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# The Density Property for Complex Manifolds and Geometric Structures

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## 1. Introduction

In his paper [3], E. Andersén proved, among several other interesting results, that every holomorphic automorphism of $\mathbb{C}^n$ whose Jacobi determinant is identically 1 can be approximated locally uniformly by finite compositions of so-called shears. Later [4], E. Andersén and L. Lempert proved that every holomorphic automorphism of $\mathbb{C}^n$ could be approximated locally uniformly by finite compositions of overshears. This work was elaborated on by F. Forstnerič and J.P. Rosay, and used by them to study *Aut*$\mathbb{C}^n$ *equivalence* [12]. Many other results concerning automorphisms of $\mathbb{C}^n$ proceeded to appear, and the list is growing. (A large collection of these results may be found in the survey [9].)

One of the major ingredients common to all of the results alluded to above is the use of the following theorem, due to Andersén and Lempert [3, 4]:[1]

*Every holomorphic vector field on $\mathbb{C}^n$ can be approximated locally uniformly by finite sums of complete (in fact, generalized shear) holomorphic vector fields. If the vector field has identically vanishing holomorphic divergence, then it can be approximated locally uniformly by finite sums of complete divergence free (in fact, shear) holomorphic vector fields.*

In an attempt to generalize the recent work on automorphisms from $\mathbb{C}^n$ to other complex manifolds, the author was led to the following definitions:

*A complex manifold $M$ is said to have the density property if every holomorphic vector field on $M$ can be approximated locally uniformly by Lie combinations[2] of complete vector fields[3] on $M$.*

It is also natural to study so called "geometric structures," i.e., Lie subalgebras of the Lie algebra $\mathcal{X}_{\mathcal{O}}(M)$ of all holomorphic vector fields on $M$.

*A geometric structure $\mathfrak{g}$ on a complex manifold $M$ is said to have the density property if every holomorphic vector field in $\mathfrak{g}$ can be approximated locally uniformly by Lie combinations of complete vector fields in $\mathfrak{g}$.*

[1] The theorem does not appear in this form in [3, 4], but is phrased in this way in [12], Lemma 1.3. The analogous theorems in [3, 4] are [3], Theorem 5.1, and [4] Proposition 3.9.

[2] See (2) below.

[3] See Section 2.3


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Clearly $M$ has the density property if and only if $\mathcal{X}_{\mathcal{O}}(M)$ does. Another important special case occurs when we specify on our complex manifold $M$ a holomorphic volume form $\omega$. Denote by $\mathcal{X}^{\omega}_{\mathcal{O}}(M)$ the Lie algebra of all holomorphic vector fields $X$ on $M$ with $\mathrm{div}_{\omega} X = 0$ (see Section 2.2).

*If the geometric structure $\mathcal{X}^{\omega}_{\mathcal{O}}(M)$ has the density property, we say that $(M, \omega)$ has the volume density property.*

We remark (see Section 4) that for a complex Lie group $G$ there is a natural choice of holomorphic volume elements, namely left (or right) invariant ones, and that the algebra $\mathcal{X}^{\omega}_{\mathcal{O}}(G)$ is independent of the choice of left invariant $\omega$. We can thus refer to a complex Lie group as having the volume density property, omitting reference to the left invariant holomorphic volume element in question. We can now state our

## Main Results:

I. 1. *If $M$ and $N$ are Stein manifolds with the density property then so is $M \times N$.*
   2. *If a Stein manifold $M$ has the density property, then so do $M \times \mathbb{C}$ and $M \times \mathbb{C}^*$.*
   3. *If $(M, \omega)$ is a Stein Manifold with holomorphic volume element such that $(M \times \mathbb{C}, \omega \wedge dz)$ has the volume density property, then $M \times \mathbb{C}$ has the density property.*

II. 1. *For any complex Lie group $G$, $G \times \mathbb{C}$ has the volume density property.*
   2. *If, moreover, $G$ is Stein, $G \times \mathbb{C}$ has the density property.*
   3. *If $G$ is a complex Lie group having the volume density property, then $G \times \mathbb{C}^*$ has the volume density property. In particular, $(\mathbb{C}^*)^k$ has the volume density property for all $k \in \mathbb{N}$.*

III. 1. *If $n > k \geq 1$, then the geometric structure $\mathfrak{g}^{n,k}_0$, consisting of holomorphic vector fields on $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$ which vanish on $\mathbb{C}^k \times \{0\}$, has the density property.*
   2. *If $n > k \geq 2$, then the geometric structure $\mathfrak{g}^{n,k}_T$, consisting of holomorphic vector fields on $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^{n-k}$ which are tangent to $\mathbb{C}^k \times \{0\}$, has the density property.*

**Remark:** Regarding the results III above, an observation about some results in [5, 8] gives the following negative result:

*For $n \geq 2$ there exist proper holomorphic embeddings $j : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ such that the geometric structures $\mathfrak{g}_T(j)$ (resp. $\mathfrak{g}_0(j)$), consisting of holomorphic vector fields which are tangent to (resp. vanish on) $j(\mathbb{C}^{n-1})$, do not have the density property.*

The basic intention of the definition of the density property is to isolate those complex manifolds for which the gap between differential topology and holomorphic geometry is considerably narrowed (that is to say, where the holomorphic geometry is somewhat flabby). The idea is that if for example a Stein manifold has the density property, then its group of holomorphic automorphisms is "very large." (The passage from the infinitesimal regime of vector fields to the global regime of automorphisms is provided by the theory of ordinary differential equations. This is explained more precisely in Section 2.3.) In contrast, the reader should compare our ideas with those discussed in the books of S. Kobayashi [22, 23]. Thus the density property for Stein manifolds should be thought of as the opposite extreme of (Kobayashi) hyperbolicity, of finite type $G$-structures, or of elliptic structures on compact manifolds, etc.

In some cases, the fact that a complex manifold or a geometric structure has the density property gives little information. For example, when the manifold is compact the density property holds trivially, but there are relatively few holomorphic vector fields [22]. However, if $M$ is a Stein manifold, then $\mathcal{X}_{\mathcal{O}}(M)$ is an infinite dimensional complex vector space. Since complete vector fields on $M$ are very sparse [6, 10], if the geometric structure in question is sufficiently large (e.g., $\mathfrak{g} = \mathcal{X}_{\mathcal{O}}(M)$ or, if $\dim_{\mathbb{C}} M \geq 2$, $\mathfrak{g} = \mathcal{X}_{\mathcal{O}}^{\omega}(M)$), the density property for $\mathfrak{g}$ may be useful in the construction of various global objects.

A well-known object of study, introduced by S. Chern, is that of $G$-structure. A $G$-structure on a manifold $M$ is a subbundle $P$ of the principle bundle $L(M)$ of frames of $M$, with structure group $G$. Some of the geometric structures which we study arise as infinitesimal automorphisms of $G$-structures, but this is not the case, for example, with the structures $\mathfrak{g}_T$ and $\mathfrak{g}_0$ of Section 5. Geometric structures in our sense represent more the geometry of the group of automorphisms in question than that of the manifold.

Finally, it should be emphasized that the understanding of the density property at this point is very poor; all we have is a collection of examples and applications. Little is known about the relationship between the density property and other, more accessible properties of complex manifolds. (For the few known facts, see [29].)

Before proceeding, we should clarify matters regarding our notation.

(1) For us, holomorphic vector fields on a complex manifold $M$ are holomorphic sections of the bundle $T^{1,0}M$, and we denote the set of holomorphic vector fields by $\mathcal{X}_{\mathcal{O}}(M)$. However, we implicitly identify $T^{1,0}M$ with $TM$. ($T^{1,0}M \ni X \mapsto 2\mathrm{Re}(X) \in TM$.) This presents no difficulties, since we restrict our attention to holomorphic vector fields. See Section 2 for the definitions and the Lie algebra structure of $\mathcal{X}_{\mathcal{O}}(M)$.

(2) A Lie combination of elements of a subset $S$ of a Lie algebra $\mathfrak{a}$ is an element of the Lie subalgebra of $\mathfrak{a}$ generated by $S$. That is to say, a Lie combination of elements of a subset $S$ is an element of $\mathfrak{a}$ which can be written as a finite sum of terms of the form

$$\left[\left[\ldots[[a_1, a_2], a_3], \ldots, a_{n-1}\right], a_n\right],$$

with $a_1, \ldots, a_n \in S$.

(3) The local flow of a holomorphic vector field $X$ on a complex manifold $M$ is the unique *local 1-parameter group* or *pseudogroup* of biholomorphisms $\{\varphi^t\}$ on $M$ which represents the "set of local solutions" of the O.D.E. (in a local coordinate chart $U \subseteq M$)

$$\frac{d}{dt}\varphi^t(z) = X \circ \varphi^t(z), \quad \varphi^0(z) = z \qquad (z \in U).$$

This local flow satisfies the local group law

$$\varphi^s \circ \varphi^t(z) = \varphi^{s+t}(z)$$

wherever and whenever both sides make sense. (If the statement makes sense for all $s, t \in \mathbb{R}$ and $z \in M$, we say $X$ is $\mathbb{R}$-complete; see also Section 2.3). This group law is a consequence, via the uniqueness theorem for solutions of O.D.E., of the fact that vector fields do not depend on time (i.e., $X$ defines an autonomous system, and so the "physical laws" which $X$ represents are "symmetric" with respect to time.)

(4) A time-dependent vector field is a special one parameter family of vector fields $\{X_t\} \subseteq \mathcal{X}_{\mathcal{O}}(M)$. The parameter $t$, called time, lies in (some subset of) $\mathbb{R}$, and it is implicitly understood that the solution (also called "time dependent flow" or "evolution operator") of the O.D.E. associated to $\{X_t\}$ has the same time parameter $t$. In particular, the solutions depend on the initial time. We

denote these local one parameter families by $\{\varphi_s^t\}$. Precisely, we have (locally)

$$\frac{d}{dt}\varphi_s^t(z) = X_t \circ \varphi_s^t(z), \quad \varphi_s^s(z) = z, \qquad (z \in U) .$$

The local group law of the autonomous system is replaced by the *determinacy law*

$$\varphi_s^t(z) = \varphi_r^t \circ \varphi_s^r(z)$$

wherever and whenever this makes sense.

(5) We think of $T$ as the "tangent functor" (see [17]), and so for a holomorphic map $f : M \to N$ we denote by $Tf : TM \to TN$ the map which, in locally trivial coordinates, is given by

$$Tf(x, v) = \left(f(x), f'(x)v\right) .$$

We prefer this notation to the more common notations $df$ and $f_*$, using the former for the linear manifold $\mathbb{C}^n$, and reserving the latter only for diffeomorphisms/biholomorphisms, when pushing forward a vector field: for a vector field $X$, $f_*X$ is the vector field given by

$$(f_*X)(x) := Tf\left(f^{-1}(x)\right) X\left(f^{-1}(x)\right) .$$

In particular, if the local flow of $X$ is $\varphi^t$, then that of $f_*X$ is $f \circ \varphi^t \circ f^{-1}$.

The organization of this article is as follows:

In Section 2 we describe the ideas in complex geometry and ordinary differential equations (dynamical systems) which motivate the definition of the density property, and are useful in applications. The status of this section as it pertains to this note is motivational. In Section 3 we define the density property and discuss its relation to automorphism groups on Stein manifolds. We then prove some general results about the density property, namely its behavior with respect to Cartesian products of complex manifolds. In Section 4 we prove that various complex Lie groups have the density property and the volume density property. In Section 5 we discuss relative geometric structures, proving both positive and negative results regarding the density property.

