The Journal of Geometric Analysis
Volume 7, Number 4, 1997

Area Minimizing Sets Subject to a Volume Constraint in a Convex Set

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ABSTRACT. For a given convex subset Ω of Euclidean n-space, we consider the problem of minimizing the perimeter of subsets of Ω subject to a volume constraint. The problem is to determine whether in general a minimizer is also convex. Although this problem is unresolved, we show that if Ω satisfies a "great circle" condition, then any minimizer is convex. We say that Ω satisfies a great circle condition if the largest closed ball B contained in Ω has a great circle that is contained in the boundary of Ω. A great circle of B is defined as the intersection of the boundary of B with a hyperplane passing through the center of B.

1. Introduction

In this paper we consider the problem of minimizing area subject to a volume constraint in a given convex set. In precise terms we have the following. Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set. Thus, $|\Omega| < \infty$ where $|\Omega|$ denotes Lebesgue measure. For a number $0 < v < |\Omega|$, let $E \subset \Omega$ denote a set with $|E| = v$ such that

$$P(E) \leq P(F)$$

for all sets $F \subset \Omega$ with $|F| = v$, where $P(E)$ denotes the perimeter of E. The main question we will investigate is whether E is convex.

It should be emphasized that the perimeter of a competitor F is taken relative to $\mathbb{R}^n$, or what is the same, the perimeter is taken relative to the closure of Ω since F is assumed to be a subset of Ω. This problem is considerably different from minimizing the perimeter relative to the interior of Ω. This was considered in [5] where it was shown that a minimizer is regular and intersects $\partial\Omega$ orthogonally.

The question of existence of a solution to our problem is resolved immediately in the context of sets of finite perimeter. Regularity questions have been considered by other authors. Tamanini [10] has shown that an area minimizing set E subject to a volume constraint has the property that $\partial E \cap \Omega$ is real analytic except for a closed set whose Hausdorff dimension does not exceed $n - 8$. Also, under the assumption that $\partial\Omega \in C^1$, it was shown in [4] that ∂E is an $(n-1)$ manifold of class C^1 in some neighborhood of each point in $\partial E \cap \partial\Omega$. In $\mathbb{R}^2$, and in R^n, $n > 2$ under an additional

Math Subject Classifications. Primary 49Q20, 49Q15; Secondary 49Q10, 52A20.
Key Words and Phrases. sets of finite perimeter, volume constraint, area minimizing, constant mean curvature.
Acknowledgements and Notes. Research supported in part by a grant from the National Science Foundation.

ISSN 1050-6926

condition on Ω, we are able to obtain regularity results and ultimately establish that a minimizer E is convex. Assuming only that Ω is bounded and convex, the convexity of E is an open question in $\mathbb{R}^n$, $n > 2$.

The additional condition we impose on Ω if $n > 2$ is the following.

$$\left\{ \begin{array}{l} \text{We assume that a largest closed ball, } B_\Omega \text{, contained in } \Omega \text{ has a great} \\ \text{circle that is a subset of } \partial\Omega. \text{ A great circle of } B_\Omega \text{ is defined as the} \\ \text{intersection of } \partial B_\Omega \text{ with a hyperplane, } T_{B_\Omega} \text{, passing through the center} \\ \text{of } B_\Omega. \text{ The equatorial "disk" is defined as } D_{B_\Omega} = T_{B_\Omega} \cap B_\Omega. \end{array} \right. \tag{1.1}$$

Also, assuming initially that $\partial\Omega \in C^2$ and strictly convex, we invoke a result of [1] to conclude that $\partial E \in C^{1,1}$ at points near $\partial\Omega$. We then show, Theorem 3.24, that E is convex. Finally, through an approximation procedure, we show that E is convex with $C^{1,1}$ boundary assuming only that Ω satisfies a great circle condition. Clearly, there is no uniqueness if v is too small. However, with H_Ω denoting the union of all largest balls in Ω, if $|H_\Omega| \leq v < |\Omega|$, then E is unique. In addition for such v we show that perimeter minimizers E are nested as a function of v. In general, for nonconvex Ω, one can expect neither uniqueness nor nestedness as indicated by examples in [3].

The nestedness of perimeter minimizers allows one to rearrange level sets of functions to create test functions useful in studying minimizers to certain variational problems. For domains Ω having certain symmetries, it is frequently possible to apply symmetrization to gain information on minimizers of functionals such as

$$\int_\Omega |\nabla u|^p + \int_\Omega F(u) + \int_0^{|\Omega|} G(u^*, u^{*\prime})$$

over appropriate function classes , where u^* is the decreasing rearrangement of u. However, this greatly restricts the collection of domains that can be considered. In Section 4 for the case $p = 1$ we construct a rearrangement which retains various useful properties of symmetrization while allowing a much larger class of domains to be considered, namely those convex domains described above. This rearrangement is useful when one has a boundary condition of the form $u = 0$ on $\partial\Omega$ and when it can be established, for instance using truncation, that $u \geq 0$ in Ω. Since this rearrangement produces functions of bounded variation, it is more accurate to replace $\int |\nabla u|$ in the functional above by the BV norm. The results of Section 3 allow one to deduce certain regularity properties for minimizers u from regularity properties of u^*. In addition, they establish the convexity of the sets $\{u > t\}$. Results in [7] show that one cannot hope for similar results if $p > 1$.

2. Notation and preliminaries

The Lebesgue measure of a set $E \subset \mathbb{R}^n$ will be denoted by $|E|$ and $H^\alpha(E)$, $\alpha > 0$, will denote its α-dimensional Hausdorff measure. If $\Omega \subset \mathbb{R}^n$ is an open set, the class of functions $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is denoted by $BV(\Omega)$ and is called the space of *functions of bounded variation in* Ω. The space $BV(\Omega)$ is endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{1;\Omega} + \|\nabla u\|(\Omega) \tag{2.1}$$

where $\|u\|_{1;\Omega}$ denotes the L^1-norm of u on Ω and where $\|\nabla u\|$ is the total variation of the vector-valued measure ∇u.

A Borel set $E \subset \mathbb{R}^n$ is said to have *finite perimeter in* Ω provided the characteristic function of E, χ_E, is a function of bounded variation in Ω. Thus, the partial derivatives of χ_E are Radon measures on Ω and the perimeter of E in Ω is defined as

$$P(E, \Omega) = \|\nabla \chi_E\| (\Omega) . \tag{2.2}$$

A set E is said to be of *locally finite perimeter* if $P(E, \Omega) < \infty$ for every bounded open set $\Omega \subset \mathbb{R}^n$.

The definition implies that sets of finite perimeter are defined only up to sets of measure 0. In other words, each set determines an equivalence class of sets of finite perimeter. In order to avoid this ambiguity, whenever a set E of finite perimeter is considered we shall always employ the measure theoretic closure as the set to represent E. Thus, with this convention, we have

$$x \in E \text{ if and only if } \limsup_{r \to 0} \frac{|E \cap B(x, r)|}{|B(x, r)|} > 0 . \tag{2.3}$$

One of the fundamental results of the theory of sets of finite perimeter is that they possess a measure-theoretic exterior normal which is suitably general to ensure the validity of the Gauss-Green theorem. A unit vector ν is defined as the exterior normal to E at x provided

$$\lim_{r \to 0} r^{-n} |B(x, r) \cap \{y : (y - x) \cdot \nu < 0, y \notin E\}| = 0$$

and

$$\lim_{r \to 0} r^{-n} |B(x, r) \cap \{y : (y - x) \cdot \nu > 0, y \in E\}| = 0 , \tag{2.4}$$

where $B(x, r)$ denotes the open ball of radius r centered at x. The measure-theoretic normal of E at x will be denoted by $\nu(x, E)$ and we define

$$\partial^* E = \{x : \nu(x, E) \text{ exists}\} . \tag{2.5}$$

Clearly, $\partial^* E \subset \partial E$, where ∂E denotes the topological boundary of E.

A set E of finite perimeter is said to be an *area minimizing in an open set* Ω if

$$\|\nabla \chi_E\| (\Omega) \leq \|\nabla \chi_F\| (\Omega) \tag{2.6}$$

for every set F with $F \Delta E \subset\subset \Omega$. Here $F \Delta E$ denotes the symmetric difference.

The regularity of ∂E will play a crucial role in our development. Suppose ∂E is an area minimizing in U and for convenience of notation, suppose $0 \in U \cap \partial E$. For each $r > 0$, let $E_r = \mathbb{R}^n \cap \{x : rx \in E\}$. It is known (cf. [9, §35], [8, §2.6]) that for each sequence $\{r_i\} \to 0$ there exists a subsequence (denoted by the full sequence) such that $\chi_{E_{r_i}}$ converges in $L^1_{\text{loc}}(\mathbb{R}^n)$ to χ_C, where C is a set of locally finite perimeter. In fact, ∂C is an area minimizing and is called the tangent cone to E at 0. Although it is not immediate, C is a cone and therefore the union of half-lines issuing from 0. It follows from [9, §37.6] that if $\overline{C}$ is contained in $\overline{H}$ where H is any half-space in $\mathbb{R}^n$ with $0 \in \partial H$, then ∂E is regular at 0. That is, there exists $r > 0$ such that

$$B(0, r) \cap \partial E \quad \text{is a real analytic hypersurface.} \tag{2.7}$$

Furthermore, ∂E is regular at all points of $\partial^* E$ and

$$H^\alpha((\partial E - \partial^* E) \cap U) = 0 \quad \text{for all } \alpha > n - 8 , \tag{2.8}$$

cf. [6, Theorem 11.8].

The notion of *excess* plays a critical role in the theory of minimal boundaries. It measures how far a set E is from being an area minimizing in a ball. Formally, it is defined by

$$\psi(x,r) = \|\nabla\chi_E\|\,(B(x,r)) - \inf\{\|\nabla\chi_F\|\,(B(x,r)) : F\Delta E \subset\subset B(x,r)\}\,.$$

Thus, $\psi \equiv 0$ when E is area minimizing. If E is an arbitrary set of finite perimeter and $\psi(x,r) \leq Cr^{n-1+2\alpha}$ for some $x \in \partial E$ and all $0 < r < R$ with given constants C, R, and $0 < \alpha < 1$, then it follows from a result of Tamanini [10] that there is an area minimizing tangent cone to ∂E at x.

Definition 2.9. Let M denote a k-dimensional C^1 submanifold of $\mathbb{R}^n$, $0 < k < n$, and let $f\colon M \to \mathbb{R}$ be an arbitrary function. We will say that f is differentiable at $x_0 \in M$ if f is the restriction to M of a function $\bar{f}\colon U \to \mathbb{R}$ where is $U \subset \mathbb{R}^n$ is some open set containing x_0 and where $\bar{f}$ is differentiable at x_0.

