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Approximation by Spherical Waves in L^p-Spaces

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ABSTRACT. The paper is devoted to the following problem. Consider the set of all radial functions with centers at the points of a closed surface in $\mathbf{R}^n$. Are such functions complete in the space $L^q(\mathbf{R}^n)$? It is shown that the answer is positive if and only if q is not less than $2n/(n+1)$. A similar question is also answered for Riemannian symmetric spaces of rank 1. Relations of this problem with the wave and heat equations are also discussed.

Introduction

The following question was posed by Lin and Pinkus [LP1, LP2]: describe sets $\Gamma \subset \mathbf{R}^n, n \geq 2$, such that the system of shifted radial functions (spherical waves)

$$\psi(|x-a|),\ a \in \Gamma,\ \psi \text{ is a function of one variable,} \tag{1}$$

is complete in different spaces of functions of n variables.

In [AQ1, AQ2] a complete solution of this problem was given for the space $C(\mathbf{R}^2)$ of all continuous functions in the plane, equipped with the topology of uniform convergence on compact sets. It was proved that a set $\Gamma \subset \mathbf{R}^2$ provides completeness if and only if $\Gamma \not\subset \Sigma_N \cup F$, where F is a finite set and Σ_N is a union of finite number of straight lines through one point, invariant under a finite Coxeter reflection group.

In particular, it follows that any closed curve $\Gamma \subset \mathbf{R}^2$ generates a complete system of spherical waves (1) in $C(\mathbf{R}^2)$. This partial result can be extended to higher dimensions and some other function spaces. In fact, completeness of the system (1) in $C(\mathbf{R}^n)$ is equivalent to the statement that any measure μ with compact support that annihilates all functions of the form (1) is equal to zero. Consider the spectral projection of such μ onto the eigenspaces of the Laplace operator [S]:

$$\psi_\lambda(x) = (\mathcal{P}_\lambda \mu)(x) = \int_{\mathbf{R}^n} \varphi_\lambda(|x-y|)\, d\mu(y),$$

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where $\varphi_\lambda(r) = (2\pi)^{\frac{n}{2}}\lambda^{\frac{n}{2}}r^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(\lambda r)$, J_k is the Bessel function of order k. Then $\Delta\psi_\lambda = -\lambda^2\psi_\lambda$ and, since $\varphi_\lambda(|x-y|)$ is of the form (1) when $x \in \Gamma$, we have $\psi_\lambda|_\Gamma = 0$. If Γ is the boundary of a bounded domain $\Omega \subset \mathbf{R}^n$, then the discreteness of the spectrum of the Dirichlet problem in Ω for the Laplacian implies that $\psi_\lambda \equiv 0$ except for a discrete set of λ and therefore $\mu = 0$.

Let us now consider a function $f \in L^q(\mathbf{R}^n)$ with $1 \le q < \frac{2n}{n+1}$ and introduce the measure $\mu = f dx$. Then ψ_λ can be defined as above and the same argument shows that $f = 0$. Thus we obtain completeness of the system (1) corresponding to the boundary Γ of a bounded domain in the predual spaces: $L^\infty(\mathbf{R}^n)$ with the weak*-topology and $L^p(\mathbf{R}^n)$ with $p > \frac{2n}{(n-1)}$. Taking f in the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ of rapidly decaying functions, we get completeness in the space of functions (or distributions) of tempered growth.

For $p \le 2n/(n-1)$ the spectral projections ψ_λ cannot be defined directly. The reason is that the Fourier transform of φ_λ is the δ-function on the sphere $S(0,\lambda)$ of center 0 and radius λ, and the Fourier transform of $f \in L^q(\mathbf{R}^n)$, $1/p + 1/q = 1$, is, generally speaking, a distribution so that its restriction to $S(0,\lambda)$ is not necessarily well defined.

For that reason we apply a different argument, which however has spectral projections behind it. The approach we use is based on the wave equation and it allows us to get a complete answer for L^p-spaces when Γ is the boundary of a bounded domain. This method was first used by the third author in order to solve the Lin-Pinkus's problem for $C(\mathbf{R}^2)$ in the case of closed curves. The problem was later solved for $C(\mathbf{R}^2)$ in full generality in [AQ1, AQ2] by a different method.

Our main result can be formulated as follows: the boundary Γ of any bounded domain $\Omega \subset \mathbf{R}^n$ generates a complete system of spherical waves (1) in the space $L^p(\mathbf{R}^n)$ as long as $p \ge 2n/(n+1)$. This property fails for $p < 2n/(n+1)$.

Thus, the estimate $p > 2n/(n-1)$ found above is not sharp and, in fact, completeness holds for p starting with $p = 2n/(n+1)$.

The critical index $q = 2n/(n-1)$, conjugate to $p = 2n/(n+1)$, is the exponent of integrability of the spherical functions in $\mathbf{R}^n$ and naturally appears as a bound in uniqueness theorems for the Pompeiu transform [RS].

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1. Main results

1.1. We consider the space $L^p(\mathbf{R}^n)$, $n \ge 2$, $1 \le p < \infty$. Given a point $a \in \mathbf{R}^n$ we denote by $W_{a,p}$ the linear subspace of $L^p(\mathbf{R}^n)$ of all functions of the form

$$\psi(|x-a|),\ \psi \in L^p(\mathbf{R}_+, r^{n-1}dr), \qquad \mathbf{R}_+ = (0,\infty).$$

For any set $\Gamma \subset \mathbf{R}^n$ we consider the linear span $W_p(\Gamma)$ generated by all $W_{a,p}$ with $a \in \Gamma$. We are interested in the density of $W_p(\Gamma)$ in $L^p(\mathbf{R}^n)$.

