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A Characterization of Complex Projective Space up to Biholomorphic Isometry

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1. Introduction

Obata [3] gave a characterization up to isometry of the standard sphere S^n in terms of the Hessian operator on a complete Riemannian manifold. With the convention Hess $u = \nabla$grad u, his result says that if M is a complete Riemannian manifold which admits a nondegenerate function u such that Hess $u = -u \cdot \mathrm{Id}$ then M is isometric to the standard sphere. Obata goes on to prove related results in conformal geometry which take advantage of the existence of a function whose Hessian has a special form. Other authors have also obtained strong geometric properties of a Riemannian manifold by exploiting the existence of a function u with Hess $u = f \cdot \mathrm{Id}$ for some function f. In particular, this last equation implies that M is a warped product. For a proof and related results, see, for example, Osgood and Stowe [5].

In the complex case, a characterization of $\mathbb{C}^n$ up to isometry was given by Stoll [6] via the complex Monge–Ampère operator. Stoll's result says that if M is a complex manifold which admits a strictly plurisubharmonic exhaustion τ: $M \to [0, \infty)$ such that $(dd^c \log \tau)^n \equiv 0$, then $(M, \tau) \simeq (\mathbb{C}^n, |z|^2)$. In other words M, with hermitian metric given by the Kähler form $dd^c\tau$, is biholomorphically isometric to $\mathbb{C}^n$. Obata also showed that a complete, connected and simply connected Kähler manifold is isometric to the complex projective space $\mathbb{P}^n$ if and only if it admits a solution to a certain linear system of third order differential equations [4]. Blair [1] subsequently showed that in some cases this characterization of $\mathbb{P}^n$ follows from a corresponding result for Riemannian manifolds and indicated that one would not expect a characterization of $\mathbb{P}^n$ by a Hessian equation analogous to that which Obata used to characterize S^n.

In this paper we give a complex analog of Obata's theorem [3]. We characterize complex projective space up to biholomorphic isometry by the existence of a solution to a system of second order equations. Since $\mathbb{P}^n$ with the Fubini-Study metric is not a warped product, there does not exist a nontrivial function u on $\mathbb{P}^n$ whose Hessian is a multiple of the identity. However, $\mathbb{P}^n$ with one point deleted is the hyperplane section bundle over $\mathbb{P}^{n-1}$ and the fibers of this bundle are totally geodesic complex lines. Thus, there is a relationship between the natural metric structure and the line bundle structure of $\mathbb{P}^n$. This relationship provides the motivation for the construction of a function u on $\mathbb{P}^n$

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whose Hessian satisfies an equation similar to Obata's equation. We are then able to show that this second order equation characterizes $\mathbb{P}^n$ up to holomorphic isometry.

The relationship between the metric structure and the fiber bundle structure we consider in this paper generalizes, in a sense, a warped product structure on a Riemannian manifold. The product structure is replaced by the "twisted product" structure of a bundle and the metric varies uniformly in the fiber direction with a correction factor to compensate for the twist. The complex manifolds $\mathbb{C}^n$ and $B^n\mathbb{C} = \{Z \in \mathbb{C}^n\colon |Z| < 1\}$ with the Euclidean metric and the complex hyperbolic metric, respectively, are other examples of this structure. Complex Euclidean space has the usual warped product structure while complex hyperbolic space has a structure analogous to the structure on $\mathbb{P}^n$. We hope to return to these topics with additional examples in subsequent work.

2. Preliminaries and notation

Let M be a complex manifold. TM denotes the real tangent bundle, $T^{\mathbb{C}}M$ the complexified tangent bundle . The complex structure is given by the operator $J\colon T^{\mathbb{C}}M \to T^{\mathbb{C}}M$. We let T^hM denotes the i-eigenspace of J, $T^{\bar{h}}M$ the $(-i)$-eigenspace. Then $T^{\mathbb{C}}M \simeq T^hM \oplus T^{\bar{h}}M$.

Suppose M is endowed with a hermitian metric g. For a vector $V \in T^{\mathbb{C}}M$, π_V will denote orthogonal projection onto the complex subspace of $T^{\mathbb{C}}M$ spanned by V; $\pi_V(X) = (g(V, \bar{V}))^{-1}\, g\,(X, \bar{V})\ V$.

Let u be a real-valued C^2 function on M. Let $\operatorname{grad} u$ denote the real vector field on M which is uniquely determined by

$$du(\zeta) = g(\zeta,\ \operatorname{grad} u)$$

for all real vectors ζ. We write $\operatorname{grad} u = \operatorname{grad}_h u + \operatorname{grad}_{\bar{h}} u$, where $\operatorname{grad}_h u \in T^hM$, $\operatorname{grad}_{\bar{h}} u \in T^{\bar{h}}M$, $\operatorname{grad}_{\bar{h}} u = \overline{\operatorname{grad}_h u}$. One may verify that $\operatorname{grad}_h u$ is the unique T^hM-valued vector field with the property that for all $V \in T^{\mathbb{C}}M$, $g(V, \operatorname{grad}_h u) = \bar{\partial} u(V)$.

Let ∇ denote the canonical complex metric connection on M associated to g. The complex Hessian of u is the section of End $(T^{\mathbb{C}}M)$ which is defined for all $V \in T^{\mathbb{C}}M$ by $\operatorname{Hess} u(V) = \nabla_V \operatorname{grad} u$. The complex Hessian is the complex linear extension of the real Hessian.

Henceforth assume that g is a Kähler metric and that $\dim_{\mathbb{C}} M = n$. We will calculate $\operatorname{grad}_h u$ and $\operatorname{Hess} u$ in terms of local holomorphic coordinates $(z_1, \ldots, z_n)$.

