The Journal of Geometric Analysis
Volume 8, Number 3, 1998

Polynomial Diffeomorphisms of $\mathbf{C}^2$: V. Critical Points and Lyapunov Exponents

By *Eric Bedford and John Smillie*

ABSTRACT. There is an invariant measure μ, which is the pluri-complex version of the harmonic measure of the Julia set for polynomial maps of **C**. *In this paper we give an integral formula for the Lyapunov exponents of a polynomial automorphism with respect to μ, analogous to the Brolin–Manning formula polynomial maps of* **C**. *Our formula relates the Lyapunov exponents to the value of a Green function at a type of critical point which we define in this paper. We show that these the critical points have a natural dynamical interpretation.*

0. Introduction

This paper deals with the dynamics of polynomial diffeomorphisms $f : \mathbf{C}^2 \to \mathbf{C}^2$. To exclude trivial cases we make the standing assumption that the dynamical degree $d = d(f)$ is greater than one (see Section 1 for a definition). It is often quite useful in dynamics to focus attention on invariant objects. A natural invariant set to consider is $K = K_f$, the set of points with bounded orbits. Pluripotential theory allows us to associate to this set the harmonic measure, $\mu = \mu_f$ of K_f. For polynomial diffeomorphisms this measure is finite and invariant, and we normalize it to have total mass one. In previous papers we have shown that this measure has considerable dynamical significance. We have shown that μ is ergodic [BS3] and that the support of μ is the closure of the set of periodic saddle orbits [BLS1]. Further, μ is the unique measure of maximal entropy [BLS1], and μ describes the distribution of periodic points [BLS2].

To any measure we can associate Lyapunov exponents. The rate of expansion and contraction of tangent vectors at a point p by f is measured by a pair of Lyapunov exponents, $\lambda^+(p)$ and $\lambda^-(p)$. In the presence of an ergodic invariant measure such as μ, these exponents are constant almost everywhere and we denote them by $\lambda^+(\mu)$ and $\lambda^-(\mu)$. By [BS3] the (complex) Lyapunov exponents of μ satisfy $\lambda^-(\mu) < 0 < \lambda^+(\mu)$. This condition is known as (nonuniform) hyperbolicity of the measure μ. Nonuniform hyperbolicity implies that at μ almost every point p there is a spitting of the tangent space into complex one-dimensional subspaces $E^u(p)$ and $E^s(p)$ so that for $v \in E^u(p)$ we have $||Df^n(v)|| \sim \exp(n\lambda^+)||v||$ and for $v \in E^s(p)$ we have $||Df^n(v)|| \sim \exp(n\lambda^-)||v||$. In this paper we will prove an integral formula for the Lyapunov exponents. In many ways our formula is analogous to the Brolin–Manning formula for Lyapunov exponents with respect to harmonic measure for polynomial maps of **C**, which we now describe.

Math Subject Classifications. 32H50, 58F23.
Key Words and Phrases. holomorphic dynamics, Lyapunov exponents, critical points.

ISSN 1050-6926

Let g be a polynomial map of $\mathbf{C}$. We let K_g denote the set of points with bounded orbits. We denote by $\mu = \mu_g$ the harmonic measure of K_g. There is a single Lyapunov exponent $\lambda(\mu)$ which gives the average rate of expansion along the orbit μ almost everywhere. The Green function of K is given by the following formula:

$$G(z) = \lim_{n \to \infty} d^{-n} \log^+ \left| g^n(z) \right| ,$$

which relates it to the superexponential rate at which an orbit escapes to infinity. The following Brolin–Manning formula relates the Lyapunov exponents to the critical c_j points of the map:

$$\lambda(\mu) = \log d + \sum G\left(c_j\right) . \tag{0.1}$$

The above formula was obtained in the case without critical points by Manning [Mn]; the present formulation appears in Przytycki [Pr] and Sibony [Si].

Formula (0.1) takes an especially simple form in the quadratic case, $g(z) = z^2 + c$. If we write G_c for the Green function of the Julia set of $g(z) = z^2 + c$, we have

$$\lambda(\mu) = \log 2 + G_c(0) .$$

Douady and Hubbard [DH] observed that $G_c(0) = \frac{1}{2}h(c)$, where $h(c)$ is the Green function of the Mandelbrot set. Thus, the Lyapunov exponent is connected to the potential theory of the parameter space. The idea of understanding the parameter space by means of potential theory sparked our interest in Lyapunov exponents. Some properties of the function $\Lambda(f) = \lambda^+(\mu)$ as a function on the parameter space of polynomial diffeomorphisms were studied in [BS3].

In this paper we do two things. First we define a notion of critical point and critical point measure for polynomial diffeomorphisms of $\mathbf{C}^2$ and explore the dynamical significance of these objects. Second, we use this measure to prove an integral formula which is the analog of (0.1).

One ingredient in our integral formula is a Green function. The function G has two analogs in $\mathbf{C}^2$:

$$G^\pm(x, y) = \lim_{n \to \infty} d^{-n} \log^+ \left| f^{\pm n}(x, y) \right| .$$

We write $K^\pm \subset \mathbf{C}^2$ for the set of points in $\mathbf{C}^2$ bounded in forward/backward time and we define $U^\pm$ to be the complement of $K^\pm$. The functions $G^\pm$ are zero on $K^\pm$ and pluriharmonic on $U^\pm$ and serve as Green functions. The function G^+ which describes the forward rate of escape is the analog of G.

The function f is a diffeomorphism and hence has no critical points in the usual sense of the word. For maps in one variable critical points of g with unbounded orbits are associated to critical points of the Green function G. This suggests that we look for critical points of G^+. Since ∇G^+ is non-zero at every point of U^+, G^+ has no critical points in the usual sense. In many situations the best analog of the set $\mathbf{C}$ for polynomial maps is not all of $\mathbf{C}^2$ but is rather the set K^- of points with bounded backward orbits. This suggests that we should look for critical points of G^+ restricted to K^-. We could make sense of this concept if K^- were a manifold. Now K^- is a rather wild set and in particular it is not a manifold. On the other hand, for μ almost every point in J the set $W^u(p)$ is a manifold and this manifold is contained in $J^- \subset K^-$. We define our set of unstable critical points, $\mathcal{C}^u$, to be the set of critical points of the restrictions $G^+|W^u(p)$.

These critical points have an interesting dynamical interpretation which does not make explicit reference to the function G^+. In the region U^+ points escape to infinity in forward time. In fact they escape at a super-exponential rate. In U^+ there is a plane field τ^+ such that for $v \in \tau^+$, $Df^n(v)$

decreases super-exponentially as $n \to \pm\infty$ (Lemma 1.2). We will show that a critical point in $W^u(p)$ as defined above is a point at which the tangent space of $W^u(p)$ concides with τ^+. In [HO] a holomorphic foliation $\mathcal{G}^\pm$ of $U^\pm$ was constructed. We will show in Proposition B.1 that the tangent space to a leaf of this foliaton is given by $\tau^\pm$, and the global leaves of $\mathcal{G}^\pm$ are super-stable/-unstable manifolds. Thus, critical points are points at which unstable manifolds $W^u(p)$ and super-stable manifolds intersect tangentially and we can think of such a critical point as a type of heteroclinic tangency.

The next ingredient in our integral formula is a critical measure μ_c^- supported on the set of critical points. In order to define this measure, we use the unstable current $\mu^- = \frac{1}{2\pi} dd^c G^+$. The unstable current μ^-, in some sense, serves as the current of integration on the unstable lamination $\mathcal{W}^u$. Locally, μ^- may be thought of as follows. There is a (pairwise disjoint) family of unstable disks D_t and a transversal measure on the space of parameters t; and μ^- is given locally as the current of integration over D_t, integrated with respect to the transverse measure. We construct the critical measure μ_c^- by replacing the current of integration over D_t by the sum of the point masses at the critical points of $G^+|D_t$.

Another way to approach the critical measure is to fix (arbitrarily) a vector α and a covector β. The variety

$$Z_k(\alpha, \beta) = \left\{ x : \beta \cdot Df_x^k(\alpha) = 0 \right\}$$

is the set of points x where Df_x^k maps the vector α to the kernel of β. If $M \subset U^+$ is a Riemann surface such that $G^+|_M$ is not locally constant, then by Lemma 5.2, $Z_k(\alpha, \beta) \cap M$ converges to the set of critical points of $G^+|_M$ as $k \to \infty$. The slice of the current μ^- by the variety $f^j Z_k(\alpha, \beta)$ is given by the wedge product $\mu^- \wedge [f^j Z_k(\alpha, \beta)]$. We show (Theorem 5.9) that the average (over α) of these slices gives the critical measure:

$$\mu_c^- = \lim_{\substack{j \to \infty \\ k-j \to \infty}} \int \sigma(\alpha)\, \mu^- \wedge \left[f^j Z_k(\alpha, \beta) \right] .$$

The set $\mathcal{C}^s$ of critical points in J^+ and the measure μ_c^+ can be defined in an analogous way.

The main result in this paper (see Theorem 6.1 and Corollary 6.6) is a formula for the Lyapunov exponents of harmonic measure:

Main Theorem.

$$\begin{aligned} \lambda^+(\mu) &= \log d + \int_{\{1 \le G^+ < d\}} G^+ \mu_c^- \\ \lambda^-(\mu) &= -\log d - \int_{\{1 \le G^- < d\}} G^- \mu_c^+ . \end{aligned} \tag{0.2}$$

□

(We omit the conventional "d" from in front of the measure in the integral in order to reduce confusion with the exterior derivative operator and with the degree.) The condition $\{1 \le G^+ < d\}$ in the formula for λ^+ has the effect of choosing a fundamental domain for the action of f on the set $\mathcal{C}^u$. This ensures that each orbit of critical points contributes only once. Other choices of fundamental domains work equally well. This is a consequence of the fact that the integrand $G^+\mu^-$ is invariant under f. The invariance of the integrand occurs because G^+ multiplies by a factor of d, and μ^- multiplies by a factor of d^{-1} under f. A geometrically appealing way of finding a fundamental domain arises naturally when f generates a real horseshoe mapping, and in this case the formula may be given in a particularly simple form; see Appendix A.

We will briefly describe some of the connections between Lyapunov exponents, dimension of harmonic measure, and connectivity for polynomial maps and polynomial diffeomorphisms.

The Hausdorff dimension of a measure is, by definition, the infimum of the Hausdorff dimensions of Borel sets with full measure. Let J be the Julia set of a polynomial map g of the complex plane. The Lyapunov exponent of g with respect to harmonic measure μ_J is related to the Hausdorff dimension of μ_J by the formula $\lambda(\mu)\mathrm{HD}(\mu) = \log d$. Formula (0.1) has the consequence that the Hausdorff dimension of the harmonic measure of the Julia set is at most one and is equal to one if and only if all critical points have bounded orbits. Thus, the Julia set is connected if and only if the harmonic measure has Hausdorff dimension one. (It is a general result of Makarov that the Hausdorff dimension of the harmonic measure of any connected set is one.)

For polynomial diffeomorphisms there is also a connection between exponents and certain planar sets. Let f be a polynomial diffeomorphism of $\mathbf{C}^2$, and let μ denote the corresponding harmonic measure. For μ almost every point p stable and unstable manifolds $W^{s/u}(p)$ exist and are complex manifolds conformally equivalent to $\mathbf{C}$. Given such a p we can consider the sets $W^u(p) \cap K^+$ and $W^s(p) \cap K^-$ (which can be viewed as subsets of the complex plane.) The "slice measures" $\mu^\pm|_{W^{s/u}(p)}$ play the role of harmonic measures for the sets $K^\pm \subset W^{u/s}$. In [BLS1] the slice measures were shown to satisfy the Ledrappier–Young [LY] formula

$$\lambda^+(\mu) = \frac{\log d}{\mathrm{HD}\left(\mu^-|_{W^u_{loc}(p)}\right)}, \qquad \lambda^-(\mu) = \frac{\log d}{\mathrm{HD}\left(\mu^+|_{W^s_{loc}(p)}\right)}.$$

We explore the relation between exponents, critical points, and connectivity for polynomial diffeomorphisms in [BS6].

Critical points play an important role in the dynamical study of polynomial maps. We have defined two sets of "critical points" namely $\mathcal{C}^s$ and $\mathcal{C}^u$, but there are other possible definitions that could be made. The critical points in $\mathcal{C}^s$ and $\mathcal{C}^u$ are points at which there is a vector v with the property that $Df^n(v)$ decreases in both forward and backward time. If we take this condition to be a characteristic of critical points, we can also ask about the set $\mathcal{C}$ of "critical points" for which both the forward and backward orbits are unbounded. These are points in $U^+ \cap U^-$ at which the super-stable and super-unstable foliations are tangent. (Such points of tangency were first considered by Hubbard.) For a given polynomial diffeomorphism f, either of the sets $\mathcal{C}^s$ or $\mathcal{C}^u$ may be empty, but we show in Proposition B.3 that the set $\mathcal{C}$ is never empty. In this paper we do not discuss the remaining case of "critical points" with bounded forward and backward orbits, which is more difficult (see [BC] and [BY]).

A question that arises for diffeomorphisms of $\mathbf{C}^2$ is the relation between stable and unstable critical points. The integral formula allows us to deduce (Propostion 6.9) that if f is dissipative, then $\mathcal{C}^s \neq \emptyset$, which by [BS6] is seen to have topological consequences.

Section 1 contains introductory material and an analysis of the growth of tangent vectors in U^+. In Section 2 we develop the laminarity of μ^+ using elementary methods. We show that for any algebraic variety X, the convergence of $cd^{-n}[f^nX]$ to μ^- induces the geometry of the laminar structure. In Section 3 we introduce results from Pesin theory concerning the stable/unstable manifolds with respect to the hyperbolic measure μ. We show (Lemma 3.3) that the laminarity of the convergence of f^nX also respects the laminar structure of the Pesin manifolds. In Section 4 we define alternative notions of average rates of growth and we relate these to the Lyapunov exponent.

In Section 5 we introduce unstable critical measure μ_c^- and establish some of its properties. In Section 6 we derive the integral formula.

1. Super-stable directions in U^+

The polynomial diffeomorphisms of $\mathbf{C}^2$ were classified up to conjugacy by Friedland and Milnor [FM], who introduced a degree d, which we call dynamical degree, and which is defined by the formula $d = \lim_{n\to\infty}(\deg(f^n))^{1/n}$. They showed that the mappings with interesting dynamics correspond to the case $d \geq 2$; and so, as in the one-dimensional case, we will consider only polynomial diffeomorphisms with $d \geq 2$.

The inverse of a polynomial diffeomorphism is again a polynomial diffeomorphism, and for this reason polynomial diffeomorphisms form a group and are often called polynomial automorphisms. Friedland and Milnor showed that a dynamically nontrivial mapping is conjugate in the group of polynomial diffeomorphisms to a mapping of the form $f = f_1 \circ \cdots \circ f_m$, where $f_j(x, y) = (y, p_j(y) - a_j x)$ and $p_j(y) = y^{d_j} + O(y^{d_j - 2})$, with $d_j \geq 2$. The algebraic degree of f is then $d = d_1 \cdots d_m$, and the iterates for $k \geq 1$ are given by

$$f^k(x, y) = \left(y^{d^k/d_1} + \cdots, y^{d^k} + \cdots\right) . \tag{1.1}$$

Thus for maps in this standard form the algebraic degree coincides with the dynamic degree. In particular it is an integer greater than or equal to 2.

Certain dynamical properties of f can be deduced simply from a consideration of the sizes of the coordinates. We recall the following standard notations and results.

$$V^+ = \{|y| \geq |x|, |y| \geq R\}, \quad V^- = \{|y| \leq |x|, |x| \geq R\}, \quad V = \{|x|, |y| \leq R\} . \tag{1.2}$$

There is an R sufficiently large that $fV^+ \subset V^+$, $fV \subset V \cup V^+$, and that for any point $(x, y) \in V^-$ the orbit $f^n(x, y)$ can remain in V^- only for finitely many $n > 0$. Further, if K^+ is the set of points with bounded forward orbits, then

$$U^+ = \mathbf{C}^2 - K^+ = \bigcup_{n=0}^{\infty} f^{-n} V^+, \quad \text{and} \quad U^- = \mathbf{C}^2 - K^- = \bigcup_{n=0}^{\infty} f^n V^- . \tag{1.3}$$

The rate of escape to infinity in forward/backward time:

$$G^{\pm} = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \left|f^{\pm n}(x, y)\right| \tag{1.4}$$

links potential theory and dynamics. The function $G^{\pm}$ is continuous and pluri-subharmonic on $\mathbf{C}^2$. We let π_1 and π_2 denote projections onto the first and second coordinates. So for a point $(x, y) \in V^+$, we have $\pi_2 f^n \sim (\pi_1 f^n)^{d_1}$ as $n \to +\infty$. Thus $|f^n| \sim |\pi_2 f^n|$ as $n \to +\infty$, and it follows that

$$G^+ = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \left|\pi_2 f^n(x, y)\right| .$$

In the remainder of this section we will analyze the growth rates of tangent vectors at points in U^+. For $(x, y) \in V^+$,

$$\left|\pi_2 f_1(x, y) - y^{d_1}\right| = O\left(y^{d_1 - 2}\right) + O(x) ,$$

where the O terms are uniform on V^+. Thus, the following estimate is uniform in $n \geq 0$ and $(x, y) \in V^+$:

$$\begin{aligned} \left|\pi_2 f^{n+1} - \left(\pi_2 f^n\right)^d\right| &\leq O\left(\left(\pi_2 f_2 \circ \cdots \circ f_m \circ f^n\right)^{d_1-2}\right) + O\left(\pi_1 f_2 \circ \cdots \circ f_m \circ f^n\right) \\ &\leq O\left(\left(\pi_2 f^n\right)^{(d_1-2)d_2 \cdots d_m}\right) + O\left(\left(\pi_2 f^n\right)^{d_2 \cdots d_m}\right) \\ &\leq O\left(\left(\pi_2 f^n\right)^{d-2}\right) . \end{aligned}$$

For $(x, y) \in V^+$ and any N, we have

$$G^+(x, y) = \frac{1}{d^N} \log\left|\pi_2 f^N\right| + \sum_{n=N}^{\infty} \left(\frac{1}{d^{n+1}} \log\left|\pi_2 f^{n+1}\right| - \frac{1}{d^n} \log\left|\pi_2 f^n\right|\right)$$

By the estimates above, the nth term in the series is dominated by

$$\begin{aligned} \frac{1}{d^{n+1}} \log\left|\frac{\pi_2 f^{n+1}}{(\pi_2 f^n)^d}\right| &\leq \frac{1}{d^{n+1}} \log\left(1 + \left|\frac{\pi_2 f^{n+1} - (\pi_2 f^n))^d}{(\pi_2 f^n)^d}\right|\right) \\ &\leq \frac{1}{d^{n+1}} \log\left(1 + \frac{O\left(|\pi_2 f^n|^{d-2}\right)}{|\pi_2 f^n|^d}\right) \\ &\leq \frac{C}{d^{n+1}\,|\pi_2 f^n|^2} . \end{aligned}$$

Summing the tail of the series, we conclude that

$$\left|G^+(x, y) - \frac{1}{d^N} \log\left|\pi_2 f^N(x, y)\right|\right| \leq \frac{C'}{d^N\left|\pi_2 f^N(x, y)\right|^2} \tag{1.5}$$

holds for all N and $(x, y) \in V^+$.