## 2. Holomorphic vector fields

In this section we introduce the basic notions needed from complex geometry and the theory of ordinary differential equations (O.D.E.).

### 2.1. Basic definitions

It is well known that there are two ($\mathbb{R}$-isomorphic) representations of the holomorphic tangent spaces of a complex manifold $M$ of $\mathbb{C}$-dimension $n$. To recall, one begins with the real tangent space $T_zM$, and obtains a complexified tangent space $T_z^{\mathbb{C}}M := T_zM \otimes_{\mathbb{R}} \mathbb{C}$. The complex structure on $M$ then gives rise to a (holomorphically well defined, or integrable) splitting

$$T_z^{\mathbb{C}}M \cong T_z^{1,0}M \oplus T_z^{0,1}M ,$$

where $T_z^{1,0}M := \text{span}\{\frac{\partial}{\partial z_j}\}$ and $T_z^{0,1}M := \text{span}\{\frac{\partial}{\partial \bar{z}_j}\}$. Writing $\pi_z : T_z^{\mathbb{C}}M \to T_z^{1,0}M$ for the projection and $j_z : T_zM \hookrightarrow T_z^{\mathbb{C}}M$ for the injection, it is then easy to see that $\varphi_z := \pi_z \circ j_z$ is a real vector space isomorphism for each $z \in M$. (More details can be found in [14].) Consequently, we obtain a map $\varphi$ which takes sections of $TM$ (i.e., the usual vector fields) to sections of $T^{1,0}M$.

Precisely, $\varphi(X)(z) := \varphi_z X(z)$, and it is easily verified that $\varphi^{-1}Z = 2\mathrm{Re}Z$. One also defines an almost complex structure $J$ on $TM$ by $J_z := (\varphi_z)^{-1}\sqrt{-1}\varphi_z$.

The sections of $TM$ and those of $T^{1,0}M$ both form Lie algebras when endowed with their respective commutator brackets. (The commutator bracket on $T^{1,0}M$ is the one inherited from $T^{\mathbb{C}}M$, which itself is the complexification of the commutator bracket on $TM$.) When restricted to so-called holomorphic vector fields, $\varphi$ is a Lie algebra isomorphism.

More precisely, the holomorphic vector fields, which we denote by $\mathcal{X}_{\mathcal{O}}(M)$, are those sections of $TM$ which are mapped by $\varphi$ to holomorphic sections of $T^{1,0}M$. It is an immediate consequence of the Cauchy-Riemann equations that for any vector field $X$ on $M$ and any holomorphic function $f$ on $M$, $\varphi(X)f = Xf$. Hence an alternate definition of holomorphic vector field is that as a derivation, it maps $\mathcal{O}(M)$ to $\mathcal{O}(M)$. This is now easily used to show that $\varphi$ is in fact a Lie algebra isomorphism from $\mathcal{X}_{\mathcal{O}}(M)$ to the holomorphic sections of $T^{1,0}M$. Note, in particular, that every holomorphic vector field $X$ commutes with $JX$.

## 2.2. The invariant notion of divergence

Let $M$ be a complex manifold of complex dimension $n$, and let $\omega$ be a nonvanishing holomorphic $(n, 0)$-form, i.e., a holomorphic volume element. Let $X \in \mathcal{X}_{\mathcal{O}}(M)$, and let $\varphi^t$ be the local flow of $X$ (see Section 1). Then we can define $\mathrm{div}_\omega X$ to be the unique holomorphic function on $M$ which satisfies

$$(\mathrm{div}_\omega X)\,\omega := \left(\varphi^{-t}\right)^* \frac{d}{dt}\left(\varphi^t\right)^* \omega = L_X\omega\,.$$

We recall H. Cartan's formula for differential forms $\alpha$ [1]:

$$L_X\alpha = X\rfloor d\alpha + d(X\rfloor\alpha)\,,$$

where for a $k$-form $\beta$, $X\rfloor\beta$ is the $(k-1)$-form defined by

$$(X\rfloor\beta)_p\,(v_1, \ldots, v_{k-1}) := \beta_p\,(X(p), v_1, \ldots, v_{k-1})\,,\ v_1, \ldots, v_{k-1} \in T_pM\,.$$

Using the fact that $\omega$ is closed, we obtain

$$(\mathrm{div}_\omega X)\,\omega = d(X\rfloor\omega)\,.$$

This formula makes sense even if $X$ is time-dependent. Moreover, it is known (see [1, Theorem 2.2.24]) that if $X_t$ is a time-dependent vector field with evolution operator $\varphi_s^t$, then the formula

$$\left(\left(\varphi_s^t\right)^{-1}\right)^* \frac{d}{dt}\left(\varphi_s^t\right)^* \omega = L_{X_t}\omega$$

holds. Consequently we obtain.

**Proposition 2.1.** *Let $X_t$ be a time-dependent vector field with evolution operator $\varphi_s^t$. Then $(\varphi_s^t)^*\omega = \omega$ if and only if $\mathrm{div}_\omega X_t \equiv 0$.*

The reader can easily verify that when $X \in \mathcal{O}(\mathbb{C})$ and $\omega = dz$,

$$\mathrm{div}_\omega\left(X\frac{\partial}{\partial z}\right) = \frac{\partial X}{\partial z}\,,$$

while when $X \in \mathcal{O}(\mathbb{C}^*)$ and $\omega = \frac{dz}{z}$,

$$\mathrm{div}_\omega\left(X(z)\cdot z\frac{\partial}{\partial z}\right) = z\frac{\partial X}{\partial z}\,.$$

More generally[4] if $G$ is a complex Lie group, $\{V_1, \ldots, V_n\}$ ($n = \dim_{\mathbb{C}} G$) is a basis of left invariant vector fields on $G$, and $\omega_G$ is the unique left invariant holomorphic volume element on $G$ such that $\omega_G(V_1, \ldots, V_n) \equiv 1$, then for

$$X = \sum_j X_j V_j \in \mathcal{X}_{\mathcal{O}}(G) \ \left(X_j \in \mathcal{O}(G)\right)$$

we have

$$\mathrm{div}_{\omega_G} X = \sum_j V_j\left(X_j\right) .$$

A result which will be very useful in the sequel is:

**Lemma 2.2.** *$div_\omega([X, Y]) = X div_\omega Y - Y div_\omega X$.*

***Proof.*** Recall that $L_{[X,Y]} = L_X L_Y - L_Y L_X$. We have

$$\begin{aligned}
(\mathrm{div}_\omega [X, Y])\,\omega &= L_{[X,Y]}\omega \\
&= L_X L_Y \omega - L_Y L_X \omega \\
&= L_X\left((\mathrm{div}_\omega Y)\,\omega\right) - L_Y\left((\mathrm{div}_\omega X)\,\omega\right) \\
&= (X \mathrm{div}_\omega Y)\,\omega + (\mathrm{div}_\omega Y)\, L_X \omega - (Y \mathrm{div}_\omega X)\,\omega + (\mathrm{div}_\omega X)\, L_Y \omega \\
&= (X \mathrm{div}_\omega Y - Y \mathrm{div}_\omega X)\,\omega + (\mathrm{div}_\omega X \mathrm{div}_\omega Y - \mathrm{div}_\omega Y \mathrm{div}_\omega X)\,\omega \\
&= (X \mathrm{div}_\omega Y - Y \mathrm{div}_\omega X)\,\omega ,
\end{aligned}$$

as desired. □

Finally, let us point out that if $(M, \omega)$ and $(N, \theta)$ are complex manifolds with holomorphic volume elements, then so is $(M \times N, (\pi_M)^*\omega \wedge (\pi_N)^*\theta)$, where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the usual projections. It is easy to verify that for $X = (U, V) \in \mathcal{X}_{\mathcal{O}}(M \times N)$ one has (writing $\omega \wedge \theta$ for $(\pi_M)^*\omega \wedge (\pi_N)^*\theta$)

$$\mathrm{div}_{\omega \wedge \theta} X = \mathrm{div}_\omega U + \mathrm{div}_\theta V .$$

### 2.3. Elementary ideas from O.D.E.

In this section we recall some results from the theory of O.D.E., regarding approximation of solutions to O.D.E. These results are the essence of the passage from the infinitesimal regime of Lie algebras of vector fields to the local and global regimes of pseudogroups of biholomorphisms and groups of automorphisms, respectively. The main reference here is the book of R. Abraham and J.E. Marsden [1]. We omit many details, and so refer to this source for background from the outset.

***Definition.*** A holomorphic vector field $X$ (time independent or not) is called $\mathbb{R}$-complete if its integral curves through any point are defined for all $t \in \mathbb{R}$. $X$ is called $\mathbb{C}$-complete if both $X$ and $iX$ are $\mathbb{R}$-complete.

In general, it is possible to extend the "time" of flows of holomorphic vector fields to the complex domain, and define $\mathbb{C}$-completeness in a different but equivalent way. Since we do not

[4] See Section 4.

make use of flows defined for complex time, we shall skip over this point. The interested reader is referred to [6, 7, 10] for more on this subject.

Perhaps the most crucial fact for us regarding holomorphic vector fields is that the flow of a complete holomorphic vector field is a one parameter group of automorphisms. (The group structure is lost if the vector field is time dependent, and we just get a one parameter family of automorphisms.) This is not true if the vector field is not complete. Generally, the flow of an incomplete vector field is a local one parameter group (or family if there is time dependence). Our aim is to understand when the time $T$ maps of this local flow can be approximated, uniformly on compacts in their domains of definition, by automorphisms. A notion which helps in this regard is that of consistent algorithms.

***Definition*** *([1]).* Let $M$ be a complex manifold, $X \in \mathcal{X}_{\mathcal{O}}(M)$, and $I \subseteq \mathbb{R}$ an interval containing 0. Suppose $\Phi : M \times I \to M$ is a continuous mapping such that $\Phi(z, \cdot)$ is $C^1$ for each $z \in M$, and $\Phi(\cdot, t)$ and $\frac{\partial \Phi}{\partial t}(\cdot, t)$ are holomorphic for each $t \in I$. We say $\Phi$ is an algorithm consistent with $X$ if

(i) $\Phi(\cdot, 0) = id_M$, and
(ii) $\frac{\partial \Phi}{\partial t}|_{t=0} = X$.

We write $\Phi_t := \Phi(\cdot, t)$ and define $\Phi_t^{(1)} := \Phi_t$, $\Phi_t^{(n)} := \Phi_t \circ \Phi_t^{(n-1)}$.

The following theorem is proved in [1, Theorem 2.1.26] in the real setting, but the same proof holds in the holomorphic category.

**Theorem 2.3 ([1]).** *Let $\Phi$ be an algorithm consistent with a vector field $X$, and let $\{\varphi^t\}$ be the flow of $X$. Then for $(t, x)$ in the domain of definition of $\{\varphi^t\}$, $\Phi_{t/n}^{(n)}(x)$ is defined for $n$ sufficiently large and converges to $\varphi^t(x)$ as $n \to \infty$. Conversely, if $\Phi_{t/n}^{(n)}$ is defined and converges for $0 \le t \le T$, then $(T, x)$ is in the domain of definition of $\{\varphi^t\}$, and*

$$\lim_{n \to \infty} \Phi_{t/n}^{(n)}(x) = \varphi^t(x) .$$

**Remark:** The notion of consistent algorithm can be applied to approximate evolution operators of time dependent vector fields by using a standard "one step method."