Lemma 2.10. *Let M be an $n-1$-dimensional C^1 submanifold of $\mathbb{R}^n$ and let $f\colon M \to \mathbb{R}$ be a Lipschitz function. Then f is differentiable at H^{n-1} almost all points of M.*

Proof. The manifold M near any of its points x_0 can be represented as the graph of a function defined on some open $n-1$-ball $B' \subset \mathbb{R}^{n-1}$. Thus, there is an open n-cylinder C of the form $C = B' \times (a,b)$ such that $C - M$ consists of two nonempty connected, open sets and that each projection of $M \cap C$ onto the top and bottom of C is a homeomorphism. Let points $x \in C$ be denoted by $x = (x', y)$ where $x' \in B'$ and $y \in (a,b)$ and define $\bar{f}\colon C \to \mathbb{R}$ by $\bar{f}(x',y) = f(x', y_M)$ where (x', y_M) is that unique point on $M \cap C$ that is the projection of (x', y). It is easy to verify that $\bar{f}\colon C \to \mathbb{R}$ is Lipschitz and therefore, by Rademacher's theorem, that $\bar{f}$ is differentiable at (Lebesgue) almost all points of C. Let N denote those points at which $\bar{f}$ is not differentiable. Clearly, if $\bar{f}$ is differentiable at a point (x', y_1), then it is differentiable at any other point of the form (x', y_2). Now define $d\colon C \to \mathbb{R}$ by $d(x',y) = |y - y_M|$. Note that d is Lipschitz and that $d^{-1}(t)$ consists of two copies of $M \cap C$, one is a vertical distance of t units above $M \cap C$ and the other is a vertical distance of t units below $M \cap C$. Now employ the co-area formula to obtain

$$0 = \int_N |\nabla d|\; dx = \int_a^b H^{n-1}[d^{-1}(t) \cap N]\, dt\,.$$

Thus, for almost every $t \in (a,b)$, $\bar{f}$ is differentiable at H^{n-1} almost all points of $d^{-1}(t)$. Consequently, $\bar{f}$ is differentiable at the corresponding points of $d^{-1}(0) = M \cap C$; that is, $\bar{f}$ is differentiable at H^{n-1} almost all points of $M \cap C$, as required. □

In view of the preceding lemma, we can define the directional derivative of f relative to M at H^{n-1}-almost all $x \in M$ in the usual manner. Given a vector τ in the tangent space to M at x, let $\gamma\colon (-1,1) \to M$ be any C^1 curve with $\gamma(0) = x$ and $\gamma'(0) = \tau$. Define

$$D_\tau f(x) = (\bar{f} \circ \gamma)'(0)$$

where it is understood that $\bar{f}$ is differentiable at x. Observe that this definition is independent of the extension $\bar{f}$.

If we are given a Lipschitz vector field $X\colon M \to \mathbb{R}^n$, by using usual methods, it now becomes clear how to define the divergence of X relative to M, denoted by $\mathrm{div}_M X$.

If the closure $\overline{M}$ of M is a C^1 manifold with boundary $\partial M = \overline{M} - M$ and if $X\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field with the property that for each $x \in M$, $X(x)$ is an element of the tangent space to M at x, then the classical divergence theorem states

$$\int_M \mathrm{div}_M X\, dH^{n-1} = \int_{\partial M} X \cdot \eta\, dH^{n-2} \tag{2.11}$$

where η is the outward pointing unit co-normal of ∂M. That is, $|\eta| = 1$, η is normal to ∂M, and tangent to M.

Definition 2.12. Let M be an oriented $n-1$-dimensional submanifold of $\mathbb{R}^n$ of class $C^{1,1}$; that is, M is of class C^1 and its unit normal ν is Lipschitz. From Lemma 2.10, we have that the components of ν are differentiable at H^{n-1} almost all points of M. Thus, $\text{div}_M \nu$ is defined H^{n-1} almost everywhere on M. At such points, we define the *mean curvature* of M at x as

$$\mathcal{H}_M(x) = \text{div}_M \nu(x) .$$

If $X: \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, consider its decomposition into its tangent and normal parts relative to M,

$$X = X^\top + X^\perp$$

where

$$X^\perp = (X \cdot \nu)\nu .$$

Then, at H^{n-1} almost all points in M, it follows that

$$\text{div}_M X^\perp = (X \cdot \nu)\text{div}_M \nu .$$

Hence,

$$\text{div}_M X^\perp = \mathcal{H}_M X \cdot \nu .$$

On the other hand, from (2.11) we have

$$\int_M \text{div}_M X^\top \, dH^{n-1} = \int_{\partial M} X \cdot \eta \, dH^{n-2} .$$

Since $\text{div}_M X = \text{div}_M X^\top + \text{div}_M X^\perp$, we obtain

$$\int_M \text{div}_M X \, dH^{n-1} = \int_M \mathcal{H}_M X \cdot \nu \, dH^{n-1} + \int_{\partial M} X \cdot \eta \, dH^{n-2} . \tag{2.13}$$

3. Main results

In this section we consider the following situation.

$$\left\{ \begin{array}{l} \text{Let } \Omega \text{ be a bounded, convex domain in } \mathbb{R}^n,\ n \geq 2. \text{ Let } E \subset \overline{\Omega} \text{ denote} \\ \text{a set which minimizes perimeter in the closure of } \Omega \text{ subject to a volume} \\ \text{constraint } |E| = v < |\Omega|. \text{ Thus,} \\ \qquad\qquad P(E, \mathbb{R}^n) \leq P(F, \mathbb{R}^n) \\ \text{for all sets } F \subset \overline{\Omega} \text{ with } |F| = v. \end{array} \right. \tag{3.1}$$

We will first establish boundary regularity and curvature properties for such perimeter minimizers under the assumption that Ω is *strictly convex* and that $\partial\Omega \in C^2$. Convexity, nestedness, and uniqueness results will then be established under the further assumption that

$$n = 2 \quad \text{or} \quad \Omega \text{ satisfies a great circle condition.}$$

The assumption of strict convexity and C^2 regularity will then be dispensed with in part through an approximation argument.

Associated with (3.1) is some further notation. We let H denote the convex hull of a minimizer E of (3.1), and we denote by H^+ that part of H that lies "above" the equatorial disk D_{B_Ω} of B_Ω as defined in (1.1). Since P divides H into two parts, we arbitrarily call one of them the part that lies "above" P.

Next, we recall some facts concerning area minimizing sets with a volume constraint. The main result of [3] is that if E is area minimizing with a volume constraint, then

$$\psi(x, r) \leq Cr^n \tag{3.2}$$

for each $x \in \partial E$ and for all sufficiently small $r > 0$. Consequently, it follows from work of Tamanini [10] that an area minimizing set E with a volume constraint possesses an area minimizing tangent cone at each point of $(\partial E) \cap \Omega$. From this it follows that $(\partial E) \cap \Omega$ enjoys the same regularity properties as an area minimizing set; that is, $(\partial E) \cap \Omega$ is real analytic except for a closed singular set S whose Hausdorff dimension does not exceed $n-8$. Furthermore, it was established in [4, Theorem 3] that ∂E is an $(n-1)$ manifold of class C^1 in some neighborhood of each point $x \in \partial E \cap \partial\Omega$.

The object of this section is to prove that E is convex and we begin by proving $C^{1,1,}$ regularity of ∂E near $\partial\Omega$. For this we will need the following result of Brézis and Kinderlehrer [1].

Theorem 3.3. *Let $a\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a C^2 vector field satisfying the condition that for each compact $C \subset \mathbb{R}^{n-1}$, there exists a constant $\nu = \nu(C) > 0$ such that*

$$(a(p) - a(q)) \cdot (p - q) \geq \nu\, |p - q|^2$$

for all $p, q \in C$. Let $U \subset \mathbb{R}^{n-1}$ be an open connected set and let $\beta \in C^2(U)$ satisfy $\beta \leq 0$ on ∂U. Let $f \in C^1(U)$. With $\mathbf{K}$=$\mathbf{K}_\beta$ denoting the convex set of Lipschitz functions v satisfying $v \geq \beta$ in U and $v = 0$ on ∂U, let $u \in \mathbf{K}$ be a solution of

$$\int_U a(\nabla u) \cdot \nabla(v - u)\, dx \geq \int_U f(v - u)\, dx$$

for all $v \in \mathbf{K}$. Then $u \in C^{1,1}(V)$ on any domain V with $\overline{V} \subset U$.

Proof. We now apply this result to obtain $C^{1,1}$ regularity of the boundary of a minimizer E of the variational problem (3.1) near $\partial\Omega$. Since ∂E is an $(n-1)$ manifold of class C^1 in some neighborhood of each point $x \in \partial E \cap \partial\Omega$, it follows that near such a point x, we may represent both ∂E and $\partial\Omega$ as graphs of functions u and β, respectively, defined on an open set $U' \in \mathbb{R}^{n-1}$ containing x' where $x = (x', y'')$, $y'' \in \mathbb{R}$. We will assume u and β chosen in such a way that $u \geq \beta$, $u = 0$ on $\partial U'$ and $\beta \leq 0$ on $\partial U'$. Using the convexity of Ω, this can be accomplished by considering a hyperplane P_0 passing through E and parallel to the tangent plane to ∂E at x. By taking P_0 sufficiently close to the tangent plane, U' can be defined as $P_0 \cap E$. Now select $v \in \mathbf{K}$ and for $0 < \varepsilon < 1$, define u_ε on U' as $u_\varepsilon = u + \varepsilon(v - u)$. We will assume ε chosen small enough so that the graph of u_ε remains in $\overline{\Omega}$. Note that $u_\varepsilon \in \mathbf{K}$. Select a point $z \in (\partial E) \cap \Omega$ at which ∂E is regular. Thus, ∂E is real analytic near z and its mean curvature is a constant K there. In a neighborhood of z, we can represent ∂E as the graph of a function w defined on some open set $V' \subset \mathbb{R}^{n-1}$ containing z' where $z = (z', z'')$. The neighborhoods about x and z where ∂E is represented as a graph are taken to be disjoint. Let $\varphi \in C_0^\infty(V')$ denote a function with the property that

$$\int_{V'} \varphi\, dH^{n-1} = \int_{U'} (v - u)\, dH^{n-1}\,, \tag{3.4}$$

and define $w_\varepsilon = w - \varepsilon\varphi$. The graphs of the functions u_ε and w_ε produce a perturbation of the set E, say E_ε. Because of (3.4), we have that $|E| = |E_\varepsilon|$. With

$$F(\varepsilon) = \int_{U'} \sqrt{1 + |\nabla u_\varepsilon|^2} + \int_{V'} \sqrt{1 + |\nabla w_\varepsilon|^2}\,,$$

the minimizing property of ∂E implies that $F(0) \leq F(\varepsilon)$ for all small ε and therefore that $F'(0) \geq 0$. Thus,

$$\int_{U'} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla(v-u) - \int_{V'} \frac{\nabla w}{\sqrt{1+|\nabla w|^2}} \cdot \nabla\varphi \geq 0\,.$$

Since w has a constant mean curvature K, we obtain

$$\int_{V'} \frac{\nabla w}{\sqrt{1+|\nabla w|^2}} \cdot \nabla\varphi = -\int_{V'} K\varphi = -K\int_{V'} \varphi = -K\int_{U'} (v-u)\,,$$

and therefore

$$\int_{U'} \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla(v-u) \geq -K\int_{U'} (v-u)\,. \tag{3.5}$$

□

If $\eta \in C_0^\infty(U')$ denotes an arbitrary nonnegative test function, then with $v - u = \eta$, (3.5) states that u is a weak solution of $\mathcal{H}_{\partial E} \leq K$. This combined with the $C^{1,1}$- regularity of u implies that $\mathcal{H}_{\partial E} \leq K$ pointwise almost everywhere in a neighborhood of $\partial\Omega$. Since $\mathcal{H}_{\partial E} = K$ in $\partial E \cap (\Omega \setminus S)$ with $H^{n-1}(S) = 0$ we have the following result.

Theorem 3.6. *Assume that Ω is bounded, convex and has a C^2 boundary. If E is a minimizer of* (3.1), *then $\partial E \in C^{1,1}$ in some neighborhood of $\partial\Omega$ and $\mathcal{H}_{\partial E} \leq K$ H^{n-1}-almost everywhere on ∂E.*

We now will exploit Theorem 3.6 to establish both regularity and a mean curvature estimate for the boundary of the convex hull of E.