Remark. The rate of convergence of numerical approximations to G^+ and ∂G^+ can be understood by (1.5) and the estimates that precede it. Suppose that $f(x, y) = (y, y^d + q(y) - ax)$, where $\deg q \leq d - 2$. We set $y_{-1} = x$, $y_0 = y$, and $y_{n+1} = y_n^d + q(y_n) - a y_{n-1}$ for $n \geq 0$. We may approximate G^+ either by $d^{-N} \log|y_N|$ or by the telescoping sum

$$\log|y_k| + \sum_{n=k}^{N} d^{-n-1} \log\left|y_{n+1}/y_n^d\right| = \log|y_k| + \sum_{n=k}^{N} d^{-n-1} \log|1 + \rho_n| ,$$

where $\rho_n = (q(y_n) - a y_{n-1}) y_n^{-d} = O(y_n^{-2})$.

Similarly, we set $\partial y_{-1} = dx$, $\partial y_0 = dy$, and $\partial y_{n+1} = (d \cdot y_n^{d-1} + q'(y_n))\partial y_n - a\partial y_{n-1}$ for $n \geq 0$. Thus, we may approximate ∂G^+ by $(2d^N y_N)^{-1} \partial y_N$ or by the telescoping sum

$$\left(2d^k y_k\right)^{-1} \partial y_k + \sum_{n=k}^{N} d^{-n} \left((2d \cdot y_{n+1})^{-1} \partial y_{n+1} - (2y_n)^{-1} \partial y_n\right) ,$$

which, after cancellation, is

$$\frac{\partial y_k}{2d^k y_k} + \sum_{n=k}^{N} d^{-n} (1 + \rho_n)^{-1} \left(\frac{-\rho_n \partial y_n}{2y_n} + \frac{q'(y_n)\, \partial y_n}{2d \cdot y_n^d} - \frac{a \partial y_{n-1}}{2d \cdot y_n^d}\right)$$

so the nth term in the summation is no larger than $O(d^{-n}y_n^{-2})$.

For a tangent vector v, we will use the notation $\partial G \cdot v$ for the pairing with the 1-form ∂G, and $Df(v)$ for the action of the differential Df.

Lemma 1.1. *There exist C and R sufficiently large that*

$$\left| \partial G^+ \cdot v - \frac{1}{d^n} \frac{\partial(\pi_2 f^n) \cdot v}{|\pi_2 f^n|} \right| \le \frac{C|v|}{d^n |f^n|^2}$$

holds for all $n \ge 0$, all $(x, y) \in V^+$, and all tangent vectors $v \in T_{(x,y)}\mathbf{C}^2$.

Proof. We have estimate (1.5) in a neighborhood of fixed radius about any point of V^+. Since these functions are harmonic, we may differentiate this estimate and have the same estimate also for the gradients. □

The following gives a dichotomy on the growth rate of $Df^n(v)$. Either it grows super-exponentially to ∞ as $n \to \infty$, or it decreases to 0 super-exponentially.

Lemma 1.2. *If $(x, y) \in U^+$ and $v \in T_{(x,y)}\mathbf{C}^2$ is a vector with $\partial G^+ \cdot v = 0$, then there are constants c and N such that*

$$\left|Df^n(v)\right| \le \frac{c|v|}{|f^n|}. \tag{1.6}$$

for $n \ge N$. If $\partial G^+ \cdot v \ne 0$, then

$$\left|Df^n(v)\right| \sim d^n \left|f^n\right| \left|\partial G^+ \cdot v\right| . \tag{1.7}$$

Further, if $(x_0, y_0) \in U^+$ and $\epsilon > 0$ are given, then there exist small $\delta > 0$ and large N such that

$$\left|Df^n(v)\right| \ge \delta d^n \left|f^n\right| \left|\partial G^+ \cdot v\right| \tag{1.8}$$

holds in a δ neighborhood of (x_0, y_0) for all $n \ge N$ and all tangent vectors v such that $|\partial G^+ \cdot v| \ge \epsilon |v|$.

Proof. To make estimates, we may identify Df^n with the pair $(\partial(\pi_1 f^n), \partial(\pi_2 f^n))$. By (1.1) and (1.5) it is sufficient to estimate $\partial \pi_2 f^n$. Thus, (1.6) follows directly from Lemma 1.1. Again by Lemma 1.1, we estimate

$$\left|\partial\left(\pi_2 f^n\right) \cdot v\right| \ge d^n \left|\partial G^+ \cdot v\right| \left|\pi_2 f^n\right| - \frac{C|v|}{|f^n|},$$

which yields (1.7) and (1.8). □

For $(x, y) \in U^\pm$ we let $\tau^\pm(x, y)$ denote the subspace of $T_{(x,y)}\mathbf{C}^2$ annihilated by $\partial G^\pm$, i.e., such that $\partial G^\pm \cdot v = 0$ for all $v \in \tau^\pm$. We will refer to $\tau^\pm$ as the forward/backward dynamical critical directions. If $v \notin \tau^\pm$, then $|Df^n \cdot v|$ grows as $n \to \pm\infty$ at the rate given in Lemma 1.2.

Corollary 1.3. *If $v \in \tau^\pm$, then $\lim_{n \to \pm\infty} \frac{1}{|n|} \log |Df^n(v)| = -\infty$.*

Proof. By (1.6),

$$\frac{1}{|n|} \log \left|Df^n(v)\right| \le \frac{1}{|n|} \left(\log |cv| - \log \left|f^n\right|\right)$$

which tends to $-\infty$ as $n \to \pm\infty$, since $|f^n| \sim e^{d^{|n|}G^\pm}$. □

Next we consider another way to measure minimal growth of Df^n. Let us fix $n \in \mathbf{Z}$ and $(x, y) \in \mathbf{C}^2$ and consider the mapping

$$T_{(x,y)}\mathbf{C}^2 \ni t \mapsto \frac{|Df^n \cdot t|}{|t|} . \tag{1.9}$$

We let $\tau_n(x, y)$ denote the subspace of $T_{(x,y)}\mathbf{C}^2$ on which this mapping is minimized.

Proposition 1.4. *$\lim_{n\to+\infty} \tau_n = \tau^+$ on U^+ and $\lim_{n\to-\infty} \tau_n = \tau^-$ on U^-.*

Proof. Let us suppose that there are vectors $v_{n_j} \in \tau_{n_j}$ which stay at positive angle from τ^+. Then there is an $\epsilon > 0$ such that $|\partial G^+ \cdot v_{n_j}| \geq \epsilon |v_{n_j}|$. By (1.8) of Lemma 1.2, $|Df^{n_j}|$ grows as $j \to \infty$. On the other hand, if $v^+ \in \tau^+$, then $Df^{n_j} v^+$ decreases to 0 as $j \to \infty$. Thus for some large j, v_{n_j} does not minimize (1.9), which is a contradiction. □

2. Laminar properties of the stable/unstable currents

In this section and the next we will discuss the laminar properties of the currents $\mu^\pm$. Laminarity is a "natural" structure for $\mu^\pm$ and has been the key for understanding the deeper properties of $\mu^\pm$ and μ. It will also be central to the definition of the critical measure. In this section we describe an explicit approach to laminarity which will be useful in Section 5. In Section 3 we describe an alternate approach to laminarity via the Pesin theory. Although we will work only with μ^-, it is evident that the analogous properties hold for μ^+.

Let us summarize some notation and terminology about currents. More details are given in [BLS1]. We let $\mathcal{D}_k$ denote the set of compactly supported k-forms (test forms). The dual space $\mathcal{D}'_k$ is the set of k-dimensional currents. A sequence $\{T_n\}$ converges in the sense of currents if $\lim_{n\to\infty} T_n(\varphi) = T(\varphi)$ for every test form $\varphi \in \mathcal{D}_k$. If X is a k-dimensional submanifold with locally finite k-dimensional area, then there is the current of integration $[X] \in \mathcal{D}'_k$, whose action on a test form is given by

$$[X](\varphi) := \int_X \varphi .$$

If S is a discrete (0-dimensional) set, then the current of integration

$$[S] = \sum_{a\in S} \delta_a$$

is the sum of point masses at S. It will be useful for us to define the mass norm of a current as

$$\mathbf{M}[T] = \sup_{|\varphi|\leq 1} |T(\varphi)| ,$$

where $|\varphi| := \sup_x |\varphi(x)|$ is the Euclidean supremum norm of a test form φ. The mass norm of T is finite if and only if T may be represented as a linear combination of k-vectors with coefficients which are finite measures.

If φ is a smooth k-form, we define $T \llcorner \varphi$ by $(T \llcorner \varphi)(\psi) = T(\varphi \wedge \psi)$. If S is a Borel set, $T \llcorner S$ will denote the restriction of T to S, i.e., $T \llcorner \chi_S$, where χ_S is the characteristic function of S. We may do this whenever the mass norm of T is locally finite.

While the stable/unstable currents $\mu^\pm := \frac{1}{2\pi} dd^c G^\pm$ are defined as positive, closed, (1,1)-currents, they also have special properties not enjoyed by general currents, and in fact it is these properties that are the most useful for studying the dynamical properties of $\mu^\pm$. If M is a 1-dimensional complex submanifold of $\mathbf{C}^2$, then $\mu^\pm$ induce measures on M, given by

$$\mu^\pm|_M = \frac{1}{2\pi}(dd^c)|_M\left(G^\pm|_M\right)$$

where $(dd^c)_M$ is the induced operator on M.

If ν is a measure on the space A, and ψ is an integrable function on A, then we denote the integral of ψ with respect to ν as $\int \psi(a)\nu(a)$ or $\int \nu(a)\psi(a)$. If $\{T_a : a \in A\}$ is a measurable family of currents, we define the (direct) integral $\int \nu(a)T_a$ by its action on a test form φ by

$$\left(\int_{a\in A}\nu(a)\,T_a\right)(\varphi) := \int T_a(\varphi)\,\nu(a)\,.$$

A current is laminar if it can be written as a direct integral of T_a as above, with the T_a being currents of integration over pairwise disjoint complex manifolds.

Our derivation of the laminar structure of μ^- will be based on the following characterization of μ^-. Let $X \subset \mathbf{C}^2$ be an algebraic variety of pure dimension 1. By [BS1, Proposition 4.2] there are positive integers n_0 and k such that f^nX has degree kd^{n-n_0} for $n \ge n_0$, and

$$\lim_{n\to\infty}\frac{1}{kd^{n-n_0}}\left[f^nX\right] = \mu^-\,. \tag{2.1}$$

All the constructions in this section will depend on the variety X and the projection $\pi_\alpha(x, y) = \alpha_1 x + \alpha_2 y$ for some choice of $\alpha \in \mathbf{C}^2$ with $|\alpha_1|^2 + |\alpha_2|^2 = 1$. We may rotate corrdinates on $\mathbf{C}^2$ so that $\pi_\alpha(x, y) = y$, and we will not include the choice of X or α explicitly in our notations.

Let $\mathcal{Q}_s$ denote the set of squares in the plane with side d^{-s} and with vertices on points of the set $d^{-s}(\mathbf{Z} + i\mathbf{Z})$. Each square $Q \in \mathcal{Q}_s$ will be half-open, i.e., $Q = [a, b) \times [c, d)$, so that $\mathcal{Q}_s$ is a partition of $\mathbf{C}$. We choose $\kappa > 0$ and let $\mathcal{Q}'_s$ denote the set of squares Q' which have side of length $(1 + 2\kappa)d^{-s}$ and which are centered about squares Q of $\mathcal{Q}_s$. There is a number $m(\kappa)$ such that each point of $\mathbf{C}$ is contained in at most $m(\kappa)$ squares of $\mathcal{Q}'_s$. We let Q_0 denote a fixed square of $\mathcal{Q}_0$.

Let $Q \subset Q_0$ be any square from $\mathcal{Q}_s$, and let $Q' \in \mathcal{Q}'_s$ denote the square centered about it. A connected component Γ' of $f^nX \cap \pi^{-1}Q'$ will be said to be *good* if the projection $\pi|_{\Gamma'} : \Gamma' \to Q'$ is a homeomorphism. We let

$$\mathcal{G}(Q, n) = \left\{\Gamma' \cap \pi^{-1}Q : \Gamma' \text{ good}\right\}\,.$$

Let us define

$$\mu^-_{\mathcal{Q}_s,n} = \frac{1}{kd^{n-n_0}}\sum_{Q\in\mathcal{Q}_s}\sum_{\Gamma\in\mathcal{G}(Q,n)}[\Gamma]\,.$$

For fixed $Q_0 \in \mathcal{Q}_0$, there is a number $R > 0$ such that

$$f^nX \cap \pi^{-1}Q'_0 \subset \{|x| < R\} \tag{2.2}$$

for all $n \ge n_0$.

Lemma 2.1. *There is a constant C, independent of $Q \subset Q_0$, s and n, such that*

$$\mathbf{M}\left[\left(k^{-1}d^{n_0-n}\left[f^nX\right] - \mu^-_{\mathcal{Q}_s,n}\right) \llcorner \pi^{-1}Q_0\right] \le C\,\mathrm{Area}(Q)\left[1 - k^{-1}d^{n_0-n}\right]m(\kappa)\,. \tag{2.3}$$

Proof. The number of components of $f^n X \cap \pi^{-1} Q'$ is no more than kd^{n-n_0}, the degree of $f^n X$. If Γ' is not good, then the number of branch points, counted with multiplicity, in Γ' is one less than the mapping degree of $\pi|_{\Gamma'}$. Thus, the sum of the mapping degrees of components that are not good is bounded above by $(kd^{n-n_0} - 1)m(\kappa)$.

Now we need to estimate $k^{-1}d^{n_0-n}$ times the area of the components that are not good. A property of analytic varieties (see Chirka [C]) is that there is a constant C depending on the R of (2.2) and κ such that the area of every component Γ' of $f^n X \cap \pi^{-1} Q'$ is bounded by

$$\mathrm{Area}(\Gamma) \le C\mu\,\mathrm{Area}(Q)$$

where μ is the mapping degree of $\pi|_{\Gamma'}$. Multiplying by d^{-2s} the area of Q, we estimate the mass in the left-hand side of (2.3) by the total of the mapping degrees coming from bad disks. □

For each good Γ' there is an analytic function $\varphi : Q' \to \mathbf{C}$ such that $\{(\varphi(y), y) : y \in Q'\} = \Gamma'$. Let $\mathcal{A}(Q, n)$ denote the set of all such analytic functions.

Let us define

$$S(Q, y, n) = \bigcup_{\varphi\in\mathcal{A}(Q,n)} \{(\varphi(y), y)\} = \bigcup_{\Gamma\in\mathcal{G}(Q,n)} \Gamma \cap \pi^{-1}(y).$$

The measures $\nu_Q(y, n) := k^{-1}d^{n_0-n}[S(Q, y, n)]$ are the slice measures of $\mu^-_{Q,n}$ with respect to the projection π. That is, $\nu_Q(y, n)$ is supported on $\pi^{-1}(y)$, and

$$\mu^-_{Q,n} \llcorner \left(\frac{i}{2} dy \wedge d\bar{y}\right) = \int_{y\in Q} \mathcal{L}^2(y)\, \nu_Q(y, n)$$

where $\mathcal{L}^2$ denotes the Lebesgue area measure on Q. Since the masses of the currents $\mu^-_{Q,n}$ are uniformly bounded by (2.2), we may choose a subsequence $\{n_j\}$ so that

$$\mu^-_Q := \lim_{j\to\infty} \mu^-_{Q,n_j} \tag{2.4}$$

exists.