The next proposition, together with Theorem 2.3, is the reason for the Lie algebra structure in the formulation of the density property in Section 3.

**Proposition 2.4.** *If $X$ and $Y$ are vector fields with flows $f$ and $g$, then*

1. $(z, t) \mapsto f^t \circ g^t(z)$ *is an algorithm consistent with* $X + Y$, *and*
2. $(z, t) \mapsto g^{-\sqrt{|t|}} \circ f^{-sgn(t)\sqrt{|t|}} \circ g^{\sqrt{|t|}} \circ f^{sgn(t)\sqrt{|t|}}(z)$ *is an algorithm consistent with* $[X, Y]$.

***Proof.*** This is just an exercise in differentiation. In particular, it is seen that locally $g^{-s} \circ f^{-t} \circ g^s \circ f^t(x) = x + st[X, Y](x) + o(s^2 + t^2)$, and hence the algorithm in 2 is $C^1$. □

More generally, this proposition can easily be used to show that given any Lie combination of vector fields, an algorithm can be constructed for this Lie combination using a finite composition of the flows of the vector fields appearing in the Lie combination.

It is a standard fact in the theory of *O.D.E.* (using Grönwall's inequality, see [1] for the latter) that approximation of vector fields leads to approximation of flows. Together with this, Theorem 2.3 and Proposition 2.4 tell us that if we are given a family of complete vector fields $\{X_\alpha\}$ on a complex manifold $M$, then we can approximate the local flow of any Lie combination of the $X_\alpha$'s by automorphisms of $M$.

There are two natural subgroups of $AutM$ associated to the collection $\{X_\alpha\}$ of complete vector fields. The first group, denoted by $\mathcal{F} = \mathcal{F}(\{X_\alpha\})$, is the group consisting of finite compositions of all time-t maps of the vector fields in $\{X_\alpha\}$. The second group $\mathcal{G} = \mathcal{G}(\{X_\alpha\})$ is the closure of $\mathcal{F}$ in $AutM$, in the topology described in [4], namely, the one in which we say $f_n \to f$ if $f_n \to f$ and $f_n^{-1} \to f^{-1}$ uniformly on compact subsets. (The first group $\mathcal{F}$ was of importance in [3, 4], where they showed that in general, $\mathcal{F} \neq \mathcal{G}$. Contrast this with the case of finite dimensional Lie groups.)

Suppose now that one is given an automorphism $\Phi \in AutM$, which is connected by a $\mathcal{C}^1$ path $\{\Phi_t\} \subseteq AutM$ to $id_M$, and one wants to know whether $\Phi \in \mathcal{G}$. In this direction, write $X_t := \frac{d\Phi_t}{dt} \circ \Phi_t^{-1}$. $X_t$ is a time dependent vector field. Let $\mathfrak{g}$ be the closure in $\mathcal{X}_{\mathcal{O}}(M)$ of the Lie algebra generated by the $X_\alpha$'s. With this notation we have

**Theorem 2.5.** *If $X_t \in \mathfrak{g}$ for each $t \in [0, 1]$, then $\Phi \in \mathcal{G}$.*

We will make use of the following well-known lemma, which we do not prove.

**Lemma 2.6.** *Let $\{f_n\}$ be a sequence of automorphisms which converges to an automorphism $f$ uniformly on compact sets. Then $\{f_n^{-1}\}$ converges to $f^{-1}$ uniformly on compact sets.*

***Proof of Theorem 2.5.*** In view of Lemma 2.6, we need only to show that given an $\epsilon > 0$ and a compact set $K \subset\subset M$, there are $f_1, \ldots, f_N \in \mathcal{F}$ such that[5]

$$\sup_{x \in K} \operatorname{dist}(f_n \circ \ldots \circ f_1(x), \Phi(x)) < \epsilon .$$

Fix $\delta > 0$, and $N \in \mathbb{Z}_+$ large enough so that $N\delta \leq 1$. Put $T := N\delta$, $T_j := j\delta$ for $0 \leq j \leq N$, $I_j := [T_{j-1}, T_j]$ for $1 \leq j \leq N$, and

$$X_{j,t} := \begin{cases} 0 & t \notin I_j \\ X_{T_{j-1}} & t \in I_j \end{cases}$$

($\sum_{j=1}^N X_{j,t}$ should be thought of as the piecewise constant approximation to $X_t$ for $0 \leq t \leq T$.) By approximation, we may assume that each $X_{T,j}$ is a Lie combination of the $X_\alpha$'s. The flow of $X_{j,t}$ is

$$g_j^t = \begin{cases} id & t < T_{j-1} \\ h_j^{t-T_{j-1}} & T_{j-1} < t < T_j \\ h_j^\epsilon & t > T_j \end{cases}$$

where $h_j$ is the local flow of the (time independent) vector field $X_{T_{j-1}}$. Then the local flow of $\sum_{j=1}^N X_{j,t}$ at time $T$ is

$$h_N^\delta \circ h_{N-1}^\delta \circ \ldots \circ h_1^\delta .$$

[5] The distance dist is with respect to some fixed complete Riemannian metric.

It is possible to show (see for example Section 4 of [3]) that this flow converges to the time-$T$ map of the flow of $X_t$ locally uniformly. The conclusion of the theorem now follows by taking $T = 1$ and applying Proposition 2.4 and Theorem 2.3 to approximate $\Phi$ by a finite composition of the $h_j$'s, and to approximate each of the $h_j$'s by finite compositions of members of $\mathcal{F}$. □

## 3. The density property

In this section, we define the density property, and develop some elementary aspects of it. While it makes sense for any complex manifold, the density property is most interesting on Stein manifolds.

### 3.1. The definition

Let $M$ be a complex manifold. A Lie subalgebra $\mathfrak{g}$ of $\mathcal{X}_{\mathcal{O}}(M)$ is said to have the density property if the Lie subalgebra of $\mathfrak{g}$ generated by all the complete vector fields in $\mathfrak{g}$ is dense in $\mathfrak{g}$ in the locally uniform topology:

$$\overline{\langle X \in \mathfrak{g} : X \text{ complete}\rangle} = \mathfrak{g} .$$

Perhaps the most important case occurs when the Lie algebra under consideration is $\mathcal{X}_{\mathcal{O}}(M)$ – the Lie algebra of all vector fields. In this case, we will say that *$M$ has the density property.* Another very important case occurs when $M$ admits a holomorphic volume element $\omega$. We will say that *$(M, \omega)$ has the volume density property* if the Lie algebra $\mathcal{X}^{\omega}_{\mathcal{O}}(M)$ of divergence zero vector fields on $M$ has the density property.

### 3.2. A remark regarding automorphism groups

In our forthcoming note [29], we explore more precisely the consequences of the density property on automorphism groups. For now, we content ourselves with the following remark.

While there is a theory of infinite dimensional groups which assigns infinite dimensional manifold structures to these groups and so on, the theory has been most successful over compact manifolds. Since we are interested mostly in noncompact manifolds, we shall avoid these details in this article, and give an operational definition based on the following facts.

Suppose we have a (finite dimensional) Lie group $\mathfrak{G}$ acting on a manifold $M$. ($M$ can be either $C^r$, $1 \leq r \leq \omega$ or complex.) Then the set of infinitesimal generators of one parameter subgroups of $\mathfrak{G}$ forms a finite dimensional Lie algebra. Conversely, the following is Theorem 3.1 of [22]

**Proposition 3.1 ([22]).** *Let $\mathfrak{G}$ be a group of differentiable transformations of a manifold $M$. Let $S$ be the set of all vector fields on $M$ which generate global 1-parameter groups $\{\varphi^t\}$ of transformations of $M$ such that $\{\varphi^t\} \subseteq \mathfrak{G}$. If the set $S$ generates a finite-dimensional Lie algebra of vector fields on $M$, then $\mathfrak{G}$ is a (finite dimensional) Lie transformation group and $S$ is the Lie algebra of $\mathfrak{G}$.*

This motivates the following definition.

***Definition.*** A group $\mathcal{G}$ of holomorphic transformations on a complex manifold $M$ is said to be infinite dimensional if the set of complete holomorphic vector fields whose flows lie entirely in $\mathcal{G}$ generates an infinite dimensional Lie algebra.

Of course, by the work in Section 2.3 above, this definition is equivalent to any reasonable definition of infinite dimensionality.

With this definition we have the following proposition.

**Proposition 3.2.** *Let $M$ be a Stein manifold.*

1. *If $M$ is of positive dimension and has the density property then $AutM$ is infinite dimensional.*
2. *If $M$ is of complex dimension $\geq 2$ and admits a holomorphic volume element $\omega$ such that $(M, \omega)$ has the density property, then*

$$Aut^{\omega} M := \{f \in AutM : f^*\omega = \omega\}$$

*is infinite dimensional.*

The proof is an immediate consequence of the definition of the density property and the following lemma, whose proof is itself an elementary application of Cartan's Theorem A and the defining properties (see [18]) of Stein manifolds. We state without proof the following.

**Lemma 3.3.** *Let $M$ be a Stein manifold.*

1. *If $M$ has positive dimension then $\mathcal{X}_{\mathcal{O}}(M)$ is an infinite dimensional vector space over $\mathbb{C}$.*
2. *If $M$ is of complex dimension $\geq 2$ and admits a holomorphic volume element $\omega$ then $\mathcal{X}^{\omega}_{\mathcal{O}}(M)$ is an infinite dimensional vector space over $\mathbb{C}$.*

We remark only that in the proof of 2, one must use the duality provided by $\omega$.

The converse of Proposition 3.2 is false. That is, there exist Stein manifolds without the density property which have infinite dimensional automorphism groups. We leave it to the reader to check that $\mathbb{C} \times \Delta$ is one such manifold, where $\Delta$ is the unit disc.

### 3.3. Product theorems

We begin with the following result.

**Theorem 3.4.** *If $M$ and $N$ are Stein manifolds with the density property, then so is $M \times N$.*

The proof is an almost immediate consequence of the following lemma.

