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Quadratic Presentations and Nilpotent Kähler Groups

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1. Introduction

It has been known for at least thirty years that certain nilpotent groups cannot be Kähler groups, i.e., fundamental groups of compact Kähler manifolds. The best known examples are lattices in the three-dimensional real or complex Heisenberg groups. It is also known that lattices in certain other standard nilpotent Lie groups, e.g., the full group of upper triangular matrices and the free k-step nilpotent Lie groups, $k > 1$, are not Kähler. The Heisenberg case was known to J-P. Serre in the early 1960's, and unified proofs of the above statements follow readily from Sullivan's theory of minimal models [6], [15], [19], Chen's theory of iterated integrals [4], [10], or more recent developments such as [9].

We recall that in [6] compact Kähler manifolds are shown to be *formal* in the sense of rational homotopy theory. This implies formality of the one-minimal model of the manifold, meaning that this object can be constructed formally from H^1 and the kernel of the cup-product map. Equivalently, the Lie algebra of the Malcev completion of the fundamental group is *quadratically presented*, i.e., is the quotient of the free Lie algebra on its abelianization by an ideal generated in degree two. It is not hard to see that the Lie algebras of the groups mentioned above do not admit quadratic presentations.

It seems to be less well known that many nilpotent Lie algebras have quadratic presentations, for instance the real or complex Heisenberg Lie algebras of dimension at least five. Thus the methods of rational homotopy theory do not exclude lattices in the corresponding Lie groups from being Kähler, contrary to an apparently commonly held impression that the only nilpotent Kähler groups are almost abelian. An exception is [12], where the problem of lattices in higher-dimensional Heisenberg groups being Kähler is explicitly posed.

This matter has finally been settled by the remarkably simple examples of nilpotent Kähler groups found by Campana [2]. Remarkable also is the fact that such examples were already in the literature [18], presumably with the connection with nilpotence passing unnoticed. Campana constructs Kähler groups that are lattices in all real Heisenberg groups of dimension at least nine. In our unpublished notes [3] we had proved that no lattice in the five- or seven-dimensional Heisenberg

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group is Kähler, so this construction is sharp for the Heisenberg groups. It is also sharp with respect to the rank of the abelianization. Namely, [3] gives (and we prove this below) that the abelianization of a nilpotent, not almost abelian Kähler group must have rank at least eight. The fact that abelianization of rank four is impossible appears in [2] by a different argument, but the mention there of the impossibility of rank six is based on [3].

The general question raised by these results is: which finitely generated nilpotent groups with quadratically presented Malcev Lie algebras are Kähler groups? A less delicate question would be: for which quadratically presented nilpotent real Lie algebras (with rational structure constants) is there a lattice in the corresponding simply connected nilpotent Lie group that is a Kähler group? For the Heisenberg Lie algebras of dimension at least five, this question is answered above.

This paper addresses two related questions. First, how wide is the class of quadratically presented Lie algebras? We give an infinite family of examples of quadratically presented three-step nilpotent Lie algebras, thus indicating that this class is quite wide. We also classify quadratically presented complex nilpotent Lie algebras with abelianization of dimension at most five. It turns out that there are only finitely many isomorphism classes, and they are at most three-step nilpotent. As part of work now in progress, S. Chen has produced examples of quadratically presented nilpotent Lie algebras of arbitrarily high class of nilpotency, beginning with a four-step nilpotent one with six-dimensional abelianization. Thus, this class of algebras is even wider than indicated by the results of this paper.

The second question we address is how restricted within the class of quadratically presented Lie algebras is the subclass of Malcev Lie algebras of nilpotent Kähler groups. Using Kähler geometry we deduce a lower bound on the rank of certain elements in the kernel of the cup-product map. This implies the lower bound on the rank of abelianization mentioned above. We also show that this excludes certain infinite families of quadratically presented Lie algebras containing algebras with abelianization of arbitrarily large dimension from being Malcev algebras of Kähler groups. This appears to be the only restriction beyond quadratic presentation that is known as of this writing.

These results raise two specific questions (also raised in [2]). First, are there nilpotent Kähler groups of class larger than two? Note that there are no known obstructions to many of our three-step nilpotent examples being Kähler. Second, can the kernel of the cup-product map of a Kähler manifold with nilpotent fundamental group have elements of type $(2, 0)$? If the answer to this question were negative, then our main theorem would imply that there are no elements of rank less than eight in the kernel of the cup-product map of a nilpotent Kähler group.

In view of the preceding discussion it appears that the difficult problem of classifying Kähler groups is wide open even in the restricted context of nilpotent groups. It is interesting to note that the situation for polycyclic groups is better, in the sense that it has been reduced to nilpotent groups. Namely, Arapura and Nori prove in [1] that a polycyclic Kähler group must be almost nilpotent.

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2. Differential algebras and Lie algebras

As noted above, a compact Kähler manifold is formal, as is the 1-minimal model of a formal space. In general, the 1-minimal model of a space X is a free, connected, minimal differential algebra $\mathcal{M}^{(1)}$, generated in degree one, which maps to the de Rham algebra of X, induces an isomorphism on H^1, and induces an injection on H^2. The 1-minimal model $\mathcal{M}^{(1)}$ of X is said to be formal if it is the same as the 1-minimal model of the differential algebra $H^*(X)$ with zero differential. Equivalent to this is the existence of a map of differential algebras $\mathcal{M}^{(1)} \longrightarrow H^*(X)$ inducing an isomorphism on H^1 and an injection on H^2. In terms of $\mathcal{M}^{(1)}$ alone, this means that there is a map of differential algebras $\mathcal{M}^{(1)} \longrightarrow H^*(\mathcal{M}^{(1)})$ inducing an isomorphism on H^1 and an injection on H^2.

We also recall from [6, p. 87] that the 1-minimal model is constructed inductively as an increasing union

$$\mathcal{M}_1^{(1)} \subset \mathcal{M}_2^{(1)} \subset \cdots \subset \mathcal{M}_i^{(1)} \subset \cdots \subset \mathcal{M}^{(1)},$$

where each $\mathcal{M}_{i+1}^{(1)}$ is a Hirsch extension of $\mathcal{M}_i^{(1)}$ defined precisely so as to kill the kernel of $H^2(\mathcal{M}_i^{(1)}) \longrightarrow H^2(X)$. More explicitly, if $V = H^1(X)$, then $\mathcal{M}_1^{(1)} = \Lambda V$, and $\mathcal{M}_2^{(1)} = \Lambda(V \oplus C_0)$, where C_0, a space homogeneous of degree one, and $d : C_0 \longrightarrow \Lambda^2 V$ are defined so that the sequence

$$0 \longrightarrow C_0 \longrightarrow \Lambda^2 V \longrightarrow H^2(X) \tag{2.1}$$

is exact, where the last map is the cup-product. Consequently, C_0 is isomorphic to the kernel of the cup-product map $\Lambda^2 H^1(X) \longrightarrow H^2(X)$. The differential on the free differential algebra $\mathcal{M}_2^{(1)}$ is determined by the requirement that it be zero on V and that on C_0 it be as just defined. Thus d has degree one in the grading of $\Lambda(V \oplus C_0)$.