Lemma 2.2. *The limit $\nu_Q(y) := \lim_{j\to\infty} \nu_Q(y, n_j)$ exists for every $y \in Q$.*

Proof. Each $\nu_Q(y, n)$ is a positive measure of mass at most one. If $\lim_{j\to\infty} \nu_Q(y, n_j)$ does not exist, then there exist subsequences $\{n_j^{(k)}\}$, $k = 1, 2$ of $\{n_j\}$ with distinct limiting measures $\nu_Q^{(k)} = \lim_{j\to\infty} \nu_Q(y, n_j^{(k)})$, $k = 1, 2$. We may assume that there is a function φ with

$$\left|\int \varphi\left(\nu_Q^1 - \nu_Q^2\right)\right| > \epsilon .$$

By the Cauchy estimate and (2.2), we have $|\psi'(y)| \le R/\kappa$ on Q for all $\psi \in \mathcal{G}(Q, n)$. Thus for any n,

$$\left|\int \varphi\left(\nu_Q(y_1, n) - \nu_Q(y_2, n)\right)\right| \le \sup\left\{ |\varphi(a_1) - \varphi(a_2)| : |a_1|, |a_2| \le R, |a_1 - a_2| \le \frac{R}{\kappa}|y_1 - y_2| \right\}. \tag{2.5}$$

Now we choose δ such that $\sup\{|\varphi(a_1) - \varphi(a_2)| : |a_1|, |a_2| \le R, |a_1 - a_2| < R\delta/\kappa\} < \epsilon/2$. Since the limit in (2.4) exists, we have the limit $\nu_Q(\hat{y}) = \lim_{j\to\infty} \nu_Q(\hat{y}, n_j)$ for almost every $\hat{y} \in Q$. Thus, we may choose $\hat{y} \in Q$ such that this limit exists and such that $|\hat{y} - y| < \delta$. It follows from (2.5) that

$$\left|\int \varphi\left(\nu_Q^{(k)} - \nu_Q\left(\hat{y}, n_j\right)\right)\right| < \epsilon/2$$

for j sufficiently large. This is a contradiction, which proves the lemma. □

Let us set $S(Q) :=$ supp $\nu_Q(c_Q)$, and let $\mathcal{A}(Q)$ denote the set of analytic functions $\varphi : Q' \to \mathbf{C}$ such that $\varphi(c_Q) \in S(Q)$, and there is a sequence of functions $\varphi_{n_j} \in \mathcal{A}(Q, n_j)$ converging to φ. Since distinct good components must be disjoint, we have $\varphi_1(y) \ne \varphi_2(y)$ for all $y \in Q'$, it follows that with the R of (2.2)

$$h(y) = \log\left(|2R|^{-1}\, |\varphi_1(y) - \varphi_2(y)|\right)$$

is a negative function on Q'. By the Harnack inequality, there is a constant independent of s, n, and Q such that

$$h(y) \le \text{const.}\, h\left(c_Q\right) \quad \text{for } y \in Q\,,$$

where c_Q denotes the center of the square Q. We conclude that there are constants R and κ (independent of s, n, and Q) such that

$$|\varphi_1(y) - \varphi_2(y)| \le R\left|\varphi_1\left(c_Q\right) - \varphi_2\left(c_Q\right)\right|^{\kappa} \tag{2.6}$$

for all $y \in Q$.

The following may be interpreted as a normal families argument for sets of functions satisfying (2.6).

Lemma 2.3. *$\mathcal{A}(Q)$ has the following properties:*

(1) For each $t \in S(Q)$ there is a unique $\varphi \in \mathcal{A}(Q)$ with $\varphi(c_Q) = t$.
(2) If $\varphi_1, \varphi_2 \in \mathcal{A}(Q)$ satisfy $\varphi_1(c_Q) \ne \varphi_2(c_Q)$, then $\varphi_1(y) \ne \varphi_2(y)$ for all $y \in Q$.
(3) For any $\epsilon > 0$, there exists J and $\delta > 0$ such that if $j \ge J$, $\varphi_1, \varphi_2 \in \mathcal{A}(Q, n_j) \cup \mathcal{A}(Q)$ satisfy $|\varphi_1(c_Q) - \varphi_2(c_Q)| < \delta$, then $||\varphi_1 - \varphi_2||_Q < \epsilon$.

Proof. We will prove (1); the assertion (2) follows from the Hurwitz Theorem, and (3) then follows from (2.6). Let us suppose that there are distinct functions φ_1 and $\varphi_2 \in \mathcal{A}(Q)$ with $\varphi_1(c_Q) = \varphi_2(c_Q)$. By By [BLS1, Lemma 6.4], we may move the point $y = c_Q$, if necessary, to have $\varphi_1'(c_Q) \ne \varphi_2'(c_Q)$. Let us write $t^{(k)} = \varphi_k'(c_Q)$, for $k = 1, 2$ and set $\epsilon = |t^{(1)} - t^{(2)}|$.

Let $\{n_j^{(k)}\}$ denote the subsequence of $\{n_j\}$ which produced φ_k. Now by (2.6) it follows that there is a neighborhood U of $(\varphi_1(c_Q), c_Q)$ and a large number J such that if $j \ge J$, then for any graph Γ from $\mathcal{A}(Q, n_j^{(k)})$, the slope of Γ is within $\epsilon/2$ of $t^{(k)}$ at all points of $U \cap \Gamma$. But this is a contradiction, for if we write μ_Q^- in polar form, as a tangent 2-vector times a measure, then on U the tangent vector must be within $\epsilon/2$ of both $t^{(1)}$ and $t^{(2)}$. □

Passing to further subsequences, we may assume that $\mathcal{A}(Q_1) \subset \mathcal{A}(Q_2)$ if $Q_1 \in \mathcal{G}_{s_1}$, $Q_2 \in \mathcal{G}_{s_2}$, and $Q_1 \supset Q_2$. Thus, if we write $\mu_{\mathcal{Q}_s}^- = \sum_{Q\in\mathcal{Q}_s} \mu_Q^-$, then $\mu_{\mathcal{Q}_s}^- \le \mu_{\mathcal{Q}_{s+1}}^-$.

Theorem 2.4. *The currents $\mu_{\mathcal{Q}_s}^-$ increase to μ^- as $s \to \infty$. Further each μ_Q^- has a uniform laminar structure given by $\mu_Q^- = \int_{a\in S(Q)} [\Gamma_a]\, \nu_Q(a)$*

Proof. We have already that $\mu_{Q,n}^- = \int_{a\in S(Q,n)}[\Gamma_a]\,\nu_Q(y,n)$ for any $y \in Q$. Now as $j \to \infty$ we have $\nu_Q(y,n_j) \to \nu_Q$ by Lemma 2.2 and $\mathcal{A}(Q,n_j) \to \mathcal{A}(Q)$ by Lemma 2.3, and thus the integral representations converge. This proves that μ_Q^- has the uniform laminar structure. We know that $k^{-1}d^{n_0-n}[f^n X]$ converges to μ^- and $\mu_{Q,n}^-$ converges to μ_Q^- as $n \to \infty$. Thus, the inequality $\mu_{Q,n}^- \le k^{-1}d^{n_0-n}[f^n X]$ yields $\mu_Q^- \le \mu^-$ and thus $\mu_{Q_s}^- \le \mu^-$. Similarly, the estimate in Lemma 2.1 converges to $\mathbf{M}[(\mu^- - \mu_{Q_s}^-)\llcorner\pi^{-1}Q_0] \le Cd^{-2s}m(\kappa)$. Thus, $\lim_{s\to\infty}\mu_{Q_s}^- = \mu^-$. □

Our derivation of laminar structure up to this point has relied on the fact that, as a current, μ^- has complex dimension 1, and thus sets of area zero inside each leaf are invisible from the point of view of μ^-. For the purpose of defining the critical measure, we will need to know that this laminar structure actually has leaves which are "complete," i.e., conformally equivalent to $\mathbf{C}$, since the critical points occur on a discrete subset of the leaf. This will be done in Section 3.

3. Pesin-theoretic properties of the stable/unstable currents

We discuss some results from smooth ergodic theory that we will apply to the structure of the currents $\mu^\pm$ and the measure μ. This includes the existence of Lyapunov exponents and the Pesin theory for stable/unstable manifolds. These dynamical methods lead us again to "laminar" properties of the stable/unstable currents $\mu^\pm$ with respect to the Pesin stable/unstable manifolds. In Lemma 3.3 we show that almost every leaf in the laminar structure obtained in Theorem 2.4 already contains the Pesin unstable manifolds.

We define Lyapunov exponents for an ergodic measure μ for a diffeomorphism in dimension 2. By the Oseledets Theorem, there is a measurable, f-invariant complex splitting $E_x^s \oplus E_x^u$ of the tangent space for μ almost every point x, and there exist numbers $\lambda^s \le \lambda^u$, such that the limits

$$\lambda^s = \lim_{k\to\pm\infty}\frac{1}{k}\log\left|Df^k\Big|_{E^s}\right|, \qquad \lambda^u = \lim_{k\to\pm\infty}\frac{1}{k}\log\left|Df^k\Big|_{E^u}\right| \tag{3.1}$$

exist. In particular, the matrix norm satisfies

$$\lim_{k\to+\infty}\frac{1}{k}\log\left\|Df_x^k\right\| = \lambda^u$$

for almost every x.

In [BS3] we showed that the Lyapunov exponents of the invariant measure μ satisfy $\lambda^s \le -\log d < 0 < \log d \le \lambda^u$; since these are nonzero, μ is a *hyperbolic measure*. In the following, we may assume more generally that μ is a hyperbolic measure of saddle type: i.e., the Lyapunov exponents satisfy $\lambda^s < 0 < \lambda^u$.

Let us recall the set $\mathcal{R}$ of Oseledets regular points. General references for the Oseledets Theorem and the Pesin Theory are Pugh and Shub [Ps] and Pollicott [Po]. A point x belongs to $\mathcal{R}$ if for each $\epsilon > 0$, there is a constant $\gamma_{x,\epsilon} > 0$ such that

$$\left|\lambda_x^{k,s}\right| = \left|Df^k\Big|_{E_x^s}\right| \le \gamma_{x,\epsilon}e^{k(\lambda_s+\epsilon)} \tag{3.2}$$

$$\left|\lambda_x^{k,u}\right| = \left|Df^{-k}\Big|_{E_x^u}\right| \le \gamma_{x,\epsilon}e^{-k(\lambda_u-\epsilon)} \tag{3.3}$$

$$\angle\left(E_{f^k x}^s, E_{f^k x}^u\right) \ge \gamma_{x,\epsilon}^{-1}e^{-|k|\epsilon}\,. \tag{3.4}$$

By the Oseledets Theorem, $\mathcal{R}$ is a Borel set of full μ measure. This means that we have strict contraction in the inequalities (3.2) and (3.3) if ϵ is small.

We note that the mapping f is said to be *uniformly hyperbolic* if inequalities (3.2) and (3.3) hold for some uniform constants γe^{-kc}, independent of the point $x \in J$. It follows in the uniform case that the angle is bounded below, independently of k and x. In general, uniformly hyperbolic diffeomorphisms are quite well behaved.

A result of the Pesin theory is that for each regular point $x \in \mathcal{R}$ the set

$$\begin{aligned} W^s(x) &= \left\{q \in \mathbf{C}^2 : \lim_{n\to\infty} \operatorname{dist}(f^n q, f^n x) = 0\right\} \\ &= \left\{q \in \mathbf{C}^2 : \lim_{n\to\infty} \frac{1}{n} \log \operatorname{dist}\left(f^n q, f^n x\right) = \lambda^s\right\} \end{aligned}$$

is a 2-dimensional imbedded submanifold. In the complex case, $W^s(x)$ is a complex manifold (Riemann surface). For μ almost every $x \in \mathcal{R}$ the manifold $W^s(x)$ is conformally equivalent to $\mathbf{C}$ (see [BLS1, Proposition 2.6] or [W]).

Let us consider a coordinate chart $\psi : U \to \Delta^2 = \{|x|, |y| < 1\}$ for some open set $U \subset \mathbf{C}^2$, and let us work on Δ^2. An analytic graph $T = \{x = g(y) : y \in \Delta\}$ will be called a vertical transversal; we define a horizontal transversal similarly. We will define a *stable box* B^s (with respect to Δ^2) to be a union of components Γ of $W^s(x) \cap \Delta^2$ for $x \in \mathcal{R}$ such that Γ is a horizontal transversal to Δ^2. Thus for any vertical transversal $T \subset \Delta^2$ there is a set $E \subset T$ such that $B^s = \bigcup_{t\in E} \Gamma^s_t$, where Γ^s_t is a horizontal transversal such that $\Gamma^s_t \subset W^s(x)$ for some $x \in \mathcal{R}$, and the point t is defined by $\{t\} = \Gamma^s_t \cap T$. It follows that distinct Γ^s_t are pairwise disjoint, and $t \mapsto \Gamma^s_t$ is continuous. We define an unstable box $B^u = \bigcup_{t\in E^u} \Gamma^u_t$ in a similar fashion, with the unstable disks taken to be vertical transversals.

If B^s and B^u are stable and unstable boxes in the same coordinate neighborhood Δ^2, then the intersection $B = B^s \cap B^u$ is called a *Pesin box*. The stable and unstable manifolds give B the structure of a topological product. By [BLS1, Theorem 4.7] the restriction $\mu \llcorner B$ is the product measure $\tau^s \otimes \tau^u$ with respect to this topological product structure.

By the Pesin theory, $\mathcal{R}$ may be covered, up to a set of μ measure zero, by a countable family of Pesin boxes $\{B_j\}$. Thus for a.e. x there exists $\epsilon(x) > 0$ such that $W^s_{loc}(x, \epsilon(x))$ is contained in the unstable box B^u_j associated with the Pesin box B_j. We let $\tilde{\mathcal{R}}$ denote the points $x \in \mathcal{R}$ such that

$$W^s(x) \subset \bigcup_{n\leq 0} f^n \left(\bigcup_j B^u_j\right).$$

The following allows us to ignore the subset of $\mathcal{W}^s$ which is not covered by stable boxes.

Proposition 3.1. *$\tilde{\mathcal{R}}$ is an f-invariant set of full μ measure. Further,*

$$\bigcup_{x\in\mathcal{R}} W^s(x) - \bigcup_{x\in\tilde{\mathcal{R}}} W^s(x)$$

has zero measure for every slice measure $\mu^+|_T$. (And thus this set has zero $|\mu^+|$-measure.)

Proof. Almost every point x is contained in a stable box B^s, and there is a number $r(x)$ such that the stable leaf in B^s containing x is the graph over a Euclidean disk of radius $r(x)$ and centered at x.

For $0 < \epsilon$ and $C < \infty$, we let $S(\epsilon, C)$ denote the set of points x such that $r(x) \geq \epsilon$ and (3.2) through (3.4) hold for $\gamma_{x,\epsilon} \leq C$. By choosing C large and ϵ small, we have $\mu(S(\epsilon, C)) > 0$. By Poincaré recurrence, almost every x has the property that $f^{n_j}x \in S(\epsilon, C)$ for infinitely many $n_j \to \infty$. Let x be such a point and set $x_j = f^{n_j}(x)$. Without loss of generality, we may assume that $\epsilon = 1$. Let D_j denote a copy of the unit disk, and let $\chi_j : D_j \to W^u(x_j)$ be a conformal coordinate chart with $\chi_j(0) = x_j$ which expresses the local stable manifold as a graph over D_j in coordinates such that the graph is flat to first order over the origin.

Now we have a family of germs of conformal mappings $\varphi_j : D_j \to D_{j+1}$ of a neighborhood of the origin which satisfy $\varphi_j = \chi_{j+1}^{-1} \circ f^{n_{j+1}-n_j} \circ \chi_j$. / Thus, $\varphi_j(0) = 0$ and $|\varphi'(0)| = \left|Df^{n_{j+1}-n_j}|_{E^s(x_j))}\right|$.

Given $0 < \rho < 1$, we choose κ such that $\kappa < (1-\rho)^2/2$. We may pass to a subsequence of $\{n_j\}$ so that $n_{j+1} - n_j \to \infty$ arbitrarily fast. By (3.3) we may assume that $|\varphi_j'(0)| \leq \kappa$ for each j. We let $D_{j,\rho}$ denote the disk of radius $\rho < 1$ inside D_j. For $R > 0$ sufficiently small, φ_j is defined on $D_{j,R}$, and by the Distortion Theorem in one complex variable, the image $\varphi_j(D_{j,\rho R})$ is contained in the disk of radius $\rho R|\varphi_j'(0)|/(1-\rho)^2$. By the choice of κ, it follows that φ_j extends to all of $D_{j,\rho}$, and $\varphi_j(D_{j,\rho})$ is contained in the disk of radius $\rho|\varphi_j'(0)|/(1-\rho)^2$. This number is less than $\rho/2$, and so the modulus of the annulus $D_{j+1,\rho}$ minus the closure of $\varphi_j(D_{j,\rho})$ is at least $\log 2$.

Now we define

$$W := \bigcup_{j=1}^{\infty} f^{-n_j} D_j \subset W^u(x) .$$

It follows that W is the increasing union of annuli of moduli at least $\log 2$, and so W is conformally equivalent to $\mathbf{C}$. Thus, $W = W^s(x)$. It follows that $x \in \tilde{\mathcal{R}}$, and so the μ measure of $\mathcal{R} - \tilde{\mathcal{R}}$ is zero.

