly complicated, and the subsequent exposition would be even more so. For this reason we will think of the stack (7.8) as a sequence instead of the actual sequence (7.5) at the cost of a small amount of precision, but with the benefit of considerable simplification. We will make no further mention of this device in the sequel, and trust the reader to remember that there will be an error in our curvature computations that can be made arbitrarily small compared to our choice of ν_m.

8. The exotic spheres have sec ≥ 1 at a point

It follows from Proposition 7.1 and the proof of the Fukaya-Yamaguchi result (Theorem 4.1) that the family of metrics (7.1) is almost nonnegatively curved.

Next we point out that (5.1), (5.3), $A_{m,-n}$ and $A_{m,0}$ are isometric with respect to the metrics (7.1). This is because each action is by symmetries of h and $\tilde{h}$ in each factor. As a consequence we see that (7.1) gives us a family of metrics on $E_{m,-n}$ for all $m, n \in \mathbb{Z}$ so that $0 \leq n \leq m-1$. It follows from O'Neill's horizontal curvature equation (equation {4}, p. 464 of [15]) that the family of metrics on $E_{m,-n}$ is almost nonnegatively curved.

Since the exotic spheres are among the bundles of type $(m, -(m-1))$, to prove Theorem A, it suffices to check that the induced metrics on these bundles have a point of positive curvature.

Proposition 8.1. *The curvature of* $E_{m,-(m-1)}$ *is nearly* ≥ 1 *at the point* $p_0 \equiv$ *orbit* $\left\{\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \cdots\cdots\right)\right\}$, *provided our metric parameters* $\tilde{\nu}_1, \nu_2, \ldots, \nu_m, c_2, c_3, \ldots, c_m$ *are chosen appropriately.*

Proof. Once again our argument can be thought of as a generalization of the one in [11].

At the points $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the vertical spaces for h and $\tilde{h}$ coincide. Therefore the vertical space for the submersion $q_{m,-(m-1)} : Sp(2,m) \longrightarrow E_{m,-(m-1)}$ at this point is the direct sum

$$(0, V_h, 0, 0 \ldots, 0) \oplus (0, -V_h, V_h, 0, \ldots, 0) \oplus (0, 0, -V_h, V_h, 0, \ldots, 0) \oplus \cdots\cdots \\ \oplus (0, 0, \ldots, 0, -V_h, V_h) \ , \tag{8.1}$$

where

$$(0, -V_h, V_h, 0, \ldots, 0) \equiv \left\{(0, -w, w, 0, \ldots, 0) \in T_{p_0}Sp(2,m) \mid w \in V_h\right\} ,$$

and the other summands of (8.1) have the obvious analogous definitions.

It follows that the horizontal space for $q_{m,-(m-1)}$ at p_0 is the direct sum

$$(V_h, 0, 0, \ldots, 0) \oplus \hat{H}_m \ , \tag{8.2}$$

where $\hat{H}_m$ is as defined on page 180.

By O'Neill's horizontal curvature equation ({4} on page 464 of [15]), it suffices to check that every plane tangent to (8.2) is positively curved with respect to the family of metrics (7.1).

To see this, observe that (8.2) is also the horizontal space for the submersion $p_{2,1} \circ p_{3,2} \circ \cdots \circ p_{m,m-1} : Sp(2,m) \longrightarrow S^7(\tilde{\nu}_1)$. The curvature of a plane in (8.2) is therefore the sum of the curvature of a plane in $S^7(\tilde{\nu}_1)$ and the appropriate A-tensor "correction term" in O'Neill's horizontal curvature equation.

$S^7(\tilde{\nu}_1)$ is positively curved for all $\tilde{\nu}_1 \leq 1$. In case the reader does not know this, we set $\tilde{\nu}_1 = 1$. Then the curvature of our plane is the sum of the curvature of a plane in the unit sphere and O'Neill's "correction term." The proof is concluded by observing that multiplying our metric parameters $\nu_2, \ldots, \nu_m$ by ε multiplies the correction term by ε^2. Therefore by choosing $\nu_2, \ldots, \nu_m$ to be sufficiently small we can guarantee not only that all curvatures of planes in (8.2) are positive, but actually that they are $\leq$ and nearly equal to 1. □

Remark on Ricci curvature: It was shown in [14] and [16] that all of the S^3-bundles over S^4 admit positive Ricci curvature. It should not be surprising therefore that all of the metrics we have constructed have positive Ricci curvature.