Theorem 3.7. *Assume that Ω is bounded, strictly convex, and has a C^2 boundary. If E is a minimizer of* (3.1) *with convex hull H, then $\partial H \in C^{1,1}$ and $\mathcal{H}_{\partial H} \leq K$ H^{n-1}-almost everywhere on ∂H.*

Proof. Note that the singular set S in ∂E is a closed subset of Ω and thus separated from $\partial\Omega$, in fact it is contained in the interior of H, for if $x \in \partial E \cap \partial H \cap \Omega$, then the tangent cone to ∂E at x must be a hyperplane because $E \subset H$ and H is convex. Consequently ∂E is regular at x. Let N be an open neighborhood of S with compact closure in the interior of H. Thus, by Theorem 3.6 and the analyticity of ∂E in $\Omega \setminus S$ we see that ∂E is $C^{1,1}$ at points in $G := \partial E \setminus N$. Therefore, for some C we have

$$|\nu(x) - \nu(z)| \leq C|x - z| \qquad x, z \in G \tag{3.8}$$

where $\nu(x)$ is the outward unit normal to ∂E at x. Also since ∂E is C^1 at points in G, there exists an ε such that for all $x \in G$ and $z \in \partial E \cap B(x, \varepsilon)$ we have

$$|\nu(x) \cdot (x - z)| \leq 1/2|x - z|\,. \tag{3.9}$$

Choose $x \in \partial E \cap \partial H \subset G$ and let $0 < \alpha < 1/2$. Then define

$$d = \alpha \min\{\varepsilon,\ \mathrm{dist}(\partial H, N),\ \frac{1}{2C},\ \mathrm{diam}\, E\}\,.$$

Let $y = x - d\nu(x)$ and observe that y is in the interior of E since ∂E cannot intersect the line segment $\overline{xy}$ at a point $z \neq x$ due to (3.9). Let $r = \operatorname{dist}(y, \partial E)$ and note that $0 < r \leq d$. Now choose any $z \in \partial E$ such that $|y - z| = r$. Note that $z \in G$, for otherwise we would have $z \in N$ and since $|x - z| \leq |x - y| + |y - z|$, it would follow that

$$2d \geq |x - z| \geq \operatorname{dist}(\partial H, N) \geq \frac{d}{\alpha} > 2d ,$$

a contradiction. Then, $|x - z| \leq |x - y| + |y - z| \leq 2d < \varepsilon$ and both (3.8) and (3.9) hold. Thus, since $x = y + d\nu(x)$ and $z = y + r\nu(z)$, we have $|d - r| \leq |\nu(x) \cdot (x - z)|$ and

$$|x - z| = |(d - r)\nu(x) + r(\nu(x) - \nu(z))| \leq (1/2 + Cr)|x - z| \leq 3/4|x - z| ,$$

(since $r \leq d \leq \alpha/(2c) \leq 1/(4c)$) which implies that $x = z$ and therefore $r = d$. This implies that for every $x \in \partial E \cap \partial H$ there exists a ball $B_x \subset E$ of radius d containing x.

Given any $p \in \partial H$ we claim that p is a convex combination of points $\{x_i\}$ in $\partial E \cap \partial H$. To see this, note that if C is a convex set with $E \subset C$, then $\overline{E} \subset C$ since if $x \in \overline{E}$ then either $x \in C$ or $x \in \partial C$; in the later case, x lies in a support plane of C so if $x \in \Omega$, regularity theory implies that $x \in E \subset C$, and if $x \in \partial\Omega$ then x is not in the singular set S of E (since S is a compact subset of Ω) so again $x \in E \subset C$. Consequently from the definition of convex hull H of E as the intersection of all convex sets containing E, we see that $\overline{E} \subset H$. Moreover, H is the convex hull of $\overline{E}$ from which we conclude by a well-known result that H is closed since $\overline{E}$ is a compact subset of $\mathbb{R}^n$. Note that the set of finite convex combinations of points from E is convex, contains E, and is contained in any convex set that contains E and so equals H. Thus, if $p \in \partial H$ we have $p \in H$, since H is closed, and consequently $p = \sum_{i=1}^k \lambda_i x_i$ for $x_i \in E$ and $\sum_{i=1}^k \lambda_i = 1$, $\lambda_i \geq 0, i = 1 \ldots k$. If we take k to be as small as possible, then either $k = 1$ and $p \in E$ and the claim is trivially true, or p lies in the k dimensional interior of the convex hull M of $\{x_i\}$ in which case no x_i can lie in the interior of H since then the same would be true of p. Consequently $x_i \in \partial E \cap \partial H$, $i = 1 \ldots k$, as claimed.

Taking the convex hull of $\cup_{i=1}^k B_{x_i}$ we see that there exists a ball $B_p \subset H$ of radius d containing p, i.e., H satisfies a uniform interior sphere condition. We claim that this implies ∂H is $C^{1,1}$. To see this, consider the problem of prescribing unit vectors $\nu_1, \nu_2 \in \mathbb{R}^n$, and finding a convex set $\tilde{H}$, satisfying the interior sphere condition noted above, and points $x, y \in \partial\tilde{H}$ with $\nu(x) = \nu_1$, $\nu(y) = \nu_2$, such that $|x - y|$ is minimized. It is clear that x, y must lie in a two-dimensional plane orthogonal to the intersection of two hyperplanes having ν_1, ν_2 as normals, i.e., one need only consider the two-dimensional case where it is easy to see that one must have $B_x = B_y$. Taking the center of this ball to be the origin, then $\nu(x) = x/d$, $\nu(y) = y/d$ and we trivially have

$$|\nu(x) - \nu(y)| \leq \frac{1}{d}|x - y| .$$

Since this is the case when $|x - y|$ is smallest for fixed $\nu(x), \nu(y)$ we have established that $\nu(x)$ is Lipschitz in general.

We now prove that $\mathcal{H}_{\partial H} \leq K$ H^{n-1}-almost everywhere in ∂H. Note that $\mathcal{H}_{\partial H} = \mathcal{H}_{\partial E}$ H^{n-1}-almost everywhere on $\partial E \cap \partial H$ by Theorem 3.6. Thus we need only consider points $p \in \partial H \setminus \partial E$. In fact since ∂H is $C^{1,1}$ we need only consider $p \in \partial H \setminus \partial E$ at which ∂H is classically twice differentiable. As above, any such p lies in the k-dimensional interior of the convex hull M of certain points $p_i \in \partial E$, $i = 1, \ldots, k$. Note that $k \neq 1$ due to $p \notin \partial E$. Choose a coordinate system such that points in $\mathbb{R}^n$ are represented as (x, y, z), $x \in \mathbb{R}^k$, $y \in \mathbb{R}^{n-k-1}$, $z \in \mathbb{R}$, with $z = 0$ the tangent plane to ∂H at p, $p_i = (x_i, 0, 0)$, $i = 1, \ldots, k$, and $z \geq 0$ in H. We will construct an analytic function g whose graph does not lie below ∂H, contains M, and has mean curvature bounded above by $K + \varepsilon$ (for any $\varepsilon > 0$) in a small neighborhood of p. This will lead to the conclusion that $\mathcal{H}_{\partial H} \le K$ at p.

Let ∂E be represented as $z = f(x, y)$ for f defined in a neighborhood in $\mathbb{R}^k \times \mathbb{R}^{n-k-1}$ of $\cup(x_i, 0)$. Thus,

$$(x_i, y, f(x_i, y)) \in \partial E \subset H$$

for small $|y|$, and consequently

$$\sum_{i=1}^{k} \lambda_i (x_i, y, f(x_i, y)) \in H \qquad \text{if} \quad \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \ge 0 \tag{3.10}$$

for small $|y|$. For any given x in N, where N is the convex hull of the points x_i, $i = 1, \dots, k$, let $\lambda = \lambda(x) = (\lambda_1(x), \dots, \lambda_k(x))$ be the unique vector such that

$$x = \sum_{i=1}^{k} \lambda_i(x) x_i, \quad \sum_{i=1}^{k} \lambda_i(x) = 1, \quad \lambda_i(x) \ge 0 .$$

Thus, if we define

$$g(x, y) = \sum_{i=1}^{k} \lambda_i(x) f(x_i, y)$$

we see from (3.10) for $x \in N$ and small $|y|$ that

$$(x, y, g(x, y)) \in H ,$$

and so the surface $z = g(x, y)$ does not lie below ∂H at such (x, y).

Note that $M \cap \partial\Omega = \emptyset$, for otherwise the plane $z = 0$, which contains M, would be a tangent plane to $\partial\Omega$, thus contradicting the strict convexity of $\partial\Omega$. Also M does not intersect the singular set of ∂E since $M \subset \partial H$. Thus, ∂E is analytic at each p_i and therefore both $f(x_i, y)$ and $g(x, y)$ are smooth for small $|y|$. Furthermore,

$$0 \le \Delta_y f(x_i, 0) \le \Delta f(x_i, 0) \le K$$

since $\nabla f(x_i, 0) = 0$, $\mathcal{H}_{\partial E}$ equals Δf at points where the gradient is zero, and the second derivatives of f are nonnegative at $(x_i, 0)$ due to the fact that $f \ge 0$, $f(x_i, 0) = 0$ for all i. Hence, for any $\varepsilon > 0$, $\Delta_y f(x_i, y) \le (K + \varepsilon)$ for small enough $|y|$ so $\Delta_y g(x, y) \le (K + \varepsilon)$ as well. However $\Delta_x g = 0$ and so $\Delta g \le (K + \varepsilon)$ for small $|y|$. Recall that ∂H is trapped between $\{z = 0\}$ and the graph of g over a region which contains p in its interior. Since $g(p) = 0$ and ∂H is twice differentiable at p we conclude that $\mathcal{H}_{\partial H}(p) \le K$ as required. □

Theorem 3.11. *Assume that Ω is bounded, strictly convex and satisfies a great circle condition. If E is a minimizer of (3.1) with $|B_\Omega| \le |E|$, then*

$$B_\Omega \subset E$$

where B_Ω is the largest ball in Ω.

Proof. If $|E| = |B_\Omega|$, then clearly E must be a ball. Since there is only one largest ball in Ω due to strict convexity, we have $E = B_\Omega$. Otherwise $|B_\Omega| < |E|$. In this case, translate the upper and lower hemispheres of B_Ω by a distance d in opposite directions orthogonal to T_{B_Ω} until H, the convex hull of the two translated hemispheres, intersects E in a set of measure $|B_\Omega|$, i.e.,

$$|H \cap E| = |B_\Omega| \,. \tag{3.12}$$

This is possible because of the great circle condition and because Ω is bounded and convex. Now translate the hemispheres back to their original positions while rigidly carrying along the parts of E lying in the exterior of H. Let $\tilde{E}$ be the union of the translated parts of E with B_Ω. Note that

$$|\tilde{E}| = |E| \quad \text{and therefore} \quad P(\tilde{E}) \geq P(E) \,. \tag{3.13}$$

Using a standard inequality, cf. [8], we have

$$P(E) + P(H) \geq P(E \cap H) + P(E \cup H)$$

where $P(S)$ denotes $P(S, \mathbb{R}^n)$. For brevity, write $D = D_{B_\Omega}$. Observe that

$$P(H) = 2dH^{n-2}(\partial D) + P(B_\Omega), \quad P(E \cup H) = P(\tilde{E}) + 2dH^{n-2}(\partial D)$$

and thus

$$P(E) + P(B_\Omega) \geq P(E \cap H) + P(\tilde{E}) \,.$$

In view of (3.13) it follows that $P(E \cap H) \leq P(B_\Omega)$. But then the isoperimetric inequality and (3.12) imply that $E \cap H$ is a ball. However, Ω contains only one largest ball and so we must have $E \cap H = B_\Omega$, i.e., $B_\Omega \subset E$. □

Suppose M is an oriented $(n-1)$-dimensional C^1 submanifold of $\mathbb{R}^n$ and $f: M \to \mathbb{R}^{n-1}$ a C^1 mapping. Let $Jf(x)$ denote the Jacobian of f at x and note that the sign of the Jacobian depends on the orientation of M. We recall the following result, cf. [2, Theorem 3.2.20]: For any H^{n-1}-measurable set $E \subset M$ and any H^{n-1}-measurable function φ,

$$\int_E \varphi[f(x)]\,|Jf(x)|\; dH^{n-1}(x) = \int \varphi(y) N(f, E, y)\, dy \tag{3.14}$$

where $N(f, E, y)$ denotes the number (possibly infinite) of points in $f^{-1}(y) \cap E$. Here equality is understood in the sense that if one side is finite, then so is the other. In our application (3.18) below, we will know the left side is finite, therefore ensuring that $N(f, E, y)$ is finite for almost all y.