The statement concerning the slice measures $\mu^+|_T$ follows because μ has a local product structure, with the factors given by the stable slices $\mu^+|_{T'}$ and the unstable slices $\mu^-|_{T''}$. □

If B^s is a stable box in the bidisk Δ^2, and if T is a vertical transversal to Δ^2, then the restriction $\mu^+|_T \llcorner (T \cap B^s)$ of the induced measure to $T \cap B^s$ will be called a transversal measure. For two vertical transversals T_1 and T_2 of Δ^2, there is a homeomorphism $\chi : T_1 \cap B^s \to T_2 \cap B^s$ obtained by following an intersection point $t_1 = T_1 \cap \Gamma_t$ along the graph of a stratum Γ_t to the intersection point $T_2 \cap \Gamma_t$. By [BLS1, Theorem 4.5], χ preserves the set of transversal measures:

$$(\chi)_* \left(\mu^+\big|_{T_1} \llcorner \left(B^s \cap T_1\right)\right) = \mu^+\big|_{T_2} \llcorner (B \cap T_2) \ . \tag{3.5}$$

If $B^s = \{\Gamma_t : t \in E\}$ is a stable box, then in [BLS1] the restriction of μ^+ to B^s was shown to be equal to

$$\mu^+ \llcorner B^s = \int \mu_\tau^+(t) \left[\Gamma_t^s\right] \tag{3.6}$$

where μ_τ^+ is any transversal measure. Likewise, for an unstable box B^u, we have a similar representation for $\mu^- \llcorner B^u$. The transformation rule $f_*\mu^+ = d^{-1}\mu^+$ corresponds to the fact that the push-forward under f_* of a transversal measure is $1/d$ times another transversal measure.

We may define a wedge product $dd^cU \wedge T$ for any bounded, continuous psh function U and positive, closed current T, where if ξ is any test form, the product $U\xi$ is a compactly supported form with continuous coefficients, so we may set

$$dd^cU \wedge T(\xi) := T\left(U dd^c \xi\right)$$

(see [BC] for further discussion of this wedge operation on currents). A related operation is the intersection product, $[Z_1]\wedge[Z_2]$, which gives the current of integration over the intersection $[Z_1\cap Z_2]$. By integration with respect to the transversal measure, we may define an intersection wedge product $\dot\wedge$ of a current of the form (3.6) and a current of integration $[Z]$. In [BLS1, Lemma 8.3] it was shown that if Z is a complex variety, and if μ^+ has the form (3.6), then these two notions of wedge product coincide, i.e.,

$$\left(\mu^+ \llcorner B\right) \wedge [Z] = \int \mu_\tau^+(t)\ [\Gamma_t \cap Z]\ . \tag{3.7}$$

Because of this, we will use intersection products whenever it is convenient, but we will just use the notation $\wedge$.

Proposition 3.2. *There are countably many unstable boxes* B_j^u *such that the splitting* $E^s \oplus E^u$ *extends continuously to* B_j^u*, there is a constant* C_j *such that (3.3) and (3.4) hold on* B_j^u *with* $\gamma_{x,\epsilon} \le C_j$ *for* $x \in B_j^u$*, and such that for any complex manifold* T*,* $T \cap \bigcup_{j,n=1}^{\infty} f^n B_j^u$ *has full measure for the slice measure* μ_T^-*.*

Proof. Let $\{B_j^u\}$ be a family of unstable boxes as in Proposition 3.1. We may choose stable boxes B_j^s such that $B_j = B_j^s \cap B_j^u$ is a Pesin box, and $E^{s/u}$ extends continuously to B_j^u and (3.3) and (3.4) hold. □

Another consequence of the Pesin theory is that there is a measurable family of Lyapunov charts. This means that almost every x is the center of a (complex) affine image $L(x)$ of a bidisk Δ^2, and there is a product metric on $L(x)$ which is strictly expanded/contracted under f (see [Pr]). (We call $L(x)$ a topological bidisk in [BLS2].) If X is a complex variety, the cutoff image of X under f, i.e., $f(X \cap L(x)) \cap L(fx)$ is stretched across $L(fx)$ in the unstable direction. In fact, if $X \cap L(x)$ intersects $W_{loc}^s(x)$ transversally at x, then there is a number $N(x)$ such that if $m \ge N(x)$, then after m stretchings and cuttings-off, we have an unstable transversal to $L(f^m x)$, i.e.,

$$f^m(X \cap L(x)) \cap f^{m-1}L(fx) \cap \cdots \cap L\left(f^m x\right) \tag{3.8}$$

is an unstable transversal to $L(f^m x)$.

Let us take a countable family of Pesin boxes B_j whose union has full measure and which has the property that the constant $\gamma_{x,\epsilon}$ in (3.2) through (3.4) satisfies $\gamma_{x,\epsilon} \le C_j$ for $x \in B_j$. Further, we may assume that the inner radius of $L(x)$ is bounded below by $r_0 > 0$ for all $x \in B_j$. Further, we may assume that the axes of the bidisk $L(x)$ are almost constant for $x \in B_j$, and we may assume that the projection π is transversal to the unstable direction, i.e., $\pi^{-1}(0)$ makes a positive angle with the unstable axis of $L(x)$. Shrinking B_j if necessary, we may assume that there is a square Q_j with $B_j \subset \pi^{-1}Q_j$, and such that $\pi^{-1}q \cap L(x)$ is a vertical transversal of $L(x)$ for all $x \in B_j$ and $q \in \bar{Q}_j$. Finally, $W_{loc}^u(x) \cap L(x)$ is an unstable transversal to $L(x)$, so we may assume that for each stratum Γ of B_j^u, Γ crosses $\pi^{-1}Q_j$ properly, i.e., the restriction of π from $\Gamma \cap \pi^{-1}Q_j$ to Q_j is a homeomorphism.

Lemma 3.3. *There are countably many Pesin boxes* $\{B_j\}$ *such that* $\bigcup B_j$ *has full* μ *measure, and for each* B_j *there is a square* $Q_j \subset \mathbf{C}$ *such that the associated unstable box* B_j^u *satisfies*

$$\mu_{Q_j}^- \ge \mu^- \llcorner \left(B_j^u \cap \pi^{-1} Q_j\right)\ .$$

Proof. We take B_j and Q_j as in the discussion above. We note that we may take B_j^u such that for each stratum Γ of B_j^u, $\Gamma \cap L(x)$ is an unstable transversal to $L(x)$ for all $x \in B_j$. Let $\{P_j\}$ be a finite family of disjoint Pesin boxes with a family of disjoint open sets V_j with $V_j \supset P_j$. Further, if $\epsilon > 0$ is given, we may assume that $\mu(\bigcup P_j) > 1 - \epsilon$.

For a fixed j, we will set $B = B_j$ and $Q = Q_j$ and will show that they have the property claimed in the lemma. Let $c > 0$ be such that $\lim_{n\to\infty} cd^{-n}[f^n X] = \mu^-$. We may suppose that

$$\int \left(\mu^+ \llcorner P_j^s\right) \wedge c\left[X \cap V_j\right] \geq (1-\epsilon)\mu\left(P_j\right) , \tag{3.9}$$

replacing X by $f^n X$ and c by cd^{-n}, n large, if necessary. For each m, let $\mathcal{G}(m, j)$ denote the set of connected components Γ of $f^m(X \cap V_j) \cap \pi^{-1}Q$ such that $\pi|_\Gamma : \Gamma \to Q$ is a homeomorpism. We let

$$\mu^-_{\mathcal{G}(m,j)} = cd^{-m} \sum_{\Gamma\in\mathcal{G}(m,j)} [\Gamma] \tag{3.10}$$

so that

$$\mu^-_Q \geq \limsup_{m\to\infty} \sum_j \mu^-_{\mathcal{G}(m,j)} .$$

The inequality arises since there are possibly good disks in $f^m(X) \cap \pi^{-1}Q$ that are lost when $f^m(X - \bigcup V_j)$ is removed. We note that since each $\Gamma \in B^u_j$ is a proper transversal to $\pi^{-1}Q$, it will suffice to show that

$$\mu^-_Q \wedge \left(\mu^+ \llcorner B^s\right) \geq \left(\mu^- \llcorner B^u\right) \wedge \left(\mu^+ \llcorner B^s\right) = \mu(B) . \tag{3.11}$$

In (3.11) it is the inequality that needs to be proved; the equality is just the product structure of μ on B.

Now we have

$$\begin{aligned} \mu^-_Q \wedge \left(\mu^+ \llcorner B^s\right) &\geq \limsup_{m\to\infty} \sum_j \mu^-_{\mathcal{G}(m,j)} \wedge \left(\mu^+ \llcorner B^s\right) \\ &\geq \sum_j (1-\epsilon)\mu\left(P_j\right)\mu(B) \\ &\geq (1-\epsilon)^2 \mu(B) , \end{aligned}$$

where the second inequality follows from Lemma 3.4 below. Thus we conclude that (3.11) holds, which completes the proof. □

Lemma 3.4. *Let B and P_j be as above. Then*

$$\lim_{m\to\infty} \int \mu^-_{\mathcal{G}(m,j)} \wedge \left(\mu^+ \llcorner B^s\right) \geq (1-\epsilon)\mu\left(P_j\right)\mu(B) .$$

Proof. We note that each unstable transversal Γ in $L(x)$ gives rise to a unique good disk $\Gamma \cap \pi^{-1}Q$. Thus, we will consider instead the current $\mu^-_{\mathcal{V}(m,j)}$, where the sum in (3.10) is replaced by $\Gamma \cap \pi^{-1}Q$ for Γ which are unstable transversal components of $f^m(X \cap V_j) \cap L(x)$ for some $x \in B$. Since $\mu^-_{\mathcal{G}(m,j)} \geq \mu^-_{\mathcal{V}(m,j)}$, it suffices to prove the lemma for $\mu^-_{\mathcal{G}(m,j)}$ replaced by $\mu^-_{\mathcal{V}(m,j)}$. By [BLS1, Lemma 6.4], we may suppose that X intersects $W^s_{loc}(x)$ transversally for each $x \in P_j$. Let $N(x)$ denote the measurable function on $X \cap V_j$ with the property (3.8).

We define $c_1 = \int \mu^+ \wedge c[X \cap V_j]$ so that $c_1 \geq (1-\epsilon)\mu(P_j)$ by (3.9). We may assume (changing V_j slightly if necessary) that $\mu^+|_X$ puts no mass on $\partial(X \cap V_j)$. Thus,

$$\lim_{m\to\infty} cd^{-m}\left[f^m\left(X \cap V_j\right)\right] = c_1\mu^- .$$

It follows that

$$\lim_{m\to\infty}\int cd^{-m}\left[f^m\left(X\cap V_j\right)\right]\wedge\left(\mu^+ \llcorner B^s\right) = c_1\int \mu^-\wedge\left(\mu^+\llcorner B^s\right)$$
$$= c_1\mu(B)\geq(1-\epsilon)\mu\left(P_j\right)\mu(B)\,.$$

Thus, if we set

$$\eta^-(m,j)=cd^{-m}\left[f^m\left(x\cap V_j\right)\right]-\mu^-_{\mathcal{V}(m,j)}$$

it will suffice to show that

$$\lim_{m\to\infty}\int \eta^-(m,j)\wedge\left(\mu^+\llcorner B^s\right)=0\,.$$

However, if we pull back to $X\cap V_j$ via f^m and recall the definiton of $N(x)$, we have

$$\int \eta^-(m,j)\wedge\left(\mu^+\llcorner B^s\right)=\int_{\{N(x)>m\}}\left[X\cap V_j\right]\wedge\left(\mu^+\llcorner P_j^s\right)\,.$$

Thus, the right-hand side tends to 0 as $m\to\infty$ since $\{N(x)>m\}$ decreases to $\emptyset$. □

4. Averaged rates of growth

Lyapunov exponents describe the behavior tangent vectors at μ a.e. point. This is not, however, the most direct way to get a hold of the value of the Lyapunov exponents. In this section we consider various alternative notions of the growth rate of vectors and we relate them to the Lyapunov exponent. We discuss a method of measuring the growth of Df^k by taking the average with respect to μ and all directions; and we show how it is related to a type of critical point. Finally, we give a formula for the averaged rate of growth by pulling back a form from projective space. This last formula (Proposition 4.6) is of interest because it involves the projectivized image of the map $x\mapsto Df^n_x\in\mathcal{L}(\mathbf{C}^2,\mathbf{C}^2)$, and thus measures the volume of the (projectivized) image rather than the size of $\|Df^n\|$. This description suggests an analogy with the definition of curvature via the Gauss map.

There is a certain symmetry between λ^+ and λ^- which can be realized by replacing f by f^{-1}. For the sake of definiteness we focus on λ^+ in this section, and our notation reflects that emphasis.

We let α, β denote constant, nonzero vector fields on $\mathbf{C}^2$, and define the quantities

$$\Lambda = \lim_{k\to\infty}\frac{1}{k}\int \log\left\|D^kf(x)\right\|\,\mu(x)\,.$$
$$\Lambda(\alpha) = \lim_{k\to\infty}\frac{1}{k}\int \log\left|Df^k_x(\alpha)\right|\mu(x)$$
$$\Lambda(\alpha,\beta) = \lim_{k\to\infty}\frac{1}{k}\int \log\left|\beta\cdot Df^k_x(\alpha)\right|\mu(x)\,.$$

The first integral arises in the proof of the Oseledec Theorem as the first step in the proof of the existence of Lyapunov exponents. From this we see that $\Lambda=\lambda^+(\mu)$. We will analyze the other two quantities.

We can identify α and β with vectors in $\mathbf{C}^2-\{0\}$. It is clear that $\Lambda(c\alpha)=\Lambda(\alpha)$ for any $c\in\mathbf{C}-\{0\}$ and $\Lambda(c_1\alpha,c_2\beta)=\Lambda(\alpha,\beta)$ for $c_1,c_2\in\mathbf{C}-\{0\}$. Thus, when it is convenient we may think of identify $\Lambda(\cdot)$ and $\Lambda(\cdot,\cdot)$ as functions on $\mathbf{P}^1$ and $\mathbf{P}^1\times\mathbf{P}^1$. On the other hand, it is sometimes convenient to assume that $|\alpha| = 1$ and $|\beta| = 1$. We let σ denote the rotation invariant measure on the unit ball in $\mathbf{C}^2$, normalized to have total mass 1. We denote by the same letter the induced measure on $\mathbf{P}^1$.

In the sequel, we will use the observation that if $\beta = (\beta_1, \beta_2) \in \mathbf{C}^2$, then

$$\int_{|\alpha|=1} \log|\alpha \cdot \beta|\, \sigma(\alpha) = \log|\beta| - \frac{1}{2} \tag{4.1}$$

depends only on $|\beta|$.

Lemma 4.1. *For σ a.e. $\beta \in \mathbf{P}^1$, $\Lambda(\alpha, \beta) = \Lambda(\alpha)$.*

Proof. For each x we have

$$\int_{\beta \in \mathbf{P}^1} \log\left|\beta \cdot Df_x^k(\alpha)\right| \sigma(\beta) = \log\left|Df_x^k(\alpha)\right| - \frac{1}{2}$$

by (4.1). Now we integrate with respect to $\mu(x)$, divide by k, and then take the limit as $k \to \infty$ to obtain

$$\int_{\beta \in \mathbf{P}^1} \Lambda(\alpha, \beta)\sigma(\beta) = \Lambda(\alpha)\,.$$

On the other hand, we may assume that $|\beta| = 1$, so $\Lambda(\alpha) \geq \Lambda(\alpha, \beta)$. Thus the lemma follows. □

Recall from Section 3 the measurable, f-invariant complex splitting $E_x^s \oplus E_x^u$ of the tangent space for μ almost every point x. We let $x \mapsto e_x^{s/u}$ be a measurable choice of unit vectors in $E_x^{s/u}$. Given a tangent vector $\alpha = \alpha_1 \partial_1 + \alpha_2 \partial_2$ (using ∂_1 and ∂_2 to denote a frame for the tangent space of $\mathbf{C}^2$ the point x) we may split it as

$$\alpha = \alpha_x^s e_x^s + \alpha_x^u e_x^u\,.$$

Thus, for μ a.e. x there are numbers $\lambda_x^{k,s/u}$ such that

$$Df_x^k(\alpha) = \lambda_x^{k,s} \alpha_x^s e_{f^k x}^s + \lambda_x^{k,u} \alpha_x^u e_{f^k x}^u\,.$$

Thus, we have represented Df^k as a diagonal matrix.

Lemma 4.2. *For σ a.e. α, we have $\Lambda(\alpha) = \Lambda$.*

Proof. The function α_x^s in the splitting above is given by the Hermitian inner product $\langle \alpha, e_x^s \rangle$ on $\mathbf{C}^2$. For x fixed, $\int \log|\langle \alpha, e_x^s \rangle| \sigma(\alpha) = -\frac{1}{2}$ as above. Since the integrand is nonpositive, it follows that $\log|\alpha_x^s|$ is integrable with respect to the product measure $\sigma \times \mu$. Reversing the order of integration, we have

$$\int \mu(x) \int \log\left|\langle \alpha, e_x^s \rangle\right| \sigma(\alpha) = \int\int \mu(x) \log\left|\langle \alpha, e_x^s \rangle\right| \sigma(\alpha)\,.$$

Thus, for almost every $\alpha \in \mathbf{P}^1$ the function $x \mapsto \log|\alpha_x^s|$ is integrable with respect to μ. Similarly, we may assume that $\log|\alpha_x^u|$ is integrable.