Theorem 1. *Let Γ be the boundary of a bounded domain in $\mathbf{R}^n$. Then the subspace $W_p(\Gamma)$ is dense in the space $L^p(\mathbf{R}^n)$ with $p \geq 2n/(n+1)$. If $p < 2n/(n+1)$, then this statement fails, for instance for $\Gamma = S^{n-1}$.*

In the particular case $\Gamma = S^{n-1}$ this result was proved earlier in [V]. If Γ is a sphere, one can use harmonic analysis on the group $SO(n)$, which of course is not applicable when Γ has no group symmetries.

1.2. Denote by $W_p^{\perp}(\Gamma)$ the annihilator of $W_p(\Gamma)$ in the dual space $L^q(\mathbf{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$. The space $W_p^{\perp}(\Gamma)$ consist of all $f \in L^q(\mathbf{R}^n)$ such that

$$(\psi * f)(a) = \int_{\mathbf{R}^n} \psi(|x-a|) f(x)\, dx = 0 \tag{2}$$

for all $a \in \Gamma$ and all $\psi \in L^p(\mathbf{R}, r^{n-1}dr)$.

The condition (2) is preserved if we replace f by the convolution $\varphi * f$ with any radial function $\varphi \in C_0^\infty(\mathbf{R}^n)$. The invariance of $W_p^{\perp}(\Gamma)$ with respect to convolutions with radial functions implies that $W_p^{\perp}(\Gamma) \cap C^\infty(\mathbf{R}^n)$ is dense in $W_p^{\perp}(\Gamma)$.

We will need the following simple result.

Lemma 1.1. *Let $g \in L^q(\mathbf{R}^n)$ and $\phi \in C_0^\infty(\mathbf{R}^n)$. Then for any bounded domain $U \subseteq \mathbf{R}^n$ and any integer $k \geq 0$, the convolution*

$$f = \phi * g$$

satisfies the following condition: the function

$$F(x, y) = f(x - y),$$

as a function of x with values in the space of functions of y, belongs to the space

$$L^q(\mathbf{R}^n, H^k(U)),$$

where $H^k(U)$ is the Sobolev space of order k in the domain U. In particular, any derivative of f belongs to $L^q(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$.

Proof of the lemma. Since derivations of the function f can be carried on ϕ, it is sufficient to prove that F belongs to the space $L^q(\mathbf{R}^n, L^\infty(U))$. Let the support of ϕ be a subset of a ball B

in $\mathbf{R}^n$. Then

$$f(x-y) = \int_B g(x-(y+z))\phi(z)\,dz = \int_{B+y} g(x-z)\phi(z-y)\,dz.$$

Thus,

$$\int \max_{y\in U} |f(x-y)|^q\,dx \le \int \max_{y\in U} \left(\int_{b_y} |g(x-z)||\phi(z-y)|\,dz \right)^q dx.$$

Applying the Hölder inequality, we conclude that this quantity can be estimated from above by

$$\int \max_{y\in U} \int_{B+y} |g(x-z)|^q\,dz \left(\int_{B+y} |\phi(z-y)|^p\,dz \right)^{q/p} dx$$

$$\le C \int_{\mathbf{R}^n} \int_{B+U} |g(x-z)|^q\,dz\,dx.$$

Switching the order of integration and using the boundedness of $B+U$, we estimate this from above by

$$C_1 \|g\|^q_{L^q(\mathbf{R}^n)},$$

which proves the first statement of the lemma. Any derivative D of order j of the function f can be expressed as $(-1)^{|j|} D_y F|_{y=0}$, so the inclusion $Df \in L^q(\mathbf{R}^n)$ follows from the first statement. Let us now divide the space $\mathbf{R}^n$ into cubes U_j, then the sequence

$$\|Df\|_{L^q(U_j)}$$

belongs to the space ℓ^q, and hence it is bounded. Using the Sobolev embedding theorems we conclude that any derivative of f is bounded. □

Let from now on $f \in L^q(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)$ satisfy the conclusion of the Lemma 1.1. It follows from (2) that $f \in W_p^\perp(\Gamma)$ if and only if

$$Rf(x,t) = 0 \text{ for all } (x,t) \in \Gamma \times \mathbf{R}_+,$$

where R is the spherical mean operator

$$Rf(x,t) = \int_{|y|=1} f(x+ty)\,dA(y),$$

and dA is the normalized area measure on S^{n-1}. We say that Γ is a *set of injectivity* for the operator R in $L^q(\mathbf{R}^n)$ if

$$Rf|_{\Gamma\times\mathbf{R}_+} = 0 \text{ for } f \in L^q(\mathbf{R}^n) \text{ implies } f = 0.$$

Due to the Hahn-Banach theorem, $W_p(\Gamma)$ is dense in $L^p(\mathbf{R}^n)$, $1 \le p < \infty$, if and only if Γ is a set of injectivity for the operator R in $L^q(\mathbf{R}^n)$, $1/p + 1/q = 1$. This shows that Theorem 1 is equivalent to the following theorem.

Theorem 2. *The boundary Γ of any bounded domain $\Omega \subset \mathbf{R}^n$ is a set of injectivity for the spherical mean operator R in $L^q(\mathbf{R}^n)$ as long as $q \le 2n/(n-1)$. The property fails for $q > 2n/(n-1)$.*

Let us comment on the last assertion. Consider the following example suggested by L. Zalcman: let

$$\varphi(x) = |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\lambda|x|),$$

where λ is a zero of the Bessel function $J_{\frac{n}{2}-1}$. Then φ is an eigenfunction of the operator R, namely,

$$R\varphi(x,t) = c\varphi(t)\varphi(x)$$

and therefore $R\varphi(x,t) = 0$ if $|x| = 1$. Since $\varphi \in L^q(\mathbf{R}^n)$ for any $q > 2n/(n-1)$, the injectivity fails on this range of values of q when Γ is the unit sphere.