Lemma 3.15. *There is a constant $C = C(n)$ such that for each $x \in (\partial E) \cap \Omega$ we have*

$$\frac{H^{n-1}((\partial E) \cap B(x, r))}{r^{n-1}} \leq C$$

for almost all sufficiently small $r > 0$.

Proof. It follows from (3.2) that we may as well assume ∂E is area minimizing. In this case, the result follows immediately from the fact that

$$\frac{H^{n-1}((\partial E) \cap B(x, r))}{r^{n-1}}$$

is nondecreasing in r, for $r > 0$ sufficiently small, cf. [2, Theorem 3.4.3]. □

Lemma 3.16. *For every $\varepsilon > 0$ and any open set $V \subset \mathbb{R}^n$ containing the singular set S of ∂E, there exists an open set W and a Lipschitz function f such that*

$$S \subset W \subset \{f = 1\}$$
$$\operatorname{spt} f \subset V$$
$$\int_{\partial E} |\nabla f| \; dH^{n-1} \leq \varepsilon \, .$$

Proof. Let V be any open set containing S and let $\delta = 1/2(\operatorname{dist} S, \mathbb{R}^n - V)$. Since $H^{n-7}(S) = 0$ and S is compact, there is a finite collection of open balls $\{B(x_i, r_i)\}_{i=1}^m$ such that $2r_i < \delta$, $B(x_i, r_i) \cap S \neq \emptyset$, $S \subset \cup_{i=1}^m B(x_i, r_i)$ and

$$\sum_{i=1}^m r_i^{n-7} < \frac{\varepsilon}{C} \, ,$$

C as in Lemma 3.15. We will assume that each ball $B(x_i, r_i)$ has been chosen so that $r_i < 1$ and that $2r_i$ satisfies Lemma 3.15. Let W denote the union of these balls and define f_i by

$$f_i(x) = \begin{cases} 1 & \text{if } |x - x_i| \leq r_i \\ 2 - \frac{|x - x_i|}{r_i} & \text{if } r_i \leq |x - x_i| \leq 2r_i \\ 0 & \text{if } 2r_i \leq |x - x_i| \, . \end{cases}$$

In view of Lemma 3.15, it follows that

$$\int_{B(x_i, r_i) \cap \partial E} |\nabla f_i| \; dH^{n-1} \leq C r_i^{n-2} < C r_i^{n-7} \, .$$

Now let $f := \max_{1 \leq i \leq m} f_i$. Then f is Lipschitz, $W \subset \{f = 1\}$, $\operatorname{spt} f \subset V$ and

$$\begin{aligned} \int_{\partial E} |\nabla f| \; dH^{n-1} &\leq \sum_{i=1}^m \int_{B(x_i, r_i) \cap \partial E} |\nabla f_i| \; dH^{n-1} \\ &< C \sum_{i=1}^m r_i^{n-7} < \varepsilon \, . \end{aligned}$$

□

Lemma 3.17. *Let T denote the $(n-1)$-rectifiable current determined by $(\partial E)^+$, the part of ∂E that lies above the equatorial disk $D := D_{B_\Omega}$ of B_Ω. Then ∂T is the $n-2$-sphere given by $\partial T = \partial D$.*

Proof. Clearly, the support of ∂T contains the $(n-2)$ sphere, but we must rule out the possibility of it containing points of S as well. For this purpose, choose $x \in S$ and let φ be any smooth differential form supported in some neighborhood of x that does not meet $(\partial E)^+ \cap \partial D$. It suffices to show that $T(d\varphi) = 0$. Let μ denote H^{n-1} restricted to $(\partial E)^+$. Appealing to Lemma 3.16, we can produce a sequence of Lipschitz functions $\{\omega_i\}$ such that

$$\omega_i \to 1 \;\; \mu \text{ a.e.}$$
$$|\nabla \omega_i| \to 0 \;\; \mu \text{ a.e.}$$
$$\omega_i \text{ vanishes in a neighborhood of } S$$
$$\int_{(\partial E)^+} |\nabla \omega_i| \; d\mu \to 0 \, .$$

Thus, we obtain

$$\begin{aligned} 0 = T(d(\varphi\omega_i)) &= T(d\varphi \wedge \omega_i) + T(\varphi \wedge d\omega_i) \\ &= \int_{(\partial E)^+} d\varphi \wedge \omega_i + \int_{(\partial E)^+} \varphi \wedge d\omega_i \, . \end{aligned}$$

The first integral tends to

$$\int_{(\partial E)^+} d\varphi = T(d\varphi)$$

while the second tends to 0. Thus, $T(d\varphi) = 0$. □

Let E denote a minimizer of (3.1), where Ω is strictly convex with C^2 boundary. Since ∂E is locally an $n-1$-manifold of class C^1 except for a singular set S whose Hausdorff dimension does not exceed $n-8$, it follows that ∂E can be regarded as an oriented $n-1$ integral current whose boundary is 0; i.e., an oriented $n-1$ integral cycle.

Let T denote the $n-1$ integral current represented by $(\partial E) \cap H^+$. Since ∂E is of class $C^{1,1}$ in a neighborhood of each point of $(\partial E) \cap (\partial\Omega)$, it follows that the tangent cone to ∂E at such points is in fact a tangent plane. Consequently, ∂E is analytic near such points and therefore the singular set S of ∂E lies in the interior of $(\partial E) \cap H^+$. We know from Lemma 3.17 that the boundary of T is the $n-2$-sphere determined by ∂D_{B_Ω}, the equator of B_Ω. Let $p\colon \mathbb{R}^n \to T_{B_\Omega}$ denote the orthogonal projection and consider the current $R := p_\#(T)$. Note that $\partial R = p_\#(\partial T) = \partial D_{B_\Omega}$. Furthermore, D_{B_Ω} is the unique current in T_{B_Ω} whose boundary is ∂D_{B_Ω} and, therefore, we conclude that $R = D_{B_\Omega}$. Let us consider the action of R operating on an $n-1$-form φ. For this we will let $\alpha(x)$ denote the Grassman $(n-1)$-vector of norm one that is in the tangent plane orthogonal to $\nu(E, x)$, the exterior normal to E at x. $\alpha(x)$ is chosen in such a way that $\alpha(x) \wedge \nu(E, x)$ forms the Grassman unit n-vector that induces a positive orientation of $\mathbb{R}^n$. Also, we let $dp(\alpha(x))$ denote the value of the differential of p operating on $\alpha(x)$. Then, with the help of (3.14), we have

$$\begin{aligned} R(\varphi) &= T(p^\#\varphi) \\ &= \int_{(\partial E)\cap H^+} p^\#\varphi \cdot \alpha \\ &= \int_{(\partial E)\cap H^+} \varphi[p(x)] \cdot dp(\alpha(x)) \, dH^{n-1}(x) \\ &= \int_{D_{B_\Omega}} \varphi(y)[N^+(p, \partial E, y) - N^-(p, \partial E, y)] \, dy \end{aligned}$$

where $N^+(p, \partial E, y)$ denotes the number of points of $p^{-1}(y) \cap \partial E$ at which Jp is positive and similarly, $N^-(p, \partial E, y)$ denotes the number of points of $p^{-1}(y) \cap \partial E$ at which Jp is negative. Since $R = D_{B_\Omega}$, we conclude that

$$N^+(p, \partial E, y) - N^-(p, \partial E, y) = 1 \tag{3.18}$$

for almost all $y \in D_{B_\Omega}$.

Lemma 3.19. *Assume that Ω is bounded, strictly convex, has a C^2 boundary, and satisfies a great circle condition. Let H denote the convex hull for any minimizer E of the variational problem* (3.1). *Then there is a constant K such that $\mathcal{H}_{\partial H} = K$ at H^{n-1} almost all points of $(\partial H) \cap \Omega$.*

Proof. First, we recall that $\partial E \cap \overline{\Omega}$ is C^1 at all of its points except for a singular set $S \subset \partial E \cap \Omega$ whose Hausdorff dimension does not exceed $n-8$. Furthermore, we know that $\partial E \cap \Omega$ is real analytic at all points away from S and that ∂H is $C^{1,1}$. Finally, we know that E contains B_Ω. Let $(\partial E)^+$ and $(\partial H)^+$ denote the parts of ∂E and ∂H respectively that lie above the equatorial plane P of B_Ω. Let $p: \mathbb{R}^n \to P$ denote the orthogonal projection. The mean curvature of ∂E is equal to a constant K at all points of $\partial E \cap (\Omega - S)$. Let X denote the vertical unit vector. We wish to apply (2.13) with $(\partial E)^+$ replacing M. Referring to the proof of Lemma 3.17, we see that this can be done in spite of the singular set $S \in (\partial E)^+$. Thus, applying (2.13), we obtain

$$\int_{(\partial H)^+} \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} = \int_{(\partial E)^+} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \tag{3.20}$$

where ν_H and ν_E denote the unit exterior normals to H and E, respectively. Let

$$\begin{aligned} A &= (\partial E)^+ \cap (\partial H)^+ \\ B &= ((\partial H)^+ - A) \cap \{x : \mathcal{H}_{\partial H}(x) < K\} \\ C &= ((\partial H)^+ - A) \cap \{x : \mathcal{H}_{\partial H}(x) = K\} \,. \end{aligned}$$

Since $\mathcal{H}_{\partial H} \leq K \ H^{n-1}$-a.e. in $(\partial H)^+ \cap \Omega$, it suffices to prove that

$$H^{n-1}(B) = 0 \,. \tag{3.21}$$

Observe that both B and C are subsets of ∂H^+. Note also that A, B, and C are mutually disjoint subsets of $(\partial H)^+$ with $H^{n-1}[(\partial H)^+ - (A \cup B \cup C)] = 0$. Thus, $p(A)$, $p(B)$, and $p(C)$ are mutually disjoint and their union occupies almost all of D_{B_Ω}. Clearly, ν_E and ν_H as well as $\mathcal{H}_{\partial H}$ and $\mathcal{H}_{\partial E}$ agree H^{n-1} almost everywhere on A. Therefore,

$$\int_A \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} = \int_A \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \,. \tag{3.22}$$

Since $X \cdot \nu_H$ is the Jacobian of the mapping $p: \partial H^+ \to D_{B_\Omega}$, it follows from (3.14) that

$$\begin{aligned} &\int_B \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} < K H^{n-1}[p(B)] \,, \\ &\int_C \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} = K H^{n-1}[p(C)] \,. \end{aligned}$$

Now let

$$\begin{aligned} A^* &= ((\partial E)^+) \cap p^{-1}[p(A)] \,, \\ B^* &= ((\partial E)^+) \cap p^{-1}[p(B)] \,, \\ C^* &= ((\partial E)^+) \cap p^{-1}[p(C)] \,. \end{aligned}$$