Letting $\gamma_{x,\epsilon}$ be as in (3.2) through (3.4), we define $S_\gamma = \{x : \gamma_{x,\epsilon} \leq \gamma\}$ for fixed $\epsilon > 0$. By the splitting above, we have

$$\log\left|Df_x^k(\alpha)\right| = \log\left|\lambda_x^{k,s} \alpha_x^s e_{f^k x}^s + \lambda_x^{k,u} \alpha_x^u e_{f^k x}^u\right| = \log|A + B|\,.$$

Given two vectors A and B, which form an angle of opening θ, the square of the sum has length

$$\begin{aligned}|A+B| &= |A|^2+|B|^2+2|A|\,|B|\cos\theta \\ &= (|A|-|B|)^2+2(\cos\theta-1)|A|\,|B| \geq 2(\cos\theta-1)|A|\,|B|\,.\end{aligned}$$

For θ small, we may estimate $2(\cos\theta - 1)$ by θ^2, so we have

$$\log\left|Df^k(\alpha)\right| \geq \log\left|\theta^2\alpha_x^s\lambda_x^{k,s}\alpha_x^u\lambda_x^{k,u}\right|$$

Thus by (3.2)), (3.3), and (4.1), and the fact that the angle between $e^s_{f^kx}$ and $e^u_{f^kx}$ is bounded below by (3.4) and $\gamma \geq \gamma_{x,\epsilon}$ for $x \in S_\gamma$ we have that the quantity

$$\begin{aligned}\frac{1}{k}\log\left|Df^k(\alpha)\right| &\geq \log\theta^2+\log\left|\lambda_x^{k,s}\lambda_x^{k,u}\right|+\log\left|\alpha_x^s\right|+\log\left|\alpha_x^u\right| \\ &\geq \left(-2\epsilon-\frac{2}{k}\log\gamma\right)+\left(\log|a|-2\epsilon-\frac{2}{k}\log\gamma\right)+\log\left|\alpha_x^s\right|+\log\left|\alpha_x^u\right|\end{aligned}$$

is bounded below by a function which is integrable with respect to μ.

For μ a.e. point x such that $\alpha_x^u \neq 0$, we have

$$\lim_{k\to\infty}\frac{1}{k}\log\left|Df_x^k(\alpha)\right| = \Lambda\,,$$

so by (4.1) and the dominated convergence theorem, we have

$$\lim_{k\to\infty}\left|\int(\Lambda-\frac{1}{k}\log\left|Df_x^k(\alpha)\right|)\mu(x)\right| \leq \mu\left(J-S_\gamma\right)(\Lambda+\log M)\,.$$

The lemma follows since $\lim_{\gamma\to\infty}\mu(J-S_\gamma)=0$. □

Lemma 4.3. *For a.e.* β *we have*

$$\lim_{k\to\infty}\frac{1}{k}\int\sigma(\alpha)\int\log\left|\beta\cdot Df_x^k(\alpha)\right|\mu(x)=\Lambda\,.$$

Proof. This follows from Lemma 4.1, 4.2, and the bounded convergence theorem. □

Another way in which a family of critical points arises is as follows. Let us define

$$Z_k(\alpha,\beta)=\left\{x\in\mathbf{C}^2:\beta\cdot Df_x^k(\alpha)=0\right\} \tag{4.2}$$

as the critical points of the scalar function $x \mapsto \beta\cdot f$ with respect to the direction ∂_α. Unlike the set of unstable critical points $\mathcal{C}^u$, which will be defined in Section 5, this set is not invariant. On the other hand, it is quite explicit.

The following computation resembles the proof of [BS3, Theorem 3.2], except that now we analyze further the integral term on the right-hand side.

Lemma 4.4. *Let* $\alpha, \beta \in \mathbf{C}^2$ *be such that the second coordinates* $\alpha_2, \beta_2 \neq 0$ *are nonvanishing. If* $T=\{x=0\}$ *is the* y*-axis, then*

$$\int\log\left|\beta\cdot Df^k\alpha\right|\mu^+\wedge\frac{1}{d^n}f_*^n[T]=\log\left|\alpha_2\beta_2 d^k\right|+\int G^+\left[Z_k(\alpha,\beta)\right]\wedge\frac{1}{d^n}f_*^n[T]\,.$$

Proof. Applying $(f^n)^*$ and treating $f^{n*}f_*^n$ as the identity transformation, we have

$$\int \log \left|\beta \cdot Df^k\alpha\right| \frac{1}{2\pi} dd^c G^+ \wedge \frac{1}{d^n} f_*^n[T] = \int f^{n*}\left(\log\left|\beta \cdot Df^k\alpha\right|\right) \frac{1}{2\pi} dd^c G^+ \wedge [T]\,,$$

where we use the functional equation $f^{n*}G^+ = G^+ \circ f^n = d^n G^+$. Furthermore, G^+ restricted to T is the Green function of $K^+ \cap T$, so that $\frac{1}{2\pi}(dd^c)_T G^+|_T$ is the harmonic measure, which we denote by $\mu^+_{K^+\cap T}$, so the equation becomes

$$\int \log \left|\beta \cdot Df^k\alpha\right| \frac{1}{2\pi} dd^c G^+ \wedge \frac{1}{d^n} f_*^n[T] = \int_T f^{n*}\left(\log\left|\beta \cdot Df^k\alpha\right|\right) \mu_{K^+\cap T}\,.$$

From formula (1.1) for f^k, we observe that

$$\beta \cdot Df^k(\alpha) = \beta_2\alpha_2 d^k y^{d^k-1} + \dots\,.$$

Since $\mu_{K^+\cap T}$ is harmonic measure, we may apply Jensen's formula [BS3, Lemma 3.1] to the monic polynomial $(\beta_2\alpha_2 d^k)^{-1}\beta \cdot Df^k(\alpha)$ (restricted to T) and obtain

$$\begin{aligned} &= \log\left|\beta_2\alpha_2 d^k\right| + \sum_{\{c\in T:\beta\cdot Df^k_{f^n c}(\alpha)=0\}} d^n G^+(c) \\ &= \log\left|\beta_2\alpha_2 d^k\right| + \int G^+\left[Z_k(\alpha,\beta)\right] \wedge \frac{1}{d^n} f_*^n[T]\,, \end{aligned}$$

where the last equation comes from pushing $[T]$ forward under f^n. This gives the desired formula. □

Corollary 4.5. *If $\alpha_2, \beta_2 \neq 0$, then*

$$\Lambda(\alpha,\beta) = \log d + \lim_{k\to\infty} \frac{1}{k} \int G^+ \mu^- \wedge [Z_k(\alpha,\beta)]\,.$$

Proof. We take the formula given in Lemma 4.4 and let $n \to \infty$. Then we divide by k and take the limit as $k \to \infty$. □

Now we find another way to replace the explicit dependence on α and β by the average over all directions. This provides an alternative approach to critical points.

The differential induces a mapping

$$\mathbf{C}^2 \ni x \mapsto Df_x \in \mathcal{L}\left(\mathbf{C}^2, \mathbf{C}^2\right)\,.$$

We may identify the dual space $\mathcal{L}(\mathbf{C}^2, \mathbf{C}^2)^* \cong \mathbf{C}^2 \otimes (\mathbf{C}^2)^*$, where $\alpha \otimes \beta \in \mathbf{C}^2 \otimes (\mathbf{C}^2)^*$ induces the functional $\mathcal{L}(\mathbf{C}^2, \mathbf{C}^2) \ni Z \mapsto \beta \cdot Z\alpha$. Let

$$(\alpha\otimes\beta)^\perp = \left\{Z \in \mathcal{L}\left(\mathbf{C}^2, \mathbf{C}^2\right) : \beta \cdot Z\alpha = 0\right\}\,,$$

and let $[(\alpha\otimes\beta)^\perp]$ denote the current of integration over $(\alpha\otimes\beta)^\perp$ as a subset of $\mathcal{L}(\mathbf{C}^2, \mathbf{C}^2)$. Now the function $V_{\alpha,\beta}(Z) = \log|\beta \cdot Z\alpha|$ satisfies the Poincaré–Lelong identity

$$\left[(\alpha\otimes\beta)^\perp\right] = \frac{1}{2\pi} dd^c V_{\alpha,\beta}\,.$$

Averaging the function $V_{\alpha,\beta}$ with respect to α and β we have

$$\tilde{V}(Z) := \int_{\alpha\in\mathbf{P}^1} \sigma(\alpha) \int_{\beta\in\mathbf{P}^1} \sigma(\beta)\, V_{\alpha,\beta}(Z)\ , \tag{4.3}$$

so that $\tilde{V}(Z)$ is continuous off the origin, plurisubharmonic, and logarithmically homogeneous. Observe that the integral on the left-hand side, as a function of Z, is invariant under the $U(2)\times U(2)$-action on $\mathcal{L}(\mathbf{C}^2,\mathbf{C}^2)$ given by $(S,T)\cdot Z \mapsto SZT^{-1}$. Thus, to evaluate the integral it suffices to consider the case where $Z = \mathrm{diag}\{\lambda_1,\lambda_2\}$ is diagonal.

In this case $V_{\alpha,\beta}(Z) = \log|\alpha_1\lambda_1\beta_1 + \alpha_2\lambda_2\beta_2|$, and by (4.1) the first integration inside (4.3) yields

$$\int_\alpha \log|\alpha_1\lambda_1\beta_1 + \alpha_2\lambda_2\beta_2|\ \sigma(\alpha) = \frac{1}{2}\log\left(|\beta_1\lambda_1|^2 + |\beta_2\lambda_2|^2\right) + C\ .$$

The (1,1) form

$$\Theta = \frac{1}{2\pi} dd^c\tilde{V} \tag{4.4}$$

on $\mathbf{C}^{2\times 2} \cong \mathcal{L}(\mathbf{C}^2,\mathbf{C}^2)$ represents the averaged current of integration. (Note that we are making an abuse of notation, representing a current as a (1,1)-form.) By the logarithmic homogeneity, we may also interpret Θ as a form on the projectivized space $\mathcal{L}(\mathbf{C}^2,\mathbf{C}^2)/\mathbf{C}^* \cong \mathbf{P}^3$, and Θ dominates a multiple of the standard Kähler form on $\mathbf{P}^3$. If we use again the notation Df^k to denote the projective image of the differential in $\mathcal{L}(\mathbf{C}^2,\mathbf{C}^2)/\mathbf{C}^*$, then the averaged critical locus is the pullback of Θ on projective space:

$$\int_{\alpha\in\mathbf{P}^1}\int_{\beta\in(\mathbf{P}^1)^*} \sigma(\alpha)\sigma(\beta)\ [Z_k(\alpha,\beta)] = \left(Df^k\right)^*\Theta\ . \tag{4.5}$$

Averaging the formula of Corollary 4.5 over $\alpha,\beta\in\mathbf{P}^1$, we obtain:

Proposition 4.6.

$$\Lambda = \log d + \lim_{k\to\infty}\frac{1}{k}\int G^+\mu^- \wedge \left(Df^k\right)^*\Theta\ .$$

5. Stable/unstable critical measures

In this section we begin by defining the unstable critical points $\mathcal{C}^u$ and the unstable critical measure μ_c^-. (The definition of the corresponding objects $\mathcal{C}^s$ and μ_c^+ should be clear.) We will show (Theorem 5.1) that if $\mu_{c,s}^-$ is the critical measure defined starting from the laminar current $\mu_{\mathcal{Q}_s}^-$, then $\mu_{c,s}^-$ converges to μ_c^- as $s\to\infty$. The rest of the section is devoted to showing (Theorem 5.9) that μ_c^- is equal to the limit of the intersection product of μ^- with the average over α and β of the critical varieties $f^jZ_k(\alpha,\beta)$ as $j,k-j\to\infty$, i.e., $\mu^-\wedge f_*^j(Df^k)^*\Theta\to\mu_c^-$.

We define the *unstable critical points* as

$$\mathcal{C}^u = \bigcup_{x\in\mathcal{R}} \mathrm{Crit}\left(G^+, W^u(x) - K\right)\ ,$$

where $\mathrm{Crit}(G^+, W^u(x)-K)$ is the set of critical points, with multiplicity, of the restriction of the function G^+ to the open subset $W^u(x)-K$ of the manifold $W^u(x)$. The restriction of G^+ to $W^u(x)$ is subharmonic on $W^u(x)$ and harmonic on $W^u(x)-K$; thus, $W^u(x)-K\neq\emptyset$. Since G^+ vanishes on $W^u(x)\cap K$ (which is nonempty since it contains x), it follows that G^+ cannot be constant on a nonempty open subset of $W^u(x) - K = \{y \in W^u(x) : G^+(y) > 0\}$. Further, since $x \in \mathcal{R}$ is a regular point, it follows from [BLS1, Proposition 2.9] that the restriction $G^+|_{W^u(x)}$ is not everywhere harmonic, and so $W^u(x) - K \neq \emptyset$. Thus, $\mathrm{Crit}(G^+, W^u(x) - K)$ is a discrete subset of $W^u(x) - K$ for each $x \in \mathcal{R}$. If f is uniformly hyperbolic, then $\mathcal{C}^u$ is a closed subset of U^+. In the general case, $\mathcal{C}^u$ is likely not to be well behaved.

We will now define the unstable measure μ_c^-. We start by defining its restriction to an unstable box B^u. For a stratum Γ_t of B^u, the critical points of $G^+|_{\Gamma_t - K}$ are discrete, as noted above. We let the current $[\mathrm{Crit}(G^+, \Gamma_t - K)]$ denote the sum of point masses (with multiplicity) at the critical points of $G^+|\Gamma_t - K$. The mapping of currents $t \mapsto [\mathrm{Crit}(G^+, \Gamma_t - K)]$ is semicontinuous and may be assumed to be bounded, so we may set

$$\mu_c^- \llcorner B^u = \int \mu_\tau^-(t) \left[\mathrm{Crit}\left(G^+, \Gamma_t - K\right)\right] .$$

It is evident that this definition of μ_c^- is independent of the stable box involved, since if we have two stable boxes, the two definitions of μ_c^- agree on the overlap. This definition of μ_c^- may be considered to give almost all of the points of $\mathcal{C}^u$, since by Proposition 3.1, we could work equally naturally with the set $\tilde{\mathcal{R}}$, in which case every critical point would lie inside an unstable box.

Defined this way, μ_c^- is evidently σ-finite, and in Section 6 we will see that it is locally finite on U^+. The set $\mathcal{C}^u$ is f-invariant. Since the transversal measures corresponding to μ^- multiply by d under push-forward by f, and the function G^+ multiplies by d^{-1} it follows that $G^+\mu_c^-$ is f-invariant:

$$f_*\left(G^+\mu_c^-\right) = G^+\mu_c^- .$$

For a square $Q \in \mathcal{Q}_s$, we let $\mu_Q^- = \int \nu_Q(a)[\Gamma_a]$ denote the laminar structure obtained in Theorem 2.4, in terms of an algebraic variety X and a projection π_α. We may define the corresponding critical measure

$$\mu_{c,s}^- = \sum_{Q \in \mathcal{Q}_s} \int \nu_Q(a) \left[\mathrm{Crit}\left(G^+, \Gamma_a\right)\right] .$$

Theorem 5.1. *For all but countably many values of* α,

$$\lim_{s \to \infty} \mu_{c,s}^- = \mu_c^- .$$

Proof. Let us choose Pesin boxes B_j whose union has full μ measure. For $\alpha \in \mathbf{C}^2$ we let $S(\alpha, j)$ denote the set of points of B_j^u where the tangent space of the corresponding stratum is annihilated by π_α. If α' and α'' define different points of $\mathbf{P}^1$, then $S(\alpha', j)$ is disjoint from $S(\alpha'', j)$. Thus, $\mu_c^-(S(\alpha, j)) > 0$ for only countably many values of α. Thus, except for countably many values of α, we have $\mu_c^-(S(\alpha, j)) = 0$ for all j. Now we may subdivide the Pesin boxes to obtain a new covering $\{B_j\}$ which satisfies the hypotheses of Lemma 3.3 with $\pi = \pi_\alpha$. Thus, for each B_j there is a Q_j such that

$$\mu_{Q_j}^- \geq \mu^- \llcorner \left(B_j^u \cap \pi^{-1} Q_j\right) .$$

It follows that if s is sufficiently large that Q_j is a union of squares from $\mathcal{Q}_s$, then we have

$$\mu_{c,s}^- \geq \mu_c^- \llcorner \left(B_j^u \cap \pi^{-1} Q\right) .$$

It follows that $\lim_{\kappa \to \infty} \mu_{c,\kappa}^- \llcorner Y \geq \mu_c^- \llcorner Y$ holds for $Y = \bigcup_{j,n} f^n \hat{B}_j^u$, where $\hat{B}_j^u = B_j^u \cap \pi^{-1} Q_j$. As in Proposition 3.1, Y has full measure with respect to all transversals, so the theorem follows. □

Now we start the sequence of lemmas that will lead to the proof of Theorem 5.9. For $\beta \in (\mathbf{C}^2)^*$, we consider $\alpha \mapsto \beta \cdot Df^k(z)(\alpha)$ and $\alpha \mapsto \partial G^+ \cdot \alpha$ as linear functionals acting on $\alpha \in \mathbf{C}^2$. We let $\langle \beta \cdot Df^k(z) \rangle$ and $\langle \partial G^+ \rangle$ denote their images in $(\mathbf{P}^1)^*$.