**Lemma 3.5.** *If a Stein manifold $M$ has the density property and $X_\lambda$ is a holomorphic vector field on $M$ depending holomorphically on a Stein parameter $\lambda \in \Lambda$, then $X_\lambda$ can be approximated locally uniformly on $M \times \Lambda$ by Lie combinations of complete holomorphic vector fields which depend holomorphically on the parameter $\lambda$.*

***Proof.*** Let $j : \Lambda \to \mathbb{C}^n$ be a proper holomorphic embedding. (Such embeddings always exist for Stein manifolds; see e.g., [18].) Consider the vector bundles $TM \times \Lambda \xrightarrow{\pi} M \times \Lambda$ defined by $\pi(v, \lambda) = (x, \lambda)$ for $v \in T_xM$, and $TM \times \mathbb{C}^n \xrightarrow{\pi'} M \times \mathbb{C}^n$ defined by $\pi'(v, z) = (x, z)$ for $v \in T_xM$. The embedding $j$ induces a bundle monomorphism $\tilde{j} : TM \times \Lambda \to TM \times \mathbb{C}^n$, i.e., $\tilde{j}(v, \lambda) = (v, j(\lambda))$. Now, the sheaf $\mathcal{S}$ of germs of holomorphic sections of $TM \times \mathbb{C}^n \xrightarrow{\pi'} M \times \mathbb{C}^n$

is known to be a coherent analytic sheaf over $M \times \mathbb{C}^n$ (see [15, 16] for this and other facts used below, regarding coherent analytic sheaves on Stein manifolds), and since $\Lambda$ is an analytic submanifold of $\mathbb{C}^n$, the subsheaf $\mathcal{I}_{M\times\Lambda}$ of germs of holomorphic sections vanishing on $M \times \Lambda$ is also known to be coherent analytic over $M \times \mathbb{C}^n$. By standard sheaf theory, the quotient sheaf $\mathcal{G}_{M\times\Lambda} := \mathcal{S}/\mathcal{I}_{M\times\Lambda}$ is also coherent analytic over $M \times \mathbb{C}^n$, and the latter is identified with the sheaf of germs of holomorphic sections of $TM \times \Lambda \xrightarrow{\pi} M \times \Lambda$. The short exact sequence

$$0 \to \mathcal{I}_{M\times\Lambda} \to \mathcal{S} \to \mathcal{G}_{M\times\Lambda} \to 0$$

gives rise to a long exact sequence in cohomology, a segment of which is

$$\ldots \to H^0\left(M \times \mathbb{C}^n, \mathcal{S}\right) \to H^0\left(M \times \mathbb{C}^n, \mathcal{G}_{M\times\Lambda}\right) \to H^1\left(M \times \mathbb{C}^n, \mathcal{I}_{M\times\Lambda}\right) \to \ldots ,$$

and since $M \times \mathbb{C}^n$ is Stein, Cartan's Theorem B says that

$$H^1\left(M \times \mathbb{C}^n, \mathcal{I}_{M\times\Lambda}\right) = 0 .$$

It follows that

$$H^0\left(M \times \mathbb{C}^n, \mathcal{S}\right) \to H^0\left(M \times \mathbb{C}^n, \mathcal{G}_{M\times\Lambda}\right)$$

is surjective. As $H^0$ is identified with global sections, we see that every section of $TM \times \Lambda \xrightarrow{\pi} M \times \Lambda$ is the restriction to $M \times \Lambda$ of a section of $TM \times \mathbb{C}^n \xrightarrow{\pi'} M \times \mathbb{C}^n$, via the identification mentioned above.

To finish the proof of the lemma, we observe that the section $X_\lambda$ of $TM \times \Lambda \xrightarrow{\pi} M \times \Lambda$ can be developed in a power series in $\lambda$ (thinking of $\lambda$ as a point in $\mathbb{C}^n$), whose coefficients are members of $\mathcal{X}_\mathcal{O}(M)$. Truncating the power series finishes the job. □

***Proof of Theorem 3.4.*** Write $X = (V, W) \in \mathcal{X}_\mathcal{O}(M \times N)$ as $X = (V, 0) + (0, W)$. Now use Lemma 3.5 to approximate $(V, 0)$ and $(0, W)$ by a Lie combination of complete holomorphic vector fields on $M \times N$ which are tangent[6] to $M$ and $N$, respectively. □

We now turn our attention to some results for Stein manifolds which are not encompassed in Theorem 3.4. (We note that the Stein manifolds in question must be of positive dimension for the results to be true, but we will neglect to mention this, assuming it from the outset.) These results rely heavily on the following lemma.

**Lemma 3.6.** *On any Stein manifold $M$ there exist vector fields $X_1, \ldots, X_N \in \mathcal{X}_\mathcal{O}(M)$ and functions $\varphi_1, \ldots, \varphi_N \in \mathcal{O}(M)$ such that*

$$\sum_{j=1}^{N} X_j \varphi_j = 1 .$$

***Proof.*** Choose $\varphi_1, \ldots, \varphi_N$ as the coordinate functions of some immersion $\varphi : M \to \mathbb{C}^N$. For a holomorphic vector bundle $Y$, denote by $\mathcal{S}(Y)$ the sheaf of germs of holomorphic sections of $Y$. We denote by $E_M$ the Whitney sum of $N$ copies of $TM$ with itself, and by $\mathbf{1}_M$ the trivial line bundle over $M$. Then $\varphi$ defines a map $\Phi : \mathcal{S}(E_M) \to \mathcal{S}(\mathbf{1}_M)$, defined by

$$\Phi\left(V_1, \ldots, V_N\right) = \sum_{1}^{N} V_j \varphi_j = \operatorname{trace}\left(T\varphi\left(V_1, \ldots, V_N\right)\right) .$$

[6] A vector field $X$ on $M \times N$ is said to be tangent to $M$ if it is of the form $X = (V, 0)$, and similarly for $N$.

$\Phi$ is clearly surjective, and hence we obtain a short exact sequence of coherent analytic sheaves

$$0 \to \ker\, \Phi \to \mathcal{S}(E_M) \to \mathcal{S}(\mathbf{1}_M) \to 0\,.$$

Passing to the induced long exact sequence in Čech cohomology and applying Cartan's Theorem B, we obtain that $H^0(M, \mathcal{S}(E_M)) \to H^0(M, \mathcal{S}(\mathbf{1}_M))$ is surjective. In particular, the global section 1 of $\mathcal{S}(\mathbf{1}_M)$ is in the image of $\Phi_*$. That is to say, there are vector fields $X_1, \dots, X_N$ such that $\sum X_j \varphi_j = 1$, as desired. □

**Remark:** In a previous version of this note, we had an approximate version of this lemma, which was sufficient for the proofs to come, but which made those proofs more cumbersome. We wish to thank Laszlo Lempert, as well as the referee of this note, for pointing out to us that this lemma is true.

Before stating the next theorem, we make the following observation. If M is a complex manifold, then any vector field on $M \times \mathbb{C}$ (resp. $M \times \mathbb{C}^*$) can be expanded in a power series (resp. Laurent series) in $z \in \mathbb{C}$ (resp. $\mathbb{C}^*$) which converges locally uniformly on $M \times \mathbb{C}$ (resp. $M \times \mathbb{C}^*$). The component of this vector field which is tangent to $M$ will be a power (resp. Laurent) series in $z$ with coefficients in $\mathcal{X}_{\mathcal{O}}(M)$. By truncating the power (resp. Laurent) series, we can approximate our vector field locally uniformly by vector fields which are polynomial (resp. Laurent polynomial) in $z$. (This is the trivial case of Lemma 3.5, whose proof basically consists in reducing to this case.) If $M$ admits a holomorphic volume element $\omega$, the reader may verify that the approximation can be done in a divergence free way. (We use the volumes $dz$ on $\mathbb{C}$ and $\frac{dz}{z}$ on $\mathbb{C}^*$; see Section 2.2.)

**Theorem 3.7.** *If $M$ is a Stein manifold with the density property, then so are $M \times \mathbb{C}$ and $M \times \mathbb{C}^*$.*

***Proof.*** As observed above, we need only to prove that every holomorphic vector field which is polynomial (or Laurent polynomial) in $z$ can be approximated, uniformly on compacts, by Lie combinations of complete vector fields. Moreover, since $M$ has the density property, it is easy to verify (say, using power or Laurent series) that the set of vector fields on $M \times \mathbb{C}$ (or $M \times \mathbb{C}^*$) which are tangent to $M$ must also have the density property. Thus it suffices to prove that for each $k \in \mathbb{Z}$ ($k \in \mathbb{Z}_+$ for the $M \times \mathbb{C}$ case) and each $\varphi \in \mathcal{O}(M)$, the vector field $z^k \varphi(x) \frac{\partial}{\partial z}$ can be approximated, uniformly on compacts, by Lie combinations of complete vector fields. In the case of $M \times \mathbb{C}$ and $k = 0$, there is nothing to prove, as the vector field $\varphi(x)\frac{\partial}{\partial z}$ is complete. For the general case, let $X_1, \dots, X_N, \varphi_1, \dots, \varphi_N$ be as in Lemma 3.6. Now

$$z^{k-1}\varphi(x) X_j(x) \left(+0 \cdot \frac{\partial}{\partial z}\right)$$

is approximable, locally uniformly, by Lie combinations of complete vector fields (since $M$ has the density property), and $\varphi_j(x) z \frac{\partial}{\partial z}$ is complete (both on $M \times \mathbb{C}$ and on $M \times \mathbb{C}^*$). Hence the vector field

$$\left[z^{k-1}\varphi(x) X_j(x), \varphi_j(x) z \frac{\partial}{\partial z}\right]$$

($k \geq 1$ in the $M \times \mathbb{C}$ case) is approximable, locally uniformly, by Lie combinations of complete vector fields. But by Lemma 3.6,

$$\sum_{j=1}^{N} \left[z^{k-1}\varphi(x) X_j(x), \varphi_j(x) z \frac{\partial}{\partial z}\right] = z^k \varphi(x) \frac{\partial}{\partial z}$$

modulo vector fields tangent to $M$. Thus the proof is finished. □

**Theorem 3.8.** *If $(M, \omega)$ is a Stein manifold with holomorphic volume element, such that $(M \times \mathbb{C}, \omega \wedge dz)$ has the volume density property, then $M \times \mathbb{C}$ has the density property.*

We shall need the following lemma.

**Lemma 3.9.** *Under the hypotheses of Theorem 3.8, let $f \in \mathcal{O}(M)$ and $k \in \mathbb{Z}_+$. Then there is a holomorphic vector field $Z \in \mathcal{X}_{\mathcal{O}}(M \times \mathbb{C})$ which is approximable by Lie combinations of complete holomorphic vector fields on $M \times \mathbb{C}$, such that*

$$(div\, Z)(x, z) = z^k f(x)\,.$$

***Proof.*** Let $X_1, \ldots, X_N, \varphi_1, \ldots, \varphi_N$ be as in Lemma 3.6, and put

$$Y_j(x, z) := z^k f(x) X_j(x) - \frac{1}{k+1} z^{k+1} \mathrm{div}_\omega \left(f X_j\right)(x) \frac{\partial}{\partial z}\,.$$

Then $\mathrm{div} Y_j = 0$, so by hypothesis $Y_j$ is approximable by Lie combinations of complete (divergence zero) vector fields on $M \times \mathbb{C}$. Since $\varphi_j(x) z \frac{\partial}{\partial z}$ is complete,

$$Z_j(x, z) := \left[Y_j(x, z), \varphi_j(x) z \frac{\partial}{\partial z}\right]$$

is approximable by Lie combinations of complete vector fields on $M \times \mathbb{C}$, and by Lemma 2.2

$$\mathrm{div}\ Z_j(x, z) = Y_j(x, z) \varphi_j(x) = z^k f(x) \left(X_j \varphi_j\right)(x)\,.$$

By Lemma 3.6,

$$Z := \sum_{j=1}^{N} Z_j$$

does the job. □

***Proof of Theorem 3.8.*** Let $V \in \mathcal{X}_{\mathcal{O}}(M \times \mathbb{C})$. By the observation preceding Theorem 3.7, we may assume $V$ is of the form

$$V(x, z) = \sum_{k=0}^{\tilde{N}} \left(z^k V_k(x) + \frac{1}{k+1} z^{k+1} \psi_{k+1}(x) \frac{\partial}{\partial z}\right) + \psi_0(x) \frac{\partial}{\partial z}\,.$$

Since $\psi_0(x) \frac{\partial}{\partial z}$ is complete, we may assume $\psi_0 = 0$. Now

$$\mathrm{div}\ V(x, z) = \sum_{k=0}^{\tilde{N}} \left(z^k \left(\mathrm{div}_\omega V_k(x) + \psi_{k+1}(x)\right)\right)\,.$$

Let $Z_k,\ 0 \le k \le \tilde{N}$ be vector fields on $M \times \mathbb{C}$, chosen so that

$$\mathrm{div}\ Z_k = z^k \left(\mathrm{div}_\omega V_k(x) + \psi_{k+1}(x)\right)\,.$$

Such vector fields, with the additional feature that they are approximable by Lie combinations of complete vector fields on $M \times \mathbb{C}$, are provided by Lemma 3.9. A computation shows that

$$\mathrm{div}\left(V - \sum Z_k\right) = 0\,.$$

It follows from the hypotheses that $V - \sum Z_k$ is approximable by Lie combinations of complete vector fields, and hence so is $V = \sum Z_k + (V - \sum Z_k)$. This completes the proof. □

## 4. Complex Lie groups

A complex Lie group $G$ is a complex manifold which has the structure of a group, such that the mappings $L_g : h \mapsto gh$ and $R_g : h \mapsto hg$ of $G$ to itself are holomorphic for each $g \in G$. For details about complex Lie groups, we refer the reader to [13], which is one of many references on the subject.