Next, observe that both B^* and C^* are subsets of Ω. To see this, consider $x \in B^*$. If it were true that $x \in B^* \cap \partial\Omega$, then $x \in (\partial H)^+$ and thus $x \in A$. This is impossible since $p(A)$ and $p(B)$ are disjoint. A similar argument holds for C^*. Referring to (3.14) and (3.18), we obtain

$$\begin{aligned} &\int_{B^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\ &= K \int_{B^* \cap \{x : X \cdot \nu_E(x) > 0\}} X \cdot \nu_E \, dH^{n-1} + K \int_{B^* \cap \{x : X \cdot \nu_E(x) < 0\}} X \cdot \nu_E \, dH^{n-1} \\ &= K \int_{p(B^*)} N^+(p, \partial E, y) - N^-(p, \partial E, y) \, dH^{n-1}(y) \\ &= K H^{n-1}[p(B^*)] \\ &= K H^{n-1}[p(B)] \,. \end{aligned}$$

Similarly,

$$\int_{C^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} = K H^{n-1}[p(C^*)] = K H^{n-1}[p(C)]$$

and

$$\int_{A^*} K X \cdot \nu_E \, dH^{n-1} = K H^{n-1}(p(A)) \, .$$

Finally, in view of the fact that $A \subset (\partial H)^+$ and therefore that $N^+(p, A, y) = 1$ and $N^-(p, A, y) = 0$ for H^{n-1}-almost all $y \in p(A)$, we obtain

$$\int_{A} K X \cdot \nu_E \, dH^{n-1} = K H^{n-1}(p(A)) \, .$$

Now, using the facts that $A^* - A \subset \Omega$ and $\mathcal{H}_{\partial E} = K$ on $A^* - A - S$, we obtain

$$\begin{aligned} \int_{A^*} & \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\ &= \int_{A^*} K X \cdot \nu_E \, dH^{n-1} + \int_{A^*} (\mathcal{H}_{\partial E} - K) X \cdot \nu_E \, dH^{n-1} \\ &= \int_{A^*} K X \cdot \nu_E \, dH^{n-1} + \int_{A} (\mathcal{H}_{\partial E} - K) X \cdot \nu_E \, dH^{n-1} \\ &= K H^{n-1}(p(A)) - K H^{n-1}(p(A)) + \int_{A} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\ &= \int_{A} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \, . \end{aligned}$$

Under the assumption $H^{n-1}(B) > 0$, we would obtain

$$\begin{aligned} \int_{(\partial H)^+} \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} &< \int_{A} \mathcal{H}_{\partial H} X \cdot \nu_H \, dH^{n-1} + K H^{n-1}[p(B)] + K H^{n-1}[p(C)] \\ &= \int_{A} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} + K H^{n-1}[p(B^*)] + K H^{n-1}[p(C^*)] \\ &= \int_{A^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} + K H^{n-1}[p(B^*)] + K H^{n-1}[p(C^*)] \\ &= \int_{A^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} + \int_{B^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\ &\qquad + \int_{C^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\ &= \int_{A^* \cup B^* \cup C^*} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \\ &\le \int_{(\partial E)^+} \mathcal{H}_{\partial E} X \cdot \nu_E \, dH^{n-1} \, , \end{aligned}$$

where we have used that A^*, B^*, and C^* are mutually disjoint. This would contradict (3.20), thus establishing (3.21). □

A function $u \in C^1(W)$ is called a *weak subsolution* (*supersolution*) *of the equation of constant K mean curvature* if

$$Mu(\varphi) = \int_W \frac{\nabla u \cdot \nabla \varphi}{\sqrt{1 + |\nabla u|^2}} - K \varphi \, dx \le 0 \quad (\ge 0)$$

whenever $\varphi \in C_0^1(W)$, $\varphi \geq 0$.

We note that if $u \in C^{1,1}$ and classically satisfies the equation of constant mean curvature equation almost everywhere, then u is a weak solution.

The following result will be stated in the context of R^{n-1} because of its applications in the subsequent development.

Lemma 3.23. *Suppose W is an open subset of R^{n-1}. If $u_1, u_2 \in C^1(W)$ are, respectively, weak super and subsolutions of the equation of constant mean curvature in W and if $u_1(x_0) = u_2(x_0)$ for some $x_0 \in W$ while $u_1(x) \geq u_2(x)$ for all $x \in W$, then*

$$u_1(x) = u_2(x)$$

for all x in some closed ball contained in W centered at x_0.

Proof. Define

$$\begin{aligned}
u_t &= tu_1 + (1-t)u_2 \text{ for } t \in [0,1]\,, \\
w &= u_1 - u_2\,, \\
a^{ij}(x) &= \int_0^1 D_{u_{x_j}}\left(\frac{D_i u_t(x)}{\sqrt{1+|\nabla u_t|^2}}\right) dt \\
&= \int_0^1 \frac{1}{\sqrt{1+|\nabla u_t|^2}}\left(\delta_{ij} - \frac{D_i u_t(x) D_j u_t(x)}{(1+|\nabla u_t|^2)}\right) dt\,.
\end{aligned}$$

Since both u_1 and u_2 are continuously differentiable in W, for each open set $V \subset\subset W$ containing x_0 there exists $M > 0$ such that $|\nabla u_t(x)| \leq M$ for all $x \in V$ and all $t \in [0,1]$. Hence,

$$\begin{aligned}
a^{ij}(x)\xi_i\xi_j &\geq \frac{1}{(1+M^2)^{1/2}}|\xi|^2\,, \text{ for all } \xi \in R^{n-1}, x \in V\,, \\
\sum_{i,j} a^{ij}(x)^2 &\leq C, \text{ for all } x \in V\,.
\end{aligned}$$

For $\varphi \in C_0^1(W)$, $\varphi \geq 0$, we have

$$\begin{aligned}
0 &\leq Mu_1(\varphi) - Mu_2(\varphi) \\
&= \int_W \int_0^1 \frac{d}{dt}\left(\frac{\nabla u_t(x) \cdot \nabla\varphi(x)}{\sqrt{1+|\nabla u_t|^2}}\right) dt\, dx \\
&= \int_W a^{ij}(x) D_j w(x) D_i \varphi(x)\, dx\,.
\end{aligned}$$

Thus, w is a weak supersolution of the equation

$$D_i(a^{ij} D_j w) = 0$$

and since $w \geq 0$, the weak Harnack inequality yields

$$\left(r^{-n}\int_{B(x_0,2r)} |w(x)|^p\, dx\right)^{1/p} \leq C \inf_{B(x_0,r)} w = 0$$

whenever $1 \leq n < n/(n-2)$ and $B(x_0, 4r) \subset W$. □

Theorem 3.24. *Suppose Ω is a bounded, strictly convex domain with C^2 boundary that satisfies a great circle condition. Then any minimizer E of the variational problem* (3.1) *is convex.*

Remark 3.25. Later we show that neither smoothness of $\partial\Omega$ nor strict convexity are required. In addition, the great circle condition is unnecessary in $\mathbb{R}^2$. The same applies to the uniqueness result below. □

Proof. It suffices to show that $H = E$ where H denotes the convex hull of E. Assume $\partial H \not\subset \partial E$ so there exists $x \in \partial H \setminus \partial E$. Thus, as in the proof of the mean curvature inequality in Theorem 3.7, we see that x lies in the convex hull M of distinct points $p_i \in \partial H \cap \partial E$, $i = 1, \dots, k$, $k > 1$. Furthermore, each p_i is an element of Ω due to the fact that they all lie in a single support plane of H; hence, if one p_i were to lie in $\partial\Omega$ then they all would, thus contradicting strict convexity. Referring to Lemma 3.23, we see that ∂H and ∂E agree in a neighborhood of the points p_i. Since M is connected, it follows again from Lemma 3.23 that $M \subset \partial E \cap \partial H$, which contradicts $x \notin \partial E$. Consequently, $\partial H \subset \partial E$ and thus $P(H) \leq P(E)$. However, $E \subset H$ so $|E| \leq |H|$. Assume $|E| < |H|$. Dilate H to obtain $\tilde{H} \subset \Omega$ satisfying $|\tilde{H}| = |E|$. But then $P(\tilde{H}) < P(H) \leq P(E)$, which contradicts the minimality of E. Thus, $|E| = |H|$ so that E and H have the same measure theoretic closure. Hence, due to our convention concerning distinguished representatives for sets of finite perimeter, $E = H$ and E is convex. □

Theorem 3.26. *If Ω is as in Theorem* 3.24, *then perimeter minimizers with measure exceeding $|B_\Omega|$ are nested and unique. That is, if E and F are perimeter minimizers, then*

$$|B_\Omega| \leq |F| < |E| \Longrightarrow E \subset F \tag{3.26.1}$$

and

$$|B_\Omega| \leq |F| = |E| \Longrightarrow F = E\,. \tag{3.26.2}$$

In addition, perimeter minimizers have disjoint boundaries relative to Ω in the sense that

$$|B_\Omega| \leq |F| < |E| \Longrightarrow \partial F \cap \partial E \subset \partial\Omega\,.$$

Remark 3.27. Note that the assumption of convexity can be relaxed. It is only required that the intersection of Ω with any vertical line is an interval. In addition, $\partial\Omega$ must not contain vertical line segments. □

Proof. To prove (3.26.1) we argue by contradiction. If E and F are perimeter minimizers satisfying $|B_\Omega| \leq |F| < |E|$, then assume F is not a subset of E. From Theorems 3.11 and 3.24 we see that E and F are convex and contain B_Ω. Since F is not a subset of E, one can employ the proof of Theorem 3.11, with F playing the role of B_Ω, to prove that there is a second perimeter minimizer E^* which contains F and satisfies $|E^*| = |E|$. Let H be the analog of H in the proof of Theorem 3.11 and let D° denote the interior of $D := D_{B_\Omega}$.

We will use the properties of perimeter minimizers to show that ∂H and $\partial(H \cup E)$ are analytic and coincide on some open set. By connectedness, this will show they are identical, thus establishing the desired contradiction.

Let O be the interior of $\partial H \setminus (H \cup E)^\circ$ relative to ∂H, and ∂O represent the boundary of O relative to ∂H. Assume there exists a point

$$x \in \partial O \cap p^{-1}(D^\circ)\,.$$

Note that $x \in \partial H \cap \partial E \cap \partial(H \cup E) \cap p^{-1}(D^\circ)$. Let y be the point on $\partial F \cap \partial E^*$ which was translated (as in the definition of H) to x. Since ∂O has positive H^{n-2} measure ($\partial O \cap p^{-1}(D^\circ) \neq \emptyset$) we can assume $y \notin S$, S being the singular set for E^*.

Since $x \in \partial E \subset \overline{\Omega}$, y lies in Ω and consequently ∂H is analytic in a neighborhood of x since ∂F is analytic in a neighborhood of y. Similarly $H \cup E$ inherits analyticity (in a neighborhood of x) from ∂E^* since $x \in \partial(H \cup E)$ and $y \notin S$. However, $\partial H \cap O \subset \partial(H \cup E)$ so ∂H and $\partial(H \cup E)$ coincide on open (relative to ∂H) subsets of any neighborhood of x so by analyticity ∂H coincides with $\partial(H \cup E)$ in some neighborhood of x. But this contradicts $x \in \partial O$ so $\partial O \cap p^{-1}(D^\circ)$ is empty.

Note that $(\partial H \setminus E) \cap p^{-1}(D^\circ)$ contains points lying both above and below D since Ω is strictly convex and H is the hull of the translated halves of F (which contain the hemispheres of the largest ball B_Ω). Thus, the same is true of $O \cap p^{-1}(D^\circ)$. Combined with $\partial O \cap p^{-1}(D^\circ) = \emptyset$, this implies $\partial H \cap p^{-1}(D^\circ) \cap E^\circ = \emptyset$, i.e., $E \subset H$. Of course this is absurd because $|E| > |F| = |H|$. Thus, the assumption that F is not contained in E is false, i.e., $F \subset E$ as required.