Lemma 5.2. *For each compact subset $U_0 \subset U^+$, the sequence $\langle \beta \cdot Df^k(z) \rangle$ converges to $\langle \partial G^+ \rangle$ as $k \to \infty$, uniformly in $z \in U_0$ and $\beta \in \mathbf{C}^2 - \{0\}$.*

Proof. If f has the form (1.1), then the coordinates $f^n = (f^n_{(1)}, f^n_{(2)})$ satisfy $f^n_{(1)} = (f_2 \circ \cdots \circ f_m \circ f^{n-1})_{(2)}$, and $f^n_{(2)} = (f^n_{(1)})^{d_1} + \cdots$, so $d^{-n} \log |f^n_{(2)}|$ and $d^{-n} d_1 \log |f^n_{(1)}|$ converge to G^+ uniformly on compact subsets of U^+. Thus, the normalizations of the gradients $\partial f^n_{(i)} |\partial f^n_{(i)}|^{-1}$, $i = 1, 2$ both converge uniformly to the normalization $\partial G^+ |\partial G^+|^{-1}$ on compact subsets of U^+. It follows that on any compact subset of U^+, the normalization of $\partial(\beta_1 f^n_{(1)} + \beta_2 f^n_{(2)})$ converges to $\partial G^+ |\partial G^+|^{-1}$ uniformly in $\beta \neq (0, 0)$. Since $\beta \cdot Df^n(\alpha)$ may be identified with $\partial(\beta_1 f^n_{(1)} + \beta_2 f^n_{(2)}) \cdot \alpha$, it follows that the projective images of these linear functionals converge uniformly. □

Let $P^u = \bigcup_{t \in T} \Gamma_t$ be an unstable box as in Proposition 3.2. For each $j \geq 0$, we define

$$\left\langle E^s \left(f^{-j} \Gamma_t \right) \right\rangle := \left\{ \langle E^s_x \rangle : x \in f^{-j} \Gamma_t \right\} \subset \mathbf{P}^1 .$$

Thus, $\langle E^s(f^{-j} \Gamma_t) \rangle$ has diameter $O(e^{-\epsilon j})$. We set

$$\mathcal{V}^s(j, t)) = \left\{ \alpha \in \mathbf{P}^1 : \operatorname{dist}\left(\alpha, \left\langle E^s \left(f^{-j} \Gamma_t \right) \right\rangle \right) < \frac{1}{4} \right\}$$

and $\mathcal{V}^u(j, t) = \mathbf{P}^1 - \mathcal{V}^s(j, t)$. It follows from (3.2) through (3.4) that $Df^j \mathcal{V}^s(j, t)$ lies in an $O(e^{-\epsilon j})$-neighborhood of $\langle E^s_x \rangle$ at all $x \in \Gamma_t$.

Lemma 5.3. *Let P and $\mathcal{V}^u(j, t)$ be as above, and let us suppose that G^+ has no critical points on $\partial \Gamma_t$ for $t \in T$. Then in terms of the Hausdorff distance we have*

$$\lim_{\substack{j \to \infty \\ k - j \to \infty}} \operatorname{dist}\left(\Gamma_t \cap f^j Z_k(\alpha, \beta), \operatorname{Crit}\left(G^+, \Gamma_t \right) \right) = 0$$

with the limit being uniform in $t \in T$, $\alpha \in \mathcal{V}^u(j, t)$, and $\beta \in (\mathbf{P}^1)^*$.

Proof. If $j, k - j \to \infty$, then there are sequences $\kappa_1(k), \kappa_2(k) \to \infty$ such that $\kappa_1(k) \leq j \leq k - \kappa_2(k)$. If $\zeta \in \Gamma_t \cap f^j Z_k(\alpha, \beta)$, then $y = f^{-j} \zeta$ satisfies $\beta \cdot Df^k_y(\alpha) = 0$. Thus, ζ satisfies $\beta \cdot Df^{k-j}_\zeta (f^j_* \alpha) = 0$.

For $\delta > 0$, we may choose κ_1 sufficiently large that if $j \geq \kappa_1$, $\zeta \in \Gamma_t$, and $\alpha \in \mathcal{V}^u(j, t)$, then $\operatorname{dist}_{\mathbf{P}^1}(f^j_* \alpha, \langle E^u_\zeta \rangle) < \delta$. Furthermore, for $\kappa_2(k)$ sufficiently large and $j \leq k - \kappa_2$, it follows from Lemma 5.2 that $\operatorname{dist}_{(\mathbf{P}^1)^*}(\partial G^+, \beta \cdot Df^{k-j}) < \delta$. Thus, the distance between the sets $\operatorname{Crit}(G^+, \Gamma_t)$ and $\{x \in \Gamma_t : \beta \cdot Df^{k-j}(f^j_* \alpha) = 0\}$ is uniformly small. □

Next we define

$$\lambda^{s/u}_{j,k}(\beta, t) = \int_{\alpha \in \mathcal{V}^{s/u}(j,t)} \sigma(\alpha) \left[\Gamma_t \cap f^j Z_k(\alpha, \beta) \right] .$$

The plan is to show that $\lambda^u_{j,k}(\beta, t)$ converges to the critical point measure $[\operatorname{Crit}(G^+, \Gamma_t)]$, and thus the integral with respect to t will converge to the critical measure $\mu^-_c \llcorner P$, and then to show that $\lambda^s_{j,k}(\beta, t)$ converges to zero as $j, k - j \to \infty$.

Lemma 5.4. *For each $t \in T$, $\lambda^u_{j,k}(\beta, t)$ converges uniformly to $[\mathrm{Crit}(G^+, \Gamma_t)]$ as $j, k - j \to \infty$; that is if ψ is any test function and $\kappa_1, \kappa_2 \to \infty$, then*

$$\lim_{k\to\infty} \max_{\kappa_1(k)\le j\le k-\kappa_2(k)} \max_{t\in T} \left| \int \psi \left(\lambda^u_{j,k}(\beta, t) - \left[\mathrm{Crit}\left(G^+, \Gamma_t\right)\right]\right) \right| = 0 .$$

Proof. As $j, k - j \to \infty$, $\Gamma_t \cap f^j Z_k(\alpha, \beta)$ converges to $\mathrm{Crit}(G^+, \Gamma_t)$ uniformly in $\alpha \in \mathcal{V}^u(j, t)$, $t \in T$ and $\beta \in (\mathbf{P}^1)^*$. The lemma follows since $\mathcal{V}^u(j, t)$ approaches full measure as $j \to \infty$. □

Lemma 5.5. *Let Γ be conformally equivalent to the unit disk in $\mathbf{C}$, and let Γ' be a relatively compact open subset of Γ. Let $h : \Gamma \to \mathbf{C}^2$ be a holomorphic function such that*

$$\max_{\zeta\in\Gamma} |h| \le C \min_{\zeta\in\Gamma} |h|$$

for some $C < \infty$. Then there is a constant $0 < b < 1$, depending only on Γ', such that

$$m = \max_{\alpha\in\mathbf{P}^1} \left\{\zeta \in \Gamma' : h(\zeta) \cdot \alpha = 0\right\} \tag{5.1}$$

satisfies either $Cb^m \ge \sqrt{3}/2$ or

$$\int_{\alpha\in\mathbf{P}^1} \#\left\{\zeta \in \Gamma' : h(\zeta) \cdot \alpha = 0\right\} \sigma(\alpha) \le m\pi \left(2Cb^m\right)^2 . \tag{5.2}$$

Proof. Without loss of generality, we may assume that $\sup_\Gamma |h| = C$ and $\inf_\Gamma |h| = 1$. There is a number $0 < b < 1$ depending only on Γ' such that for any holomorphic function ψ on Γ with m zeros in Γ',

$$\max_{\Gamma'} |\psi| \le b^m \max_{\Gamma} |\psi| .$$

Let us fix α_0 with $|\alpha_0| = 1$ such that the maximum is attained in (5.1). It follows that

$$\max_{\zeta\in\Gamma'} |h(\zeta) \cdot \alpha_0| \le b^m C .$$

If $\theta(\zeta)$ is the angle between $\mathrm{Ker}(h(\zeta))$ and $\alpha_0 \in \mathbf{C}^2$, then

$$|h| \sin\theta(\zeta) = |h(\zeta) \cdot \alpha_0| .$$

Since $|h(\zeta)| \ge 1$, it follows that $|\sin\theta(\zeta)| \le Cb^m$. It follows that $h(\zeta) \cdot \alpha \ne 0$ for $\zeta \in \Gamma'$ if the sine of the angle between α and α_0 is greater than Cb^m. If $|\sin\theta| < \sqrt{3}/2$, then $\theta/2 < |\sin\theta|$. Thus, if $Cb^m < \sqrt{3}/2$, then $|\theta(\zeta)| \le 2Cb^m$, and so $\alpha \mapsto \#\{\zeta \in \Gamma' : h \cdot \alpha = 0\}$ is supported in a disk of radius $2Cb^m$ about α_0. In this case, the integral in (5.2) is bounded by $m\pi(2Cb^m)^2$. □

Lemma 5.6. *Let $\Gamma' \subset \Gamma$, h, b, and C be as in Lemma 5.5. If $\mathcal{V} \subset \mathbf{P}^1$ is contained in a disk of radius δ, then*

$$\int_{\alpha\in\mathcal{V}} \#\left\{\zeta \in \Gamma' : h(\zeta) \cdot \alpha = 0\right\} \sigma(\alpha) \le \delta^2 C' \log\left(\frac{1}{\delta}\right) ,$$

where C' depends only on b and C.

Proof. Let us choose α_0 which maximizes m in (5.1). If $Cb^m < \sqrt{3}/2$, then

$$\int_{\alpha\in\mathcal{V}} \#\left\{\zeta \in \Gamma' : h(\zeta) \cdot \alpha = 0\right\} \sigma(\alpha) \le m\pi\delta^2 < \frac{\pi \log\left(\sqrt{3}/2C\right)}{\log b} \delta^2 .$$

If $Cb^m \geq \sqrt{3}/2$, then by Lemma 5.5 the integral is bounded by $m\pi(2Cb^m)^2$. We also have the trivial upper bound $m\pi\delta^2$. Thus,

$$\int_{\alpha\in\mathcal{V}} \#\{\zeta \in \Gamma' : h(\zeta)\cdot\alpha = 0\}\,\sigma(\alpha) \leq \min\left(m\pi\left(2Cb^m\right)^2, m\pi\delta^2\right) = \frac{\log(\delta/2C)\,\pi\delta^2}{\log b}$$

since the minimum is attained when $2Cb^m = \delta$. □

Lemma 5.7. *For κ_0 sufficiently large, there exists a constant C such that for $k - j \geq \kappa_0$, $h = (\beta\circ f^{k-j})^{-1}\beta\circ Df^{k-j} : \mathbf{C}^2 \to \mathbf{C}^2$ satisfies*

$$\max_{\Gamma_t} |h| \leq C \min_{\Gamma_t} |h|$$

for all $t \in T$ and $\beta \in (\mathbf{P}^1)^*$.

Proof. This is a direct consequence of Lemma 5.2. □

Lemma 5.8.

$$\lim_{\substack{j\to\infty\\ k-j\to\infty}} \lambda^s_{j,k}(\beta, t) = 0\,.$$

Proof. Let $\Gamma'_t \subset \Gamma_t$ be a relatively compact open subset with no critical points in the boundary. By Lemmas 5.5, 5.6, and 5.7, we have

$$\lambda^s_{j,k}(\beta,t)\llcorner\Gamma'_t \leq \max\left(\frac{1}{b}\left(\log\frac{\pi}{4C}\right)\pi\delta_j^2, \frac{\log\left(\delta_j(4C)^{-1}\right)}{\log b}\pi\delta_j^2\right)$$

where δ_j is chosen so that $\mathcal{V}^s_j(\beta, t)$ is contained in a disk of radius δ_j. The lemma now follows since $\delta_j \to 0$ as $j\to\infty$ and since Γ'_t can be chosen to exhaust Γ_t. □

Define

$$\hat{Z}_k(\beta) = \int_{\alpha\in\mathbf{P}^1} \sigma(\alpha)\ [Z_k(\alpha,\beta)]\ .$$

Theorem 5.9. *As $j, k-j\to\infty$ the restrictions of $\mu^-\wedge f^j_*\hat{Z}_k(\beta)$ to U^+ converge to μ^-_c in the sense of currents on U^+.*

Proof. We let $P^u = P$ be an unstable box as above. We choose an unstable box $P' \subset P$ such that Γ'_t is relative compact in Γ_t, and there are no critical points on $\partial\Gamma'_t$. Further, we may assume that $\mu^-_c(\bigcup_{t\in T}\partial\Gamma_t) = 0$. By Proposition 3.1, it suffices to show that

$$\lim_{\substack{j\to\infty\\ k-j\to\infty}} \left(\mu^-\wedge f^j_*\hat{Z}_k(\beta)\right)\llcorner P = \mu^-_c\llcorner P\ .$$

Using the notation above

$$\begin{aligned}
\mu^-\wedge f^j_*\hat{Z}_k(\beta)\llcorner P &= \left(\mu^-\llcorner P\right)\wedge f^j_*\hat{Z}_k(\beta)\\
&= \int_{t\in T}\mu^-_\tau(t)\ [\Gamma_t]\wedge f^j_*\hat{Z}_k(\beta)\\
&= \int_{t\in T}\mu^-_\tau(t)\int_{\alpha\in\mathbf{P}^1}\sigma(\alpha)\ [\Gamma_t]\wedge f^j_*\ [Z_k(\alpha,\beta)]\\
&= \int_{t\in T}\mu^-_\tau(t)\int_{\alpha\in\mathbf{P}^1}\sigma(\alpha)\ \left[\Gamma_t\cap f^j Z_k(\alpha,\beta)\right]\ .
\end{aligned}$$

If we break up the inner integral as $\mathbf{P}^1 = \mathcal{V}_j^s(\beta, t) \cup \mathcal{V}_j^u(\beta, t)$, then we have

$$\mu^- \wedge f_*^j \hat{Z}_k(\beta) \llcorner P = \int_{t \in T} \mu_\tau^-(t) \lambda_{j,k}^s(\beta, t) + \int_{t \in T} \mu_\tau^-(t) \lambda_{j,k}^u(\beta, t) .$$

It follows from Lemma 5.8, then, that the first integral on the right-hand side converges to zero, and from Lemma 5.4 that the second integral converges to $\mu_c^- \llcorner P$. □

We observe that as in (4.5) $(Df^k)^*\Theta = \int \sigma(\beta) \hat{Z}_k(\beta)$, we may integrate the previous result with respect to β to obtain:

Corollary 5.10. *Let Θ be as in (4.4). Then as $j, k - j \to \infty$ the restrictions of the currents $\mu^- \wedge f_*^j (Df^k)^*\Theta$ to U^+ converge to μ_c^- in the sense of currents on U^+.*

6. The integral formula

The main goal of this section is to prove Theorem 6.1, which gives the Main Theorem. In fact, Theorem 6.1 is a consequence of Theorem 6.2, relating the rate of expansion to the unstable critical measure. This may be viewed as applying Corollary 5.10 inside the integral formula of Proposition 4.6.

For a set P, we put $\tilde{P} = \bigcup_{n \in \mathbf{Z}} f^n P$. We will say that a Borel set P is a *fundamental domain* for $\mathcal{C}^u$ if $\tilde{P} \supset \mathcal{C}^u$ and if $P \cap f^n P = \emptyset$ for all $n \neq 0$.

Theorem 6.1. *Let $P \subset \mathcal{C}^u$ be a fundamental domain for $\mathcal{C}^u$. Then*

$$\lambda^+(\mu) = \log d + \int_P G^+ \mu_c^- .$$

Remark. A convenient choice for fundamental domain is $\{1 \le G^+ < d\} \cap \mathcal{C}^u$. This choice gives us the Main Theorem.

For a domain P satisfying $P \cap f^n P = \emptyset$ for all $n \neq 0$, every point $x \in \tilde{P}$ may be written uniquely as $x = f^n y$, so we have a projection $\pi_P : \tilde{P} \to P$ given by $\pi_P(x) = y$; it is evident that π_P is Borel measurable.

Theorem 6.2. *Let $P \subset J^-$ be a Borel set such that $\mu_c^-(\partial P) = 0$, where ∂P denotes the boundary relative to J^-. If $P \cap f^n P = \emptyset$ for all $n \neq 0$, then*

$$G^+ \mu_c^- \llcorner P = \lim_{k \to \infty} (\pi_P)_* \left(\frac{1}{k} G^+ \mu^- \wedge \left(Df^k \right)^* \Theta \llcorner \tilde{P} \right) . \tag{6.1}$$

Remark. Both sides of the equation put no mass on $J^- \cap K$, so without loss of generality we may assume that $P \subset J^- - K$. Indeed, the general case follows from the case where P is a fundamental domain.

Proof of Theorem 6.1. We will show how Theorem 6.1 is deduced from Theorem 6.2. We first prove

$$\lambda^+(\mu) = \log d + \int_{\{t \le G^+ < td\}} G^+ \mu_c^-$$

for some value of t. For this, we note that for every $t > 0$, $P_t = \{t \le G^+ < td\} \cap J^-$ is a fundamental domain for $J^- - K$. By the fact that G^+ is pluriharmonic, we have $\partial P_t = \{G^+ = t\} \cup \{G^+ = td\}$, so that the boundaries ∂P_t are disjoint for $0 < t < d$. Now since μ_c^- is σ-finite, we have $\mu_c^-(\partial P_t) = 0$ for all but countably many values of t. So we may apply Proposition 4.6 and Theorem 6.2 to conclude that the formula above holds for such t.