1.3. Let us now relate the previous problem to the wave equation. We consider the Cauchy problem for the wave equation in $\mathbf{R}^n$ for $u = u(x,t)$, $x \in \mathbf{R}^n$, and $t > 0$:

$$\begin{aligned} u_{tt} &= \Delta u, \\ u(x,0) &= 0, \\ u_t(x,0) &= f(x). \end{aligned} \tag{3}$$

One can extend the solution u uniquely to the whole time axis by assuming that $u(x,-t) = -u(x,t)$ for all $t \in \mathbf{R}$.

The initial velocity f is supposed to belong to $L^q(\mathbf{R}^n) \cap C(\mathbf{R}^n)$. Using convolutions with smooth compactly supported radial functions one can reduce all subsequent considerations to the case where f belongs to $L^q(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)$ and satisfies the conclusions of Lemma 1.1. The key point is the following lemma.

Lemma 1.2. *If $Rf|_{\Gamma\times\mathbf{R}_+} = 0$, then $u|_{\Gamma\times\mathbf{R}_+} = 0$.*

Proof. It follows from the Kirchhoff-Poisson formula that

$$u(x,t) = \text{const}(\partial_t)^{n-2} F(x,t) \tag{4}$$

where

$$F(x,t) = \int_0^t (t^2 - r^2)^{(n-3)/2} r Q(x,r)\,dr,$$
$$Q(x,r) = \int_{|y|=1} f(x+ry)\,dA(y).$$

Now, one just needs to note that for $x \in \Gamma$ the values $F(x,t)$ and $u(x,t)$ can be expressed in terms of $Rf|_{\Gamma\times\mathbf{R}_+}$. □

In fact, one can also show that the converse of the statement in Lemma 1.2 is also true, although we will not use this. The proof employs the inversion of the Abel transform.

Lemma 1.2 says that Theorem 1 and Theorem 2 follow from (and, in fact, are equivalent to) the next statement.

Theorem 3. *Let Γ be the boundary of a bounded domain in $\mathbf{R}^n$, $n \geq 2$. Suppose that the solution $u(x,t)$ of the Cauchy problem* (3) *with initial data $f \in L^q(\mathbf{R}^n)$ satisfies the condition*

$$u(x,t) = 0 \text{ for all } x \in \Gamma \text{ at any time } t > 0.$$

Then $u \equiv 0$ as long as $q \leq 2n/(n-1)$. This theorem fails for $q > 2n/(n-1)$.

Remark. For $n = 2$, Theorem 3 can be interpreted as follows: an oscillating infinite membrane cannot remain stationary on a closed curve as long as the initial velocity satisfies

$$\int_{\mathbf{R}^2} |f(x)|^q\,dx < \infty \text{ with } q \leq 4.$$

If $q > 4$, then closed curves (for instance, circles) can remain stationary. An example is the following solution of the wave equation

$$u(x,t) = \sin(\lambda_k t) J_0(\lambda_k |x|), \text{ with } J_0(\lambda_k r) = 0.$$

This effect can be qualitatively described as follows: If a closed curve remains fixed, this means that the energy stays constant in this region. If the initial energy distribution dies out at infinity fast enough, then the energy must decay locally, which is a contradiction. On the other hand, if the decay of the initial energy distribution at infinity is not fast enough, then sufficient energy can come from the regions located far away to support a fixed energy in our bounded region.

Another interpretation of our result is that for the values of q described in the Theorem 3, one can observe the motion of the membrane over any closed curve, and the obtained data uniquely determines the motion of the whole membrane. The observations, however, must be made for all values of time $t > 0$. (This is in the spirit of [E].)

A result similar to the Theorem 3 holds also for the heat equation.

Let us mention that if f has compact support, then it is proved in [AQ1, AQ2] that the conditions to be satisfied by the nodal sets are more restrictive. Namely, not only closed curves but any *nonlinear* curve cannot remain stationary in this case. We do not know what is the general answer for the case when $f \in L^p$.

2. Wave equation

In this section we prove Theorem 3 which, as we have shown, implies (and is in fact equivalent to) Theorems 1 and 2.

Let us consider the Cauchy problem (3) with initial data $f \in L^q, q \leq 2n/(n-1)$. After convolution with a radial function we can assume by Lemma 1.1 that $f \in L^q(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)$ and $f(x-y) \in L^q(\mathbf{R}^n, H^k(U))$ for any nonnegative integer k and any bounded region U. We extend the solution $u(x,t)$ for $t < 0$ by $u(x,-t) = -u(x,t)$.

Let Γ be a closed hypersurface in $\mathbf{R}^n$ for which $u(x,t) = 0$, $x \in \Gamma$, for any $t \in \mathbf{R}$.

Lemma 2.1. *For any bounded domain U in $\mathbf{R}^n$ and any nonnegative integer k, the solution $u(x,t)$ of (3) has polynomial growth in t as a function with values in $H^k(U)$. In particular, $u \in \mathcal{S}'(\mathbf{R}, H^k(U))$. For any $h \in \mathcal{S}(\mathbf{R})$ the function*

$$(h *_t u)(x,t) = \int_{\mathbf{R}} h(t-s)u(x,s)\,ds$$

belongs to $L^q(\mathbf{R}^n)$ for any fixed t.

Proof. Polynomial growth in the variable t follows from the Kirchhoff-Poisson formula (4) since F is bounded in U along with all its derivatives.