Let $g_{i\bar{j}} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$ and define $g^{\bar{j}k}$ by

$$\sum_{j=1}^{n} g_{i\bar{j}} g^{\bar{j}k} = \delta_{ik} = \begin{cases} 0 \ \text{if}\ i \neq k \\ 1 \ \text{if}\ i = k\,. \end{cases}$$

Let $u_i = \frac{\partial u}{\partial z_i}$ and $u_{\bar{i}} = \frac{\partial u}{\partial \bar{z}_i}$. Christoffel symbols are denoted by Γ^k_{ij}. From the equation $g(\frac{\partial}{\partial z_i},\ \operatorname{grad}_h u) = u_i$, we obtain the expression

$$\operatorname{grad}_h u = \sum_{i,j=1}^{n} u_{\bar{i}} g^{\bar{i}j} \frac{\partial}{\partial z_j}\,.$$

To calculate $\operatorname{Hess} u$, we write

$$\operatorname{Hess} u = \sum_{i,j=1}^{n} \left(a_i^j dz_i \otimes \frac{\partial}{\partial z_j} + b_i^j dz_i \otimes \frac{\partial}{\partial \bar{z}_j} + c_i^j d\bar{z}_i \otimes \frac{\partial}{\partial z_j} + d_i^j d\bar{z}_i \otimes \frac{\partial}{\partial \bar{z}_j} \right)\,.$$

Since u is real-valued, $d_i^j = \overline{a_i^j}$ and $c_i^j = \overline{b_i^j}$. One computes a_i^j as follows:

$$\operatorname{Hess} u\left(\frac{\partial}{\partial z_i}\right) = \nabla_{\frac{\partial}{\partial z_i}} \operatorname{grad} u = \nabla_{\frac{\partial}{\partial z_i}} \operatorname{grad}_h u + \nabla_{\frac{\partial}{\partial z_i}} \operatorname{grad}_{\bar{h}} u = \sum_j a_i^j \frac{\partial}{\partial z_j} + \sum_j b_i^j \frac{\partial}{\partial \bar{z}_j}.$$

Therefore,

$$\begin{aligned}
\sum_j a_i^j \frac{\partial}{\partial z^j} &= \nabla_{\frac{\partial}{\partial z_i}} \operatorname{grad}_h u \\
&= \nabla_{\frac{\partial}{\partial z_i}} \left(\sum_{k,j} u_{\bar{k}} g^{\bar{k}j} \frac{\partial}{\partial z_j}\right) \\
&= \sum_{k,j}\left[\left(u_{i\bar{k}} g^{\bar{k}j} + u_{\bar{k}} \frac{\partial g^{\bar{k}j}}{\partial z_i}\right)\frac{\partial}{\partial z_j} + u_{\bar{k}} g^{\bar{k}j} \sum_l \Gamma_{ij}^l \frac{\partial}{\partial z_l}\right] \\
&= \sum_{k,j} u_{i\bar{k}} g^{\bar{k}j} \frac{\partial}{\partial z_j}.
\end{aligned}$$

The last equality follows from the fact that for a Kähler manifold,

$$\Gamma_{ij}^l = \sum_{m=1}^n g^{\bar{m}l} \frac{\partial g_{j\bar{m}}}{\partial z_i} = -\sum_{m=1}^n g_{j\bar{m}} \frac{\partial g^{\bar{m}l}}{\partial z_i}.$$

To compute b_i^j, we first note that $\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial \bar{z}_j} = 0$ for all i and j. Therefore,

$$\begin{aligned}
\sum_j b_i^j \frac{\partial}{\partial \bar{z}_j} &= \nabla_{\frac{\partial}{\partial z_i}} \operatorname{grad}_{\bar{h}} u \\
&= \nabla_{\frac{\partial}{\partial z^i}} \left(\sum_{k,j} u_k g^{\bar{j}k} \frac{\partial}{\partial \bar{z}_j}\right) \\
&= \sum_j \frac{\partial}{\partial z_i}\left(\sum_k u_k g^{\bar{j}k}\right)\frac{\partial}{\partial \bar{z}_j}.
\end{aligned}$$

As an immediate consequence of this calculation we obtain the following lemma.

Lemma 2.1. *$b_i^j = 0$ for all i and j if and only if $\operatorname{grad}_h u$ is a holomorphic vector field, that is, if and only if the coefficients of $\operatorname{grad}_h u$ are holomorphic.*

To summarize,

$$\begin{aligned}
\operatorname{Hess} u &= \sum_{i,j}\left(\sum_k u_{i\bar{k}} g^{\bar{k}j} dz_i \otimes \frac{\partial}{\partial z_j} + \sum_k \frac{\partial}{\partial z_i}\left(u_k g^{\bar{j}k}\right) dz_i \otimes \frac{\partial}{\partial \bar{z}_j}\right. \\
&\quad \left. + \sum_k \frac{\partial}{\partial \bar{z}_j}\left(u_{\bar{k}} g^{\bar{k}j}\right) d\bar{z}_i \otimes \frac{\partial}{\partial z_j} + \sum_k u_{\bar{i}k} g^{\bar{j}k} d\bar{z}_i \otimes \frac{\partial}{\partial \bar{z}_j}\right).
\end{aligned}$$

3. Main theorem

We now state our main theorem, which may be thought of as a complex version of Obata's theorem characterizing the sphere.

Theorem. *Let M be a complex manifold of dimension $n \geq 2$, g a complete Kähler metric on M. Then M is biholomorphically isometric to complex projective space $\mathbb{P}^n$ with the Fubini-Study metric if and only if there is a nonconstant real-valued function $u \in C^2(M)$ such that on $\{p \in M\colon \operatorname{grad} u(p) \neq 0\}$*

$$Hess\, u = -u\, Id + \frac{1}{2}(u-1)(Id - \pi)\,, \tag{3.1}$$

where $\pi = \pi_{grad_h u} + \pi_{grad_{\bar{h}} u}$.