Now, *if X is formal*, then the construction of the remaining $\mathcal{M}_i^{(1)}$ is purely formal, in the sense that it makes no further use of X. It can easily be checked that the construction of $\mathcal{M}^{(1)}$ described in [6] proceeds as follows in the formal case. Define C_1 to be a space of homogeneous elements of degree one and define $d : C_1 \longrightarrow V \otimes C_0$ so that the sequence

$$0 \longrightarrow C_1 \longrightarrow V \otimes C_0 \xrightarrow{m} \Lambda^3 V \tag{2.2}$$

is exact, where m is defined by $m(v \otimes c) = v \wedge dc$ for all $v \in V$ and $c \in C_0$. For $i \geq 2$, define a space C_i of homogeneous degree one and $d : C_i \longrightarrow V \otimes C_{i-1}$ inductively by the requirement that the sequence

$$0 \longrightarrow C_i \longrightarrow V \otimes C_{i-1} \xrightarrow{m} \Lambda^2 V \otimes C_{i-2} \tag{2.3}$$

is exact, where m is defined to be the composition

$$V \otimes C_{i-1} \xrightarrow{1 \otimes d} V \otimes V \otimes C_{i-2} \xrightarrow{\wedge \otimes 1} \Lambda^2 V \otimes C_{i-2}.$$

Then $\mathcal{M}_i^{(1)}$ is the free minimal differential algebra generated in degree one by the space

$$V \oplus C_0 \oplus \cdots \oplus C_{i-2}.$$

It is therefore the exterior algebra

$$\Lambda(V \oplus C_0 \oplus \cdots \oplus C_{i-2}) \tag{2.4}$$

with differential determined by the conditions that on V, $d = 0$, and that on C_i, $i \geq 0$, d is defined as above.

Observe that in [19, Theorem 12.1], it is proved that formality is independent of the ground field. For this reason we will sometimes be vague on the field of definition ($\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$).

Finally, we recall the relation of the 1-minimal model to the Malcev Lie algebra $\mathcal{L}$ of $\pi_1(X)$. First, $\mathcal{L}$ is a tower of graded, nilpotent Lie algebras $\{\mathcal{L}_i\}$:

$$\cdots \longrightarrow \mathcal{L}_3 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{L}_1 \longrightarrow 0.$$

Each element of this tower $\mathcal{L}_i$ is dual to the differential algebra $\mathcal{M}_i^{(1)}$. This means that

$$\mathcal{L}_i^* \approx V \oplus C_0 \oplus C_1 \oplus \cdots \oplus C_{i-2},$$

and that, under this isomorphism, the restriction to $\mathcal{L}_i^*$ of the differential d defined above is the map $d : \mathcal{L}_i^* \longrightarrow \Lambda^2 \mathcal{L}_i^*$ dual to the Lie bracket $\Lambda^2 \mathcal{L}_i \longrightarrow \mathcal{L}_i$. Formality of $\mathcal{M}^{(1)}$ is equivalent to *quadratic presentation* of $\mathcal{L}$ in the sense that the tower $\{\mathcal{L}_i\}$ is isomorphic to the tower of maximal i-step nilpotent quotients of $\mathcal{F}(V^*)/I$, where $\mathcal{F}(V^*)$ denotes the free Lie algebra [17] on V^* and I is a homogeneous ideal generated by I_2, its homogeneous component of degree 2. Moreover, $I_2 \subset \mathcal{F}_2(V^*) \approx \Lambda^2 V^*$ is the annihilator of $C_0 \subset \Lambda^2 V$. Equivalently, I_2 is the image of the map $H_2(X) \longrightarrow \Lambda^2 V^* = \Lambda^2 H_1(X)$ dual to the cup-product map $\Lambda^2 H^1(X) \longrightarrow H^2(X)$. This is a special case of results in [15] and can be seen by a discussion similar to, but simpler than, that surrounding the proof of Theorem 9.4 of that paper. Namely, drop all discussions of bi-grading and replace his generating space A by our simpler V^*.

Since we are only interested in formal spaces X with $\pi_1(X)$ nilpotent, we are only interested in the case in which the tower $\mathcal{L}$ is finite; equivalently, $\mathcal{M}^{(1)} = \mathcal{M}_j^{(1)}$ for some j, or $C_i = 0$ for some i. We collect this summary of known facts in the following proposition:

Proposition 2.1. *There is a one-to-one correspondence between nilpotent, quadratically presented Lie algebras and finite-dimensional, formal, free minimal differential algebras generated in degree one. This correspondence assigns to a Lie algebra $\mathcal{L} = \mathcal{F}(V^*)/I$, where $\mathcal{F}(V^*)$ is the free Lie algebra on V^* and I is a homogeneous ideal generated by I_2, its homogeneous component of degree two, the differential algebra (2.4). The space $d(C_0) \subset \Lambda^2 V$ is the annihilator of I_2, and C_i is defined as above for $i \geq 1$. If $C_j \neq 0$ and $C_{j+1} = 0$, the class of nilpotency of $\mathcal{L}$ is $j + 2$.*

The Malcev Lie algebra of $\pi_1(X)$, where X is a formal space with $\pi_1(X)$ nilpotent, is obtained in this way with $V = H^1(X)$ and $d(C_0)$ the kernel of the cup-product map $\Lambda^2 V \longrightarrow H^2(X)$.

Using this proposition, we will sometimes describe a Lie algebra by describing the corresponding differential algebra. This correspondence also gives us a very useful invariant of a quadratically presented Lie algebra, which we now define.

Definition 2.2. If $\mathcal{L}$ is a nilpotent Lie algebra as in the proposition (or a quadratically presented tower as explained above), its *characteristic subspace* $C \subset \Lambda^2 V$ is the annihilator of $I_2 \subset \Lambda^2 V^* \approx \mathcal{F}_2(V^*)$. The *characteristic classes* of L are the elements of C.

Remark. If $\mathcal{L}$ is the Malcev Lie algebra of $\pi_1(X)$, then its characteristic subspace $C \subset \Lambda^2 V = \Lambda^2 H^1(X)$ is the kernel of the cup-product map. For formal X we have seen that C determines the 1-minimal model of X and hence the Malcev Lie algebra $\mathcal{L}$. □

We illustrate this proposition and the concept of the characteristic subspace with the simplest examples of non-abelian, quadratically presented nilpotent Lie algebras.

Example 2.3 (The Higher-Dimensional Heisenberg Algebras). Let V^* be a symplectic vector space of *dimension at least four* with symplectic form $\omega = x_1 \wedge y_1 + \cdots + x_n \wedge y_n$, $n \geq 2$. Let $C \subset \Lambda^2 V$ be the one-dimensional space with basis ω. The kernel of the map m in (2.2) is $\{v \otimes \omega : v \wedge \omega = 0\} = 0$ (because $n \geq 2$) and so this map is injective. Thus, if we let C_0 be a one-dimensional space with basis z of degree one and let $d : C_0 \longrightarrow C$ defined by $dz = \omega$, then the process gives $C_1 = 0$. Consequently, the minimal differential algebra

$$\Lambda(x_1, y_1 \ldots, x_n, y_n, z), \qquad dz = x_1 \wedge y_1 + \cdots + x_n \wedge y_n \tag{2.5}$$

is formal and corresponds to a two-step nilpotent, quadratically presented Lie algebra, the $2n+1$-dimensional *Heisenberg Lie algebra*, which we denote by $\mathcal{H}(n)$. This is the Lie algebra of the Heisenberg group $H(n)$ of matrices of the form

$$g = \begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.6}$$

where I is the $n \times n$ unit matrix, x is an n-dimensional row vector, y is an n-dimensional column vector, and z is a scalar. A quadratic presentation of $\mathcal{H}(n)$ is obtained by taking the dual basis $\{X_1, Y_1, \ldots, X_n, Y_n\}$ of V^* and writing down the annihilator of ω. Doing so, we obtain the relations

$$[X_i, X_j] = 0, \qquad [Y_i, Y_j] = 0, \qquad [X_i, Y_j] = 0, \qquad i \neq j$$

and

$$[X_i, Y_i] = [X_j, Y_j].$$

From these and the Jacobi identity, one easily finds that $Z = [X_1, Y_1] = \cdots = [X_n, Y_n]$ is central and that the algebra is two-step nilpotent. (The assumption $n \geq 2$ is critical here.)