Now we conclude with the observation that if the theorem holds for one choice of Borel measurable fundamental domain, it holds for any other. Given the fundamental domain P, the restriction of the mapping $\pi_P : \{t \le G^+ < td\} \cap C^u \to P$ is one to one and onto. Since $G^+\mu_c^-$ is f-invariant, it follows that it is invariant under π_P, and thus

$$\int_P G^+\mu_c^- = \int_{\{t \le G^+ < td\}} G^+\mu_c^- ,$$

which completes the proof. □

By (1.1), $\beta \cdot Df^k(\alpha) = \beta_2\alpha_2 d^k y^{d^k - 1} + \cdots$, so that if $\alpha_2\beta_2 \ne 0$, then the total mass of the intersection current is

$$\int \mu^- \wedge [Z_k(\alpha, \beta)] = d^k - 1 . \tag{6.2}$$

Lemma 6.3. *If* $\kappa_2(k)$ *satisfies* $\lim_{k\to\infty}(\kappa_2(k) - \log_d k) = -\infty$, *then*

$$\lim_{k\to\infty} \frac{1}{k} \int_{\{G^+ < d^{-k+\kappa_2}\}} G^+\mu^- \wedge \left(Df^k\right)^* \Theta = 0 .$$

Proof. By (6.2), the total mass of $\mu^- \wedge [Z_k(\alpha, \beta)]$ is $d^k - 1$ for almost every $\alpha, \beta \in \mathbf{C}^2 - \{0\}$. Thus,

$$\frac{1}{k} \int_{\{G^+ < d^{-k+\kappa_2}\}} G^+\mu^- \wedge [Z_k(\alpha, \beta)] \le \frac{1}{k} d^{-k+\kappa_2} d^k$$

so the lemma follows from the condition on κ_2 after integrating with respect to α and β. □

For a tangent vector $\alpha \in \mathbf{C}^2$ we define

$$Z_\infty(\alpha) = U^+ \cap \left\{\partial G^+ \cdot \alpha = 0\right\} .$$

We note that since Df^k and ∂G^+ are nonsingular $Z_k(\alpha', \beta) \cap Z_k(\alpha'', \beta) = \emptyset$ and $Z_\infty(\alpha') \cap Z_\infty(\alpha'') = \emptyset$ for all α', α'' which define distinct elements of $\mathbf{P}^1$.

Lemma 6.4. *For each nonzero* $\alpha \in \mathbf{C}^2$ *the currents* $[Z_k(\alpha, \beta)]$ *converge to* $[Z_\infty(\alpha)]$ *as currents on* U^+, *uniformly in* β. *That is, if* ψ *is a test form with compact support in* U^+, *then*

$$\lim_{j\to\infty} \max_\beta \left| \int \psi \wedge ([Z_k(\alpha, \beta)] - [Z_\infty(\alpha)]) \right| = 0 .$$

Proof. Since by Lemma 5.2 the projective images of the defining functions of $Z_k(\alpha, \beta)$ converge uniformly, this gives the uniform convergence of the currents. □

Let $V^+(R) = \{|y| > |x|, |y| > R\}$. Since $G^+(x, y) = \log|y| + O(|y|^{-1})$ on $V^+(R)$, it follows that

$$\partial G^+ \cdot \alpha = \frac{\alpha_2}{y} + O\left(|y|^{-2}\right) . \tag{6.3}$$

Multiplying this by y^2, we see that for R large and $\alpha = (1, \alpha_2)$

$$V^+(R) \cap Z_\infty(\alpha) = \{\alpha_2 y + A_1(x, y) + \alpha_2 A_2(x, y) = 0\}$$

where A_1, A_2 are bounded and holomorphic in $V^+(R)$. Thus, we have $|dy/dx| \leq c|y^{-1}|$ on $V^+(R) \cap Z_\infty(\alpha)$, and for $|\alpha_2|$ sufficiently small $V^+(R) \cap Z_\infty(\alpha)$ is a complex disk $\{y = \varphi_\alpha(x) : x \in D_\alpha\}$ satisfying

$$\frac{c'}{|\alpha_2|} \leq |\varphi_\alpha(x)| \leq \frac{c''}{|\alpha_2|} . \tag{6.4}$$

Lemma 6.5. *For any* $c > 0$

$$\lim_{k\to\infty} \int_{\{G^+>c\}} G^+\mu^- \wedge \left(Df^k\right)^* \Theta = \int_{\alpha\in\mathbf{P}^1} \int_{\{G^+>c\}} G^+\mu^- \wedge [Z_\infty(\alpha)] < \infty .$$

Proof. We will first show that for every β

$$\lim_{k\to\infty} \int_{\alpha\in\mathbf{P}^1} \sigma(\alpha) \int_{\{G^+>c\}} G^+\mu^- \wedge [Z_k(\alpha,\beta)] = \int_{\alpha\in\mathbf{P}^1} \int_{\{G^+>c\}} G^+\mu^- \wedge [Z_\infty(\alpha)] < \infty .$$

Let us consider the regions $\{G^+ > c\} \cap \{|y| \leq R\}$ and $\{G^+ > c\} \cap V^+(R)$ separately. The currents $[Z_\infty(\alpha)]$ put no mass on $\{G^+ = c\} \cup \{|y| = R\}$. Thus, by Lemma 6.4, the integrals over the first region converge to the desired limit as $k \to \infty$.

For the second region, we first check that the integral on the right-hand side is finite. For R large, $Z_\infty(\alpha) \cap V^+(R)$ is a complex disk as in (6.4). Thus, $[Z_\infty(\alpha) \cap V^+(R)]$ has total mass 1. Thus, for $\alpha = (1, \alpha_2)$ with $|\alpha_2| \leq \epsilon$, the integral over the second region is no larger than

$$\int_{|\alpha_2|<\epsilon} \log\left(\frac{c}{|\alpha_2|}\right) \sigma(\alpha) < \infty .$$

The convergence of the integrals holds because the disks $Z_k(\alpha, \beta) \cap V^+(R)$ are close to the disks $Z_\infty(\alpha) \cap V^+(R)$ throughout $V^+(R)$, uniformly in k. Since this convergence as $k \to \infty$ holds uniformly in β, we may integrate with respect to β to complete the proof of the lemma. □

Proof of Theorem 6.2. Let us choose P to be an unstable box for which $\mu_c^-(\partial' P) = 0$, and let us write $\lambda_k = G^+\mu^- \wedge (Df^k)^*\Theta \llcorner \tilde{P}$. Since we may exhaust the P in the hypothesis of the theorem by a countable family of such stable boxes, it suffices to show that

$$\lim_{k\to\infty} \frac{1}{k} (\pi_P)_* \lambda_k = G^+\mu_c^- \llcorner P .$$

We may choose $\kappa_2(k)$ as in Lemma 6.3 so that

$$\lim_{k\to\infty} \frac{1}{k} (\pi_P)_* \left(\lambda_k \llcorner \left\{G^+ < d^{-k+\kappa_2}\right\}\right) = 0 .$$

Now for any positive integer κ_1 we set $c = d^{-\kappa_1}$, so by Lemma 6.5 the integrals $\int G^+\mu^- \wedge (Df^k)^*\Theta$, with $k \geq 0$, are all bounded by a number $m(\kappa_1)$. We may define a function $\kappa_1(k)$ to increase to infinity sufficiently slowly that $k^{-1}m(\kappa_1(k)) \to 0$ as $k \to \infty$. It follows, then, that

$$\lim_{k\to\infty} \frac{1}{k} (\pi_P)_* \left(\lambda_k \llcorner \left\{G^+ > d^{-\kappa_1}\right\}\right) = 0 .$$

Choosing κ_1 possibly smaller, we also have $\lim_{k\to\infty} k^{-1}(\kappa_2+\kappa_1)=0$. Now for $j=j_k$ satisfying $\kappa_1 \le j \le k-\kappa_2$ it follows from Theorem 5.9 that

$$\lim_{k\to\infty} f_*^j\left(\lambda_k \llcorner f^{-j}P\right) = G^+\mu_c^- \llcorner P\ .$$

Thus from the uniformity of the convergence in Theorem 5.9 we have

$$\lim_{k\to\infty}\frac{1}{k}(\pi_P)_*\left(\lambda_k \llcorner \left\{d^{-k+\kappa_2} \le G^+ \le d^{-\kappa_1}\right\}\right)$$
$$= \lim_{k\to\infty}\frac{1}{k}\sum_{j=\kappa_1}^{k-\kappa_2} f_*^j\left(\lambda_k \llcorner f^{-j}P\right) = G^+\mu_c^- \llcorner P\ ,$$

which completes the proof. □

Corollary 6.6.

$$\lambda^-(f) = -\log d - \int_{\{1\le G^- < d\}} G^-\mu_c^+\ .$$

Proof. We can apply the integral formula to f^{-1}. The corresponding invariant measure is the same, i.e., $\mu_f = \mu_{f^{-1}}$. Replacing f by f^{-1} interchanges the role of stable and unstable directions and changes the signs of the exponents. If we write $\lambda^+(f)$ and $\lambda^-(f)$ for the Lyapunov exponents of f, then we observe that $\lambda^-(f) = -\lambda^+(f^{-1})$. Thus, the integral formula applied to f^{-1} yields the formula above. □

The following characterization is a consequence of the integral formula:

Corollary 6.7. *The following are equivalent:*

1. $\lambda^+(\mu) = \log d$.
2. $\mu_c^- = 0$.
3. *For* μ *a.e.* x, $G^+|_{W^u(x)-K^+}$ *has no critical points.*

Proof. The measure μ_c^- has all of its mass on the set $G^+ > 0$, so by Theorem 6.1 and the remark following, if $\Lambda_+ = \log d$, then $\mu_c^- = 0$. Thus, (1) implies (2). The construction of the measure shows that if the measure vanishes, then $G^+|_{W^u(x)-K^+}$ can have no critical points for μ a.e. x. So (2) implies (3). Similarly, if $G^+|_{W^u(x)-K^+}$ has no critical points, then the measure μ_c^- is zero, so (3) implies (1) by Theorem 6.1 and the remark following. □

Applying Corollary 6.7 to f^{-1} gives:

Corollary 6.8. *The following are equivalent:*

1. $\lambda^-(\mu) = -\log d$.
2. $\mu_c^+ = 0$.
3. *For* μ *a.e.* x, $G^-|_{W^s(x)-K^-}$ *has no critical points.*

In [BS6] we will explore further the topological consequences of the nonexistence of critical points. In particular we will show that if $\mu_c^\pm = 0$, then $\mathcal{W}^u \cap U^+$ is in fact a locally trivial lamination, so that the critical points satisfy $\mathcal{C}^{s/u} = \emptyset$ in a strong pointwise sense (not just on unstable manifolds of Pesin regular points).

We close by noting some relations between the existence of stable and unstable critical points and the Jacobian determinant of f. Recall that the Jacobian determinant of a polynomial diffeomorphism, $\det Df_p$, depends only on f and not on the point p. If $|\det Df| < 1$, f is said to be dissipative. If $|\det Df| = 1$, f is said to be volume preserving.

Proposition 6.9. *If f is dissipative, then $\mathcal{C}^s \neq \emptyset$. If f is volume preserving, then $\mathcal{C}^s = \emptyset$ if and only if $\mathcal{C}^u = \emptyset$.*

Remark. When $\det Df = 1$, then f is conjugate to its inverse so that the equivalence of the conditions $\mu_c^+ = 0$ and $\mu_c^- = 0$ is clear.

Proof. It is a general property of Lyapunov exponents that the sum of the exponents is related to the Jacobian determinant. We have $\lambda^+(\mu) + \lambda^-(\mu) = \int \log|\det f| d\mu = \log|\det f|$. Combining this fact with the integral formulas for $\lambda^\pm(\mu)$ gives:

$$\int_{\{1 \le G^+ < d\}} G^+ \mu_c^- - \int_{\{1 \le G^- < d\}} G^- \mu_c^+ = \log|\det Df| \, .$$

The contribution of each integral is non-negative and is positive when the corresponding measure is non-zero. When f is dissipative, the right-hand side of the equation is negative. It follows that the value of the second integral must be non-zero; hence, the equivalent conditions of Corollary 6.8 are all false. When f is volume preserving, then the right-hand side of the equation is zero. It follows that the value of the integrals are equal. Hence, the equivalent conditions of Corollary 6.7 are equivalent to those of Corollary 6.8. □

A. Appendix: Lyapunov exponent of real horseshoes

Let f_R be a polynomial automorphism of degree d with real coefficients, so $f_R : \mathbf{R}^2 \to \mathbf{R}^2$ has a real polynomial inverse. Let us suppose that there is a topological square $D \subset \mathbf{R}^2$ such that f_R maps D across itself d times. A heuristic version of the case $d = 3$ is shown in Figure 1; the horizontal lines represent stable manifolds. This situation occurs for the mapping

$$f : (x, y) \mapsto \left(y, y^d + c_{d-2}y^{d-2} + \ldots + c_0 - ax\right)$$

in a non-empty real parameter region, for instance, if $d = 2$ and $-c_0 \gg 0$ or if $d = 3$ and $-c_1 \gg |c_0|^{2/3}$. In this case, f has a weak d-fold horseshoe, and if follows (see Friedland and Milnor [FM]) that f_R is topologically conjugate on the set $K_R := \bigcap_{n \in \mathbf{Z}} f^n B$ to the bilateral shift on d symbols. In this case f_R has topological entropy equal to $\log d$, and by [BLS1] $K_R = J_C = K_C \subset \mathbf{R}^2$, where J_C and K_C denote the sets J and K for the complexified mapping $f_C : \mathbf{C}^2 \to \mathbf{C}^2$.

We let $V_1, \ldots, V_d$ denote the (vertical) components of $D \cap fD$. Then there are components $B_1, \ldots, B_{d-1}$ of $fD - J^+$ with the property that B_j intersects two distinct vertical components. These are the fundamental bends; the case $d = 3$ is depicted in Figure 1. We let $\mathcal{C}_{0,j}$ denote the set of critical points lying in the jth fundamental bend, i.e., $\mathcal{C}_{0,j} = B_j \cap \mathcal{C}^u$. Thus, $\mathcal{C}_0 := \mathcal{C}_{0,1} \cup \cdots \cup \mathcal{C}_{0,d-1}$ are all the critical points that lie in the fundamental bends. The critical points of the nth image under f, $n \in \mathbf{Z}$, are defined as $\mathcal{C}_n = f^n \mathcal{C}_0$.

Lemma A.1. *Let f be a d-fold real horsehoe, as above. For every $x \in J$, the restriction $G^+|W^u(x)$ has the property that every component of $\{G^+|W^u(x) < c\}$ is relatively compact in $W^u(x)$.*

Proof. Let $W^u(p)$ be the unstable manifold of a periodic point p. Since $\lim_{\zeta \to J} G^+(\zeta) = 0$, and since $W^u(p) \cap J$ is a Cantor set, there is a $\lambda_0 > 0$ such that the component of ω_0 of $W^u(p) \cap \{G^+ < \lambda_0\}$ containing p is relatively compact. If ω is any component of $W^u(p) \cap \{G^+ < \lambda\}$, then $f^{-n}\omega \subset \omega_0$ for n sufficiently large. Thus ω is relatively compact. Now the result follows since f is hyperbolic, and J has a local product structure. □

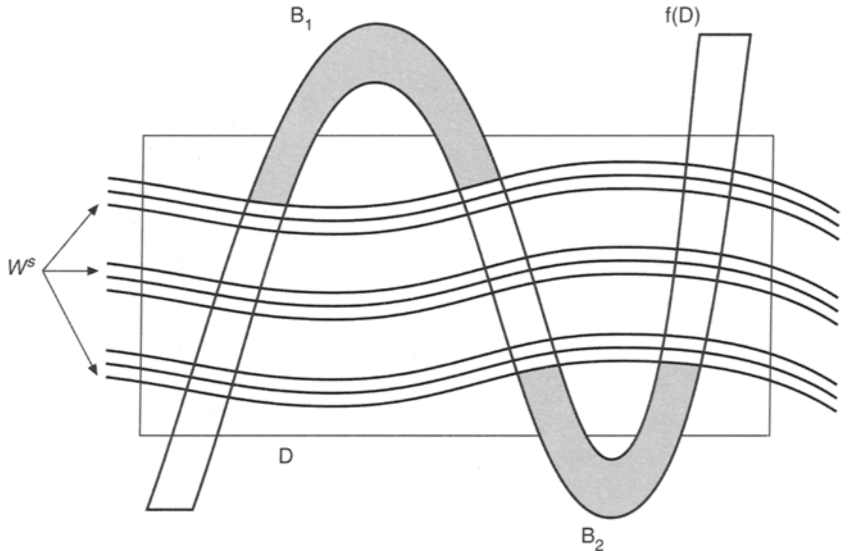

FIGURE 1

Lemma A.2. *Let E be a closed subset of $\mathbf{C}$, and let $L_j \subset \mathbf{R}$ be disjoint, open intervals such that $\mathbf{R} - E = \bigcup L_j$. Let $h \geq 0$ with $h(x+iy) = h(x-iy)$ be continuous on $\mathbf{C}$ and harmonic on $\mathbf{C} - E$, and let $E = \{h = 0\}$. If each connected component of $\{h < c\}$ is bounded, then for each j there exists a unique critical point $c_j \in L_j$. Further, the $\{c_j\}$ are all of the critical points of h.*

Proof. Let ω be a component of $\{h < \lambda\}$. Since ω is relatively compact, it follows from the maximum principle that $E \cap \omega \neq \emptyset$. Since $\tilde{\omega} = \{\bar{z} : a \in \omega\}$ is also a component of $\{h < \lambda\}$, and since $\emptyset \neq \tilde{\omega} \cap E = \omega \cap E \subset \mathbf{R}$, it follows that $\tilde{\omega} = \omega$.

Now we claim that $\omega \cap \mathbf{R}$ is an interval. It is nonempty, and if it contains two components, then by the fact that ω is connected and $\omega = \tilde{\omega}$, we have that $\mathbf{C} - \omega$ contains a compact component. But this contradicts the maximum principle since h is subharmonic on $\mathbf{C}$.

Next suppose that there is a critical point $c \notin \mathbf{R}$. Let ω', ω'' be two components of $\{h < h(c)\}$ which contain c in their boundaries (possibly $\omega' = \omega''$). Since these sets are invariant under complex conjugation, it follows that $\bar{c}$ is also in their boundaries. Thus, the complement of $\omega' \cup \omega'' \cup \{c, \bar{c}\}$ in $\mathbf{C}$ contains a compact component, which violates the maximum principle. Thus, all critical points are real.