The operator π is the projection onto the complex subspace in $T_p^{\mathbb{C}}M$ spanned by $\operatorname{grad}_h u$ and $\operatorname{grad}_{\bar{h}} u$, or, equivalently, by grad u and J grad u. In local coordinates, equation $(*)$ is a system of $2n^2$ equations: for all i, j,

$$\begin{aligned} \left(\sum u_{\bar{k}} u_l g^{\bar{k}l}\right)\left(2u_{i\bar{j}} - (u+1)g_{i\bar{j}}\right) &= (1-u)u_i u_{\bar{j}} \\ u_{ij} &= \sum u_l \Gamma^l_{ij} \end{aligned}$$

4. Proof of necessity

On $\mathbb{P}^n$, let $Z = [Z_0 : Z_1 : \cdots : Z_n]$ represent the homogeneous coordinates. Let $Z \mapsto (Z_0, Z_1, \ldots, Z_n)$ be a local holomorphic choice of representative. The Kähler form of the Fubini-Study metric on $\mathbb{P}^n$ is

$$\omega = -4i\partial\bar{\partial}\log\left(|Z_0|^2 + \cdots + |Z_n|^2\right)$$

and the corresponding hermitian metric is given by $g(X, Y) = \omega(JX, Y)$.

On $U_0 = \{Z \in \mathbb{P}^n\colon Z_0 \neq 0\}$ we use the usual inhomogeneous holomorphic coordinates $z_i = \frac{Z_i}{Z_0}$ for $i = 1, \ldots, n$. Let $|z|^2 = \sum_{i=1}^n |z_i|^2$.

In these coordinates,

$$\begin{aligned} g_{i\bar{j}} &= \frac{2}{\left(1+|z|^2\right)^2}\left(\delta_{ij}(1+|z|^2\right) - z_j\bar{z}_i) \text{ and} \\ g^{\bar{j}k} &= \frac{1+|z|^2}{2}\left(\delta_{jk} + \bar{z}_j z_k\right) \end{aligned}$$

Define $u(Z) = \frac{|Z_0|^2 - |Z_1|^2 - \cdots - |Z_n|^2}{|Z_0|^2 + |Z_1|^2 + \cdots + |Z_n|^2}$. On U_0, $u(z) = \frac{1-|z|^2}{1+|z|^2}$. One may check directly that $\{\operatorname{grad} u \neq 0\} = U_0 \setminus \{[1 : 0 : \cdots : 0]\}$ and that on this set u satisfies equation (3.1).

Note that for $n = 1$, u is the "height function" on $S^2 \simeq \mathbb{P}^1$ and satisfies Obata's theorem [3]. While S^n is the most natural generalization of S^2 in the context of real manifolds, $\mathbb{P}^n$ is the appropriate complex generalization. One should think of u as extending the idea of the height function to $\mathbb{P}^n$.

5. Geometric properties of the foliations associated to grad u and grad$_h$ u

We now begin the proof of sufficiency for the main theorem. Let M, g, and u be as in the theorem. We use the notation $\xi = \operatorname{grad} u$ and $X = \operatorname{grad}_h u$. From ξ one obtains a foliation of $\{\xi \neq 0\}$ by real curves, the integral curves of ξ, and from X one obtains a foliation by complex curves. Equation (3.1) imposes strong geometric properties of these foliations.

We first show that the integral curves of ξ, when reparametrized by arc length, are geodesics. Let $f = |\xi|$ and where $\xi \neq 0$, let $\tau = \frac{1}{|\xi|}\xi$, so that $g(\tau, \tau) \equiv 1$ and $\xi = f\tau$. We will show $\nabla_\tau \tau = 0$. Calculate:

$$\nabla_\tau \tau = \frac{1}{f}\nabla_\xi\left(\frac{1}{f}\xi\right) = -\frac{1}{f^3}\xi(f)\xi + \frac{1}{f^2}\nabla_\xi \xi\ .$$

By equation (3.1), $\nabla_\xi \xi = \operatorname{Hess} u(\xi) = -u\xi$. Thus, $\nabla_\tau \tau$ is a multiple of ξ, hence, a multiple of τ. However, since τ is of constant length, $0 = \tau(g(\tau, \tau)) = 2g(\nabla_\tau \tau, \tau)$, so $\nabla_\tau \tau = 0$.

We next show that each leaf of the foliation associated to X is totally geodesic. Observe that the lemma combined with equation (3.1) implies that X is a holomorphic vector field.

Let $p \in M$, $X_p \neq 0$, L the leaf through p of the foliation defined by X. To show that L is totally geodesic, it suffices to show that the second fundamental form of L vanishes. A real vector ζ is tangent to L if and only if $\frac{1}{2}(\zeta - iJ\zeta)$, which will be denoted by $p_h(\zeta)$, is a complex multiple of X. Therefore, we must show that if $\zeta \in T_pM$ and τ is a real vector field near p such that $p_h(\zeta) = \lambda X$, $p_h(\tau) = \phi X$, then $p_h(\nabla_\zeta \tau)$ is a multiple of X.

Observe that $\zeta = \lambda X + \bar{\lambda}\bar{X}$, $\tau = \phi X + \bar{\phi}\bar{X}$. Since X is holomorphic, $\nabla_X \bar{X} = 0$ and $\nabla_{\bar{X}} X = 0$. Thus,

$$\begin{aligned}\nabla_\zeta \tau &= \nabla_{\lambda X + \bar{\lambda}\bar{X}}\left(\phi X + \bar{\phi}\bar{X}\right)\\ &= \lambda X(\phi)X + \lambda\phi\nabla_X X + \lambda X\left(\bar{\phi}\right)\bar{X} + \bar{\lambda}\bar{X}(\phi)X\\ &\quad + \bar{\lambda}\bar{X}\left(\bar{\phi}\right)\bar{X} + \bar{\lambda}\bar{\phi}\nabla_{\bar{X}}\bar{X}\ .\end{aligned}$$

Now $\nabla_{\bar{X}}\bar{X} = \overline{\nabla_X X}$ and from equation (3.1) we obtain $\nabla_X X = \nabla_X(X + \bar{X}) = \nabla_X \xi = \operatorname{Hess} u(X) = -uX$. Therefore,

$$\begin{aligned}\nabla_\zeta \tau &= [\lambda X(\phi) - \lambda\phi u + \bar{\lambda}\bar{X}(\phi)]X\\ &\quad + \left[\lambda X\left(\bar{\phi}\right) + \bar{\lambda}\bar{X}\left(\bar{\phi}\right) - \bar{\lambda}\bar{\phi}u\right]\bar{X} \quad \text{and}\\ p_h\left(\nabla_\zeta \tau\right) &= \left[\lambda X(\phi) - \lambda\phi u + \bar{\lambda}\bar{X}(\phi)\right]X\ ,\end{aligned}$$

which is a multiple of X.