Remark. If in the above example we set $n = 1$, we obtain the three-dimensional Heisenberg Lie algebra $\mathcal{H}(1)$ which is well-known not to be quadratically presented. One proof is as follows. Let x and y be a basis for V, and let X, Y be the dual basis. Since $[X, Y]$ is nonzero, the image of $x \wedge y$ is nonzero. Therefore, C_0 is the one-dimensional space with basis $x \wedge y$. But the image of $x \otimes (x \wedge y)$ and $y \otimes (x \wedge y)$ under the map m in (2.2) is zero, so C_1 is nonzero. Thus, if $\mathcal{H}(1)$ were formal, it would have index of nilpotence greater than two, a contradiction. In other words, the formal 1-minimal model determined by the same cup-product relations as $\mathcal{H}(1)$ is infinite-dimensional: for each $i \geq 1$, the elements $(z_{j_1} \otimes \cdots \otimes z_{j_i}) \otimes (x \wedge y)$, where each z equals either x or y, give linearly independent elements of C_i. Thus for all i, $C_i \neq 0$, and so the dual Lie algebra is not nilpotent. □

This way of thinking about $\mathcal{H}(1)$ leads to a very useful lemma:

Lemma 2.4. *Let $C \subset \Lambda^2 V$ be the characteristic subspace of a nilpotent, quadratically presented Lie algebra $\mathcal{L} = \mathcal{F}(V^*)/I$, and let $v \in C \otimes \mathbb{C}$, $v \neq 0$. Then v is not decomposable.*

Proof. Suppose, on the contrary, that $\mathcal{L}$ is quadratically presented and $v = x \wedge y \in C \otimes \mathbb{C}$. Apply the last argument of the remark to contradict the nilpotency of $\mathcal{L} \otimes \mathbb{C}$. An alternative argument would be to complete x, y to a basis of $V \otimes \mathbb{C}$, let $X, Y \in V^* \otimes \mathbb{C}$ be dual to x, y, and to observe that the elements of I_2 contain no occurrence of the monomial $[X, Y]$, so that the free Lie algebra $\mathcal{F}(X, Y)$ has zero intersection with I, hence injects into $\mathcal{L} \otimes \mathbb{C}$, contradicting the nilpotency of $\mathcal{L}$. In this connection, compare with Theorem 4.1 of [4]. □

Corollary 2.5. *The full algebra of strictly upper-triangular $(n+1) \times (n+1)$ matrices, $n > 1$, is not quadratically presented.*

Proof. This Lie algebra is generated by the space V^* consisting of matrices with only the entries directly above the main diagonal not zero, which has basis $X_1, \ldots, X_n$, where X_i has all entries zero except for position $(i, i+1)$ where it has a one. Thus we can write the algebra as $\mathcal{F}(V^*)/I$, and it is easy to see that the homogeneous component of degree two, I_2, has basis

$$[X_i, X_j], \qquad j - i > 1.$$

Therefore, if $x_1, \ldots, x_n$ denotes the dual basis of V, then the characteristic space C has basis

$$x_i \wedge x_{i+1}, \qquad i = 1, \ldots, n-1,$$

and thus, by the lemma, I_2 cannot generate I. □

Example 2.6 (Three-Step Nilpotent Algebras). We now produce, for each integer m of the form $m = k(k+3)/2$, $k \geq 2$, a three-step nilpotent quadratically presented Lie algebra with abelianization of dimension m. Let V be a vector space of dimension m, with a basis that we divide in two subsets:

$$\{x_1, \dots x_k\}, \qquad \{y_{ij} : i, j = 1, \dots, k;\ y_{ij} = y_{ji}\}.$$

Let $c_i \in \Lambda^2 V$ be defined by

$$c_i = \sum_j x_j \wedge y_{ij},$$

and let $C \subset \Lambda^2 V$ be the subspace with basis $c_1, \dots, c_k$. Set $C_0 = C$, with the homogeneous degree shifted down by one, and define $d : C_0 \longrightarrow C$ to be the identity map. For $i \geq 1$, let C_i be defined by the inductive process described above. For the corresponding differential algebra, let $\mathcal{L}(k)$ denote the corresponding Lie algebra, nilpotent according to Proposition 2.7, and let $L(k)$ denote the corresponding simply connected nilpotent Lie group. Since the structure constants of $\mathcal{L}(k)$ are rational, this group contains lattices [13, Theorem 7].

Proposition 2.7. *The Lie algebra $\mathcal{L}(k)$, just defined for each $k \geq 2$, is a quadratically presented three-step nilpotent Lie algebra of dimension $(k^2 + 5k + 2)/2$. Its abelianization has dimension $k(k+3)/2$ and its center is one-dimensional.*

Proof. By construction, $\mathcal{L}(k)$ is quadratically presented and its abelianization is the vector space V of the asserted dimension. To prove the remaining assertions we must show that $\dim(C_1) = 1$ and $C_2 = 0$. Recall that C_1 is defined to be the kernel of the restriction to $V \otimes C_0$ of the wedge product map $V \otimes \Lambda^2 V \longrightarrow \Lambda^3 V$.

We claim that C_1 is one-dimensional with basis $b = x_1 \otimes c_1 + \cdots + x_k \otimes c_k$. First, note that $b \in C_1$. Next, suppose $a \in C_1$. Since $c_1, \dots, c_k$ is a basis for C_0, there exist $v_1, \dots, v_k \in V$ such that $a = v_1 \otimes c_1 + \cdots + v_k \otimes c_k$. Write $v_i = \sum_l \alpha_{il} x_l$. Then $a \in C_1$ means that

$$\sum_{i,j,l} \alpha_{il}\, x_l \wedge x_j \wedge y_{ij} = 0.$$

If $i \neq l$, the coefficient of $x_l \wedge x_i \wedge y_{ii}$ in this expression is α_{il}, so that $\alpha_{il} = 0$ for $i \neq l$. Moreover, if $i \neq j$, the coefficient of $x_i \wedge x_j \wedge y_{ij}$ is $\alpha_{ii} - \alpha_{jj}$, hence $\alpha_{ii} = \alpha_{jj}$. Thus there is a scalar λ so that $v_i = \lambda x_i$ for $i = 1, \dots, k$, i.e., $a = \lambda b$, and C_1 is one-dimensional with basis b, as asserted.