It is sufficient to prove the second assertion for $t = 0$. The t-derivatives in (4) can be carried on h, so we disregard them. Now,

$$\|Q(x,r)\|_{L^q(\mathbf{R}^n_x)} \leq C \int_{|y|=1} \|f\|_{L^q}\, dA(y) = C\|f\|_{L^q},$$

$$\|F(x,t)\|_{L^q(\mathbf{R}^n_x)} \le \int_0^t (t^2-r^2)^{(n-3)/2} r\,dr,$$

and

$$\begin{aligned}\|(h*F)(0,\cdot)\|_{L^q} &\le \int_0^\infty (|h(s)|+|h(-s)|)\|F(x,s)\|_{L^q(\mathbf{R}^q_x)}\,ds\\ &\le C\|f\|_{L^q}\int_0^\infty (|h(s)|+|h(-s)|)\int_0^s (s^2-r^2)^{(n-3)/2} r\,dr\,ds\\ &\le C_1\|f\|_{L^q}.\end{aligned}$$

□

Lemma 2.2. *Let $b(x,t)$ be a solution to the wave equation*

$$b_{tt} = \Delta b, \qquad x\in\mathbf{R}^n, \quad t>0,$$

$$b(x,0)=0, \qquad b_t(x,0)=f(x),$$

where $f(x-y)\in L^q(\mathbf{R}^n, H^k(U))$ for any $k\ge 0$ and any bounded open set U. If there exists an open ball $B(x_0,\epsilon)$ such that $u(x,t)=0$, for $x\in B(x_0,\epsilon)$ and all $t>0$, then $u(x,t)\equiv 0$ and correspondingly $f(x)\equiv 0$.

Proof. The proof of this statement is actually contained in the Section 17, Chapter VI of [CH]. We will provide a brief sketch of a proof different from the one in [CH]. Extend b to $\mathbf{R}^n\times\mathbf{R}$ as a solution u of the wave equation which is odd in t. We can take the Fourier transform $\hat{u}(x,\lambda)$ of $u(x,t)$ with respect to t in the sense of tempered distributions because of Lemma 2.1. Thus $\hat{u}(x,\lambda)\in\mathcal{S}'(\mathbf{R}, H^k(U))$ for any bounded U. Since $u(x,t)$ satisfies the wave equation, we obtain

$$(-\Delta-\lambda^2)\hat{u}(x,\lambda)\equiv 0.$$

Moreover, $\hat{u}(x,\lambda)=0$ for $x\in B(x_0,\epsilon)$. Now the local uniqueness theorem for the last equation shows that $\hat{u}(x,\lambda)\equiv 0$. □

Proof of Theorem 3. Denote by Ω the domain in $\mathbf{R}^n$ bounded by Γ. The negative part of the theorem was discussed already at the end of Section 1. So we assume $q\le 2n/(n-1)$. We can also assume that the initial data f satisfies the conclusions of Lemma 1.1. We want to conclude that if u vanishes on $\Gamma\times\mathbf{R}$, then $u\equiv 0$. First of all, let us notice that the Dirichlet Laplacian in Ω generates a self-adjoint operator D in $L^2(\Omega)$ with the discrete spectrum $\{-\lambda_k^2\}$ (see Theorem 1.4 in Chapter VI of [EE]). Besides, due to smoothness and zero conditions, $u(\cdot,t)$ belongs to the domain of D for any t. Let now $\{\psi_k\}_{k=1}^\infty$ be the corresponding orthonormal basis in $L^2(\Omega)$ of eigenfunctions of D.

For any fixed t decompose $u(x, t)$ into a L^2-convergent series

$$u(x,t) = \sum_{k=0}^{\infty} c_k(t)\psi_k(x).$$

Since u belongs to the domain of D, we have

$$\Delta u(x,t) = \sum_{k=0}^{\infty}(-\lambda_k^2)c_k(t)\psi_k(x).$$

On the other hand, since $u \in \mathcal{S}'(\mathbf{R}, H^k(U))$, we conclude that

$$u''_{tt}(x,t) = \sum_{k=0}^{\infty} c''_k(t)\psi_k(x).$$

Hence,

$$c''_k(t) = -\lambda_k^2 c_k(t).$$

Therefore,

$$c_k(t) = c_k^+ e^{i\lambda_k t} + c_k^- e^{-i\lambda_k t} \tag{5}$$

and $c_k^- = -c_k^+$ since u is odd with respect to t.

Let us fix an arbitrary index k. Choose a real-valued function $h \in \mathcal{S}(\mathbf{R})$ with the following properties:

(a) $h(-t) = h(t)$.

(b) The Fourier transform $\hat{h}$ belongs to $C_0^\infty(\mathbf{R})$ and for some small $\epsilon > 0$ we have

$$\operatorname{supp}\hat{h} \subset (\lambda_k - \epsilon, \lambda_k + \epsilon) \cup (-\lambda_k - \epsilon, -\lambda_k + \epsilon).$$

(c) $\hat{h}(\lambda_k) = \hat{h}(-\lambda_k) = 1$.

(d) $\operatorname{supp}\hat{h}$ does not contain any of the points $\pm\lambda_j$ for $j \neq k$.

Due to Lemma 2.1, the convolution

$$v = h *_t u$$

is well defined and $v(\cdot, t) \in L^q(\mathbf{R}^n)$ for any fixed $t \in \mathbf{R}$. Now

$$\langle v, \psi_j\rangle_{L^2(\Omega)} = \int_\Omega v(x,t)\psi_j(x)\,dx = \int_\Omega \left(\int_{\mathbf{R}} h(t-s)u(x,s)\,ds\right)\psi_j(x)\,dx$$

$$= \int_{\mathbf{R}} h(t-s)\left(\int_{\Omega} u(x,s)\psi_j(x)\,dx\right) ds = \int_{\mathbf{R}} h(t-s)c_j(s)\,ds$$
$$= c_j^+\hat{h}(\lambda_j)e^{i\lambda_j t} + c_j^-\hat{h}(-\lambda_j)e^{-i\lambda_j t}.$$

Due to the choice of h, we conclude that

$$v(x,t) = (c_k^+ e^{i\lambda_k t} + c_k^- e^{-i\lambda_k t})\psi_k(x) \text{ for } x \in \Omega \text{ and } t \in \mathbf{R}. \tag{6}$$

Let us fix a point $x_0 \in \Omega$ and consider the "radialization" $v_{x_0}^{\#}$ of v with respect to x_0, namely,

$$v_{x_0}^{\#}(x,t) = \int_{SO(n)} v(x_0 + \sigma(x - x_0), t)\,d\sigma.$$

We can apply a translation and assume for the sake of simplicity that $x_0 = 0$. Note that radialization preserves conditions of the type "v belongs to L^p." Set $v^{\#} = v_0^{\#}$.