Appendix

Topological computations for the bundles of type $(4, \cdot)$

A.1. $E_{4,0}$

Consider the $S^3 \times S^3 \times S^3$ action on $Sp(2,4)$ that is given by

$$(q_1, q_2, q_3)(u, v, w, x) = (q_1 u, q_1 v \bar{q}_2, q_2 w \bar{q}_3, q_3 x) \ .$$

The quotient $E_{4,0}$ is an S^3 bundle over S^4. Indeed the map

$$p_{4,0} : \text{orbit}\,(u, v, w, x) \mapsto \tilde{h}(x)$$

is a bundle map. This can be seen via an argument analogous to the one we gave for $E_{3,0}$.

Proposition A.1. *$(E_{4,0}, p_{4,0})$ is the S^3-bundle over S^4 of type $(4, 0)$.*

Proof. We define charts $h_1, h_2 : \mathbb{R}^4 \times S^3 \longrightarrow E$ for $p_{4,0}$ by

$$h_1(u,q) = \text{orbit}\left(\begin{pmatrix} q \\ -uq \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u), \begin{pmatrix} 1 \\ -\bar{u} \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u)\right), \qquad (A.1)$$

and

$$h_2(v,r) = \text{orbit}\left(\begin{pmatrix} \bar{v}r \\ -r \end{pmatrix}\phi(v), \begin{pmatrix} 1 \\ v \end{pmatrix}\phi(v), \begin{pmatrix} v \\ -1 \end{pmatrix}\phi(v), \begin{pmatrix} 1 \\ v \end{pmatrix}\phi(v)\right). \qquad (A.2)$$

h_1 and h_2 are embeddings onto the open dense sets

$$U_1 = \left\{\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right) \middle| \delta \neq 0\right\} \text{ and}$$

$$U_2 = \left\{\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right) \middle| \beta \neq 0\right\}.$$

In fact their inverses are given by

$$h_1^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right)\right) = \left(\frac{\bar{\beta}\delta}{|\delta|^2}, \frac{\bar{\delta}\bar{\alpha}\bar{d}a}{|\delta||\alpha||d||a|}\right)$$

and

$$h_2^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right)\right) = \left(\frac{\bar{\beta}\delta}{|\beta|^2}, \frac{\bar{\beta}\bar{\gamma}\bar{b}c}{|\beta||\gamma||b||c|}\right).$$

Therefore

$$\begin{aligned} h_2^{-1} \circ h_1(u,q) &= h_2^{-1}\left[\text{orbit}\left(\begin{pmatrix} q \\ -uq \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u), \begin{pmatrix} 1 \\ -\bar{u} \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u)\right)\right] \\ &= \left(\frac{u}{|u|^2}, \frac{uuuuq}{|v|^4}\right) \\ &= \left(\frac{u}{|u|^2}, \frac{u^4 q}{|u|^4}\right). \end{aligned}$$

□

A.2. $E_{4,-1}$

$S^3 \times S^3 \times S^3$ also acts on $Sp(2,4)$ by

$$(q_1, q_2, q_3)\,(u, v, w, x) = (q_1 u \bar{q}_3, q_1 v \bar{q}_2, q_2 w \bar{q}_3, q_3 x).$$

The quotient $E_{4,-1}$ is an S^3 bundle over S^4. Indeed the map

$$p_{4,-1} : \text{orbit}\,(u, v, w, x) \mapsto \tilde{h}(x)$$

is a bundle map. Again we can see this via an argument analogous to the one we gave for $E_{3,0}$.

Proposition A.2. *$(E_{4,-1}, p_{4,-1})$ is the S^3-bundle over S^4 of type* $(4, -1)$.

Proof. Let $h_1, h_2 : \mathbb{R}^4 \times S^3 \longrightarrow E$ be defined as in (A.1) and (A.2).

The inverses are now given by

$$h_1^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right)\right) = \left(\frac{\bar{\beta}\delta}{|\delta|^2}, \frac{\bar{\delta}\bar{\alpha}\bar{d}a\delta}{|\delta|^2|\alpha||d||a|}\right)$$

and

$$h_2^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right)\right) = \left(\frac{\bar{\beta}\delta}{|\beta|^2}, \frac{\bar{\beta}\bar{\gamma}\bar{b}c\beta}{|\beta|^2|\gamma||b||c|}\right).$$

Therefore

$$\begin{aligned} h_2^{-1} \circ h_1(u, q) &= h_2^{-1}\left[\text{orbit}\left(\begin{pmatrix} q \\ -uq \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u), \begin{pmatrix} 1 \\ -\bar{u} \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u)\right)\right] \\ &= \left(\frac{u}{|u|^2}, \frac{uuuuq\bar{u}}{|u|^5}\right) = \left(\frac{u}{|u|^2}, \frac{u^4 q\bar{u}}{|u|^5}\right). \end{aligned}$$