Every complex Lie group has trivial canonical bundle, and in fact admits a canonical one parameter family of holomorphic volume elements, constructed as follows: Let $V_1, \dots, V_n$ be a basis of left invariant vector fields on $G$, and let $\alpha_1, \dots, \alpha_n$ be a dual basis of left invariant 1-forms. Then the holomorphic volume element $\omega_G := \alpha_1 \wedge \cdots \wedge \alpha_n$ is a left invariant holomorphic volume element. Every left invariant volume element is a constant multiple of $\omega_G$. We note that the Lie algebra of vector fields $X \in \mathcal{X}_{\mathcal{O}}(G)$ such that $L_X \omega_G = 0$ (i.e., $\mathrm{div}_{\omega_G} X = 0$) is independent of the choice of left invariant $\omega_G$. As we pointed out in Section 2.2, if

$$X = \sum_1^n X_i V_i \in \mathcal{X}_{\mathcal{O}}(G)$$

then

$$\mathrm{div}_{\omega_G} X = \sum_1^n V_i X_i \ .$$

Thus, in particular, left invariant vector fields are of divergence zero.

Let us now turn our attention to Stein Lie groups, i.e., complex Lie groups whose underlying manifold is Stein. It is known [21, 28] that $G$ is Stein if and only if $H^1(G, \mathcal{O}) = 0$.

**Proposition 4.1.** *If $G$ is Stein, then every $X \in \mathcal{X}_{\mathcal{O}}(G)$ is $\mathbb{C}$-complete if and only if it is $\mathbb{R}$-complete.*

***Proof.*** Since $G$ admits one to one immersions of either $\mathbb{C}$ or $\mathbb{C}^*$ tangent to any direction at any point (use the flows of left invariant vector fields), there are no nonconstant negative plurisubharmonic functions on $G$. The result follows from a theorem of Forstnerič [7]. □

Let us give some examples of Stein Lie groups.

1. Every simply connected complex Lie group is Stein [25]. (Recall that there is for every finite dimensional complex Lie algebra $\mathfrak{a}$ a unique (as Lie group, but not as complex manifold) simply connected complex Lie group whose Lie algebra is $\mathfrak{a}$.) Some examples are $SL(n, \mathbb{C})$, $\mathrm{Spin}(n, \mathbb{C})$, and $Sp(n, \mathbb{C})$, among many others.
2. $Gl(n, \mathbb{C})$ $(n \geq 1)$ and $So(n, \mathbb{C})$ $(n \geq 2)$ are Stein. ($So(n, \mathbb{C})$ is the subgroup of $Gl(n, \mathbb{C})$ consisting of matrices $A$ satisfying $A^t = A^{-1}$.) None of these are simply connected (and they are distinct except for $Gl(1, \mathbb{C}) = So(2, \mathbb{C}) = \mathbb{C}^*$). $Gl(n, \mathbb{C})$ is Stein because it is the complement in $\mathbb{C}^{n^2}$ of the closed subvariety $\{\det = 0\}$. $So(n, \mathbb{C})$ is Stein because it is a closed complex submanifold of the Stein manifold $Gl(n, \mathbb{C})$.
3. The only commutative Stein Lie groups are $(\mathbb{C}^*)^k \times \mathbb{C}^l$.

There are, of course, other examples.

### 4.1. Density theorems

We now come to our first results regarding the density property. When we refer to a complex Lie group as having the volume density property, it is with respect to (any) left invariant holomorphic volume element.

**Theorem 4.2.** *Let G be a complex Lie group.*

*(1) $G \times \mathbb{C}$ has the volume density property.*
*(2) If G is Stein and of positive dimension, then $G \times \mathbb{C}$ has the density property.*

**Remarks:**

1. If $G = \mathbb{C}^{n-1}$, $n \geq 2$, then Theorem 4.2 (1) was proved by E. Andersén [3] and Theorem 4.2 (2) by E. Andersén and L. Lempert [4].[7]
2. $\mathbb{C}$ has the volume density property, but not the density property. Indeed, $\mathrm{div}_{dz} X = dX/dz = 0$ implies that $X$ is constant, and all constant vector fields are complete. On the other hand, all complete vector fields on $\mathbb{C}$ are affine linear.

The requirement in Theorem 4.2 (2) that $G$ is Stein cannot be dropped completely. For example, if $\mathbb{T}$ is any compact complex Lie group (these are all diffeomorphic to tori) then $\mathbb{T} \times \mathbb{C}$ cannot have the density property. Indeed we leave it as an exercise to prove that:

**Proposition 4.3.** *Aut $(\mathbb{T} \times \mathbb{C})$ consists of all mappings of the form*

$$(t, z) \mapsto (F_z(t), az + b)\,, \qquad F_z \in Aut\mathbb{T},\ a \in \mathbb{C}^*,\ b \in \mathbb{C}\,.$$

Consequently every Lie combination of complete vector fields on $\mathbb{T} \times \mathbb{C}$ is of the form

$$(t, z) \mapsto (f(z), \alpha z + \beta), \qquad f \in \mathcal{O}(\mathbb{C}),\ \alpha, \beta \in \mathbb{C}\,,$$

and so $\mathbb{T} \times \mathbb{C}$ cannot have the density property.

***Proof of Theorem 4.2.*** We begin by pointing out two things. First, as the reader can verify, every vector field on $G \times \mathbb{C}$ which has zero divergence can be approximated, locally uniformly, by divergence zero vector fields which are polynomial in $z \in \mathbb{C}$. Second, the vector fields $(g, z) \mapsto \varphi(g)\frac{\partial}{\partial z}$ ($\varphi \in \mathcal{O}(G)$) are complete and have divergence zero.

Assuming Theorem 4.2 (1), 4.2 (2) follows immediately from Theorem 3.8. To prove Theorem 4.2 (1), let $V_1, \ldots, V_n$ ($n = \dim_{\mathbb{C}} G$) be a basis of left invariant vector fields on $G$. A computation shows that for any $\varphi \in \mathcal{O}(G)$ there is a function $\psi \in \mathcal{O}(G \times \mathbb{C})$ such that

$$\left[\varphi(g)\frac{\partial}{\partial z}, \frac{1}{k+1} z^{k+1} V_j\right] = z^k \varphi(g) V_j + \psi(g, z)\frac{\partial}{\partial z}$$

is a Lie combination of complete holomorphic vector fields on $G \times \mathbb{C}$. Hence, taking any vector field $P \in \mathcal{X}_{\mathcal{O}}^{\omega}(G \times \mathbb{C})$ ($\omega = \omega_{G \times \mathbb{C}}$) which is polynomial in $z \in \mathbb{C}$,

$$P(g, z) = \sum_{j=1}^{n} \left(\sum_k \varphi_{kj}(g) z^k\right) V_j(g) + \sum_l f_l(g) z^l \frac{\partial}{\partial z}\,,$$

[7] See footnote 1.

there is a Lie combination $X \in \mathcal{X}^{\omega}_{\mathcal{O}}(G \times \mathbb{C})$ of complete holomorphic vector fields such that $P - X = h\frac{\partial}{\partial z}$ for some $h \in \mathcal{O}(G \times \mathbb{C})$. And since $\mathrm{div}(P - X) = 0$, $h$ is independent of $z \in \mathbb{C}$, whence $P - X$ is complete (by our second observation above). Thus $P = X + (P - X)$ is a Lie combination of complete vector fields. By our first observation above, the theorem is proved. □

We cannot, at this time, prove a result like Theorem 4.2 (1) with $G \times \mathbb{C}^*$ in place of $G \times \mathbb{C}$. However, we have the following.

**Theorem 4.4.** *Let $G$ be a complex Lie group.*

(1) *If $G$ has the volume density property, then $G \times \mathbb{C}^*$ has the volume density property.*
(2) *If $G$ is Stein and has the density property, then $G \times \mathbb{C}^*$ has the density property.*

From this we have, by induction, the following special case.

**Corollary 4.5.** *Let $k \in \{2, 3, \dots\}$.*

1. *$(\mathbb{C}^*)^k$ has the volume density property.*
2. *If $(\mathbb{C}^*)^2$ has the density property, then $(\mathbb{C}^*)^k$ has the density property.*

**Remark:** At this point it not known whether $(\mathbb{C}^*)^2$ has the density property, or what is more, if there are any complete holomorphic vector field on $(\mathbb{C}^*)^2$ with $\frac{1}{zw}dz \wedge dw$-divergence not identically zero.

***Proof of Theorem 4.4.*** The second statement is a special case of Theorem 3.8. The proof of the first is virtually the same as that of Theorem 4.2 except that there is one additional detail which must be taken care of first. This detail is precisely the reason that one needs additional hypotheses on $G$.

Suppose $(M, \omega)$ is a complex manifold with holomorphic volume element, and

$$X \in \mathcal{X}^{\omega \wedge \frac{dx}{x}}_{\mathcal{O}} \left(M \times \mathbb{C}^*\right)$$

is of the form

$$X(p, x) = \sum_{k \in \mathbb{Z}} x^k V_k(p) + \left(\sum_{k \in \mathbb{Z}} x^k \varphi_k(p)\right) x \frac{\partial}{\partial x} .$$

If

$$\begin{aligned} \mathrm{div} X &= \mathrm{div}_{\omega}\left(\sum_{k \in \mathbb{Z}} x^k V_k(p)\right) + x\frac{\partial}{\partial x}\left(\sum_{k \in \mathbb{Z}} x^k \varphi_k(p)\right) \\ &= \sum_{k \in \mathbb{Z}} x^k \left(\mathrm{div}_{\omega} V_k(p) + k\varphi_k(p)\right) \\ &= 0 \end{aligned}$$

then

$$\mathrm{div}_{\omega} V_0 = 0 .$$

Returning to our proof, suppose $G$ has the volume density property. We note that, similar to the proof of Theorem 4.2, those divergence zero holomorphic vector fields on $G \times \mathbb{C}^*$ which are

Laurent polynomials in $x \in \mathbb{C}^*$ form a dense subset of the divergence zero holomorphic vector fields on $G \times \mathbb{C}^*$. Let

$$P(g, x) = \sum_{j=1}^{n} \left( \sum_{k} \varphi_{kj}(g) x^k \right) V_j(g) + \sum_{l} \left( x^l f_l(g) \right) x \frac{\partial}{\partial x}$$

be a divergence zero holomorphic vector fields on $G \times \mathbb{C}^*$ which is Laurent polynomial in $x \in \mathbb{C}^*$. By the above computation,

$$\operatorname{div} \left( \sum_{j=1}^{n} \varphi_{0j}(g) V_j(g) \right) = 0 .$$