If $x \in C_2$, then there exists a $v \in V$ such that $x = v \otimes b$ and $\sum (v \wedge x_i) \otimes c_i = 0$. It follows that $v \wedge x_i = 0$ for $i = 1, \dots, k$; hence $v = 0$ since $k \geq 2$. The proof of the proposition is complete. $\square$

Example 2.8. We write down in detail the Lie algebra $\mathcal{L}(2)$. Let $x_1, \ldots, x_5$ be the basis for V, and let

$$c_1 = x_1 \wedge x_3 + x_2 \wedge x_4, \qquad c_2 = x_1 \wedge x_4 + x_2 \wedge x_5.$$

Let $X_1, \ldots, X_5$ be the dual basis of V^*. Then a basis for the space I_2 of quadratic relations is given by

$$[X_1, X_2], \quad [X_1, X_5], \quad [X_2, X_3], \quad [X_3, X_4], \quad [X_3, X_5], \quad [X_4, X_5]$$

and

$$[X_1, X_3] - [X_2, X_4], \qquad [X_1, X_4] - [X_2, X_5].$$

There is a single nonvanishing triple commutator formed with the basis elements, namely

$$[X_1, [X_1, X_3]] = [X_1, [X_2, X_4]] = [X_2, [X_1, X_4]] = [X_2, [X_2, X_5]] \bmod(I),$$

which generates the center of $L(2)$, and all four-fold commutators are in I.

3. Classification for V of small dimension

The purpose of this section is to classify nilpotent quadratically presented Lie algebras $\mathcal{L}$ over $\mathbb{C}$ such that the dimension of the abelianized algebra $\mathcal{L}_{ab}$ is at most five. We take them to be of the form $\mathcal{L} = \mathcal{F}(V^*)/I$, where the ideal I is generated by $I_2 \subset \mathcal{F}_2(V^*) \approx \Lambda^2 V^*$, its homogeneous component of degree two, and where $\dim(V) \leq 5$. Recall that $\mathcal{L}$ is determined by its characteristic subspace $C \subset \Lambda^2 V$, the annihilator of I_2.

If V is a vector space of dimension at most 3, then every element of $\Lambda^2 V$ is decomposable, so by Lemma 2.4 it follows that $C = 0$, and so $\mathcal{L}$ is abelian. If $n \geq 4$, the lines determined by decomposables constitute the Plücker embedding of the Grassmann manifold $G(2, V)$ of two-dimensional subspaces of V in the projective space of $\Lambda^2 V$. The former has dimension $2(n-2)$, where $n = \dim(V)$, while the latter has dimension $n(n-1)/2 - 1$.

If $\dim(V) = 4$, then the Grassmannian has dimension four and the projective space has dimension five, so if $\dim C > 1$, then C would contain a decomposable element, contrary to Lemma 2.4. Consequently, if $\mathcal{L}$ is finite-dimensional, then $\dim C = 0$ or $\dim C = 1$. In the first case, $\mathcal{L}$ is abelian. In the second case, there is a single characteristic class ω that is not decomposable and so is nondegenerate, i.e., symplectic. Thus, $\mathcal{L}$ is the five-dimensional Heisenberg Lie algebra $\mathcal{H}(2)$.

If $\dim(V) = 5$, then the Grassmannian has dimension six and the projective space dimension nine. If $\dim C > 3$, then C would contain a decomposable element, contrary to Lemma 2.4. Thus we must have $\dim C \leq 3$. If $\dim C = 0$, then $\mathcal{L}$ is abelian, and if $\dim C = 1$, then $\mathcal{L} = \mathcal{H}(2) \oplus \mathbb{C}$, the direct sum of the five-dimensional Heisenberg algebra and a one-dimensional abelian algebra. It remains to analyze the cases $\dim C = 2$ and $\dim C = 3$. This will be done in the following three lemmas.

Lemma 3.1. *Let V be a five-dimensional vector space over $\mathbb{C}$, and let $C \subset \Lambda^2 V$ be a two-dimensional subspace that contains no decomposables. Then*

(a) *The map $S^2C \longrightarrow \Lambda^4 V$ defined by $c_1 c_2 \longrightarrow c_1 \wedge c_2$ is injective.*

(b) *For any basis c_1, c_2 of C, there exists a basis $x_1, \ldots, x_5$ of V so that*

$$c_1 = x_1 \wedge x_3 + x_2 \wedge x_4, \qquad c_2 = x_1 \wedge x_4 + x_2 \wedge x_5$$

is a basis for C. In particular, all such spaces are equivalent under automorphisms of V.

Proof. Recall that $c \in \Lambda^2 V$ is decomposable if and only if $c \wedge c = 0$. Thus $c \wedge c \neq 0$ for all nonzero c in C. Define the *carrier* of an element of $\Lambda^2 V$ to be the vector space obtained by contracting it with the dual of V. For any nonzero element $c \in C$, the carrier is four-dimensional and $c \wedge c$ is a volume element for it. Consider a basis c_1, c_2 of C and suppose for a moment that the volume forms of c_1 and c_2 are proportional. Then the carriers of these elements coincide and so constitute a single four-dimensional vector space U. But then C is a two-dimensional subspace of $\Lambda^2 U$ and so must contain decomposables, a contradiction. Now suppose that $c_1 \wedge c_2 = 0$. Then $(c_1 + c_2) \wedge (c_1 + c_2) = (c_1 - c_2) \wedge (c_1 - c_2)$. But $\{c_1 - c_2,\ c_1 + c_2\}$ is also a basis for C and our previous argument applies to exclude proportionality of the associated volume forms. It follows that $c_1 \wedge c_2 \neq 0$, for any pair of nonzero elements $c_1, c_2 \in C$, and this proves the first assertion.

For the second assertion, let c_1, c_2 be a basis of C, and choose a nonzero element $w \in \Lambda^5 V$. Then there is a unique $z \in V^*$ so that $c_1 \wedge c_2 = \iota(z)w$, where ι denotes interior product. Since ι is a derivation, $(\iota(z)c_1) \wedge c_2 + c_1 \wedge \iota(z)c_2 = 0$. Thus, if $x_1 = \iota(z)c_2$ and $x_2 = \iota(z)c_1$, then

$$x_1 \wedge c_1 + x_2 \wedge c_2 = 0.$$

It follows that x_1 and x_2 are independent (otherwise a linear combination of c_1 and c_2 would be decomposable). Then it is well known and easy to check that there exist elements $x_3, x_4, x_5 \in V$ so that

$$c_1 = x_1 \wedge x_3 + x_2 \wedge x_4, \qquad c_2 = x_1 \wedge x_4 + x_2 \wedge x_5.$$

If the set $x_1, \ldots, x_5$ were dependent, C would be a subspace of two-vectors in a vector space of dimension ≤ 4; thus it would contain decomposables. Therefore $x_1, \ldots, x_5$ forms a basis for V, and the proof is complete. □

Corollary 3.2. *Let $\mathcal{L}$ be a nilpotent, quadratically presented Lie algebra over $\mathbb{C}$ with $\dim \mathcal{L}_{ab} = 5$ and $\dim(C) = 2$. Then $\mathcal{L}$ is isomorphic to the three-step nilpotent Lie algebra $\mathcal{L}(2)$ of Example 2.8.*

It remains to consider the case of $\dim(C) = 3$. To this end, we introduce the following:

Lemma 3.3. *Let V be a five-dimensional complex vector space, and let $C \subset \Lambda^2 V$ be a three-dimensional subspace that contains no decomposables. Then there is a basis $x_1, \ldots, x_5$ for V so that*

$$c_1 = x_1 \wedge x_3 + x_2 \wedge x_4, \qquad c_2 = x_1 \wedge x_4 + x_2 \wedge x_5, \qquad c_3 = x_2 \wedge x_3 + x_5 \wedge x_4$$

is a basis for C. In particular, all such spaces C are equivalent under automorphisms of V.