6. Calculation of u

We will first calculate u along an integral curve of ξ. Let $p \in M$ be a point at which $\xi_p \neq 0$. Let $\gamma\colon \mathbb{R} \to M$ be the unique geodesic with $\gamma(0) = p$, $\dot{\gamma}(0) = \frac{\xi_p}{|\xi_p|}$. Since the metric is complete, $\gamma(t)$ is defined for all $t \in \mathbb{R}$. We denote $\xi_{u(t)}$ by ξ_t, $u(\gamma(t))$ by $u(t)$ and $\dot{\gamma}(u)$ by u'. Because the integral curves of ξ are geodesics, we may write $\dot{\gamma}(t) = \frac{\xi_t}{|\xi_t|}$ when $\xi_t \neq 0$.

It will be useful to compute $|\xi_t|$. We have $g(\xi, \dot{\gamma}) = du(\dot{\gamma}) = \dot{\gamma}(u) = u'$. Also $g(\xi, \dot{\gamma}) = |\xi|g(\dot{\gamma}, \dot{\gamma}) = |\xi|$. Thus, $|\xi| = u'$, and $\xi_t = u'(t)\dot{\gamma}_t$ wherever $\xi_t \neq 0$. Since $g(\xi_t, \dot{\gamma}_t) = u'(t)$ holds for all t, we see that $\xi_t = 0$ if and only if $u'(t) = 0$, so $\xi_t = u'(t)\dot{\gamma}_t$ for all $t \in \mathbb{R}$.

We now use equation (3.1) to compute $u(t)$. For $\xi_t \neq 0$,

$$\nabla_\xi \xi = \text{Hess}\, u(\xi) = -u\xi \, .$$

Thus,

$$g\left(\nabla_\xi \xi, \xi\right) = -ug(\xi, \xi) \, .$$

Now

$$g(\xi, \xi) = \left(u'\right)^2 g\left(\dot\gamma, \dot\gamma\right) = \left(u'\right)^2$$

and

$$2g\left(\nabla_\xi \xi, \xi\right) = \xi(g(\xi, \xi)) = \xi\left(\left(u'\right)^2\right) = u'\dot\gamma\left(\left(u'\right)^2\right) = 2\left(u'\right)^2 u'' \, .$$

Therefore, $(u')^2 u'' = -u(u')^2$ and we obtain

$$u''(t) + u(t) = 0 \tag{6.1}$$

for all t such that $\xi_t \neq 0$. The general solution to (6.1) is $u(t) = A\cos t + B\sin t,\ A, B \in \mathbb{R}$. In fact (6.1) holds for all t, as shown by the following lemma.

Lemma 6.2. *Along the geodesic γ, $\xi = 0$ only at isolated points.*

Proof. Suppose not. Then there is a limit point t_0 of $\{t\colon\ \xi_t = 0\}$ which is also a limit point of $\{t\colon\ \xi_t \neq 0\}$. Since u is C^2, $u'(t_0) = u''(t_0) = 0$. But $u(t) = A\cos t + B\sin t$ on $\{t\colon\ \xi_t \neq 0\}$, so it is impossible for $u'(t)$ and $u''(t)$ to simultaneously approach zero at t_0 unless $A = B = 0$. But this is not the case; if it were, then ξ_t would vanish at all points in a neighborhood of t_0. □

By continuity, then, $u(t) = A\cos t + B\sin t$ for all $t \in \mathbb{R}$. Therefore, u achieves a positive maximum and a negative minimum along γ. Suppose $u(t)$ attains its maximum at a point t_0. For convenience, we reparametrize γ so that $t_0 = 0$ and denote the point $\gamma(0)$ by P.

P is a critical point of u since $\xi_P = 0$. We will show that it is an isolated critical point for u on M. To do this, we will show that Hess $u(P)$ is nonsingular.

For t near 0, $t \neq 0$, we have $\xi_t \neq 0$ and $\text{Hess}\, u(\xi_t) = -u(t)\xi_t$. Thus, $\text{Hess}\, u(\dot\gamma_t) = -u(t)\dot\gamma_t$ by linearity. By continuity, this equation holds for $t = 0$ also. One may show in an analogous fashion that $\text{Hess}\, u(J\dot\gamma_0) = -u(0)J\dot\gamma_0$. Let ζ_t be any continuous real vector field along γ with ζ_t orthogonal to $\dot\gamma_t$ and to $J\dot\gamma_t$. Then $\text{Hess}\, u(\zeta_t) = -u\zeta_t + \frac{1}{2}(u-1)\zeta_t = -\frac{1}{2}(u+1)\zeta_t$ for t near 0. Again by continuity, the equation holds for $t = 0$ also. Since $u(P) > 0$, we see that Hess $u(P)$ is negative definite, so P is an isolated critical point of u.

We next show that $u(P) = 1$. For points in a neighborhood of P but not equal to P, equation (3.1) holds: $\text{Hess}\, u + u\,\text{Id} = \frac{1}{2}(u-1)(\text{Id} - \pi)$. The left hand side is defined and continuous at P, so it provides a continuous extension of the right side to P. However, since P is an isolated maximum, ξ always "points toward" P near P, so there can be no continuous extension of $\text{Id} - \pi$ to P when $n \geq 2$. Therefore, $u(P) = 1$ must hold.