Thus, since $G$ has the volume density property, we may assume without loss of generality that

$$\sum_{j=1}^{n} \varphi_{0j}(g) V_j(g) = 0 .$$

Now, as $x^k V_j$ and $\frac{1}{k} \varphi_{jk}(g) x \frac{\partial}{\partial x}$ are complete, and

$$\left[ x^k V_j, \frac{1}{k} \varphi_{jk}(g) x \frac{\partial}{\partial x} \right] = x^k \varphi_{jk}(g) V_j + (*) x \frac{\partial}{\partial x}, \quad k \in \mathbb{Z} \backslash \{0\} ,$$

we see that there is a vector field $X$ on $G \times \mathbb{C}^*$ which is a Lie combination of complete divergence zero holomorphic vector fields on $G \times \mathbb{C}^*$, such that $P - X = hx \frac{\partial}{\partial x}$ for some $h \in \mathcal{O}(G \times \mathbb{C}^*)$. Now $0 = \operatorname{div}(P - X) = x \frac{\partial}{\partial x} h$, which implies that $h$ is independent of $x \in \mathbb{C}^*$. It is thus a simple matter to check that $P - X$ is complete. Hence, $P = X + (P - X)$ is a Lie combination of complete holomorphic divergence zero vector fields, as required. □

## 5. Relative geometric structures in $C^n$

### 5.1. Main results

Let us begin with the definitions of the geometric structures in question. Suppose that $j : M \hookrightarrow \mathbb{C}^n$ is a holomorphically embedded Stein manifold. (For us, an embedding is a proper 1-1 immersion.) We define the geometric structures (on $\mathbb{C}^n$) $\mathfrak{g}_T(j)$ and $\mathfrak{g}_0(j)$ as the families of holomorphic vector fields on $\mathbb{C}^n$ which are tangent to $j(M)$, and which vanish on $j(M)$, respectively. In case $M = \mathbb{C}^k$ and $j_0 : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ is the standard embedding $j_0(z_1, \ldots, z_k) = (z_1, \ldots, z_k, 0, \ldots, 0)$, we put $\mathfrak{g}_T^{n,k} := \mathfrak{g}_T(j_0)$, and $\mathfrak{g}_0^{n,k} := \mathfrak{g}_0(j_0)$. It is easy to verify that $\mathfrak{g}_T(j)$ and $\mathfrak{g}_0(j)$ are subalgebras of $\mathcal{X}_\mathcal{O}(\mathbb{C}^n)$, the Lie algebra of holomorphic vector fields on $\mathbb{C}^n$ with the usual Lie bracket $[X, Y] := XY - YX$. Hence these are geometric structures in the sense of Section 1, and as such, one can investigate whether or not they have the density property.

For general Stein manifolds $M$, $\mathfrak{g}_T(j)$ and $\mathfrak{g}_0(j)$ will not have the density property. For instance, in the case of $\mathfrak{g}_T(j)$ one can encounter obstructions which can be seen via the following considerations:

For $X \in \mathfrak{g}_T(j)$ denote by $j^*X$ the restriction of $X$ to $j(M)$. $j^*X$ can be identified in a natural way with an intrinsically defined tangent vector field on $M$, and it is a standard fact that $j^*[X, Y] = [j^*X, j^*Y]$, the bracket on the right being that of $\mathcal{X}_\mathcal{O}(M)$. Now, it is a consequence of Cartan's theorems that every vector field on $M$ is the restriction to $M$ of some vector field in $\mathbb{C}^n$, that is, $j^*(\mathfrak{g}_T(j)) = \mathcal{X}_\mathcal{O}(M)$. With this, and with the obvious but important fact that every complete vector field on $\mathbb{C}^n$ gives rise, via the Lie algebra epimorphism $j^*$, to a complete vector field on

$M$, we see that the density property for $\mathcal{X}_{\mathcal{O}}(M)$ is a necessary condition for $\mathfrak{g}_T(j)$ to have the density property. Thus if $AutM$ is finite dimensional (for example if $M = \mathbb{C}$ or $\mathbb{C}^*$ or if $M$ is Kobayashi hyperbolic) then $\mathfrak{g}_T(j)$ will fail to have the density property (see Proposition 3.2).

**Remark:** These arguments do not hold for $\mathfrak{g}_0(j)$, and in this regard, one is led to ask the following very basic question: Is there an embedding $j : \Delta \hookrightarrow \mathbb{C}^n$ of the disc in $\mathbb{C}^n$ (for any $n \geq 2$) such that $\mathfrak{g}_0(j)$ has the density property? More basically, is there such an embedding with the property that there is a non-zero complete holomorphic vector field on $\mathbb{C}^n$ which vanishes precisely on $j(\Delta)$? This is of course a very special case of the more general question: What are the zero sets of complete holomorphic vector fields?

More remarkable, however, is the fact that even if $\mathcal{X}_{\mathcal{O}}(M)$ has the density property, the particular embedding in question may still force that $\mathfrak{g}_T(j)$ (and $\mathfrak{g}_0(j)$) do not have the density property. This reveals the "relative nature" of the geometric structures $\mathfrak{g}_T(j)$ and $\mathfrak{g}_0(j)$, i.e., their dependence on the embedding $j$ as well as on $M$. The examples displaying this phenomenon will be given shortly.

From here on we restrict ourselves to the case $M = \mathbb{C}^k$. The following theorem is the main result of this section.

**Theorem 5.1.** *Let $n$ and $k$ be integers with $1 \leq k < n$.*

*(1)* $\mathfrak{g}_0^{n,k}$ *has the density property.*

*(2) If $k \geq 2$,* $\mathfrak{g}_T^{n,k}$ *has the density property.*

**Remark:** As pointed out above, $Aut\mathbb{C}$ is finite dimensional, and hence $\mathfrak{g}_T^{n,1}$ cannot have the density property.

Before proving Theorem 5.1, we turn our attention to the study of its dependence on the fact that the embedding $j_0$ defines $\mathfrak{g}_T^{n,k}$ and $\mathfrak{g}_0^{n,k}$, as opposed to some other embedding. The first observation is that for any $\Phi \in Aut\mathbb{C}^n$, $\mathfrak{g}_T(\Phi \circ j)$ (resp. $\mathfrak{g}_0(\Phi \circ j)$) has the density property if and only if $\mathfrak{g}_T(j)$ (resp. $\mathfrak{g}_0(j)$) does. Thus, Theorem 5.1 holds if $j_0$ is replaced by any straightenable[8] embedding $j : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$. Abyankhar and Moh [2] showed that every polynomial embedding $\mathbb{C} \hookrightarrow \mathbb{C}^2$ is straightenable by (polynomial) automorphisms. The same result for polynomial embeddings $\mathbb{C} \hookrightarrow \mathbb{C}^n$ holds true for $n \geq 3$, and is somewhat more elementary [19, 20].[9] However, there are embeddings of $\mathbb{C}^k$ in $\mathbb{C}^n$ which cannot be straightened out by automorphisms, for $1 \leq k \leq n-1$ [27, 11, 5, 8]. In fact [5, 8], there exist embeddings $j' : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ with the property that every immersion $f : \mathbb{C}^{n-k} \to \mathbb{C}^n$ has image which intersects $j'(\mathbb{C}^k)$ infinitely often. As a corollary of the latter results, one has the following.

**Proposition 5.2.** *There do not exist $n-k$ complete independent (as vector fields, but not necessarily pointwise) holomorphic vector fields in $\mathbb{C}^n$ which are in involution, and which are tangent to $j'(\mathbb{C}^k)$.*

***Proof.*** If such vector fields exist, we obtain, via the holomorphic Frobenius theorem, an immersion $\mathbb{C}^{n-k} \to \mathbb{C}^n$ which is in the complement of $j'(\mathbb{C}^{n-k})$, contradicting [5, 8]. □

[8] An embedding $j : \mathbb{C}^k \hookrightarrow \mathbb{C}^n$ is *straightenable* (also called *tame*) if there exists $\Phi \in Aut\mathbb{C}^n$ such that $\Phi \circ j(z) = (z, 0)$ for all $z \in \mathbb{C}^k$.

[9] For the case $n = 3$ it is not known that the straightening automorphism can be taken polynomial.

Consequently one immediately obtains the following.

**Corollary 5.3.** *There are embeddings $j : \mathbb{C}^{n-1} \hookrightarrow \mathbb{C}^n$ $(n \geq 2)$ such that $\mathfrak{g}_T(j)$ and $\mathfrak{g}_0(j)$ do not have the density property.*

## 5.2. Auxiliary algebras and the proof of Theorem 5.1

The proof of Theorem 5.1 is somewhat technical, and requires several cases. The general scheme, however, is not markedly different from that in [4], where it is proved that $\mathcal{X}_{\mathcal{O}}(\mathbb{C}^n)$ has the density property. There are two major differences. The first is that the introduction of the Lie algebra structure of $\mathcal{X}_{\mathcal{O}}(\mathbb{C}^n)$ into the game simplifies the technicalities. The second is that technicalities require the introduction of several auxiliary Lie algebras. In [4] the auxiliary Lie algebra used is that of divergence zero vector fields, which is in itself quite interesting (see [3]). In our case, different auxiliary algebras are needed for different proofs, and not all (but most) are defined as kernels of divergence-type operators. And though the algebras we define may be of intrinsic interest, we do not focus our attention on them.

***Definition.*** A Lie subalgebra $\mathfrak{a} \subseteq \mathcal{X}_{\mathcal{O}}(\mathbb{C}^n)$ is called **densely polynomial** if $\{X \in \mathfrak{a} : X \text{ is polynomial}\}$ is dense in $\mathfrak{a}$, in the topology of uniform convergence on compacts in $\mathbb{C}^n$.

All the Lie algebras which come up in this section are densely polynomial. This fact is based on the observations that all of our linear differential operators map homogeneous polynomial maps of degree $j$ to homogeneous polynomial functions of degree $j-1$, and that holomorphic maps of the form $\mathbb{C}^n \ni z \mapsto z_j X(z) \in \mathbb{C}^n$ can be approximated uniformly on compact subsets by polynomials of the form $\mathbb{C}^n \ni z \mapsto z_j P(z) \in \mathbb{C}^n$. We shall neglect to mention this again, leaving it to the reader to verify the statement $\mathfrak{a}$ *is densely polynomial* whenever it arises.

We now list a collection of complete vector fields on $\mathbb{C}^n$ which will be used to prove that the various Lie algebras in question have the density property. These are all shear fields, with obvious conditions added so as to force these fields to lie in the particular algebra in question. The latter conditions will be clear when the particular algebras are defined. For $z \in \mathbb{C}^n$ we write $z = (z', z'')$ with $z' \in \mathbb{C}^k$ and $z'' \in \mathbb{C}^{n-k}$. Also $f, g \in \mathcal{O}(\mathbb{C}^n)$. Here is our list:

$$\begin{array}{llll}
(S_{1,f,j}) & z \mapsto f(z)\frac{\partial}{\partial z_j} & \frac{\partial f}{\partial z_j} \equiv 0 & 1 \leq j \leq k \\
(G_{1,f,j}) & z \mapsto z_j f(z)\frac{\partial}{\partial z_j} & \frac{\partial f}{\partial z_j} \equiv 0 & 1 \leq j \leq k \\
(S_{2,g,j}) & z \mapsto g(z)\frac{\partial}{\partial z_j} & \frac{\partial g}{\partial z_j} \equiv 0,\ g(z',0) \equiv 0 & k+1 \leq j \leq n \\
(G_{2,g,j}) & z \mapsto z_j g(z)\frac{\partial}{\partial z_j} & \frac{\partial g}{\partial z_j} \equiv 0 & k+1 \leq j \leq n
\end{array}$$

We note that when $k < n-1$, these shears all lie in $\mathfrak{g}_T^{n,k}$, that $(S_{2,g,j})$ and $(G_{2,f,j})$ lie in $\mathfrak{g}_0^{n,k}$, and that if $f(z',0) \equiv 0$, $(S_{1,f,j})$ and $(G_{1,f,j})$ also lie in $\mathfrak{g}_0^{n,k}$. When $n = k+1$, $(S_{2,g,j}) \equiv 0$.