Proof. Let G denote the Grassmann manifold of three-dimensional subspaces of $\Lambda^2 V$, which is a manifold of dimension 21. We compute the dimension of the $SL(V)$ orbit of C in G. To this end, consider the map $\phi : S^2 C \longrightarrow \Lambda^4 V$ given by $\phi(c_1 c_2) = c_1 \wedge c_2$. By the first assertion of Lemma 3.1, for any two-dimensional subspace $C' \subset C$, the restriction of this map to $S^2 C'$ is injective. In other words, the kernel of ϕ contains no nonzero symmetric tensors of rank ≤ 2. Now the subvariety of symmetric tensors of rank ≤ 2 is the cone on the secant variety of the Veronese variety and so has codimension one. If the kernel of ϕ is of dimension greater than one it must intersect the Veronese variety, a contradiction. We conclude that ϕ has one-dimensional kernel and hence is surjective. The kernel is generated by an element q of rank three, which we take to be of the form $q = c_1^2 + 2c_2 c_3$ for some basis c_1, c_2, c_3 of C.

Let $H \subset SL(V)$ denote the identity component of the isotropy group of C. Then the action of H on C preserves q up to scalars. On the other hand, the action of H on C contains no homotheties by λ, where $\lambda^2 \neq 1$. Indeed, such a homothety would map by the surjection $\phi : S^2 C \longrightarrow \Lambda^4 V$ to a homothety by λ^2 on $\Lambda^4 V$. Since $SL(V)$ acts unimodularly on $\Lambda^4 V$, this is impossible. It follows that the action of H on C preserves q. Since the group that preserves a quadratic tensor in dimension three is three-dimensional, $\dim(H) \leq 3$. Since $SL(V)$ has dimension 24 and G has dimension 21, the orbit of C in G has dimension 21 and hence is open. Moreover, it follows that H is isomorphic to an orthogonal group in three variables, hence it operates transitively on the set of homothety classes of bases for C satisfying the relation $c_1^2 + 2c_1 c_2 = 0$. Consequently, all such spaces C and all such choices of basis c_1, c_2, c_3 are equivalent (up to homothety) under the action of $SL(V)$.

If $x_1, \ldots, x_5$ is a basis for V and c_1, c_2, c_3 are defined as in the statement of the lemma, then it is easy to check that the space $C = \text{span}\{c_1, c_2, c_3\}$ contains no decomposables and that the relation $c_1^2 + 2c_2 c_3 = 0$ is satisfied. This completes the proof of the lemma. □

Lemma 3.4. *Let V be a five-dimensional complex vector space with basis $x_1, \ldots, x_5$, let $C \subset \Lambda^2 V$ be the three-dimensional subspace with basis*

$$c_1 = x_1 \wedge x_3 + x_2 \wedge x_4, \qquad c_2 = x_1 \wedge x_4 + x_2 \wedge x_5, \qquad c_3 = x_2 \wedge x_3 + x_5 \wedge x_4,$$

and let $C' \subset V \otimes C$ be the five-dimensional subspace with basis

$$x_1 \otimes c_1 + x_2 \otimes c_2, \qquad x_3 \otimes c_1 + x_4 \otimes c_3, \qquad x_4 \otimes c_2 - x_2 \otimes c_3,$$

$$x_2 \otimes c_1 + x_5 \otimes c_2 + x_1 \otimes c_3, \qquad x_4 \otimes c_1 + x_3 \otimes c_2 + x_5 \otimes c_3.$$

Then the sequences

$$0 \longrightarrow C' \longrightarrow V \otimes C \longrightarrow \Lambda^3 V \longrightarrow 0 \tag{3.1}$$

and

$$0 \longrightarrow V \otimes C' \longrightarrow \Lambda^2 V \otimes C \tag{3.2}$$

are exact.

Proof. To see that the last map in (3.1) is surjective, observe that if $x_i \wedge x_j \wedge x_k, i < j < k$, is a basis element for $\Lambda^3 V$ other than $x_1 \wedge x_3 \wedge x_5$, then it is of the form $\pm x_l \wedge c_m$ for some l, m. This is because c_1, c_2, c_3 are nondegenerate two-vectors in the four-dimensional subspaces of V obtained by deleting the basis vector x_5, x_3, x_1, respectively. But $x_1 \wedge x_3 \wedge x_5 = c_1 \wedge x_5 + c_2 \wedge x_4$, so the surjectivity of this map follows.

By counting dimensions, we see that the kernel of this last map is five-dimensional. One checks easily that C' is contained in this kernel, and so the exactness of (3.1) follows.

To prove the exactness of (3.2), let $\gamma_1, \ldots, \gamma_5$ denote the basis elements of C', written as listed in the lemma. We must show that if $v_1, \ldots, v_5 \in V$ and $a = v_1 \otimes \gamma_1 + \cdots + v_5 \otimes \gamma_5$ is in the kernel of the map $V \otimes C' \longrightarrow \Lambda^2 V \otimes C$, then $v_1 = \cdots = v_5 = 0$. Writing explicitly the coefficients of c_1, c_2, c_3 in the image of a under this map, we obtain the system of equations

$$\begin{aligned} v_1 \wedge x_1 + v_2 \wedge x_3 + v_4 \wedge x_2 + v_5 \wedge x_4 &= 0 \\ v_1 \wedge x_2 + v_3 \wedge x_4 + v_4 \wedge x_5 + v_5 \wedge x_3 &= 0 \\ v_2 \wedge x_4 - v_3 \wedge x_2 + v_4 \wedge x_1 + v_5 \wedge x_5 &= 0 \end{aligned}$$

for $v_1, \ldots, v_5$. At this point it is convenient to relabel the unknowns as $w_1, \ldots, w_5$, where $w_1 = v_2,\ w_2 = v_5,\ w_3 = v_1,\ w_4 = v_4,\ w_5 = v_3$. The equations then become

$$\begin{aligned} w_3 \wedge x_1 + w_1 \wedge x_3 + w_4 \wedge x_2 + w_2 \wedge x_4 &= 0 \\ w_3 \wedge x_2 + w_5 \wedge x_4 + w_4 \wedge x_5 + w_2 \wedge x_3 &= 0 \\ w_1 \wedge x_4 - w_5 \wedge x_2 + w_4 \wedge x_1 + w_2 \wedge x_5 &= 0. \end{aligned} \tag{3.3}$$

Write $w_j = \sum_i a_{ij} x_i$. Then, since the indices 5, 1, 3 do not appear in the first (second, and third, respectively) equation, the matrix a_{ij} must have the form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}. \tag{3.4}$$