We can now calculate u along γ. Since $u(0) = 1$ and $u'(0) = 0$, $u(t) = \cos t$.

The integral curve of ξ through any point near P passes through P. Thus, every geodesic γ through P is tangent to ξ; that is, $\xi = |\xi|\dot\gamma$. The above analysis of u then shows that $u(\gamma(t)) = \cos t$ for any unit speed geodesic γ with $\gamma(0) = P$. Because the metric is complete, every point of M lies on such a geodesic; $u(p) = \cos(\text{dist}_M(p, P))$

We next show that P is the unique maximum point of u on M. Let L be a leaf of X, $\tilde u = u|_L$. Since L is totally geodesic, equation (3.1) becomes $\text{Hess}\,\tilde u = -\tilde u\,\text{Id}$ when restricted to $L \setminus \{\xi = 0\}$.

By continuity, this equation holds on all of L. By Obata's theorem [3], L is isometric to $\mathbb{P}^1$ with the Fubini-Study metric. (One may instead compute the holomorphic sectional curvature of L to show that L is isometric to $\mathbb{P}^1$; a similar curvature computation is carried out in Section 7.)

From this we see that u has a unique maximum on each leaf at P, so P is the unique maximum of u on M.

Leaves of X can intersect only at points where $\xi = 0$. Therefore,

$$\exp_P\colon\ T_P M \to M$$

is a diffeomorphism when restricted to the open ball of radius π and maps the closed ball of radius π onto M.

7. Definition of Ψ: $\mathbb{P}^n \to M$

Using the information we have obtained concerning the geometry of M, we now define a map from $\mathbb{P}^n$ to M.

Let $P_0 = [1 : 0 : \cdots : 0] \in \mathbb{P}^n$. The hyperplane $\{Z\colon\ Z_0 = 0\} \subset \mathbb{P}^n$ will be denoted by H_0. Let

$$u_0(Z) = \frac{|Z_0|^2 - |Z_1|^2 - \cdots - |Z_n|^2}{|Z_0|^2 + |Z_1|^2 + \cdots + |Z_n|^2}, \quad \xi_0 = \operatorname{grad} u_0, \quad X_0 = \operatorname{grad}_h u_0 \,.$$

Observe that $\exp_{P_0}$ is a diffeomorphism when restricted to the open ball of radius π in $T_{P_0}\mathbb{P}^n$. We denote the inverse of this restriction by $\exp_{P_0}^{-1}$.

Fix once and for all a complex linear isometric identification $\iota\colon\ T_{P_0}\mathbb{P}^n \to T_P M$. This identification will sometimes be used without explicit mention. Let $H \subset M$ be the set of all points of distance π from P. Observe that $H = \{p \in M\colon\ u(p) = -1\}$.

Define a diffeomorphism $\Psi\colon\ \mathbb{P}^n \setminus H_0 \to M \setminus H$ by

$$\Psi = \exp_P \circ\, \iota\, \circ \exp_{P_0}^{-1} \,.$$

Note that $u \circ \Psi = u_0$ and $\Psi_* \xi_0 = \xi$, so Ψ maps leaves of ξ_0 to leaves of ξ.

We extend Ψ to a map from $\mathbb{P}^n$ to M (also called Ψ) as follows. For $Z \in H_0$, let $\{Z^\nu\}$ be a sequence of points in a leaf of ξ_0 through Z (that is, in a geodesic containing Z and P_0) such that $Z^\nu \notin H_0$ and $\lim_{\nu\to\infty} Z^\nu = Z$. Let $\Psi(Z) = \lim_{\nu\to\infty} \Psi(Z^\nu)$. Since Ψ maps leaves of ξ_0 to leaves of ξ, $\Psi(Z)$ is well-defined; it is the unique point of the corresponding leaf of ξ which lies in H. We will show that Ψ is biholomorphic and isometric.

8. Proof of the main theorem

To prove the theorem, we must show that Ψ is a biholomorphism and an isometry. We first show that $\Psi|_{\mathbb{P}^n \setminus H_0}$ has these properties.

Step 1: $\Psi|_{\mathbb{P}^n \setminus H_0}$ is biholomorphic.

We will show that $M \setminus H$ is Stein and that $\Psi_*(J_0 \xi_0) = J\xi$. Then the argument in [2, p. 359] implies that $\Psi|_{\mathbb{P}^n \setminus H_0}$ is biholomorphic.

On $M \setminus H$, let $\phi = \frac{1}{u+1}$. Then ϕ is a C^∞ exhaustion function for $M \setminus H$.

$$\partial\bar{\partial}\phi = \frac{1}{(u+1)^3}\left[-(u+1)\partial\bar{\partial}u + 2\partial u \wedge \bar{\partial}u\right] .$$

For $W \in T^h M$,

$$\partial\bar{\partial}u\left(W, \bar{W}\right) = g\left(\text{Hess}\, u(W), \bar{W}\right) ,$$
$$\left(\partial u \wedge \bar{\partial}u\right)(W, \bar{W}) = \left|g\left(W, \bar{X}\right)\right|^2 .$$

We write $W = aX + Y$, with $g(X, \bar{Y}) = 0$. Then

$$\begin{aligned}\partial\bar{\partial}\phi\left(W, \bar{W}\right) &= \frac{1}{(u+1)^3}\left[\frac{1}{2}(u+1)^2 g\left(Y, \bar{Y}\right)\right.\\ &\quad \left. + |a|^2 g\left(X, \bar{X}\right)\left((u+1)u + 2g\left(X, \bar{X}\right)\right)\right] .\end{aligned}$$

Now $g(X, \bar{X}) = \frac{1}{2}g(\xi, \xi) = \frac{1}{2}(u')^2 = \frac{1}{2}\sin^2 t$, where $t(p) = \text{dist}_M(p, P)$. Therefore, $2g(X, \bar{X}) + (u+1)u = 1 + \cos t$, which is positive for $0 \le t < \pi$. Since $(u+1) > 0$ on $M \setminus H$, $\partial\bar{\partial}\phi(W, \bar{W}) > 0$ for all $W \neq 0$. Thus, ϕ is strongly plurisubharmonic and $M \setminus H$ is Stein.