Let $\epsilon > 0$ be such that the ball $B(0,\epsilon)$ is contained in Ω. Then (6) implies that

$$v^{\#}(x,t) = (c_k^+ e^{i\lambda_k t} + c_k^- e^{-i\lambda_k t})\psi_k^{\#}(x) \text{ for } x \in B(0,\epsilon). \tag{7}$$

Here $\psi_k^{\#}$ is the radialization of ψ_k. Let us denote the right-hand side of (7) by $\Psi(x,t)$. The function $\psi_k^{\#}$ is a radial eigenfunction of the Laplace operator in $B(0,\epsilon)$ with no singularities at $x = 0$, and hence it is a solution of the corresponding Bessel equation. Therefore

$$\psi_k^{\#}(x) = \text{const}\,.|x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(\lambda_k |x|) \tag{8}$$

in $B(0,\epsilon)$, and thus the function $\Psi(x,t)$ extends to a global solution of the wave equation in $\mathbf{R}^n$.

On the other hand, $v^{\#}(x,t)$ is also a global solution of the wave equation, namely,

$$v_{tt}^{\#} = \Delta v^{\#},$$

and it satisfies the initial condition

$$v^{\#}(x,0) = 0$$

since

$$v^{\#}(x,0) = \int_{\mathbf{R}} h(-s)u^{\#}(x,s)\,ds = 0$$

because u is odd in the variable s and h is even. The other initial condition is

$$v_t^{\#}(x,0) = \int_{\mathbf{R}} h'(-s)u^{\#}(x,s)\,ds.$$

Hence the function

$$b := v^{\#} - \Psi$$

satisfies the hypotheses of Lemma 2.2 and we conclude that

$$v^{\#}(x,t) = \Psi(x,t), \quad \text{for all } (x,t) \in \mathbf{R}^n \times \mathbf{R}.$$

As we said earlier, since $v(\cdot, t) \in L^q(\mathbf{R}^n)$ then $v^{\#}(\cdot, t) \in L^q(\mathbf{R}^n)$ for any fixed t. However, the asymptotic expansion of the Bessel function yields

$$\psi_k^{\#}(x) \sim \text{const} \cdot \cos\left(|x| - \frac{\pi}{2} - \frac{n-2}{2}\right) |x|^{\frac{1-n}{2}}$$

and

$$\cos\left(|x| - \frac{\pi}{2} - \frac{n-2}{2}\right) |x|^{\frac{1-n}{2}} \notin L^q(\mathbf{R}^n)$$

when $q \le 2n/(n-1)$, which is exactly our situation. We must conclude that the constant factor in (8) is equal to zero, and hence $v^{\#} = 0$. This implies in turn that $v(x_0, t) = 0$ for any t. Since the center $x_0 \in \Omega$ used for averaging had been chosen arbitrarily, this implies $u(x,t) = 0$ for $x \in \Omega$, $t \in \mathbf{R}$. Therefore, $c_k^+ = c_k^- = 0$, and due to arbitrariness of the index k we have $u(x,t) \equiv 0$. □

Remark. In the definition (1), we can restrict ourselves to functions $\psi = e_k$, where e_k is any basis in $L^p(\mathbf{R}, r^{n-1}dr)$, in order to get a dense subspace in the space spanned by (1). For instance, choosing $e_k(r) = e^{-kr^2}$ we obtain the following result.

Corollary 1. *The system of Gaussian functions*

$$e_{k,a}(x) = e^{-k|x-a|^2}, \qquad k \in \mathbf{N}_+$$

where a runs over the points of the boundary Γ of an arbitrary bounded domain, is complete in $L^p(\mathbf{R}^n)$ as long as $p \ge 2n/(n+1)$, but it may be incomplete if $p < 2n/(n+1)$.

3. The case of $L^2(\mathbf{R}^n)$

3.1. We want to make use of the fact that for functions in $L^2(\mathbf{R}^n)$ the spectral projections on eigenspaces of the Laplace operator can be constructed explicitly. This section is based on the results of Strichartz [S].

If $f \in L^2(\mathbf{R}^n)$, then the spectral projections appear if we write the Fourier inversion formula in polar coordinates:

$$f(x) = \int_0^\infty (\mathcal{P}_\lambda f)(x)\, d\lambda, \tag{9}$$

where

$$\mathcal{P}_\lambda f(x) = (2\pi)^{-n} \lambda^{n-1} \int_{S^{n-1}} \hat{f}(\lambda u) e^{i\lambda x \cdot u}\, du \tag{10}$$

is defined for almost every $\lambda \in (0, \infty)$.

Suppose f is continuous and $(\varphi * f)(x) = 0$ for all $x \in \Gamma$, where Γ is a fixed set in $\mathbf{R}^n$, and $\varphi \in L^2(\mathbf{R}^n)$ is any radial function. Then

$$\mathcal{P}_\lambda(\varphi * f) = \hat{\varphi}(\lambda)\mathcal{P}_\lambda f,$$

with the obvious abuse of language for $\hat{\varphi}$, and from (9) we have

$$\int_0^\infty \hat{\varphi}(\lambda)\mathcal{P}_\lambda f(x)\, d\lambda = 0 \quad \text{for all } x \in \Gamma.$$

This implies that $\mathcal{P}_\lambda f(x) = 0$, $x \in \Gamma$, for almost every $\lambda \in (0, \infty)$. Conversely, the last identity implies that $\varphi * f|_\Gamma = 0$ for any radial $\varphi \in L^2(\mathbf{R}^n)$.