Recall the fact that if $y_1, \dots y_5 \in V$ and $y_1 \wedge x_1 + \cdots + y_5 \wedge x_5 = 0$, then the matrix expressing the y_j in terms of the basis x_i is *symmetric*. Thus each equation imposes some condition on an appropriately rearranged 4×4 submatrix of a_{ij}. Explicitly, the first equation implies that the matrix

$$\begin{pmatrix} 0 & 0 & a_{11} & 0 \\ a_{23} & a_{24} & a_{21} & a_{22} \\ a_{33} & 0 & 0 & 0 \\ a_{43} & a_{44} & a_{41} & a_{42} \end{pmatrix}$$

is symmetric. Thus we must have $a_{23} = a_{21} = a_{43} = a_{41} = 0$. Set these coefficients equal to zero in the matrix (3.4), and then write down the condition imposed by the second equation, namely that the matrix

$$\begin{pmatrix} 0 & a_{22} & a_{25} & a_{24} \\ a_{33} & 0 & 0 & 0 \\ 0 & a_{42} & a_{45} & a_{44} \\ 0 & 0 & a_{55} & 0 \end{pmatrix}$$

is symmetric, hence $a_{25} = a_{24} = a_{42} = 0$. Set these coefficients also equal to zero in the matrix (3.4), and write down the condition resulting from the third equation, namely that the matrix

$$\begin{pmatrix} 0 & 0 & a_{11} & 0 \\ 0 & 0 & 0 & a_{22} \\ a_{44} & -a_{45} & 0 & 0 \\ 0 & -a_{55} & 0 & 0 \end{pmatrix}$$

is symmetric, hence $a_{45} = 0$. This leaves the coefficients a_{11}, a_{22}, a_{33}, a_{44}, a_{55}. From the symmetry of the first two submatrices, we see that all these coefficients are equal, and from the symmetry of the third, $a_{22} = -a_{55}$. Thus, all coefficients also vanish, and the proof is complete. □

Example 3.5. Let V be a 5-dimensional vector space over any field (say $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) with basis $x_1, \dots, x_5$, and let C, and C' be the spaces defined by the bases in the last lemma. Let $C_0 = C$ except that C_0 is declared to have homogeneous degree one, let $d : C_0 \longrightarrow C$ be the identity, and for $i \geq 1$, define C_i by the process described in Section 2. Then the lemma yields $C_1 \cong C'$ and $C_2 = 0$. Let $\mathcal{K}$ denote the corresponding nilpotent Lie algebra and K the corresponding simply connected nilpotent Lie group. Then $\mathcal{K}$ is a three-step quadratically presented nilpotent Lie algebra of dimension 13 with five-dimensional abelianization and five-dimensional center. Note that the orthogonal group of the form q defined in the proof of Lemma 3.1 acts on $\mathcal{K}$ by Lie algebra automorphisms. Note also that K contains lattices since $\mathcal{K}$ has rational structure constants [13, Theorem 7].

Corollary 3.6. *Let $\mathcal{L}$ be a nilpotent, quadratically presented Lie algebra over $\mathbb{C}$ with $\dim \mathcal{L}_{ab} = 5$ and $\dim(C) = 3$. Then $\mathcal{L}$ is isomorphic to the three-step nilpotent Lie algebra $\mathcal{K}$ of Example 3.5.*

We can summarize the discussion of this section as the following classification theorem.

Theorem 3.7. *Let $\mathcal{L}$ be a nilpotent, quadratically presented Lie algebra over $\mathbb{C}$ with $\dim \mathcal{L}_{ab} \leq 5$. Then*

(a) *If $\dim \mathcal{L}_{ab} \leq 3$, then $\mathcal{L}$ is abelian.*

(b) *If $\dim \mathcal{L}_{ab} = 4$, then $\mathcal{L}$ is either abelian or the 5-dimensional Heisenberg algebra $\mathcal{H}(2)$ of Example 2.3.*

(c) *If $\dim \mathcal{L}_{ab} = 5$, then $\mathcal{L}$ is either abelian, or $\mathcal{H}(2) \oplus \mathbb{C}$, or the algebra $\mathcal{L}(2)$ of Example 2.8 or the algebra $\mathcal{K}$ of Example 3.5.*

In particular, these algebras are classified up to isomorphism by the dimensions of $\mathcal{L}_{ab}$ and of the characteristic space C.

4. Restrictions on nilpotent Kähler groups

In this section we derive restrictions on nilpotent Kähler groups that use Kähler geometry and go beyond the conclusions of rational homotopy theory. We work first in the context of a formal topological space X with nilpotent fundamental group Φ. By passing to a finite cover of X, we may assume that Φ is torsion free; in fact, Φ is a lattice in the simply connected real nilpotent Lie group L with Lie algebra $\mathcal{L}$ the Malcev completion of Φ [13].

Let T be a torus of dimension $\operatorname{rank}(\Phi_{ab})$ and let $f : X \longrightarrow T$ be a continuous map that induces an isomorphism $\Phi_{ab} \longrightarrow \pi_1(T)$. Then $f^* : H^1(T, \mathbb{R}) \longrightarrow H^1(X, \mathbb{R}) = V$ is an isomorphism, where V is as in the notation of Section 2. Because f^* preserves the multiplicative structure of cohomology, the spaces $\ker(f^*) \subset H^2(T, \mathbb{R}) \cong \Lambda^2 H^1(T, \mathbb{R})$ and $C \subset \Lambda^2 V$ correspond.

Now suppose X is a compact Kähler manifold, which we denote by M. Then the torus T can be given a complex structure as the *Albanese torus* of M, namely

$$Alb(M) = H^{1,0}(M)^*/H_1(M, \mathbb{Z}),$$

and the map f can be chosen to be holomorphic, namely the *Albanese map*. Then we have:

Lemma 4.1. *Let M be a compact Kähler manifold, let T be its Albanese torus, let $f : M \longrightarrow T$ be the Albanese map, and let $\mathcal{L} = \mathcal{F}(V^*)/I$ be the real Malcev Lie algebra of $\pi_1(M)$. Then the canonical isomorphism $H^2(T, \mathbb{R}) \approx \Lambda^2 V$ takes $\ker(f^*)$ to the characteristic subspace C. In particular, V has an integral structure and a Hodge structure that induce integral and Hodge structures on $\Lambda^2 V$. With respect to these structures, C is a subspace of $\Lambda^2 V$ defined over the integers and has a Hodge structure.*

Now consider a real (p, p) form α on a complex manifold M. At each point x of M one can find a coordinate system that diagonalizes α in the sense that α_x is a linear combination of basic monomials, i.e., of products of the forms

$$\sqrt{-1}dz_k \wedge d\bar{z}_k.$$

If the coefficients of the basic monomials are nonnegative, we say that α is semipositive at x. If α is semipositive at all points of M and has at least one positive coefficient at one point, we call it *quasipositive*. The following lemma is basic:

Lemma 4.2. *If a closed (p, p)-form α is quasipositive and M is Kähler, then α is nonzero in cohomology.*

For the proof, consider $\beta = \alpha \wedge \omega^{m-p}$, where m is the dimension of M and ω is its Kähler form. Then β is a quasipositive (m, m)-form, hence a form with positive integral. Therefore it, and so α as well, is nonzero in cohomology. The next result is an immediate consequence of the preceding two lemmas.