Now assume that $|B_\Omega| < |F| = |E| = v$. Choosing a sequence of perimeter minimizers F_i of measure $v_i \uparrow v$, it follows from (3.26.1) that $F_i \subset E \cap F$. Consequently, $|E \cap F| = v$ and so $E = F$.

To prove that minimizers are strictly nested in the sense defined above, assume that $|B_\Omega| \leq |F| < |E|$ and so $F \subset E$. Assume in addition that $G := (\partial F \cap \partial E \cap \Omega)^\circ$ is not empty. Since F, E are analytic in Ω and nested, it is clear that $\mathcal{H}_{\partial F} \geq \mathcal{H}_{\partial E}$ at points in G. Given that $\mathcal{H}_{\partial F}$, $\mathcal{H}_{\partial E}$ are constants, say k_f, k_e, in Ω and equal almost everywhere on $\partial F \cap \partial E \cap \partial\Omega$, we may derive a contradiction from $k_f \geq k_e$ through the use of (2.13). In fact, we obtain

$$\int_D \mathcal{H}'_{\partial F}\, dH^{n-1} = H^{n-2}(\partial D) = \int_D \mathcal{H}'_{\partial E}\, dH^{n-1} \tag{3.28}$$

where $\mathcal{H}'_{\partial F}(x) := \mathcal{H}_{\partial F}(p^{-1}(x) \cap \partial F)$ and $\mathcal{H}'_{\partial E}(x) := \mathcal{H}_{\partial E}(p^{-1}(x) \cap \partial E)$. However, with $A := \partial F \cap \partial\Omega$ and $B := \partial F \cap \Omega$, we see that

$$\begin{aligned}
\int_D \mathcal{H}'_{\partial E}\, dH^{n-1} &= \int_{p(A)} \mathcal{H}'_{\partial F}\, dH^{n-1} + \int_{p(B)} \mathcal{H}'_{\partial E}\, dH^{n-1} \\
&\leq \int_{p(A)} \mathcal{H}'_{\partial F}\, dH^{n-1} + \int_{p(B)} \mathcal{H}'_{\partial F}\, dH^{n-1} \\
&= \int_D \mathcal{H}'_{\partial F}\, dH^{n-1},
\end{aligned} \tag{3.29}$$

and thus we have equality due to (3.28). Therefore, $\mathcal{H}'_{\partial E} = k_e = k_f = \mathcal{H}'_{\partial F}$ on $p(B)$. However, since $\mathcal{H}'_{\partial E} = \mathcal{H}'_{\partial F}$ almost everywhere on $p(A)$, we obtain

$$\mathcal{H}'_{\partial E} = k_e = k_f = \mathcal{H}'_{\partial F} \qquad H^{n-1}\text{-almost everywhere on } D\,.$$

Thus, for $x \in \partial F \cap \partial E$, apply Lemma 3.23 to conclude that ∂F and ∂E coincide in a neighborhood of x. Thus, $p(\partial F \cap \partial E)$ is both open and closed relative to D°, and therefore contains D°, a contradiction since $|F| < |E|$. □

We now dispense with the assumptions of strict convexity and smoothness of $\partial\Omega$. When the assumption of strict convexity is dropped, complications arise because there is no longer a unique largest ball in Ω. Eliminating the smoothness assumption on the boundary forces us to take limits of perimeter minimizers, and to establish convexity of all perimeter minimizers through a uniqueness theorem.

One interesting observation is that a perimeter minimizer can be thought of as a smooth approximation of Ω, especially when its measure is close to that of Ω. This is due to the fact that even after we have dispensed with the smoothness assumption on $\partial\Omega$, perimeter minimizers still have $C^{1,1}$ boundaries.

For the proof of Theorem 3.31, we need the following lemma.

Lemma 3.30. *Let $a < c < b$ and let I_1, I_2 denote the closed intervals $[a, c]$ and $[c, b]$, respectively. Let f_1 and f_2 be functions such that $f_i \in C^2(I_i)$, $i = 1, 2$, with $f_1(c) = f_2(c)$. Furthermore, assume there are constants c_1, c_2, and c_3 such that*

(i) $f_i'' \leq c_1 < 0$ *on* I_i, $i = 1, 2$,
(ii) $f_1' \geq c_2 > 0$ *on* I_1 *and* $f_2' \leq c_3 < 0$ *on* I_2.

Then, there exists a C^2, strictly concave function g on $[a, b]$ such that g is uniformly close to f on $[a, b]$ and that $g = f$ on the complement of any given open interval containing c.

Proof. A given open interval containing c in turn contains an open interval $I = (a', b')$ with $c \in I$ determined by the constants c_1, c_2, and c_3 such that the following three conditions hold:

(i) There are points $x_1, x_2 \in I$ with $x_1 < c < x_2$ such that $f_1(x_1) = f_2(x_2)$.
(ii) There are polynomials p_i of degree 2 (i=1,2) such that $p_i(x_i) = f_i(x_i)$ and such that the functions

$$h_1(x) := \begin{cases} f_1(x) & \text{for } a \leq x \leq x_1 \\ p_1(x) & \text{for } x_1 \leq x \leq c \end{cases} \qquad h_2(x) := \begin{cases} f_2(x) & \text{for } x_2 \leq x \leq b \\ p_2(x) & \text{for } c \leq x \leq x_2 \end{cases}$$

are C^2 and strictly concave on I_i.
(iii) There is a point $c' \in I$ such that $h_1(c') = h_2(c')$.

Thus, the function

$$h := \begin{cases} h_1 & \text{on } [a, c'] \\ h_2 & \text{on } [c', b] \end{cases}$$

is strictly concave on $[a, b]$. We will now mollify h restricted to I by using a smooth mollifying kernel φ with the property that

$$\varphi_\varepsilon * p(x) = p(x)$$

whenever p is a polynomial of degree 2, $\varepsilon > 0$, and $x \in \mathbb{R}$, cf. [11, Lemma 3.5.6]. Thus, for sufficiently small $\varepsilon > 0$, $\varphi_\varepsilon * h(x) = h(x)$, for $x \in (a' + \varepsilon, c' - \varepsilon) \cup (c' + \varepsilon, b' - \varepsilon)$. Also, $\varphi_\varepsilon * h$ is strictly concave since h is. Thus, our desired function g is defined by

$$g(x) = \begin{cases} h(x) & \text{for } a \leq x \leq a' + \varepsilon \\ \varphi_\varepsilon * h(x) & \text{for } a' + \varepsilon < x < c' - \varepsilon \\ h(x) & \text{for } c' - \varepsilon \leq x \leq b. \end{cases}$$

□

We define H_Ω to be the union of all largest balls in Ω. Thus, H_Ω is the convex hull of the two largest balls which are furthest apart. H_Ω essentially plays the role of B_Ω.

Theorem 3.31. *Suppose Ω is a bounded, convex domain that satisfies a great circle condition. Given v, $|H_\Omega| \leq v < |\Omega|$ there is a unique minimizer E with $|E| = v$ of the variational problem (3.1).*

E is convex with $C^{1,1}$ boundary. Such minimizers are nested with disjoint boundaries relative to Ω as in Theorem 3.26. If $|B_\Omega| < v \le |H_\Omega|$, then any minimizer E is the convex hull of two largest balls (clearly uniqueness is lost for $v < |H_\Omega|$).

Proof. We first smooth Ω and then establish the existence of a nested family of convex perimeter minimizers by taking limits. We finish by adapting the uniqueness result of Theorem 3.26 and the proof of disjointness of boundaries.

Let T_{B_Ω} be the hyperplane which intersects orthogonally the midpoint of the line segment joining the centers of the two largest balls whose hull forms H_Ω. Think of the "vertical" axis as coinciding with this line segment and take the origin of our coordinate system to be the midpoint just mentioned. As defined previously, p is orthogonal projection onto T_{B_Ω}. Let B_Ω be the largest ball in Ω with equatorial plane in T_{B_Ω}. Let $D_{B_\Omega} = p(\Omega)$ so D_{B_Ω} is an $(n-1)$-ball. Let C be the interior of the union of a closed right circular cone with base D_{B_Ω} with its reflection across T_{B_Ω}. Let B be the largest ball in C and note that $C \setminus B$ has three components (four in $\mathbb{R}^2$). Let C_0 denote the component (or union of two components in $\mathbb{R}^2$) which intersects D_{B_Ω} and consider the set $C_1 = C \setminus \overline{C_0}$.

First we show that Ω can be approximated arbitrarily closely by strictly convex sets satisfying a great circle condition, then we will approximate the latter by sets with C^2 boundary of the same type. Note that $\Omega \cap C_1$ is convex and satisfies a great circle condition with B being the largest ball. Also $\partial(\Omega \cap C_1)$ consists of the union of the graphs of functions f_i, $i = 1, 2$, $f_1 \ge 0$, $f_2 \le 0$. Let Ω' be the set whose boundary is the union of the graphs of $f_1 + \varepsilon b$, $f_2 - \varepsilon b$ where $\varepsilon > 0$ and b is the function whose graph is the upper hemisphere of B. Note that Ω' is strictly convex and satisfies a great circle condition. Also, as $\varepsilon \to 0$, C approaches a cylinder, and $\Omega' \to \Omega$ in the Hausdorff sense.

We now may assume without loss of generality that Ω is strictly convex. Consider $G = \Omega \cap C$. Note that ∂G is the union of graphs of $f_i : \overline{D}_{B_\Omega} \to \mathbb{R}$, $i = 1, 2$ with $f_1 \ge 0$, $f_2 \le 0$. Given $r > 0$ let B_r be the ball of radius r concentric to B_Ω, $D_r = D_{B_\Omega} \cap B_r$,and R the radius of B_Ω. Also let $\bar{r}$ be the distance from $\partial B_\Omega \cap \partial C$ to the vertical axis.

Consider ε, $0 < \varepsilon << R$. For a smooth radially symmetric approximate identity η_ε supported in B_ε let $f_\varepsilon = f_1 * \eta_\varepsilon$. Thus, f_ε is defined in $D_{R-\varepsilon}$ and is a surface of revolution in $A_\varepsilon = D_{R-\varepsilon} \setminus D_{\bar{r}+\varepsilon}$

Now consider $\delta > 0$ such that $\bar{r} < R - \delta$ but $\partial B_{R-\delta}$ does not intersect ∂C. Take ε small enough that the graph of f_ε does not intersect $\partial B_{R-\delta}$. Let $g_\varepsilon : [\bar{r} + \varepsilon, R - \varepsilon] \to \mathbb{R}$ be the function the rotation of whose graph around the vertical axis produces the graph of f_ε over A_ε. In the r, z plane let C_2 be a circle of radius $s >> R$ with center on the negative r axis which passes through $(R - \delta, 0)$. Let $c : [\bar{r} + \varepsilon, R - \varepsilon] \to \mathbb{R}$ be the function whose graph lies in the upper half of C_2 and define $h_\varepsilon = \min(g_\varepsilon, c)$ on $[\bar{r} + \varepsilon, R - \delta]$. Note that h_ε is a strictly concave function and is smooth except at the point q of intersection of the graphs of g_ε and c (which exists if s is large enough). Now employ Lemma 3.30 to alter h_ε in a small neighborhood of q to produce a C^2 function which is still strictly concave.