The spectral projections $f_\lambda = \mathcal{P}_\lambda f$ of functions $f \in L^2(\mathbf{R}^n)$ were completely characterized by Strichartz [S, Th. 3.3]. Besides, the completeness of the system (1) in $L^2(\mathbf{R}^n)$ is equivalent to the triviality of its orthogonal complement. Thus we have

Theorem 4. *Let Γ be a subset of $\mathbf{R}^n$. The system*

$$\psi(|x-a|), a \in \Gamma, \psi \in L^2(\mathbf{R}_+, r^{n-1}dr)$$

is incomplete in $L^2(\mathbf{R}^n)$ if and only if there exists a nonzero measurable function $f_\lambda(x)$ on $(0, \infty) \times \mathbf{R}^n$ such that

(a) $\Delta f_\lambda = -\lambda^2 f_\lambda$ for a.e. λ.

(b) $\sup_{z,t} \int_0^\infty \frac{1}{\lambda} \int_{|x-z|<t} |f_\lambda(x)|^2\, dx\, d\lambda < \infty$.

(c) $f_\lambda \in C(\mathbf{R}^n)$ for a.e. λ and $f_\lambda(x) = 0$ for all $x \in \Gamma$.

If Γ is the boundary of a bounded domain, then $f_\lambda = 0$ for a.e. λ due to the discreteness of the spectrum of the Laplace operator and we get completeness in accordance with Theorem 3.

In order to obtain a more constructive description of the sets Γ corresponding to complete systems of spherical waves, we have to characterize in explicit terms sets Γ satisfying the conditions (a), (b), (c), which seems to be a difficult problem.

3.2. We know from Theorem 3 that any function $f \in L^p(\mathbf{R}^n)$, $p \geq \frac{2n}{n+1}$ can be approximated by linear combinations of functions $\psi(|x-a|)$, $\psi \in L^2((0,\infty), r^{n-1}dr)$, where a belongs to the boundary Γ of a bounded domain $\Omega \subset \mathbf{R}^n$.

In the case of $p = 2$ the approximating functions ψ can be obtained from f by averaging.

For any $a \in \mathbf{R}^n$ denote by S_a the averaging operator

$$S_a f(x) = \int_{SO(n)} f(a + u(x-a))\, du,$$

which is related to the spherical mean operator R by $S_a f(x) = Rf(a, |x-a|)$.

Clearly $S_a f \in L^2(\mathbf{R}^n)$ if $f \in L^2(\mathbf{R}^n)$ and the function $S_a f(x)$ depends only on the distance to the point a.

Proposition 3.1. *Let $f \in L^2(\mathbf{R}^n)$ and let Γ be the boundary of a bounded domain in $\mathbf{R}^n$. Then $f \in X(\Gamma) = \operatorname{cl\,span}\{S_{a_1}\cdots S_{a_m} f, a \in \Gamma, m \in \mathbf{N}\}$—the smallest closed subspace in $L^2(\mathbf{R})$, containing the function f and invariant with respect to the operators S_a, $a \in \Gamma$.*

Proof. Let $f = f_1 + f_2$ be the orthogonal decomposition, $f_1 \in X(\Gamma)$, $f_2 \in X(\Gamma)^\perp$. Since $S_a f, S_a f_1 \in X(\Gamma)$ we have $S_a f_2 \in X(\Gamma), a \in \Gamma$. Then $\|S_a f_2\|^2 = \langle S_a f_2, S_a f_2\rangle = \langle f_2, S_a f_2\rangle = 0$. Hence $Rf_2(a,t) = 0$ for all $(a,t) \in \Gamma \times \mathbf{R}_+$ and Theorem 2 implies $f_2 = 0$. □

It is worthwhile to note that such kind of approximation is possible not in all spaces. Let us consider, for example, the space $H(\mathbf{R}^n)$ of all harmonic functions. Then $S_a f(x) \equiv f(a)$ for each $f \in H(\mathbf{R}^n)$ and all $a \in \mathbf{R}^n$ and therefore $H(\mathbf{R}^n)$ is not spanned by its projections $S_{a_1}\cdots S_{a_m}(H(\mathbf{R}^n))$ (which consists of constants) even if a_j are arbitrary points in $\mathbf{R}^n$.

3.3. As it has been established in Theorem 2, the spherical mean operator

$$\begin{aligned} R&: \ L^2(\mathbf{R}^n) \cap C(\mathbf{R}^n) \to C(\Gamma \times \mathbf{R}_+), \\ R&: \ f(x) \to Rf(a,t) \end{aligned}$$

is injective if Γ is the boundary of a bounded domain in $\mathbf{R}^n$.

The natural question arises: What is the range of the operator R and how to reconstruct f from Rf?

4. Symmetric spaces

Now we want to study the analogous problem in non-Euclidean spaces, specifically, symmetric spaces of noncompact type.

Let $X = G/K$ be a symmetric space, where G is a real semisimple Lie group, and K is a maximal compact subgroup in G. Denote by dx the G-invariant volume on X, $o = eK$ the origin in X.

Denote by A the Cartan subgroup of G, $\mathcal{A}$ the Cartan subalgebra of the Lie algebra $\mathcal{G}$, $\mathcal{A}^*$ the dual space, $\mathcal{A}^+$ a Weyl chamber, Σ_+ the set of positive roots, M the centralizer of A in K, $r = \dim \mathcal{A}$ the rank of X.

Let $\Gamma = K \cdot x_o$ be a K-orbit of some point $x_o \in X$. The orbit Γ is *generic* if it has maximal possible dimension $\dim \Gamma = \dim X - r$. The decomposition $X = K\overline{A^+} \cdot o$, $A^+ = \exp\mathcal{A}$, ([He], Prop. 1.4), shows that the orbit Γ is generic if the point x_o is *regular*. The latter means that x_o is in the open Weyl chamber A^+ after applying an appropriate element from K, i.e., $x_o = ka \cdot o$, for some $k \in K$, $a \in A^+$. Degenerate orbits correspond to points on the boundary $\overline{A^+} \setminus A^+$ of the Weyl chamber.