Corollary 4.3. *Let ω be a characteristic class of a Kähler group. Then the $(1, 1)$ component of ω is not (positive or negative) definite.*

Recall that if $\omega \in \Lambda^2 V$, then the rank of ω (necessarily even) is the dimension of the smallest subspace $W \subset V$ such that $\omega \in \Lambda^2 W$. Observe that $\operatorname{rank}(\omega) = 2$ if and only if ω is decomposable. With these preliminaries we can state our main theorem:

Theorem 4.4. *Let M be a compact Kähler manifold with nilpotent fundamental group Φ which is not almost abelian, let C and V be as above, and let $\omega \in C \subset \Lambda^2 V$, $\omega \neq 0$ be integral and of type $(1, 1)$ (in the sense of Lemma 4.1). Then $\operatorname{rank}(\omega) \geq 8$.*

Proof. Let ω be as in the hypothesis. We may view ω as an integer-valued skew form on the real vector space $V^* = H_1(T, \mathbb{R})$, the universal cover of T. Let J be the complex structure on V^*, that is, the pullback of the complex structure on T. Since ω is of type (1,1), it is invariant under J, so its null space is J-invariant, i.e., a complex subspace, and defined over $\mathbb{Z}$. It therefore covers a complex subtorus T' of T and determines an exact sequence

$$0 \longrightarrow T' \longrightarrow T \xrightarrow{q} T'' \longrightarrow 0$$

of complex tori, and so a holomorphic map $g = q \circ f$ of M to T''. Moreover, $\dim_{\mathbb{R}}(T'') = \text{rank}(\omega)$, and ω is the pullback of an integral $(1, 1)$ class ω'' on T'' of maximum rank.

We already know from Lemma 2.4 that $\text{rank}(\omega) \geq 4$. Thus, to prove the theorem we only need consider two cases:

Case 1. $\text{rank}(\omega) = 4$. Then, the map $g : M \longrightarrow T''$ has real rank 2 or 4. In the latter case it is surjective, $g^*\omega''^2$ is quasipositive, hence nonzero in cohomology, which contradicts $g^*\omega'' = 0$ in $H^2(M)$. In the former case, the image of g is a (possibly singular) Riemann surface Y. Since g^* factors through the one-dimensional space $H^2(Y)$, its kernel has codimension one and hence contains decomposables, in contradiction with Lemma 2.4.

Case 2. $\text{rank}(\omega) = 6$. Then, the map $g : M \longrightarrow T''$ has real rank 2, 4, or 6, and the case of rank 2 can be disposed of as above. The case of maximum rank 6 is also as above: $g^*\omega''^3$ is quasipositive, hence nonzero in cohomology, which contradicts $g^*\omega'' = 0$ in $H^2(M)$.

It remains to treat the case in which the real rank of g is 4. Then $Y = g(M)$ is a possibly singular algebraic surface and T'' is an algebraic torus. Consider first the case in which Y is ample and smooth. Factor g as $j \circ k$ where $j : Y \longrightarrow T''$ is the inclusion. By the Lefschetz hyperplane theorem [14, p. 41], $j^* : H^2(T'') \longrightarrow H^2(Y)$ is injective, so that $j^*\omega''$ is nonzero. Let η be a class in $H^2(Y)$ such that $j^*\omega'' \cup \eta$ is a positive multiple of the fundamental class. Then $k^*(j^*\omega'' \cup \eta)$ is quasipositive of rank 2 and so is nonzero in cohomology. But $k^*(j^*\omega'' \cup \eta) = g^*\omega'' \cup k^*\eta$, so $g^*\omega'' \neq 0$, and so we arrive at the by now familiar contradiction.

For the singular case, a slightly more elaborate argument is required, but the conclusion is the same: $g^*\eta \neq 0$. To see this, we begin with the observation that the Lefschetz theorem [14, p. 41] still applies to give $j^*\eta \neq 0$. Next, consider a resolution of singularities $p : \hat{Y} \longrightarrow Y$. By a mixed Hodge-theoretic argument [5, Proposition 8.2.7], we know that the kernel of p^*j^* is the same as the kernel of j^*. Therefore, $p^*j^*\eta \neq 0$. Now let $\hat{M}$ be a desingularization of the fiber product $\hat{Y} \times_Y M$, and let $\hat{k} : \hat{M} \longrightarrow \hat{Y}$, $q : \hat{M} \longrightarrow M$ be the natural maps. Since $p^*j^*\eta \neq 0$, there is a class α such that $p^*j^*\eta \cup \alpha$ represents a multiple of the fundamental class. Therefore, $\hat{k}^*(p^*j^*\eta \cup \alpha) \neq 0$, and so $\hat{k}^*p^*j^*\eta \neq 0$. But then $g^*\eta = q^*k^*j^*\eta \neq 0$, which is what was claimed.

Let us now consider the case in which Y is not ample. Then T'' is reducible, and Y is the pullback of a divisor on a quotient torus [16, pp. 25–26]. In addition there is a finite unramified cover of T'' which splits as $\mathcal{E} \times A$, where $\mathcal{E}$ is an elliptic curve and A is an abelian variety of dimension 2. Replace M and T'' by the corresponding unramified covers so as to reduce the problem to one in which T'' splits. Then the divisor Y has the form $\mathcal{E} \times C$, where C is a genus-two curve, or $\mathcal{E} \times \mathcal{E}'$, or $\{p\} \times A$, where p is a point. The second and third cases are excluded, otherwise g_* would not be surjective on H_1. The first case is also excluded, for it yields a nonconstant map of M to a Riemann surface, thus contradicting Lemma 2.4, as in Case 1. The proof of the theorem is complete. □

Corollary 4.5. *Let Φ be a nilpotent, not almost abelian, Kähler group. Then* $\text{rank}(\Phi_{ab}) \geq 8$.

Proof. We only need to show that if Φ were a nilpotent Kähler group with $\text{rank}(\Phi_{ab}) \leq 6$ and $\omega \in C$, $\omega \neq 0$, then ω has type $(1, 1)$. But if ω had a nonzero $(2, 0)$-component $\omega^{(2,0)}$, then $\omega^{(2,0)} \in \Lambda^2 H^{2,0}(T)$. Since $\dim H^{2,0}(T) \leq 3$, $\omega^{(2,0)}$ would be decomposable, in contradiction to Lemma 2.4. Thus ω is of type $(1,1)$, in contradiction to the theorem, and the proof is complete. □

Corollary 4.6. *Let Φ be a nilpotent Kähler group that is not almost abelian, and let C be as above. Then there exists $x \in C$ with* $\text{rank}(x) \geq 8$. *If* $\dim C \leq 2$, *then* $\text{rank}(x) \geq 8$ *for all $x \in C$, $x \neq 0$.*

Proof. If Φ is not almost abelian, then $C \neq 0$. If C is of type $(1,1)$, i.e., $C \otimes \mathbb{C} = C^{1,1}$, then the theorem gives us that every nonzero element of C has rank at least 8. Otherwise, $C^{2,0} \neq 0$, and if α is a nonzero element of $C^{2,0}$, then by Lemma 2.4, $\text{rank}(\alpha) \geq 4$. Thus, there are complex linear coordinates $z_1, \ldots, z_n$ for V so that

$$\alpha = dz_1 \wedge dz_2 + \cdots + dz_{2k-1} \wedge dz_{2k}, \qquad k \geq 2.$$

It is then easy to check that any nontrivial linear combination of the real and imaginary parts of α has rank $4k \geq 8$. Thus, there are elements of C of rank ≥ 8, and the first assertion follows. For the second assertion, if $\dim C = 1$, then C is of type $(1,1)$, and if $\dim C = 2$, then the only possibilities for C is that it be either all of type $(1, 1)$ or all of type $(2, 0) + (0, 2)$. In either case, we have seen that all its elements have to have rank at least 8.