Consider the surface obtained by taking the union of the surface of revolution formed by rotating the graph of the smoothed h_ε with the graph of f_ε over $D_{\bar{r}+\varepsilon}$. This is a C^2 surface and when combined with a similarly constructed surface for f_2 produces the boundary of a strictly convex set Ω_ε. Note that $\partial\Omega_\varepsilon$ is C^2 and that Ω_ε satisfies a great circle condition with $B_{R-\delta}$ being the largest ball. Also, as C approaches a cylinder and $\delta, \varepsilon \to 0$ we have $\Omega_\varepsilon \to \Omega$ in the Hausdorff sense as required. To make the process of taking limits easier in the following we can dilate the sets Ω_ε a small amount so they contain Ω.

Thus, there exists a sequence of C^2 strictly convex sets Ω_n which contain Ω, satisfy a great circle condition, and which converge to Ω in the Hausdorff sense. For v, $|B_\Omega| < v \le |\Omega|$ (and n large enough so $|B_{\Omega_n}| < v$) let $E_n(v)$ be the unique perimeter minimizer in Ω_n of measure v. It is easy to see that for a dense set of v_is we can, by repeatedly extracting subsequences and diagonalizing, construct a subsequence of E_n such that for all i, $E_n(v_i)$ converges (on the subsequence) to $E(v_i)$, a subset of Ω, in the Hausdorff sense. Nestedness and convexity are clearly inherited. Thus taking intersections of appropriate $E(v_i)$ we extend the definition of $E(v)$ to all v, $|B_\Omega| < v < |\Omega|$. Nestedness allows us to extend convergence to all such v.

We claim that the sets $E(v)$ are perimeter minimizers relative to Ω. To see this note that given any set $F \subset \Omega$ with $|F| = v$ we have $F \subset \Omega_n$ since $\Omega \subset \Omega_n$; consequently by lower semicontinuity of perimeter we have

$$P(E(v)) \le \liminf P(E_n(v)) \le P(F)$$

(with the liminf taken over the subsequence), i.e., $E(v)$ is a perimeter minimizer.

For v, $0 \le v \le |H_\Omega|$ we can characterize perimeter minimizers. Assume E is a perimeter minimizer of measure v. If $0 < v \le |B_\Omega|$, then E is clearly a ball. If $|B_\Omega| < v \le |H_\Omega|$, we claim that E is the convex hull of two largest balls in $\overline{\Omega}$. In proving this we will also prove for $v \ge |H_\Omega|$ that any perimeter minimizer E satisfies $H_\Omega \subset E$. Assume $|B_\Omega| < v$. Consider the following extension of the proof of Theorem 3.11. As it stands, the proof of Theorem 3.11 implies that E contains a largest (in $\overline{\Omega}$) ball. In fact one can conclude much more. Let B_1, B_2 be the closed balls whose convex hull is H_Ω, let ℓ be the line through their centers, and consider any set H which is the convex hull of two translates of B_1 with centers on ℓ such that $|H \cap E| = |B_1|$ and $H \cap H_\Omega$ contains a translate of B_1. A mild variation in the proof of Theorem 3.11 shows that $H \cap E$ is a translate of B_1. We claim that this implies that $E \cap H_\Omega$ is the convex hull of two translates of B_1. To see this let B_3, $B_4 \subset E$ be distinct translates of B_1 with x being the midpoint between their centers. Since the hull of B_3, B_4 has measure larger than $|B_1|$, construct H as above using translates of B_1 placed symmetrically with respect to x. However, $H \cap E$ is a translate of B_1. Thus there is a translate of B_1 contained in E lying strictly between two such balls. Therefore, the centers of such balls form an interval in ℓ.

Now take ℓ to be the vertical axis with B_1 lying above B_2, let B_u, B_l be the uppermost and lowest translates of B_1 in E, and E_u, E_l the parts of E strictly above and below B_u, B_l, respectively. Assume E_u is not empty so $|E_u| \ne 0$. If $B_u \ne B_1$, construct H as above by translating hemispheres of B_1 so that H contains subsets of positive measure from both $E \cap H_\Omega$ and E_u. However, this is a contradiction since by the above $E \cap H$ is a translate of B_1 which cannot possibly intersect E_u. Thus, E_u not empty implies $B_u = B_1$. Similarly E_l not empty implies $B_l = B_2$. This establishes the claim.

Moreover, one can conclude that $v \ge |H_\Omega|$ implies $H_\Omega \subset E$. To see this note if $v \ge |H_\Omega|$ then at least one of E_u, E_l is nonempty. If both are nonempty, then $H_\Omega \subset E$ as claimed. If only one is nonempty, say E_u, then translate E as far down as possible while remaining in $\overline{\Omega}$ to form a set E^* which contains B_2 (E_l is empty). Note that E^* is also a perimeter minimizer of measure v. Thus, E_u^* nonempty, i.e., $E^* = E$ with $H_\Omega \subset E$ as required.

Now that we have characterized perimeter minimizers for v, $0 < v \le |H_\Omega|$ we can redefine $E(v)$ so that $E(v)$ is the convex hull of two translates of B_1, symmetrically placed in H_Ω, if $|B_1| < v \le |H_\Omega|$, and $E(v)$ is a symmetrically placed ball if $0 < v \le |B_1|$. Thus, we have a nested collection of convex perimeter minimizers which can be used to establish uniqueness. Given $\bar{v}$, $|H_\Omega| < \bar{v}$, assume that E is a perimeter minimizer with measure $\bar{v}$. Recall from above that $H_\Omega \subset E$. Before proceeding we define an auxiliary collection $\{H(v) : |H_\Omega| \le v \le \bar{v}\}$, $H(v)$ defined analogously to H in Theorem 3.11 by translating the halves of $E(v)$ the least possible amount such that the resultant hull $H(v)$ satisfies $|H(v) \cap E| = v$. Note that the sets $H(v)$ are nested since if $|H_\Omega| \le v < w$ and one translates the halves of $E(w)$ the same distance as for $E(v)$ in the definition of $H(v)$, and calls the hull of the translated halves $\tilde{H}$, then $|\tilde{H}\setminus H(v)| = w - v$ so $|\tilde{H}\cap E| = |(\tilde{H}\setminus H(v))\cap E| + |H(v)\cap E| \leq (w-v)+v = w$, i.e., $H(v)\subset \tilde{H}\subset H(w)$ as required.

Let $v_0 = \sup\{v : E(v)\subset E\}$. If $v_0 = |E|$, then $E = E(v_0)$, otherwise $v_0 < |E|$ so $E^\circ\setminus E(v_0)$ is not empty. Let B be a closed ball of positive radius in $E^\circ\setminus E(v_0)$, $v_1 = \sup\{v : H(v)\cap B \text{ is empty }\}$, and $v_2 = \inf\{v : B\subset H(v)\}$. Clearly $v_2 = |H(v_2)\cap E| \geq |H(v_1)\cap E| + |B| = v_1 + |B|$ so choosing v, $v_1 < v < v_2$ we see that B contains points in $H(v)$ and its complement. Consequently, $\partial H(v)$ intersects B. One can now proceed as in the proof of Theorem 3.26 with H replaced by $H(v)$ with the following modifications. In proving that $(\partial H(v)\setminus E)\cap p^{-1}(D^\circ_{B_\Omega})$ is not empty, one uses the fact proved above that $H_\Omega\subset F$ so that $H(v)$ contains a convex hull of "largest balls" which is larger than H_Ω and thus must intersect the complement of Ω. Finally we see that the conclusion $\partial H(v)\cap p^{-1}(D^\circ_{B_\Omega})\cap E^\circ = \emptyset$ is absurd due to our construction in which $\partial H(v)$ intersects E°. Thus, the assumption that $v_0 < |E|$ must be false and consequently $E = E(|E|)$ as required.

It remains only to prove the disjointness result. The proof is identical to that in Theorem 3.26 once we have established that minimizers have $C^{1,1}$ boundaries and satisfy the same mean curvature properties as before. Assume $|H_\Omega| \leq v < |\Omega|$. Let $E_n(v)$ be as above and note that k_n, the constant mean curvature associated with $\partial E_n(v)$, is bounded uniformly in n since as in the proof of Theorem 3.26 we have

$$H^{n-1}(\partial D_{B_\Omega}) = \int_{D_{B_\Omega}} \mathcal{H}'_{\partial E_n} \geq k_n H^{n-1}(p(\partial E_n(v)\cap\Omega^\circ))$$

where $H^{n-1}(p(\partial E_n(v)\cap\Omega^\circ))$ is uniformly bounded from zero (on a subsequence) for geometrical reasons since $E_n(v)$ is convex, contains B_Ω, and converges (on a subsequence) to $E(v)$. Consequently $0\leq \mathcal{H}_{\partial E_n}\leq k_n\leq M$ almost everywhere and we see that $\partial E_n(v)$ is uniformly $C^{1,1}$ from which we see that $E(v)$ is $C^{1,1}$ as well. Note that tangent planes converge almost everywhere so that locally first derivatives converge almost everywhere and consequently one can take limits in the weak definition of mean curvature to show that if $k_n\to k$ (on a subsequence), then $\partial E(v)$ has mean curvature k in the interior of Ω, and that $\mathcal{H}_{\partial E}\leq k$ as required. □

Theorem 3.32. *If $n = 2$, and Ω is as in Theorem 3.31, except that the great circle condition is not assumed, then the results of Theorem 3.31 still hold. Furthermore,*

(i) *if $|H_\Omega| \leq |E| < |\Omega|$, then a perimeter minimizer E is the union of all balls in $\overline{\Omega}$ of curvature equal to the curvature of $\partial E\cap\Omega$,*
(ii) *if $|B_\Omega| < |E| < |H_\Omega|$, then E is the union of two largest balls in Ω,*
(iii) *if $0 < |E| \leq |B_\Omega|$, then E is a ball.*

Proof. Smooth $\partial\Omega$ as before but without requiring the great circle condition. The same regularity properties hold as before for perimeter minimizers E_n in the smoothed domains Ω_n. Note that there is no singular set since $n = 2$. Also $\partial E_n\cap\Omega$ consists of circular arcs. Thus if $x\in\Omega$ is a limit point of points $x_n\in\partial E_n$, then it is easy to see geometrically that the curvatures of the circular arcs in $\partial E_n\cap\Omega$ must be uniformly bounded in n. Regularity and curvature results for the limiting perimeter minimizer follows as before.

We claim that any perimeter minimizer E must be convex. First note that E cannot have an infinite number of components since otherwise ∂E would contain a limit point of points in the boundaries of distinct components of E which would violate the regularity of ∂E. In addition, each component must be simply connected because otherwise one could add a bounded component of the complement of E to E which would reduce the perimeter of E and increase its measure. Thus, a scaling argument as in the proof of Theorem 3.24 would violate the fact that E is a perimeter minimizer.

Also each component must be convex. To see this note that locally ∂E is a graph of a $C^{1,1}$ function f. Thus f' is Lipschitz continuous, monotone increasing (if axes are chosen properly) on $f^{-1}(\partial E \cap \partial\Omega)$, and monotone increasing on each component of $f^{-1}(\partial E \cap \Omega)$ from which the claim easily follows.

Finally, given two components considering the two unique lines which are support lines for both components one sees that one of the components can be translated without leaving Ω until it first touches another component. This translation does not change the measure of the overall set and does not increase perimeter so a new perimeter minimizer is created. Due to the regularity of ∂E the point of contact lies in Ω. However, this contradicts the fact that the boundary of a perimeter minimizer must be a circular arc locally in Ω. Consequently, there must be only one component which we have already shown to be convex so E is convex as claimed.