To show that $\Psi_*(J_0\xi_0) = J\xi$, we use an argument similar to that of Burns [2, p. 359–360].

Let γ_0 be a unit speed geodesic in $\mathbb{P}^n$ with $\gamma_0(0) = P_0$. Then we know that $\xi_0(t) = -(\sin t)\dot{\gamma}_0(t)$. Let $\rho(t) = \exp_{P_0}^{-1}(\gamma_0(t))$ be the corresponding ray in $T_{P_0}\mathbb{P}^n$ and let $\gamma(t) = \Psi(\gamma_0(t))$. Then $\gamma(0) = P$ and γ is a unit speed geodesic with $\xi(t) = -(\sin t)\dot{\gamma}(t)$.

Fix $s \in (0, \pi)$. To calculate $\Psi_*(J_0\xi_0(s))$, we first calculate $(\exp_{P_0}^{-1})_*(J_0\xi_0)$. To do this, we seek the Jacobi field $V_0(t)$ along γ_0 with $V_0(0) = 0$ and $V_0(s) = J_0\xi_0(s)$. Since $J_0\dot{\gamma}_0$ is parallel along γ_0, V_0 is of the form

$$V_0(t) = f_0(t) J_0\dot{\gamma}_0(t) .$$

Also note that V_0 satisfies the Jacobi equation

$$\nabla_{\dot{\gamma}_0}\nabla_{\dot{\gamma}_0}V_0 + R\left(V_0, \gamma_0\right)\dot{\gamma}_0 = 0 .$$

Since the leaves of X_0 are totally geodesic and since the holomorphic sectional curvature of the Fubini-Study metric on $\mathbb{P}^n$ is identically 1, $R(V_0, \dot{\gamma}_0)\dot{\gamma}_0 = V_0 = f_0 J_0\dot{\gamma}_0$. Furthermore $\nabla_{\dot{\gamma}_0}\nabla_{\dot{\gamma}_0}V_0 = \dot{\gamma}_0(\dot{\gamma}_0(f))J_0\dot{\gamma}_0 = f_0'' J_0\dot{\gamma}_0$. The Jacobi equation reduces to $f_0''(t) + f_0(t) = 0$. The initial conditions on V_0 become $f_0(0) = 0$ and $f_0(s) = -\sin s$. Therefore, $f_0(t) = -\sin t$ and $V_0(t) = -\sin t\; J_0\dot{\gamma}_0(t)$. From this we obtain $(\nabla_{\dot{\gamma}_0}V_0)(0) = -J_0\dot{\gamma}_0(0)$.

Identify $T_{P_0}(T_{P_0}\mathbb{P}^n)$ with $T_{P_0}\mathbb{P}^n$ and consider $-J_0\dot{\gamma}_0(0) \in T_{P_0}(T_{P_0}\mathbb{P}^n)$. Parallel transport $-J_0\dot{\gamma}_0(0)$ along $\rho(t)$ and let $Y(t) = -t\; J_0\dot{\gamma}_0(0)$ along $\rho(t)$. Then $(\exp_{P_0}^{-1})_*(J_0\xi_0(s)) = Y(s) = -s\; J_0\dot{\gamma}_0(0)$.

Under the identification ι, $\iota_*(-s\; J_0\dot{\gamma}_0(0)) = -sJ\dot{\gamma}(0)$. We follow a similar procedure to calculate $(\exp_P)_*(-s\; J\dot{\gamma}(0))$. We seek a Jacobi field of the form $V(t) = f(t)J\dot{\gamma}(t)$ along γ with $V(0) = 0$ and $(\nabla_{\dot{\gamma}}V)(0) = Y'(0) = -J\dot{\gamma}(0)$. As shown in Section 5, each leaf of X is totally geodesic and has constant curvature 1. Therefore, $R(V, \dot{\gamma})\dot{\gamma} = V = fJ\dot{\gamma}$ and the Jacobi equation becomes $f'' + f = 0$, with initial conditions $f(0) = 0$, $f'(0) = -1$. Then $f(t) = -\sin t$ and $V(t) = -\sin t\; J\dot{\gamma}(t)$. Finally,

$$\Psi_*\left(J_0\xi_0(s)\right) = \left(\exp_P\right)_* \iota_* \left(\exp_{P_0}^{-1}\right)_* \left(J_0\xi_0(s)\right)$$

$$\begin{aligned} &= \left(\exp_P\right)_* (-s\, J\dot\gamma(0)) \\ &= V(s) = -\sin s\, J\dot\gamma(s) = J\xi(s)\,. \end{aligned}$$

Step 2: $\Psi|_{\mathbb{P}^n \setminus H_0}$ is an isometry.

We now know $\Psi_*\xi_0 = \xi$ and $\Psi_* J_0\xi_0 = J\xi$. Since the metric is hermitian, we have $g_0(\xi_0, J\xi_0) = g(\xi, J\xi) = 0$ and $g_0(\xi_0, \xi_0) = g_0(J_0\xi_0, J_0\xi_0) = \sin^2 t = g(\xi, \xi) = g(J\xi, J\xi)$. Where $\xi_p \neq 0$, ξ and $J\xi$ span the leaf of X through p, so Ψ is an isometry on the leaves of X_0.

Now let $\zeta_0 \in T(\mathbb{P}^n \setminus H_0)$ be orthogonal to ξ_0 and to $J_0\xi_0$. In order to show that $g_0(\zeta_0, \zeta_0) = g(\Psi_*\zeta_0, \Psi_*\zeta_0)$, we use an argument analogous to that of Step 1 to calculate $\Psi_*\zeta_0$.