Corollary 4.7. *Let Φ be a cartesian product of one of the following forms:*

(a) $\Phi = \Phi_1 \times \Phi_2$, *where Φ_1 is a lattice in the the five- or seven-dimensional real Heisenberg group $H(2)$ or $H(3)$, and Φ_2 is a free abelian group;*

(b) $\Phi = \Phi_1 \times \Phi_2 \times \Phi_3$, *where Φ_1 and Φ_2 are as in (a) and Φ_3 is a lattice in a Heisenberg group $H(k)$, $k \geq 2$;*

(c) $\Phi = \Phi_1 \times \Phi_2$, *where Φ_1 is a lattice in one of the real Lie groups $L(2)$ or $L(3)$ of Example 2.8 or the group K of Example 3.5 and Φ_2 is free abelian.*

Then the Malcev Lie algebra of Φ is quadratically presented. By suitable choice of Φ_2 or Φ_3, the rank of Φ_{ab} can be made even and arbitrarily large, yet Φ is not a Kähler group.

Proof. By construction, either $\dim C \leq 2$ and some characteristic class of Φ has rank at most 6, or, in the cases involving $L(3)$ and K, $\dim C = 3$ and all elements of C have rank at most six. Thus, the last corollary excludes these as Kähler groups. □

Remark 4.8. It is not known whether characteristic classes of type $(2, 0)$ can occur for nilpotent Kähler groups. If it were true that C always has type $(1, 1)$, then our theorem would give much stronger restrictions on nilpotent Kähler groups. □

5. Remarks on the examples

Finally, we examine the characteristic classes of Campana's examples of nilpotent Kähler groups [2]. For the construction, it is helpful to consult Section 9 of [7] and Part II of [9]. Let A and B be Abelian varieties of dimension n, and let $f : A \longrightarrow \mathbb{P}^n$ and $g : B \longrightarrow \mathbb{P}^n$ be finite maps. Set $X = A \times B$ and let $h = f \times g : X \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$, which is also a finite map. Choose two copies $\bar{A}$ and $\bar{B}$ of $\mathbb{P}^n$ in $\mathbb{P}^{2n+1}$ which are in general position, hence disjoint. Let V be the complement of these, and note that it fibers over $\bar{A} \times \bar{B}$ with fiber $\mathbb{C}^*$. Indeed, if x is a point of V, let a be the unique point of intersection of the $(n+1)$-dimensional linear space spanned by x and $\bar{B}$ with $\bar{A}$. Reverse the roles of the two $\mathbb{P}^n$'s to construct a point b on $\bar{B}$, and define $p(x) = (a, b)$. The characteristic class of this principal $\mathbb{C}^*$ bundle is $(\omega, -\omega)$, where ω is the class of the hyperplane section. Let X^* be the pullback of V to $A \times B$ via h. Note that X^* is a quasi-projective variety, and is a principal $\mathbb{C}^*$ bundle with characteristic class $\eta = (f^*\omega, -g^*\omega)$. Let $\Phi = \pi_1(X^*)$. The exact sequence of the fibration gives

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Phi \longrightarrow \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \longrightarrow 0.$$

In fact, Φ is a central extension of $\mathbb{Z}$ by $\mathbb{Z}^{4n}$ with characteristic class η. Since $\eta^{2n} \in H^{4n}(X, \mathbb{Z})$ is a nonzero multiple (namely, the degree of h) of the positive generator, η is a symplectic form on the vector space $H_1(X, \mathbb{Q})$. This implies that the rational Malcev completion of Φ is the rational Heisenberg group $H(2n)$ of Example 2.3 and that Φ is a subgroup of finite index in the *integral Heisenberg group*, namely the group of matrices (2.6) with *integral* entries. In fact, it is easy to see that if $d_1, \ldots, d_{2n}$ are the elementary divisors of η as a skew form on $H_1(X, \mathbb{Z}) \approx \mathbb{Z}^{4n}$, then Φ is isomorphic to the subgroup of (2.6), where x_i is divisible by d_i and where $x = (x_1, \ldots, x_{2n})$.

Observe that Φ cannot be isomorphic to the full integral Heisenberg group. If this were the case, all elementary divisors of $f^*\omega$ and $g^*\omega$ would be one, in which case the cohomology classes would define principal polarizations of A and B respectively. However, the line bundle of a principal polarization has a one-dimensional space of sections, and so it cannot be induced from a map to $\mathbb{P}^n$.

Note that Φ is the fundamental group of the quasiprojective variety X^*. Now let $\tilde{h} : X^* \longrightarrow V$ be the map of $\mathbb{C}^*$ bundles covering $h : X \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$, and let $L \subset \mathbb{P}^{2n+1}$ be a linear space of dimension 2 that is in general position with respect to $\bar{A}$ and $\bar{B}$. It is contained in V and so we can form the *projective* variety $Y = \tilde{h}^{-1}(L)$. The generalized Lefschetz theorem of [9, p. 195] implies that

$$\pi_i(X^*, \tilde{h}^{-1}(L)) = 0$$

for $i \leq 2$ provided that $n \geq 2$. Thus

$$\pi_1(Y) \cong \pi_1(X^*) = \Phi,$$

and therefore Φ is the fundamental group of a projective manifold provided $n \geq 2$. This situation is similar in spirit to that of [20], [21], in which the fundamental groups of certain quasiprojective varieties turn out to be the fundamental groups of certain projective varieties, namely of suitable linear sections.

A concrete example can be obtained as follows. Let E be an elliptic curve, and let $E \longrightarrow \mathbb{P}^1$ be the standard map of degree 2. Let $E \times E \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the Cartesian product of two such maps, and let $f : E \times E \longrightarrow \mathbb{P}^2$ be the composition of this map with the standard degree two map $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^2$. Let $A = B = E \times E$ and $f = g$. Then f and g are of degree 8; consequently, h is of degree 64. It can easily be checked that all the elementary divisors of η are 2; thus Φ is isomorphic to the subgroup of (2.6) with all x_i even, where $x = (x_1, \ldots, x_4)$. The subgroup of Φ of index two where in addition z is required to be even is isomorphic to the integral Heisenberg group. Another example would be to take polarizations with elementary divisors 1, 5 on A and B and f and g generic projections to $\mathbb{P}^2$ of their embeddings in $\mathbb{P}^4$ [11]. In this case, f and g have degree 10 and Φ is isomorphic to the subgroup of (2.6) with x_2 and x_4 divisible by 5. J. Kollár has pointed out to us that polarizations with elementary divisors 1, 3 on generic abelian surfaces also give finite maps to $\mathbb{P}^2$. In this case, f and g would have degree 6, and Φ would be isomorphic to the subgroup of (2.6), where x_2 and x_4 are divisible by 3. This is the subgroup of smallest index in the integral Heisenberg group, which we know to be a Kähler group.

Note that the characteristic class of each Φ in the examples just constructed is of type $(1, 1)$. It is also indefinite (in accordance with Corollary 4.3) and of rank at least eight, in accordance with Theorem 4.4. It is not difficult to extend the construction just given to one for Kähler groups, which are central extensions of $\mathbb{Z}^k$ by $\mathbb{Z}^{2m}$ for various m and k. (For instance, take more factors of $\mathbb{P}^n$, take factors of different dimensions, etc.) The characteristic classes are still of type $(1, 1)$; cf. Remark 4.8.

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