Given a point $a \in X$ we denote by $W_{a,p}$ the linear space in $L^p(X) = L^p(X, dx)$ consisting of shifted K-invariant functions

$$f^a(x) = \varphi(g_a^{-1}x),$$

$\varphi(kx) = \varphi(x)$ for all $k \in K$ and $g_a \in G$ is such that $g_a(o) = a$ (clearly, f does not depend on the choice of g_a).

For any set $\Gamma \subset X$ we denote $W_p(\Gamma)$ the linear span of all $W_{a,p}$ with $a \in \Gamma$.

Theorem 5. *Let $\Gamma = K \cdot x_o$ be a generic K-orbit in X (i.e., x_o is a regular point).*[1] *Then $W_p(\Gamma)$ is dense in $L^p(X)$ if $p \geq 2$. The property fails to be true for $p < 2$.*

When the symmetric space X has rank $= 1$ we are able to avoid requiring K-symmetry for Γ:

Theorem 6. *Let rank $X = 1$. Then the boundary Γ of any bounded domain generates a complete system $W_p(\Gamma)$ as long as $p \geq 2$. This is not true when $p < 2$.*

[1]The authors thank Prof. S. Helgason, who pointed out to them the necessity of the regularity condition.

4.1. The reason for $p = 2$ to be the critical index for the denseness $W_p(\Gamma)$ in $L^p(X)$ is the fact that, in contrast with the Euclidean case, spherical functions belong to $L^q(X)$ for any $q > 2$. Let us comment on this assertion.

The following inequality is a particular case of the Harish-Chandra inequality (cf. [He], p. 485):

$$|\varphi_\lambda(a \cdot o)| \leq C_\lambda e^{-\rho(\log a)}, \quad a \in \mathcal{A}, \tag{11}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} m_\alpha \alpha$, m_α is the multiplicity of α.

Then the "polar decomposition" of integration on X ([He], Theorem 5.8, p. 186) yields

$$\int_X |\varphi_\lambda(x)|^q \, dx = \text{const} \int_{A^+} |\varphi_\lambda(a \cdot o)|^q \delta(a) \, da,$$

where

$$\delta(\exp H) = \prod_{\alpha \in \Sigma^+} (\sinh \alpha(H))^{m_\alpha}, \quad H \in \mathcal{A}^+.$$

Thus

$$\int_X |\varphi_\lambda(x)|^q \, dX \leq \text{const} \int_{A^+} e^{-q\rho(\log a)} \prod_{\alpha \in \Sigma^+} (e^{\alpha(\log a)} - e^{-\alpha(\log a)})^{m_\alpha} \, da$$

and the right-hand side is finite if $q > 2$.

On the other hand,

$$\|\varphi_\lambda\|_{L^2} = \int_X \varphi_\lambda(x) \overline{\varphi_\lambda(x)} \, dx = \infty$$

since the integral coincides with the value of the spherical Fourier transform of φ_λ at the point λ, i.e., with ∞. Since for $\lambda \in \mathcal{A}^*$ the functions φ_λ are bounded, $\varphi_\lambda \notin L^q$ for any $q \leq 2$.

Now the last (i.e., the negative) assertion in Theorem 5 and Theorem 6 follows from the duality discussed in Section 2. Indeed, if a K-spherical function $\varphi_\lambda \not\equiv 0$ vanishes on a K-orbit Γ, then $\varphi_\lambda \in W_p^\perp(\Gamma)$ for any $p < 2$, and hence $W_p(\Gamma)$ is not dense in $L^2(X)$.

End of the proofs of Theorems 5 and 6. Let $p \geq 2$ and $f \in W_p^\perp(\Gamma) \subset L^q(X)$, $1/p + 1/q = 1$. Using convolutions with K-invariant functions we can assume f to be bounded and therefore $f \in L^2(X)$. For the same reason f can also be assumed to be continuous. Dealing with $L^2(X)$ allows us to avoid considering the wave equation and directly use the following restatement of Helgason's Fourier inversion formula as given in [S, (4.33) and (4.34)]:

$$f(x) = \int\limits_{\mathcal{A}_+^*} Q_\mu(x)\, d\mu \tag{12}$$

where

$$Q_\mu(x) = c_\mu \int\limits_X f(y)\varphi_\mu(x, y)\, dy \tag{13}$$

is a measurable function in the μ-variable, which is a joint eigenfunction for the G-invariant differential operators $D \in \mathbf{D}(G/K)$ in the x-variable.

The functions φ_μ and Q_μ are defined for all $\mu \in \mathcal{A}^*$ so that $\varphi_{w\mu} = \varphi_\mu$ and $Q_{w\mu} = Q_\mu$ for any w in the Weyl group W.

The function $\varphi_\mu(x, y)$ can be given by the formula

$$\varphi_\mu(x, y) = \int\limits_{K/M} e^{(i\mu+\rho)(A(x,b))} e^{-(i\mu+\rho)(A(y,b))}\, db,$$

due to the symmetries of the spherical functions [He, Theorem 1.1]. Here $A(gK, kM) = A(k^{-1}g)$ is the component in the Iwasawa representation $k^{-1}g = n \exp A(k^{-1}g)u$, $n \in N$, $u \in K$. The function $\varphi_\mu(x, y)$ has the property $\varphi_\mu(x, gy) = \varphi_\mu(x, y)$ whenever $g \in G$ satisfies $gx = x$. Therefore $\varphi_\mu(x, y) = \psi_\mu(g_x^{-1}y)$, where $g_x \cdot o = x$, for an appropriate K-invariant function ψ_μ.