To establish the uniqueness and nestedness properties, it is sufficient to characterize perimeter minimizers. In fact we claim that if E is a perimeter minimizer with $|H_\Omega| \leq |E| < |\Omega|$, then it is the union of all balls in $\overline{\Omega}$ of curvature given by the curvature of $\partial E \cap \Omega$. We prove the claim in two parts. We first establish that if a point x lies in E, then x lies in a ball contained in $\overline{\Omega}$ whose boundary has the same curvature as $\partial E \cap \Omega$. We finish by proving that if $|H_\Omega| \leq |E| < |\Omega|$, then E contains all balls with the same curvature as $\partial E \cap \Omega$.

Assume $x \in E$ and let $d = \operatorname{dist}(x, \partial E)$, $r = \frac{1}{k}$ where k is the curvature of $\partial E \cap \Omega$. If $d \geq r$, then x is clearly in a ball of radius r contained in $\overline{\Omega}$ as claimed. If $d < r$, then choose a point $y \in \partial E$ closest to x. Choose axes so that y is the origin, x lies on the positive horizontal axis, and the vertical axis is tangent to ∂E at y. Let C be the upper half of the circle of radius r containing y with center on the positive horizontal axis. Let $(0, a)$ be the largest subinterval of $(0, 2r)$ over which the part of ∂E lying above the horizontal axis is a graph. Let $f : (0, a) \to \mathbb{R}$ be the function having such a graph. Let $g : [0, 2r] :\to \mathbb{R}$ be the function with C as its graph. Integrating the divergence form for curvature over (ε, t) for $t < a$, $\varepsilon > 0$ and then letting $\varepsilon \to 0$, one obtains

$$1 - J(f'(t)) = -\int_0^t [J(f'(s))]'\,ds \leq \int_0^t k\,ds = -\int_0^t [J(g'(s))]'\,ds = 1 - J(g'(t))$$

where $J(x) = x/(\sqrt{1+x^2})$ since $f'(0) = g'(0) = \infty$ (recall ∂E is $C^{1,1}$). Since $J(x)$ is monotone increasing, this implies that $g'(t) \leq f'(t)$ on $(0, a)$. However, $0 = g(0) \leq \lim_{s\to 0} f(s)$ so $g(t) \leq f(t)$ on $(0, a)$. From the estimate on f' and the convexity of E we see that $a = 2r$. A similar argument shows that the part of ∂E lying below the horizontal axis in fact lies below the other half of the circle of radius r mentioned above. Thus from the convexity of E we see that this circle lies in E as claimed. Consequently E lies in the union of all balls of radius r which lie in $\overline{\Omega}$.

To prove our second claim let B be a ball of radius r contained in E (such a ball exists by the above argument). Let D be any other ball of radius r contained in Ω and let H be the convex hull of B, D. Assume that D is not a subset of E so there exists $x \in D \setminus E$. Thus ∂E separates x from B. However H°, the interior of H, lies in Ω so $\partial E \cap H^\circ$ is locally a circular arc of radius r. The only way a circular arc of radius r can separate x from B is if it is a half circle C tangent at its end points to the line segments in ∂H. In such a case ∂E must contain C and the (possibly empty) line segments in ∂H with endpoints in C and ∂B. Since $x \notin E$ one can translate E towards x while remaining in Ω due to the geometrical relationship between E and H. The translated set is thus still a perimeter minimizer with end opposite D lying in Ω. Thus the end opposite D is a circular arc and E is the convex hull of two (possibly identical) balls of radius r.

If $|E| \leq |B_\Omega|$ then clearly E is a ball. If $|B_\Omega| < |E| \leq |H_\Omega|$, then Ω satisfies a great circle condition since the line segments in ∂H must lie in $\partial\Omega$. Thus we can use the characterization of E in Theorem 3.31 as the convex hull of two largest balls in $\overline{\Omega}$. If $|H_\Omega| < |E|$, then E cannot be a ball or a hull of two balls in Ω as concluded in the last paragraph. Consequently the assumption that D was not a subset of E is false and $D \subset E$. Since D was an arbitrary ball of radius r, we see that the union of all such balls lies in E. Combining this with our earlier conclusion we see that E is in fact equal to the union of all such balls.

The disjointness property for boundaries of perimeter minimizers follows from nestedness of minimizers and the fact that the curvature of the boundary of a perimeter minimizer in $\overline{\Omega}$ strictly increases as a function of the measure v of the minimizer if $|H_\Omega| \leq v$, a fact which follows directly from the characterization of minimizers. If E, F are minimizers with $E \subset F$, and $\partial E \cap \partial F \cap \Omega$ is not empty, then geometrically the curvature of $\partial F \cap \Omega$ cannot be larger than the curvature of $\partial E \cap \Omega$. However, this contradicts the monotonicity of curvature as a function of measure mentioned above. □

4. Eqimeasurable convex rearrangement

Various standard symmetrizations have the useful property of rearranging functions in an equimeasurable fashion while reducing various norms such as $\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$ and $\|u\|_{BV(\Omega)}$ (see (2.1)). However, they alter Ω, the domain of definition of u, unless Ω has appropriate symmetries. This is unfortunate from the point of view of studying minimizers to certain variational problems. Using the results of Section 2 we introduce an equimeasurable rearrangement which preserves convex domains, reduces $\|u\|_{BV(\Omega)}$, and creates level sets which are boundaries of convex sets, when $u \in BV(\mathbb{R}^n)$ with $u \geq 0$ and $u = 0$ in $\mathbb{R} \setminus \Omega$. Results of [7] imply that such a rearrangement cannot exist for the norm $\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$, $p > 1$. Any equimeasurable rearrangement clearly fixes the first term in the BV norm (2.1). From the co-area formula we will see that a rearrangement which minimizes the perimeter of sets $\{u > t\}$ will minimize the BV norm over an appropriate class of equimeasurable functions.

In minimizing functionals such as

$$\|u\|_{BV(\Omega)} + \int_\Omega F(u) + \int_0^{|\Omega|} G(u^*, u^{*\prime}) \tag{4.1}$$

over appropriate function classes, where u^* is the decreasing rearrangement of u, $u^*(v) = \sup\{t : |\{u > t\}| \geq v\}$, it is sometimes straightforward to derive regularity estimates for u^*. Assuming continuity of u^*, the results of Theorem 4.2 imply continuity for minimizers of (4.1) in $\Omega \setminus H_\Omega$, using the continuity and uniqueness properties of $\tilde{u}$. Of course to apply Theorem 4.2 it is necessary that $u = 0$ on Ω is a boundary condition for the variational problem and that one can establish $u \geq 0$ in Ω for minimizers, for instance, by using a truncation argument. Behavior in H_Ω is also highly constrained by the characterization of level sets up to translation. It is fairly straightforward but more delicate to prove partial regularity results for ∇u if $\Omega \subset \mathbb{R}^2$ by analyzing interactions between boundaries of perimeter minimizers and $\partial\Omega$. However, in higher dimensions, this is a difficult open problem.

Assume that Ω is a bounded convex set in $\mathbb{R}^n$. In addition assume that $n = 2$, or Ω satisfies a great circle condition. Thus, from Section 2 we have a family of convex nested perimeter minimizers $E(v)$ defined as follows. If B_Ω is a largest ball in Ω and H_Ω is the union of all such balls, then if $0 < v \leq |B_\Omega|$ let $E(v)$ be a ball of measure v centered symmetrically in H_Ω, if $|B_\Omega| < v \leq |H_\Omega|$ (in which case Ω satisfies a great circle condition) then let $E(v)$ be the convex hull of two largest balls symmetrically centered in H_Ω and of measure v, finally if $|H_\Omega| < v < |\Omega|$ let $E(v)$ be the unique perimeter minimizer of measure v shown to exist in Section 2.

Define

$$BV_0^+(\Omega) = \{u \in BV(\mathbb{R}^n) : u \geq 0,\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$$

and define the convex rearrangement of a function $u \in BV_0^+(\Omega)$ by

$$\tilde{u}(x) = \inf\{s \geq 0 : x \notin E(|\{u > s\}|)\}\,.$$

Theorem 4.2. *If Ω is as above and $u \in BV_0^+(\Omega)$, then $\tilde{u}$ is upper semicontinuous in $\mathbb{R}^n$, continuous in Ω if u^* is continuous (equivalently $|\{u > t\}|$ is strictly increasing), $\tilde{u} \in BV_0^+(\Omega)$,*

$$|\{\tilde{u} > t\}| = |\{u > t\}|$$

for all t, and

$$\|\tilde{u}\|_{BV(\mathbb{R}^n)} \leq \|u\|_{BV(\mathbb{R}^n)}\,. \tag{4.2.1}$$

If there is equality in (4.2.1), then $\tilde{u} = u$ in $\mathbb{R}^n \setminus H_\Omega$ in the BV sense, and in H_Ω the sets $\partial^*\{\tilde{u} > t\}$ are translations of $\partial\{u > t\}$.

Remark 4.3. From the remark after Theorem 3.26 one sees that it is possible to create a rearrangement even if the convexity assumption is relaxed. However, it is unclear that one can in this context establish qualitative information analogous to convexity of $\{\tilde{u} > t\}$. □

Proof. Semicontinuity and continuity results are clear from the definition of $\tilde{u}$ and the disjointness results on boundaries of perimeter minimizers in Ω. It is also clear that $\tilde{u} \in BV_0^+(\Omega)$. Due to the convexity and nestedness (which is strict in Ω) of the sets $E(v)$ we see that

$$E^\circ(|\{u > t\}|) \subset \{\tilde{u} > t\} \subset E(|\{u > t\}|)\,;$$

thus,

$$|\{u > t\}| = |E(|\{u > t\}|)| = |\{\tilde{u} > t\}|$$

and

$$P(\{\tilde{u} > t\}) = P(E(|\{u > t\}|)) \leq P(\{u > t\})\,.$$

The result on BV norms then follows from the co-area formula.

If one has equality in the BV norm expression, then from the co-area formula and the minimization property of the sets $E(v)$ it is clear that $P(\{\tilde{u} > t\}) = P(\{u > t\})$, and consequently $\{\tilde{u} > t\}$ is a perimeter minimizer for almost all t. Let $t_0 = \sup\{t : |\{\tilde{u} > t\}| \geq |H_\Omega|\}$ so applying the uniqueness result for perimeter minimizers we see that $\{\tilde{u} > t\}$ and $\{u > t\}$ have the same measure theoretic closure for almost every t, $0 \leq t < t_0$. For $t \geq t_0$ we have $|\{\tilde{u} > t\}| < |H_\Omega|$ so this is true only up to translation within H_Ω in which case $\partial^*\{\tilde{u} > t\}$ is a translation of $\partial\{u > t\}$ (recall $\{u > t\}$ is convex) as claimed. This is easily justified for all t, $t \geq t_0$ by a limit argument.

Returning to the case $0 \leq t < t_0$ let E be an arbitrary measurable subset of Ω and $d\mu = \chi_E\, dx$ where dx represents Lebesgue measure. From Fubini's theorem we see that

$$\int_0^{t_0} \mu(\{u > t\})\, dt = \int \int_0^{t_0} \chi_{\{u>t\}}\, dt\, d\mu = \int_E \min(u,\, t_0)\,.$$

Using the fact that $\{\tilde{u} > t\}$ and $\{u > t\}$ have the same measure theoretic closure for almost every t, $0 \leq t < t_0$ we conclude that $\min(u,\, t_0) = \min(\tilde{u},\, t_0)$ almost everywhere. Recalling that $\{u > t\}$ and the set theoretic closure of $\{u > t\}$ are subsets of H_Ω for $t > t_0$, it is clear that $\tilde{u} = u$ almost everywhere in $\mathbb{R} \setminus H_\Omega$. □

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Received September 12, 1994
Revision received March 17, 1995

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Communicated by F. Almgren