Suppose $\zeta_0 \in T_Z\mathbb{P}^n$. Let γ_0 be the unit speed geodesic from P_0 to Z with $\gamma_0(0) = P_0$, $\gamma_0(s) = Z$, $0 < s < \pi$. Let $\zeta_0(t)$ denote the parallel translate of ζ_0 along γ_0 to $\gamma_0(t)$. Then $\zeta_0(t)$ is orthogonal to $\xi_0(t)$ and to $J_0\xi_0(t)$.

We seek a Jacobi field $V_0(t) = f_0(t)\zeta_0(t)$ such that $V_0(0) = 0$ and $V_0(s) = \zeta_0(s)$. Since ζ_0 is parallel, $\nabla_{\dot\gamma_0}\nabla_{\dot\gamma_0} V_0 = f_0''\zeta_0$. To calculate $R(V_0, \dot\gamma_0)\dot\gamma_0$ we use the definition of the curvature operator R, equation $(*)$, and the fact that $\pi_X(\zeta_0(t)) = \pi_{\bar X}(\zeta_0(t)) = 0$, so that Hess $u_0(\zeta_0) = -\frac{1}{2}(u_0+1)\zeta_0$. We could instead use our explicit knowledge of the Fubini-Study metric, but we prefer this apparently more general calculation because it will also apply to the manifold M. For convenience, we write

$$R\left(V_0, \dot\gamma_0\right)\dot\gamma_0 = R\left(f_0\zeta_0, \frac{1}{u_0'}\xi_0\right)\frac{1}{u_0'}\xi_0 = \frac{f_0}{(u_0')^2}R\left(\zeta_0, \xi_0\right)\xi_0\,.$$

We also observe that $\nabla_{\xi_0}\xi_0 = \nabla_{u_0'\dot\gamma_0}\zeta_0 = 0$ since ζ_0 is parallel and that $\zeta_0(u_0) = g_0(\xi_0, \zeta_0) = 0$. By definition,

$$R\left(\zeta_0, \xi_0\right)\xi_0 = \nabla_{\zeta_0}\nabla_{\xi_0}\xi_0 - \nabla_{\xi_0}\nabla_{\zeta_0}\xi_0 - \nabla_{[\zeta_0,\xi_0]}\xi_0\,.$$

Now

$$\begin{aligned} \nabla_{\zeta_0}\nabla_{\xi_0}\xi_0 &= \nabla_{\zeta_0}\left(\text{Hess}\, u_0\left(\xi_0\right)\right) \\ &= -u_0\nabla_{\zeta_0}\xi_0 \\ &= \frac{1}{2}u_0\left(u_0+1\right)\zeta_0\,. \end{aligned}$$

Also

$$\begin{aligned} \nabla_{\xi_0}\nabla_{\xi_0}\xi_0 &= \nabla_{\xi_0}\left(-\frac{1}{2}\left(u_0+1\right)\zeta_0\right) \\ &= -\frac{1}{2}\left[\xi_0\left(u_0\right)\zeta_0\right] \\ &= -\frac{1}{2}\left(u_0'\right)^2\zeta_0\,. \end{aligned}$$

To calculate $\nabla_{[\xi_0,\xi_0]}\xi_0$, notice that $[\zeta_0, \xi_0] = \nabla_{\zeta_0}\xi_0 - \nabla_{\xi_0}\zeta_0 = -\frac{1}{2}(u_0+1)\zeta_0$, since the metric is torsion-free. Therefore,

$$\nabla_{[\zeta_0,\xi_0]}\xi_0 = -\frac{1}{2}\left(u_0+1\right)\nabla_{\zeta_0}\xi_0 = \frac{1}{4}\left(u_0+1\right)^2\zeta_0\,.$$

Adding the three terms, we obtain

$$\begin{aligned} R(\zeta_0, \xi_0)\xi_0 &= \left(\frac{1}{2}u_0(u_0+1) + \frac{1}{2}\left(u_0'\right)^2 - \frac{1}{4}(u_0+1)^2\right)\zeta_0 \\ &= \left(\frac{1}{2}\cos t(\cos t + 1) + \frac{1}{2}\sin^2 t - \frac{1}{4}(\cos t + 1)^2\right)\zeta_0 \\ &= \frac{1}{4}\sin^2 t\,\zeta_0 \\ &= \frac{1}{4}\left(u_0'\right)^2\zeta_0 \end{aligned}$$

Thus, $R(V_0, \dot\gamma_0)\dot\gamma_0 = \frac{f_0}{4}\zeta_0$ and the Jacobi equation becomes $f_0'' + \frac{f_0}{4} = 0$. The initial conditions on V_0 imply $f_0(0) = 0$ and $f_0(s) = 1$, so $f_0(t) = (\csc\frac{s}{2})(\sin\frac{t}{2})$ and $V_0(t) = (\csc\frac{s}{2})(\sin\frac{t}{2})\zeta_0(t)$.

As before, we identify the vector $(\nabla_{\dot\gamma_0}V_0)(0) = \frac{1}{2}(\csc\frac{s}{2})\zeta_0(0) \in T_{P_0}\mathbb{P}^n$ with a vector in $T_{P_0}(T_{P_0}\mathbb{P}^n)$ and parallel translate it along $\rho(t) = \exp_{P_0}^{-1}(\gamma_0(t))$. Define $Y(t) = \frac{1}{2}(\csc\frac{s}{2})t\,\zeta_0$ along $\rho(t)$. Then $(\exp_{P_0}^{-1})_*(\zeta_0(s)) = Y(s)$.