□

A.3. $E_{4,-2}$

$S^3 \times S^3 \times S^3$ also acts on $Sp(2, 4)$ by

$$(q_1, q_2, q_3)(u, v, w, x) = (q_1 u\bar{q}_2, q_1 v\bar{q}_2, q_2 w\bar{q}_3, q_3 x).$$

The quotient $E_{4,-2}$ is an S^3 bundle over S^4. Indeed the map

$$p_{4,-2} : \text{ orbit } (u, v, w, x) \mapsto \tilde{h}(x)$$

is a bundle map. Again we can see this via an argument analogous to the one we gave for $E_{3,0}$.

Proposition A.3. *$(E_{4,-2}, p_{4,-2})$ is the S^3-bundle over S^4 that is of type* $(4, -2)$.

Proof. Let $h_1, h_2 : \mathbb{R}^4 \times S^3 \longrightarrow E$ be defined as in (A.1) and (A.2).

The inverses are now given by

$$h_1^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right)\right) = \left(\frac{\bar{\beta}\delta}{|\delta|^2}, \frac{\bar{\delta}\bar{\alpha}\bar{d}a\alpha\delta}{|\delta|^2|\alpha|^2|d||a|}\right)$$

and

$$h_2^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right)\right) = \left(\frac{\bar{\beta}\delta}{|\beta|^2}, \frac{-\bar{\beta}\bar{\gamma}\bar{b}c\gamma\beta}{|\beta|^2|\gamma|^2|b||c|}\right).$$

Therefore

$$\begin{aligned} h_2^{-1} \circ h_1(u, q) &= h_2^{-1}\left[\text{orbit}\left(\begin{pmatrix} q \\ -uq \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u), \begin{pmatrix} 1 \\ -\bar{u} \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u)\right)\right] \\ &= \left(\frac{u}{|u|^2}, \frac{uuuuq\bar{u}^2}{|u|^6}\right) = \left(\frac{u}{|u|^2}, \frac{u^4 q\bar{u}^2}{|u|^6}\right). \end{aligned}$$

□

A.4. $E_{4,-3}$

The $S^3 \times S^3 \times S^3$ action on $Sp(2,4)$ whose quotient is the exotic sphere of type $(4,-3)$ is

$$(q_1, q_2, q_3)(u, v, w, x) = (q_1 u \bar{q}_1, q_1 v \bar{q}_2, q_2 w \bar{q}_3, q_3 x) \ .$$

The submersion from the quotient to S^4 is again given by

$$p_{4,-3} : \text{ orbit } (u, v, w, x) \mapsto \tilde{h}(x) \ .$$

Proposition A.4. *$(E_{4,-3}, p_{4,-3})$ is the S^3-bundle over S^4 of type $(4,-3)$. In particular, $E_{4,-3}$ is an exotic 7-sphere, and is not diffeomorphic to the example of Gromoll and Meyer or to $E_{3,-2}$.*

Proof. As before we define charts for $p_{4,-3}$ using the formulas (A.1) and (A.2). h_1 and h_2 are again embeddings onto U_1 and U_2, only now their inverses are given by

$$h_1^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right)\right) = \left(\frac{\bar{\beta}\delta}{|\delta|^2}, \frac{\bar{\delta}\bar{\alpha}\bar{d}ad\alpha\delta}{|\delta|^2|\alpha|^2|d|^2|a|}\right)$$

and

$$h_2^{-1}\left(\text{orbit}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}, \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \begin{pmatrix} \beta \\ \delta \end{pmatrix}\right)\right) = \left(\frac{\bar{\beta}\delta}{|\beta|^2}, -\frac{\bar{\beta}\bar{\gamma}\bar{b}cb\gamma\beta}{|\beta|^2|\gamma|^2|b|^2|c|}\right) .$$

Therefore

$$\begin{aligned} h_2^{-1} \circ h_1(u,q) &= h_2^{-1}\left[\text{orbit}\left(\begin{pmatrix} q \\ -uq \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u), \begin{pmatrix} 1 \\ -\bar{u} \end{pmatrix}\phi(u), \begin{pmatrix} \bar{u} \\ 1 \end{pmatrix}\phi(u)\right)\right] \\ &= \left(\frac{u}{|u|^2}, \frac{uuuuq\bar{u}^3}{|u|^7}\right) = \left(\frac{u}{|u|^2}, \frac{u^4 q\bar{u}^3}{|u|^7}\right) . \end{aligned}$$

□

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