***Definition.*** A vector field in $\mathbb{C}^n$ is called **n-basic** (resp. **1-basic**) if it is of the form

$$z^\alpha \frac{\partial}{\partial z_j} + f\frac{\partial}{\partial z_n}\ (1 \leq j \leq n-1) \quad \left(\text{resp. } f\frac{\partial}{\partial z_1} + z^\alpha \frac{\partial}{\partial z_j}\ (2 \leq j \leq n)\right)$$

for some $f \in \mathcal{O}(\mathbb{C}^n)$. That is to say,

(i) All but one of its last (resp. first) n-1 components are zero, and
(ii) this non-zero component is a monomial.

***Proof of Theorem 5.1 (1).*** Let $1 \le k \le n-1$. Define $\delta = \delta_{n,k} : \mathfrak{g}_0^{n,k} \to \mathcal{O}(\mathbb{C}^n)$ by

$$\delta_{n,k}\left(\sum_{j=1}^{n} X_j \frac{\partial}{\partial z_j}\right) := \begin{cases} \frac{\partial X_1}{\partial z_1} + \ldots + \frac{\partial X_n}{\partial z_n}, & \text{k < n-1} \\ \frac{\partial X_1}{\partial z_1} + \ldots + \frac{\partial X_n}{\partial z_n} - \frac{X_n}{z_n}, & k = n-1 \end{cases}$$

and let $\mathfrak{h}^{n,k} := \ker \delta_{n,k}$. We note that $\delta_{n,k}$ is well defined. Indeed, the only possible problem is when $k = n-1$, and in this case

$$X \in \mathfrak{g}_0^{n,k} \Rightarrow X_n(z', 0) \equiv 0 \iff z_n \text{ divides } X_n .$$

It is a straightforward matter[10] to verify that

$$\delta([X, Y]) = X(\delta Y) - Y(\delta X)$$

from which it follows that $\mathfrak{h}^{n,k}$ is a Lie subalgebra of $\mathfrak{g}_0^{n,k}$, i.e., it is a geometric structure. □

**Lemma 5.4.** *For all $j$ such that $1 \le j \le n-1$ and all $\alpha \in (\mathbb{Z}_+)^n$ such that $\alpha_{k+1} + \ldots + \alpha_n > 0$, there is an $n$-basic vector field $z^\alpha \frac{\partial}{\partial z_j} + (*)\frac{\partial}{\partial z_n}$ which is a Lie combination of complete vector fields in $\mathfrak{h}^{n,k}$.*

***Proof.***

*Case 1:* $1 \le k < n-1$ (Note: $n > 2$)

(a) Suppose $k < j \le n-1$. If $\alpha_j = 0$, $Z^\alpha \frac{\partial}{\partial z_j}$ is complete. If $\alpha_j 0$, then

$$X_1(z) := z_j^{\alpha_j} \frac{\partial}{\partial z_n}, \quad X_2(z) := \frac{1}{(\alpha_n + 1)} \frac{z^\alpha z_n}{z_j^{\alpha_j}} \frac{\partial}{\partial z_j}$$

are complete vector fields in $\mathfrak{h}^{n,k}$, and

$$[X_1, X_2](z) = z^\alpha \frac{\partial}{\partial z_j} + (*)\frac{\partial}{\partial z_n} .$$

(b) Suppose $1 \le j \le k$. Then one of $\alpha_{k+1}, \ldots, \alpha_n$ must be positive. Say that $l \in \{k+1, \ldots, n\}$ is the smallest integer such that $\alpha_l 0$.

(b') If $l < n$, take

$$X_1(z) := z_j^{\alpha_j} z_l^{\alpha_l} \frac{\partial}{\partial z_n}, \quad X_2(z) := \frac{1}{(\alpha_n + 1)} \frac{z^\alpha z_n}{z_j^{\alpha_j} z_l^{\alpha_l}} \frac{\partial}{\partial z_j}$$

to get

$$[X_1, X_2](z) = z^\alpha \frac{\partial}{\partial z_j} + (*)\frac{\partial}{\partial z_n} .$$

(b'') If $l = n$, take

$$X_1(z) := z_j^{\alpha_j} z_n^{\alpha_n} \frac{\partial}{\partial z_{n-1}}, \quad X_2(z) := \frac{1}{(\alpha_{n-1} + 1)} \frac{z^\alpha z_{n-1}}{z_j^{\alpha_j} z_n^{\alpha_n}} \frac{\partial}{\partial z_j} .$$

[10] This can be done directly, but is most easily seen by noticing that the divergence with respect to the left invariant volume form associated to the complex Lie groups $\mathbb{C}^n$ (when $k < n-1$) and $\mathbb{C}^{n-1} \times \mathbb{C}^*$ (when $k = n-1$) is precisely $\delta$, and then applying Lemma 2.2.

Then $X_1, X_2 \in \mathfrak{h}^{n,k}$ and

$$[X_1, X_2](z) = z^\alpha \frac{\partial}{\partial z_j} - \left(\frac{\alpha_j}{\alpha_{n-1}+1}\right) \frac{z_{n-1} z^\alpha}{z_j} \frac{\partial}{\partial z_{n-1}},$$

and by (a) there are complete vector fields $Y_1, Y_2 \in \mathfrak{h}^{n,k}$ such that

$$[Y_1, Y_2](z) = \left(\frac{\alpha_j}{\alpha_{n-1}+1}\right) \frac{z_{n-1} z^\alpha}{z_j} \frac{\partial}{\partial z_{n-1}} + (*)\frac{\partial}{\partial z_n}.$$

Thus

$$[X_1, X_2](z) + [Y_1, Y_2](z) = z^\alpha \frac{\partial}{\partial z_j} + (*)\frac{\partial}{\partial z_n},$$

and Case 1 is proved.

*Case 2:* $k = n-1$. In this case $\alpha_n > 0$. We take

$$X_1(z) := z_n z_j^{\alpha_j} \frac{\partial}{\partial z_n}, \quad X_2(z) := \frac{1}{\alpha_n} \frac{z^\alpha}{z_j^{\alpha_j}} \frac{\partial}{\partial z_j}.$$

Then

$$[X_1, X_2](z) = z^\alpha \frac{\partial}{\partial z_j} + (*)\frac{\partial}{\partial z_n}.$$

This proves Case 2, and hence Lemma 5.4. □

**Corollary 5.5.** *$\mathfrak{h}^{n,k}$ has the density property for $1 \le k \le n-1$.*

***Proof.*** In view of Lemma 5.4, for each $\alpha \in (\mathbb{Z}_+)^n$ such that $\alpha_{k+1} + \ldots + \alpha_n > 0$ and each $j \in \{1, 2, \ldots, n-1\}$ there is an n-basic vector field $z^\alpha \frac{\partial}{\partial z_j} + (*)\frac{\partial}{\partial z_n}$ which is a Lie combination of complete vector fields in $\mathfrak{h}^{n,k}$. It then follows that, given a polynomial vector field $X \in \mathfrak{h}^{n,k}$, there is a Lie combination $Y \in \mathfrak{h}^{n,k}$ of complete vector fields such that $X - Y = (*)\frac{\partial}{\partial z_n}$. Now, $\delta(X - Y) = 0$, and it is a simple matter to see that any n-basic vector field in $\ker \delta$ is complete. It follows that $X = Y + (X - Y)$ is a Lie combination of complete vector fields, and since $\mathfrak{h}^{n,k}$ is densely polynomial, the proof is complete. □

**Lemma 5.6.** *If $X \in \mathfrak{g}_0^{n,n-1}$, then $\delta X|_{\mathbb{C}^{n-1}\times\{0\}} = 0$.*

***Proof.*** Let

$$X = X_1 \frac{\partial}{\partial z_1} + \ldots + X_{n-1} \frac{\partial}{\partial z_{n-1}} + z_n \xi \frac{\partial}{\partial z_n} \in \mathfrak{g}_0^{n,n-1}.$$

Then $X_j|_{\mathbb{C}^{n-1}\times\{0\}} = 0$ for $1 \le j \le n-1$, and since $\delta X = \frac{\partial X_1}{\partial z_1} + \ldots + \frac{\partial X_{n-1}}{\partial z_{n-1}} + z_n \frac{\partial \xi}{\partial z_n}$, the lemma is proved. □

**Lemma 5.7.** *Given $\alpha \in (\mathbb{Z}_+)^n$ ($\alpha_n > 0$ if $k = n-1$; see Lemma 5.6), there is a Lie combination $X$ of complete polynomial vector fields in $\mathfrak{g}_0^{n,k}$ with $\delta X(z) = z^\alpha$.*

***Proof.***

*Case 1:* $k < n-1$ ($\Rightarrow n > 2$).

If $\alpha_l = 0$ for some $l$ with $k+1 \le l \le n$, take $X(z) = z^\alpha z_l \frac{\partial}{\partial z_l}$. Then $(\delta X)(z) = z^\alpha$. Otherwise $\alpha_l > 0$ for all $l \in \{k+1, \dots, n\}$. In that case, take

$$X_1(z) = z_1^{\alpha_1} \dots z_{n-1}^{\alpha_{n-1}} \frac{\partial}{\partial z_n}, \quad X_2(z) = \frac{1}{(\alpha_n + 1)} z_n^{\alpha_n+1} z_{n-1} \frac{\partial}{\partial z_{n-1}} .$$

(Note that these are both in $\mathfrak{g}_0^{n,k}$ for $k < n-1$.) Then

$$\delta X_1 = 0, \quad \delta X_2(z) = \frac{z_n^{\alpha_n+1}}{(\alpha_n + 1)} ,$$

and so

$$\delta\left([X_1, X_2]\right)(z) = (X_1 \delta X_2)(z) = z^\alpha .$$

Hence $X = [X_1, X_2]$ does the job.

*Case 2:* $k = n-1$.

In this case simply take

$$X_1(z) = z_1^{\alpha_1} z_n \frac{\partial}{\partial z_n}, \quad X_2(z) = \frac{1}{\alpha_n} \frac{z_1 z^\alpha}{z_1^{\alpha_1}} \frac{\partial}{\partial z_1} .$$

Then

$$\delta X_1 = 0, \quad \delta X_2(z) = \frac{z^\alpha}{\alpha_n z_1^{\alpha_1}} ,$$

and

$$\delta\left([X_1, X_2]\right)(z) = (X_1 \delta X_2)(z) = z^\alpha .$$

Again, $X = [X_1, X_2]$ does the job, and Lemma 5.7 is proved. □

***Conclusion of the Proof of Theorem 5.1 (1).*** By Lemmas 5.6 and 5.7, given a polynomial vector field $X \in \mathfrak{g}_0^{n,k}$, there is a polynomial Lie combination $Y$ of complete vector fields in $\mathfrak{g}_0^{n,k}$ such that $\delta X = \delta Y$. Hence $X - Y \in \mathfrak{h}^{n,k}$, and since $X - Y$ is polynomial, by Lemma 5.5 $X = Y + (X - Y)$ is a Lie combination of complete vector fields in $\mathfrak{g}_0^{n,k}$. Since $\mathfrak{g}_0^{n,k}$ is densely polynomial, Theorem 5.1 (1) is proved. □

The tangential phenomena seem to present more technicalities than the vanishing phenomena which we have just dealt with.