We again identify $T_{P_0}\mathbb{P}^n$ and T_PM using ι, set $\zeta = \iota(\zeta_0(0))$, and calculate $(\exp_P)_*(Y(s)) = (\exp_P)_*(\frac{1}{2}(\csc\frac{s}{2})s\zeta_0(0))$. Let $\gamma(t) = \exp_P(\rho(t))$ and let $\zeta(t)$ be the parallel translate of ζ along γ. We set $V(t) = f(t)\zeta(t)$ and find a function f such that V is a Jacobi field with $V(0) = 0$ and $(\nabla_{\dot\gamma}V)(0) = Y'(0) = \frac{1}{2}(\csc\frac{s}{2})\zeta_0(0)$. Since (∗) holds on M, one follows a calculation similar to the one above to show that $f(t) = (\csc\frac{s}{2})(\sin\frac{t}{2})\zeta_0(t)$. Thus, $\Psi_*(\zeta_0(s)) = V(s) = \zeta(s)$.

Since ζ_0 and ζ are parallel along γ_0 and γ, respectively, $g_0(\zeta_0(s), \zeta_0(s)) = g_0(\zeta_0(0), \zeta_0(0))$ and $g(\zeta(s), \zeta(s)) = g(\zeta(0), \zeta(0))$. The identification ι was chosen to be an isometry, so $g(\zeta(0), \zeta(0)) = g_0(\zeta_0(0), \zeta_0(0))$ and Ψ is an isometry.

Step 3: Ψ: $\mathbb{P}^n \to M$ is a biholomorphic isometry.

It is clear from the definition of Ψ that it is surjective. To see that it is injective, suppose $Z, \tilde Z \in \mathbb{P}^n$, $\Psi(Z) = \Psi(\tilde Z)$. By the definition of Ψ, there exist sequences

$$\{Z^\nu\}, \left\{\tilde Z^\nu\right\} \subset \mathbb{P}^n \setminus H_0 \text{ with } \lim_{\nu\to\infty} Z^\nu = Z, \lim_{\nu\to\infty} \tilde Z^\nu = \tilde Z\,,$$
$$\lim_{\nu\to\infty} \Psi(Z^\nu) = \Psi(Z),\ \lim_{\nu\to\infty} \Psi\left(\tilde Z^\nu\right) = \Psi\left(\tilde Z\right).$$

Since each complex leaf of the foliation associated to X intersects H at exactly one point, $\mathrm{dist}_{M\setminus H}(Q_1, Q_2) = \mathrm{dist}_M(Q_1, Q_2)$ for all $Q_1, Q_2 \in M \setminus H$. Furthermore, $\mathrm{dist}_{\mathbb{P}^n\setminus H_0}(Q_1, Q_2) = \mathrm{dist}_{\mathbb{P}^n}(Q_1, Q_2)$ for all $Q_1, Q_2 \in \mathbb{P}^n \setminus H_0$. Since Ψ is an isometry on $\mathbb{P}^n \setminus H_0$,

$$\begin{aligned} \mathrm{dist}\left(Z, \tilde Z\right) &= \lim_{\nu\to\infty} \mathrm{dist}\left(Z^\nu, \tilde Z^\nu\right) = \lim_{\nu\to\infty} \mathrm{dist}\left(\Psi(Z^\nu), \Psi\left(\tilde Z^\nu\right)\right) \\ &= \mathrm{dist}\left(\Psi(Z), \Psi\left(\tilde Z\right)\right) = 0\,. \end{aligned}$$

Therefore, $Z = \tilde Z$.

Ψ is a homeomorphism. In fact, the above argument shows that for all

$$Z, \tilde Z \in \mathbb{P}^n, \mathrm{dist}\left(\Psi(Z),\ \Psi\left(\tilde Z\right)\right) = \mathrm{dist}\left(Z, \tilde Z\right),$$

so Ψ is continuous. It follows from the compactness of $\mathbb{P}^n$ that Ψ is an open map, so Ψ^{-1} is also continuous.

To show that Ψ is a diffeomorphism, fix $Z \in H_0$. We will find a neighborhood U of Z and construct a diffeomorphism $\Phi\colon U \to \Phi(U) \subset M$. We will show that $\Psi = \Phi$ on a dense subset of U, which implies that $\Psi = \Phi$ on U.

To construct Φ, fix $W \in \mathbb{P}^n \setminus H_0$ close enough to Z so that there is an open set U containing Z which is the diffeomorphic image under $\exp_W$ of an open set in $T_W\mathbb{P}^n$ and so that $\Psi(U)$ is the diffeomorphic image under $\exp_{\Psi(W)}$ of an open set in $T_{\Psi(W)}M$. For $\zeta_0 \in T_W\mathbb{P}^n$, let $\tau_0 \in T_{P_0}\mathbb{P}^n$ be its parallel translation to P_0 along the unique minimizing geodesic γ_0 from W to P_0. Let $\iota\colon T_{P_0}\mathbb{P}^n \to T_PM$ be as previously defined, $\tau = \iota(\tau_0) \in T_PM$, and $\zeta \in T_{\Psi(W)}M$ be the parallel translation of τ along the geodesic $\Psi \circ \gamma_0$ to $\Psi(W)$. Let $\iota_W(\zeta_0) = \zeta$ and define $\Phi\colon U \to \Psi(U)$ by $\Phi = \exp_{\Psi(W)} \circ \iota_W \circ \exp_W^{-1}$. Then Φ is a diffeomorphism.

Since Ψ is an isometry on $\mathbb{P}^n \setminus H_0$, $\Psi = \Phi$ on a dense subset of U, namely on the set of all $Q \in U$ such that the minimal geodesic from W to Q does not intersect H_0. By continuity of Ψ, $\Psi = \Phi$ on U, so Ψ is a diffeomorphism on U. Therefore, $\Psi\colon \mathbb{P}^n \to M$ is diffeomorphic.

By the continuity of Ψ_* and the fact that Ψ is holomorphic and isometric on $\mathbb{P}^n \setminus H_0$, we conclude that $\Psi\colon \mathbb{P}^n \to M$ is a holomorphic isometry, which completes the proof of the main theorem.

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