Let us fix an interval $L_j = (a_j, b_j)$ and let $\omega_{a_j}(\lambda)$ (resp. $\omega_{b_j}(\lambda)$) denote the component of $\{h < \lambda\}$ containing a_j (resp. b_j). For $\lambda > 0$ sufficiently small, $\bar{\omega}_{a_j}(\lambda) \cap \bar{\omega}_{b_j}(\lambda) = \emptyset$. If there is a critical point $c \in \bar{\omega}_{b_j}(\lambda) \cap L_j$, then we may decrease λ so that $c \in \partial\omega_{b_j}(\lambda) \cap L_j$. Since $\omega_{b_j}(\lambda) = \tilde{\omega}_{b_j}(\lambda)$, there must be a distinct component ω of $\{h < \lambda\}$ such that $c \in \partial\omega$. But since

$\omega \cap \mathbf{R}$ is an interval, and $\omega \cap E \neq \emptyset$, we have $a_j \in \omega$. This is a contradiction, so we have no critical points in $L_j \cap (\omega_{a_j}(\lambda) \cup \omega_{b_j}(\lambda))$ if $\bar{\omega}_{a_j}(\lambda) \cap \bar{\omega}_{b_j}(\lambda) = \emptyset$.

On the other hand, if λ is the supremum of numbers such that $\bar{\omega}_{a_j}(\lambda) \cap \bar{\omega}_{b_j}(\lambda) = \emptyset$, then $\bar{\omega}_{a_j}(\lambda) \cap \bar{\omega}_{b_j}(\lambda) = \{c_j\}$ is the unique critical point in L_j. □

The following theorem shows that the critical points for a horseshoe are arranged in the fashion given schematically in Figure 2.

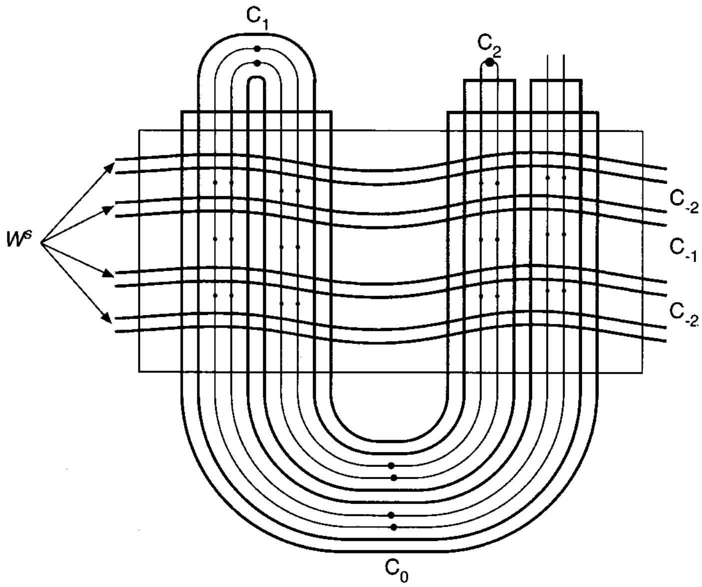

FIGURE 2

Theorem A.3. *Let $x \in J$ be given, and let $W^u_R(x)$ denote the (real) unstable manifold passing through x. For each connected component γ of $W^u_R(x) \cap B_j$ there is a unique critical point c_γ for the complexified mapping f_C. The union of all such critical points gives $\mathcal{C}_0$, and $\mathcal{C}^u = \bigcup_{n \in \mathbf{Z}} \mathcal{C}_n$. In particular, all complex critical points are real.*

Proof. Let $W^u(x)$ denote the complex stable manifold through x, and let $\psi : \mathbf{C} \to W^u(x)$ denote a uniformization. Since f is real, we may replace $\psi(\zeta)$ by $\psi(e^{i\theta}\zeta)$ so that $\psi_{\mathbf{R}} : \mathbf{R} \to W^u_R(x)$. Let $h = G^+ \circ \psi$ and let $E = \psi^{-1}(W^u(x) \cap J)$ so that $\mathbf{R} - E = \bigcup L_j$. Then h is a subharmonic function on $\mathbf{C}$, and by Lemma A.1 it satisfies the hypotheses of Lemma A.2. It follows that all of the critical points of h are real, and so $\mathcal{C}^u \cap W^u(x) \subset \mathbf{R}$. Thus, $\mathcal{C}^u = \bigcup_{x \in J} \mathcal{C}^u \cap W^u(x) \subset \mathbf{R}$. Also by Lemma A.2, we have that each critical point $c \in \mathcal{W}^u(x)$ corresponds uniquely to an interval L_j, and $\psi(L_j)$ corresponds to a connected component γ_c of $W^u_R(x) - \mathcal{W}^s = W^u_R(x) - J$. Now it is a property of the horseshoe that for each component γ_c, there is an $n \in \mathbf{Z}$ such that $f^n \gamma_c \subset B_j$ for some j. □

Remark. If f_R has the form above, then the line $\{x = 0\}$ will intersect the image of any non-horizontal line exactly once. Under iteration, this yields that $\{x = 0\}$ will intersect each component of an unstable manifold in a bend exactly once. Since the total mass of the intersection measure $\mu^- \wedge [\{x = 0\}]$ is 1, we see by Theorem A.3 that $\mu^-_c(\mathcal{C}_{0,j}) = 1$. Further, μ^-_c has a balanced property that allows us to define it in terms of the "generational" structure. It suffices to define μ_c^- on $\mathcal{C}_{0,j}$, i.e., inside one of the fundamental bends. For this, we note that $B_j \cap f^n D$ has d^{n-1} connected components. The intersection of any of these components with $\mathcal{C}_0 \cap B_j$ has mass d^{-n+1}, and this defines μ_c^- on all Borel subsets of $\mathcal{C}_0 \cap B_j$.

Theorem A.4. *If f is a real horseshoe mapping as above, then the Lyapunov exponent is given by*

$$\Lambda = \log d + \int_{\mathcal{C}_0} G^+ \mu_c^- .$$

Further, we have the estimate

$$(d-1) \min_{\mathcal{C}_0} G^+ < \Lambda - \log d < (d-1) \max_{\mathcal{C}_0} G^+ .$$

Proof. The integral formula follows from Theorem 6.1 and Theorem A.3. The inequalities follow since $\mu_c^-(\mathcal{C}_0)) = d - 1$, as was remarked above. □

B. Appendix: Heteroclinic tangencies in $U^+ \cap U^-$

We discuss the behavior of f on $U^+ \cap U^-$. Conversations with Hubbard have been helpful for our understanding of the critical locus in this region. The map $G^+ : U^+ \to \mathbf{R}^+$ has been studied as a fibration in [H, HO], where it was shown that the level sets $\{G^+ = c\}$ are foliated by complex manifolds which are dense and conformally equivalent to $\mathbf{C}$.

It is shown in [H, HO] that we have an analytic function φ^+ on V^+ given by the formula

$$\varphi^+(x, y) = \lim_{n\to\infty} \left(\pi_2 \circ f^n(x, y)\right)^{\frac{1}{d^n}} ,$$

where we take the d^nth root so that $\varphi^+(x, y) = y + o(1)$ holds on V^+. It is immediate that $\varphi^+ \circ f = (\varphi^+)^d$ and $\log|\varphi^+| = G^+$ hold on V^+. In particular, φ^+ is locally constant on the leaves of $\mathcal{G}^+$.

For $|\xi| > R$,

$$\Delta_\xi := \left\{p \in V^+ : \varphi^+(p) = \xi\right\}$$

is a complex disk, and $f\Delta_\xi \subset \Delta_{\xi^d}$. By the trapping property of V^+ the global leaf L_ξ of $\mathcal{G}^+$ which contains Δ_ξ has the form

$$L_\xi = \bigcup_{n=1}^{\infty} f^{-n} \Delta_{\xi^{d^n}} ,$$

and it is evident that $L_\xi \cap V^+ = \bigcup L_{\xi'}$, where the union is taken over all ξ' such that $\xi'\xi^{-1}$ is a d^nth root of unity.

Proposition B.1. *The global leaves of $\mathcal{G}^+$ are the super-stable manifolds of f.*

Proof. By Lemma 1.2, $Df^n|_{T\mathcal{G}^+}$ decreases super-exponentially as $n \to +\infty$. Thus, any two points ζ', ζ'' in the same global leaf of $\mathcal{G}^+$ approach each other super-exponentially as $n \to +\infty$. Conversely, suppose that $\zeta', \zeta'' \in U^+$ are not in the same global leaf U^+. Then for $n \geq n_0$, $f^n\zeta', f^n\zeta'' \in V^+$, but

$$\varphi^+\left(f^n\zeta'\right) = \varphi^+\left(f^{n_0}\zeta'\right)^{n-n_0} \neq \varphi^+\left(f^{n_0}\zeta''\right)^{n-n_0} = \varphi^+\left(f^n\zeta''\right) .$$

Thus $\varphi^+(f^n\zeta')$ does not tend to $\varphi^+(f^n\zeta'')$, and since $\varphi^+ \approx y$ on V^+, it follows that $f^n\zeta'$ does not tend to $f^n\zeta''$. □

The 2-form $\partial G^+ \wedge \partial G^-$ is invariant under f, and its zero locus defines the dynamical critical locus of f:

$$\mathcal{C} := \left\{(x, y) \in U^+ \cap U^- : \tau^+ = \tau^-\right\} = \left\{\partial G^+ \wedge \partial G^- = 0\right\} .$$

Thus, the critical locus consists of the points where the forward and backward critical directions coincide; thus, it can be thought of as the set of heteroclinic tangencies of the super-stable and super-unstable manifolds.

For $\epsilon > 0$ there exists R_ϵ such that

$$\left(G^+, G^-\right) \approx (\log|y|, \log|x|), \quad \text{and} \quad \partial G^+ \wedge \partial G^- \approx \frac{dx \wedge dy}{4xy} \tag{B.1}$$

for $\epsilon|x| < |y| < \epsilon^{-1}|x|,\ |x| > R_\epsilon$, and so

$$\mathcal{C} \cap \left\{\epsilon|x| < |y| < \epsilon^{-1}|x|,\ |x| > R_\epsilon\right\} = \emptyset\,. \tag{B.2}$$

The inclusions of sets $\iota_\pm : U^+ \cap U^- \to U^\pm$ induce mappings on homology

$$\iota_{\pm *} : H_1\left(U^+ \cap U^-, \mathbf{Z}\right) \to H_1\left(U^\pm, \mathbf{Z}\right) . \tag{B.3}$$

Lemma B.2. *The mapping (B.3) is surjective, and $H_1(U^+ \cap U^-, \mathbf{Z})$ is not finitely generated.*

Proof. For R large, consider the curve $\gamma : \theta \mapsto (Re^{i\theta}, Re^{i\theta})$, which is contained in $V^+ \cap V^- \subset U^+ \cap U^-$. Now $\varphi^+_*\gamma$ is approximately the circle of radius R in $\mathbf{C} - \bar{\Delta}$, so it defines a nontrivial homology class, and thus γ defines a nontrivial element of $H_1(U^\pm, \mathbf{Z})$ in the range of $\iota_{\pm *}$. Since the range is nonzero and invariant under f_*^k for $k \in \mathbf{Z}$, the maps $\iota_{\pm *}$ are onto. Finally, $H_1(U^+ \cap U^-, \mathbf{Z})$ is not finitely generated because its image is not. (See [HO] for these last two facts about $H_1(U^\pm, \mathbf{Z})$.) □

Proposition B.3. $\mathcal{C} \neq \emptyset$.

Proof. We consider the fibration

$$G = \left(G^+, G^-\right) : U^+ \cap U^- \to \mathbf{R}^+ \times \mathbf{R}^+ ,$$

which has compact fiber. By (B.1), the fiber of G over points of $\{\epsilon|x| < |y| < \epsilon^{-1}|x|,\ |x| > R_\epsilon\}$ is a 2-torus. If $\mathcal{C} = \emptyset$, then $dG^+ \wedge dG^- \neq 0$, and the fibration is locally trivial. Since the base of the fibration is topologically trivial, it follows that

$$H_1\left(U^+ \cap U^-, \mathbf{Z}\right) \cong H_1\left(\mathbf{T}^2, \mathbf{Z}\right) \cong \mathbf{Z}^2 .$$

But this is not possible since, by Lemma B.2, $H_1(U^+ \cap U^-, \mathbf{Z})$ is not finitely generated. Thus, $\mathcal{C} \neq \emptyset$. □

Proposition B.4. *$\bar{\mathcal{C}} \cap J^+ \cap U^- \neq \emptyset$ and $\bar{\mathcal{C}} \cap J^- \cap U^+ \neq \emptyset$.*

Proof. If $\bar{\mathcal{C}} \cap J^- \cap U^+ = \emptyset$, then $\mathcal{C}$ is a closed subvariety of U^+. Let $\mathcal{C}' = \mathcal{C} \cap \{|y| > R,\ |y| > |x|\}$. By (B.2), $\pi_2|_{\mathcal{C}'} : \mathcal{C}' \to \{|y| > R\}$ is proper, so it has some degree δ. This degree multiplies by d under the mapping f. But since $f\mathcal{C} = \mathcal{C}$, this degree must stay constant. Thus, we conclude that $\bar{\mathcal{C}} \cap J^- \cap U^+ \neq \emptyset$. The argument to show $\bar{\mathcal{C}} \cap J^+ \cap U^- \neq \emptyset$ is the same. □

References

[BS1] Bedford, E. and Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$: Currents, equilibrium measure and hyperbolicity, *Invent. Math.,* **87**, 69–99, (1990).

[BS3] Bedford, E. and Smillie, J. Polynomial diffeomorphims of $\mathbf{C}^2$, III: Ergodicity, exponents and entropy of the equilibrium measure, *Math. Ann.,* **294**, 395–420, (1992).

[BS6] Bedford, E. and Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$, VI: Connectivity of J. *Annals of Math.,* **148**, 695–735, (1998).

[BS7] Bedford, E. and Smillie, J. Polynomial diffeomorphisms of $\mathbf{C}^2$, VII: Hyperbolicity and external rays. *Annales Scientifiques de L'Ecole Normale Supérieure,* to appear.

[BLS1] Bedford, E., Lyubich, M., and Smillie, J. Polynomial diffeomorphims of $\mathbf{C}^2$, IV: The measure of maximal entropy and laminar currents, *Invent. Math.,* **112**, 77–125, (1993).

[BLS2] Bedford, E., Lyubich, M., and Smillie, J. Distribution of periodic points of polynomial diffeomorphisms of $\mathbf{C}^2$, *Invent. Math.,* **114**, 277–288, (1993).

[BC] Benedicks, M. and Carleson, L. The dynamics of the Hénon map, *Ann. Math.,* **133**, 73–169, (1991).

[BY] Benedicks, M. and Young, L-S. Sinai–Bowen–Ruelle measures for certain Hénon maps, *Invent. Math.,* **112**, 541–576, (1993).

[Br] Brolin, H. Invariant sets under iteration of rational functions, *Ark. Mat.,* **6**, 103–144, (1965).

[C] Chirka, E. *Complex Analytic Sets,* Kluwer, (1985).

[DH] Douady, A. and Hubbard, J.H. Itération des polynômes quadratiques complexes, *C.R. Acad. Sci. Paris Série I,* **294**, 123–126, (1982).

[FM] Friedland, S. and Milnor, J. Dynamical properties of plane polynomial automorphisms, *Ergodic Theory Dyn. Syst.,* **9**, 67–99, (1989).

[H] Hubbard, J.H. Hénon mappings in the complex domain, in *Chaotic Dynamics and Fractals,* Barnsley, M. and Demko, S., Eds., Academic Press, 101–111, (1986).

[HO] Hubbard, J.H. and Oberste-Vorth, R. Hénon mappings in the complex domain I: The global topology of dynamical space, *Inst. Hautes Études Sci. Publ. Math.,* **79**, 5–46, (1994).

[LY] Ledrappier, F. and Young, L.–S. The metric entropy of diffeomorphisms, I & II, *Ann. Math., 122,* 509–539 and 540–574, (1985).

[Mn] Manning, A. The dimension of the maximal measure for a polynomial map, *Ann. Math.,* **119**, 425–430, (1984).

[Po] Pollicott, M. Lectures on ergodic theory and Pesin theory on compact manifolds, London Mathematical Society Lecture Note Series 180, Cambridge University Press, (1993).

[Pr] Przytycki, F. Hausdorff dimension of harmonic measure on the boundary of an attractive basin for a holomorphic map, *Invent. Math.,* **80**, 161–179, (1985).

[Ps] Pugh, C. and Shub, M. Ergodic attractors, *Trans. AMS,* **312**, 1–54, (1989).

[Si] Sibony, N. Iteration of polynomials, U.C.L.A. Course Lecture Notes.

[T] Tortrat, P. *Aspects potentialistes de l'itération des polynômes,* In *Séminaire de Théorie du Potentiel Paris,* No. 8, (Lect. Notes Math., 1235), Springer-Verlag, (1987).

[W] Wu, H. Complex stable manifolds of holomorphic diffeomorphisms, *Indiana U. Math. J.,* **42**, 1349–1358, (1993).

Received June 21, 1996

Department of Mathematics, Indiana University, Bloomington, IN 47405
e-mail: bedford@indiana.edu

Department of Mathematics, Cornell University, Ithaca, NY 14853
e-mail:smillie@math.cornell.edu

Communicated by John Erik Fornæss