***Proof of Theorem 5.1 (2).*** We begin with the definition of an auxiliary Lie algebra. A straightforward calculation shows the following. □

**Lemma 5.8.** *The set of vector fields*

$$\mathfrak{n}^{n,k} := \left\{ X \in \mathfrak{g}_T^{n,k} : z_j \text{ divides } X_j,\ k+1 \le j \le n \right\}$$

*is a Lie subalgebra of* $\mathfrak{g}_T^{n,k}$.

**Remark:** $\mathfrak{n}^{n,n-1} = \mathfrak{g}_T^{n,n-1}$, but for $1 \le k < n-1$, $\mathfrak{n}^{n,k} \subsetneq \mathfrak{g}_T^{n,k}$.

We define $\delta : \mathfrak{n}^{n,k} \to \mathcal{O}(\mathbb{C}^n)$ by

$$\delta\left(X_1\frac{\partial}{\partial z_1}+\ldots+X_n\frac{\partial}{\partial z_n}\right) := \frac{\partial X_1}{\partial z_1}+\ldots+\frac{\partial X_n}{\partial z_n}-\frac{X_{k+1}}{z_{k+1}}-\ldots-\frac{X_n}{z_n}$$

and put

$$\mathfrak{l}_{n,k} := \ker\left(\delta : \mathfrak{n}^{n,k} \to \mathcal{O}\left(\mathbb{C}^n\right)\right) .$$

As before, one verifies that $\delta([X, Y]) = X\delta Y - Y\delta X$, whence $\mathfrak{l}_{n,k}$ is a Lie algebra.

**Lemma 5.9.** *For $2 \le k \le n-1$, $\mathfrak{l}_{n,k}$ has the density property.*

***Proof.*** For the same reasons as in the proof of Lemma 5.5, it suffices to prove that for any $j \in \{2, \ldots, n\}$, $\alpha \in (\mathbb{Z}_+)^n$ ($\alpha_j > 0$ if $j > k$), there is a 1-basic vector field $(*)\frac{\partial}{\partial z_1} + z^\alpha \frac{\partial}{\partial z_j}$ which is a Lie combination of complete vector fields in $\mathfrak{l}_{n,k}$.

If $j > k$ then $\alpha_j > 0$, and we take

$$X_1(z) = z_j^{\alpha_j - 1}\frac{\partial}{\partial z_1}, \quad X_2(z) = \frac{1}{\alpha_1+1}\frac{z_1 z_j z^\alpha}{z_j^{\alpha_j}}\frac{\partial}{\partial z_j} .$$

Then

$$[X_1, X_2](z) = (*)\frac{\partial}{\partial z_1} + z^\alpha\frac{\partial}{\partial z_j} .$$

Otherwise $j \le k$ (here we use $k \ge 2$), and we simply take

$$X_1(z) = z_j^{\alpha_j}\frac{\partial}{\partial z_1}, \quad X_2(z) = \frac{1}{\alpha_1+1}\frac{z_1 z^\alpha}{z_j^{\alpha_j}}\frac{\partial}{\partial z_j}$$

and again get

$$[X_1, X_2](z) = (*)\frac{\partial}{\partial z_1} + z^\alpha\frac{\partial}{\partial z_j} .$$

This proves Lemma 5.9. □

**Lemma 5.10.** *$\mathfrak{n}^{n,k}$ has the density property for $2 \le k \le n-1$.*

***Proof.*** It suffices, by Lemma 5.9 and the linearity of $\delta$, to show that given any monomial $z^\alpha$, there is a polynomial vector field $X$ which is a Lie combination of complete vector fields in $\mathfrak{n}^{n,k}$ such that $(\delta X)(z) = z^\alpha$. For this take

$$X_1(z) = \frac{z^\alpha z_n}{z_n^{\alpha_n}}\frac{\partial}{\partial z_n} \quad X_2(z) = \frac{1}{\alpha_n} z_1 z_n^{\alpha_n}\frac{\partial}{\partial z_1} .$$

Then

$$\delta\,[X_1, X_2](z) = X_1\delta X_2 - X_2\delta X_1 = X_1\delta X_2 = z^\alpha ,$$

and the proof is finished. □

We note that Theorem 5.1 (2) is now proved if $k = n-1$.

**Lemma 5.11.** *Let $2 \le k \le n-2$. Given any polynomial vector field $X \in \mathfrak{g}_T^{n,k}$, there is a polynomial vector field $X'$ which is a Lie combination of complete vector fields in $\mathfrak{g}_T^{n,k}$, such that $X - X' \in \mathfrak{n}^{n,k}$.*

***Proof.*** Let $j \in \{k+1,\dots,n\}$ and $l \in \{k+1,\dots,n\}\backslash\{j\}$. (This is possible because $|\{k+1,\dots,n\}| \geq 2$.) Set

$$X_1(z) := z_j^{\alpha_j}\frac{\partial}{\partial z_1} \quad X_2(z) := \frac{1}{\alpha_1+1}\frac{z_l z^{\alpha} z_1}{z_j^{\alpha_j}}\frac{\partial}{\partial z_j}.$$

Then

$$[X_1, X_2](z) = (*)\frac{\partial}{\partial z_1} + z_l z^{\alpha}\frac{\partial}{\partial z_j},$$

and Lemma 5.11 follows as is by now usual. □

***Conclusion of the Proof of Theorem 5.1 (2).*** Let $X \in \mathfrak{g}_T^{n,k}$ be a polynomial holomorphic vector field. By Lemma 5.11 there is a polynomial vector field $X' \in \mathfrak{g}_T^{n,k}$ such that $X - X' \in \mathfrak{n}^{n,k}$. Since $\mathfrak{n}^{n,k}$ has the density property (Lemma 5.10), and since $\mathfrak{g}_T^{n,k}$ is densely polynomial, Theorem 5.1 (2) follows. □

### 5.3. More than one submanifold: a prelude

In many constructions, one might need to prove results about vector fields in $\mathbb{C}^n$ which vanish in more than one submanifold. We shall restrict ourselves to affine complex hyperplanes (i.e., zero sets of holomorphic polynomials of degree 1).

***Example 5.12.*** Two parallel hyperplanes.

If we fix two parallel hyperplanes in $\mathbb{C}^n$, the set of vector fields on $\mathbb{C}^n$ which vanish on these hyperplanes will not have the density property. We prove this now.

**Proposition 5.13.** *Aut* $(\mathbb{C}^n \times \mathbb{C}\backslash\{0,1\})$ *consists of maps of the form*

$$\left(\mathbb{C}^n \times \mathbb{C}\backslash\{0,1\}\right) \ni (z,w) \mapsto (F_w(z), \gamma(w)),$$

*where* $\gamma \in Aut\,(\mathbb{C}\backslash\{0,1\})$.

**Remark:** Actually, $(z,w) \mapsto F_w(z)$ is holomorphic, and for each $w \in \mathbb{C}\backslash\{0,1\}$ $F_w \in Aut\,\mathbb{C}^n$, but we won't need these facts.

***Proof.*** Let $\Phi \in Aut\ \mathbb{C}^n$, and write $\Phi = (f,g)$. Fixing $w \in \mathbb{C}\backslash\{0,1\}$, we get a map $\mathbb{C}^n \ni z \mapsto g(z,w) \in \mathbb{C}\backslash\{0,1\}$, which must be constant. Write $\gamma(w) := g(z,w)$. Repeating the same argument for $\Phi^{-1}$ we get $\Phi^{-1}(z,w) = (\tilde{f}(z,w), \tilde{\gamma}(w))$. Computing $\Phi \circ \Phi^{-1}$ and $\Phi^{-1} \circ \Phi$, both of which are the identity, we get the result. □

Now, *Aut* $(\mathbb{C}\backslash\{0,1\})$ consists of all Möbius transformations which permute $\{0,1,\infty\} \subseteq S^2$, and since the values of a Möbius transformation on $\{0,1,\infty\}$ uniquely determine this transformation, *Aut* $(\mathbb{C}\backslash\{0,1\})$ is a finite group. It follows from this and Proposition 5.13 that every complete vector field in $\mathbb{C}^n$ which vanishes on the two hyperplanes $\{z_n = 0\} \cup \{z_n = 1\}$ must have its $z_n$ component identically zero. Since this is clearly true of Lie brackets of complete vector fields as well, and since there are plenty of vector fields whose $z_n$ component is not identically zero, it follows that the Lie algebra of vector fields on $\mathbb{C}^n$ which vanish on $\{z_n = 0\} \cup \{z_n = 1\}$ does not have the density property.

***Example 5.14.*** $k$ hyperplanes through zero.

If we have $k \geq n+1$ hyperplanes through the origin in $\mathbb{C}^n$, then as soon as $k = k(n)$ is large enough (for example, if $n = 2$, $k = 3$ will do) we obtain "little Picard theorem"-type obstructions to the density property. Denote the union of these hyperplanes by $S$. We quote a theorem of Fujimoto and Green from [24] to the effect that any mapping $f : \mathbb{C} \to \mathbb{CP}^n$ which misses $n + p$ hyperplanes has image which is contained in a projective linear subspace of dimension $\leq [\frac{n}{p}]$. In particular, it must be the case that every complete holomorphic vector field on $\mathbb{C}^n$ which vanishes on $S$ must be tangent to every hyperplane through 0 in $\mathbb{C}^n$. Since being tangent to a hyperplane through zero is a "Lie-closed" condition, it follows that any Lie combination of complete holomorphic vector fields on $\mathbb{C}^n$ which vanishes on $S$ must be tangent to every hyperplane through 0 in $\mathbb{C}^n$. However, there are many vector fields on $\mathbb{C}^n$ which vanish on $S$ and which are transverse to many hyperplanes through the origin. Thus the theorem of Fujimoto and Green gives obstructions to the density property.

On the other hand, if $1 \leq k \leq n$, there are some positive results. Since the methods used to prove these results are analogous to those used in the proof of Theorem 5.1, we will content ourselves with stating results, and omit all proofs. Replacing $k$ by $n - k$, we denote by $\mathfrak{a}^{n,k}$ the Lie algebra of vector fields which vanish on $\{z_{k+1} = 0\} \cup \ldots \cup \{z_n = 0\}$. We have.

**Proposition 5.15.** *With the above notation,*

1. *If* $1 \leq k \leq n-2$, $\mathfrak{a}^{n,k}$ *has the density property.*
2. *The Lie algebra of vector fields in* $\mathfrak{a}^{n,n}$ *which satisfy*

$$\sum_{j=1}^{n} \left( \frac{\partial X_j}{\partial z_j} - \frac{X_j}{z_j} \right) = 0$$

*has the density property.*

In connection with the remark following Corollary 4.5, it is not known whether $\mathfrak{a}^{n,n}$ has the density property.

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