Let $\hat{K}$ denote the set of equivalence classes of unitary irreducible representations of K. Fix a representation $\delta \in \hat{K}$ and let V_δ be the vector space in which the representation δ is realized. Define

$$b_{\mu,\delta}(x) = (\delta * Q_\mu)(x) = \int_K Q_\mu(k^{-1}x)\delta(k)\, dk, \tag{14}$$

which is a C^∞ map from X to Hom (V_δ, V_δ).

Since $f \in W_p^\perp(\Gamma)$ and ϕ_μ is a shift of K-invariant function ψ_μ, the right-hand side of (13) is zero whenever $x \in \Gamma$, by the definition of the space $W_p(\Gamma)$, and we have

$$Q_\mu(x) = 0 \text{ for all } x \in \Gamma \text{ and a.e. } \mu \in \mathcal{A}^*. \tag{15}$$

Since Γ is a K-orbit, we obtain from (14) for $x \in \Gamma$:

$$b_{\mu,\delta}(x) = 0 \quad \text{for a.e. } \mu \in \mathcal{A}^*.$$

However, the function Q_μ is a joint eigenfunction for operators in $\mathbf{D}(G/K)$ and it follows from (14) that

$$b_{\mu,\delta}(kx) = \delta(k)b_{\mu,\delta}(x), \quad k \in K.$$

Therefore the function $b_{\mu,\delta}$ is proportional to a generalized spherical function ([He], p. 233), namely to

$$\Phi_{\mu,\delta}(x) = \int_K e^{(i\mu+\rho)(A(x,kM)}\delta(k)\,dk,$$

i.e.,

$$b_{\mu,\delta}(x) = c(\mu,\delta)\Phi_{\mu,\delta}(x) \tag{16}$$

for all $x \in X$ and a constant $c(\mu,\delta) \in \mathbf{C}$.

Now we consider the map $\mu \to$ Trace $\Phi_{\mu,\delta}(x_o)$ from $\mathcal{A}^*$ to $\mathbf{C}$. It is real analytic and hence either

(i) Trace $\Phi_{\mu,\delta}(x_o) \neq 0$ for almost every $\mu \in \mathcal{A}^*$, or

(ii) Trace $\Phi_{\mu,\delta}(x_o) = 0$ for all $\mu \in \mathcal{A}^*$.

However the following proposition shows that the case (ii) is impossible due to regularity of the point x_o. The proposition and its proof was kindly suggested to us by Prof. S. Helgason.

Proposition 4.1 (S. Helgason). *If $x_o \in X$ is a point where Trace $\Phi_{\mu,\delta}$ vanishes for all $\mu \in \mathcal{A}^*$ and L is the subgroup of K which fixes x_o, then the operators $\delta(l)$, $l \in L$, have no common nonzero fixed vector in V_δ.*

In particular, the Trace $\Phi_{\mu,\delta}$, *$\mu \in \mathcal{A}^*$, have no regular common zero in X.*

Proof. Let $D(X)$ be the space of C^∞ functions on X of compact support. and $D_{\check{\delta}}$ the subspace of K-finite functions in $D(X)$ of type $\check{\delta}$, where $\check{\delta}$ is the contragredient representation.

The inversion formula for δ-spherical transform, [He, Theorem 5.16]:

$$g(x) = const \cdot Tr\left[\int_{\mathcal{A}^*} \Phi_{\lambda,\delta}(x)\tilde{g}(\lambda)|c(\lambda)|^{-2}\,d\lambda\right]$$

yields $g(x_o) = 0$ for every $g \in D_{\check{\delta}}$.

If h is an arbitrary function from $D(X)$ then the convolution

$$g = \chi_\delta * h$$

is in $D_{\bar\delta}$ and hence

$$\int_K \chi_\delta(k)h(k^{-1}x_o)\,dk = g(x_o) = 0.$$

Due to arbitrariness of h this immediately implies

$$\int_L \chi_\delta(lk)\,dl = 0$$

for all $k \in K$.

However this amounts to Trace $(\delta(k)E) = 0$, for all $k \in K$, where E is the projection on the fixed point space of the operators $\delta(l)$, $l \in L$. Hence $E = 0$ and Proposition 4.1 is proved. □

Now we are able to complete the proof of Theorem 5. We know from Proposition 4.1 that Trace $\Phi_{\mu,\delta}(x_o) \neq 0$ for a.e. $\mu \in \mathcal{A}^*$. But Trace $b_{\mu,\delta}(x_o) = 0$ for a.e. μ and we obtain from (16) that the constants $c_{\mu,\delta} = 0$ for a.e. μ and hence Trace $b_{\mu,\delta} \equiv 0$ for a.e. $\mu \in \mathcal{A}^*$.

The irreducible representation $\delta \in \hat{K}$ is chosen to be abitrary and the definition (14) of $b_{\mu,\delta}$ shows that restriction of the functions Q_μ to any K-orbit is orthogonal to all characters $\chi_\delta =$ Trace δ. Then the Peter-Weyl theorem implies that $Q_\mu \equiv 0$ for a.e. μ.

Finally, we conclude from (12) that $f = 0$ and therefore $W_p^\perp = 0$. This completes the proof of Theorem 5. □

In the case rank $X = 1$ (Theorem 6), we can replace a K-orbit by the boundary Γ of an arbitrary bounded domain since in this situation we can use the discreteness of the spectrum of the Laplace operator with Dirichlet conditions on Γ in order to conclude that the $Q_\mu \equiv 0$ for all μ except a discrete set. Then (12) gives $f = 0$. □

Remark. One can show that Theorem 5 fails for degenerate (nongeneric) orbits Γ.

In order to drop the K-invariance condition in Theorem 5 it would be sufficient to understand what closed submanifolds of codim $\leq r$ can be contained in the set of common zeros of the joint eigenfunctions Q_μ. We were not able to do this yet. □

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