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Ginzburg-Landau Equation and Motion by Mean Curvature, I: Convergence

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ABSTRACT. In this paper we study the asymptotic behavior ($\epsilon \to 0$) of the Ginzburg-Landau equation:

$$u^\epsilon_t - \Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon) = 0,$$

where the unknown u^ϵ is a real-valued function of $[0, \infty) \times \mathcal{R}^d$, and the given nonlinear function $f(u) = 2u(u^2 - 1)$ is the derivative of a potential $W(u) = (u^2 - 1)^2/2$ with two minima of equal depth. We prove that there are a subsequence ϵ_n and two disjoint, open subsets $\mathcal{P}, \mathcal{N}$ of $(0, \infty) \times \mathcal{R}^d$ satisfying

$$u^{\epsilon_n} \to \mathbf{1}_{\mathcal{P}} - \mathbf{1}_{\mathcal{N}}, \qquad \text{as } n \to \infty,$$

uniformly in $\mathcal{P}$ and $\mathcal{N}$ (here $\mathbf{1}_A$ is the indicator of the set A). Furthermore, the Hausdorff dimension of the interface

$$\Gamma = \text{complement of } (\mathcal{P} \cup \mathcal{N}) \subset (0, \infty) \times \mathcal{R}^d$$

is equal to d and it is a weak solution of the mean curvature flow as defined in [13, 92]. If this weak solution is unique, or equivalently if the level-set solution of the mean curvature flow is "thin," then the convergence is on the whole sequence. We also show that u^{ϵ_n} has an expansion of the form

$$u^{\epsilon_n}(t, x) = q\left(\frac{d(t, x) + O(\epsilon_n)}{\epsilon_n}\right),$$

where $q(r) = \tanh(r)$ is the traveling wave associated to the cubic nonlinearity f, $O(\epsilon) \to 0$ as $\epsilon \to 0$, and $d(t, x)$ is the signed distance of x to the t-section of Γ. We prove these results under fairly general assumptions on the initial data, u_0. In particular we do *not* assume that $u^\epsilon(0, x) = q(d(0, x)/\epsilon)$, nor that we assume that the initial energy, $\mathcal{E}^\epsilon(u^\epsilon(0, \cdot))$, is uniformly bounded in ϵ. Main tools of our analysis are viscosity solutions of parabolic equations, weak viscosity limit of Barles and Perthame, weak solutions of mean curvature flow and their properties obtained in [13] and Ilmanen's generalization of Huisken's monotonicity formula.

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Key Words and Phrases phase transitions, viscosity solutions, weak viscosity limits, monotonicity formula, Ginzburg-Landau equation, mean curvature flow.

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1. Introduction

The equation

$$u_t^\epsilon - \Delta u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon) = 0, \qquad \text{in } (0,\infty)\times\mathcal{R}^d, \tag{1.1}$$

with

$$f(u) = 2u(u^2-1) = W'(u), \qquad W(u) = (u^2-1)^2/2$$

is the gradient flow of the energy functional

$$\mathcal{E}^\epsilon(u^\epsilon(t,\cdot)) = \int_{\mathcal{R}^d} \left[\frac{\epsilon}{2}|Du^\epsilon(t,x)|^2 + \frac{1}{\epsilon} W(u^\epsilon(t,x))\right] dx,$$

since

$$\mathcal{E}^\epsilon(u^\epsilon(t,\cdot)) = \mathcal{E}^\epsilon(u^\epsilon(0,\cdot)) - \epsilon \int_{\mathcal{R}^d}\int_0^t [u_t^\epsilon(s,x)]^2\, ds\, dx.$$

The term W/ϵ forces the solution u^ϵ to take the values ± 1. Indeed, Bronsard and Kohn [19] proved that if $\mathcal{E}^\epsilon(u^\epsilon(0,\cdot))$ is uniformly bounded in ϵ, then u^ϵ converges to a function u in L^1_{loc} and $|u| = 1$ (also see Section 5 in [43]). Thus the asymptotic behavior of u^ϵ is determined by the interface Γ that separates the two regions $\mathcal{P}$ and $\mathcal{N}$ on which u^ϵ converges $+1$ and -1, respectively. In the limit the interface Γ moves by mean curvature. The precise formulation and the proof of this statement was the content of several papers [13, 27, 43, 62, 68, 76, 86], and in this paper we will prove a convergence result that is global in time, for general initial data with no assumption on the limiting geometric flow.

We continue with a description of earlier work on this problem. In 1979, Allen and Cahn proposed equation (1.1) as a model for the motion of a curved antiphase boundary [1]. In their paper Allen and Cahn also gave a short, formal argument indicating that in the limit, the interfacial velocity V, is proportional to its mean curvature, K:

$$V = K.$$

This geometric equation was proposed earlier by Mullins to model an idealized grain boundary movement [78]. For a detailed account of these models, we refer the reader to the recent articles of Gurtin [57, 58] and the monographs of Fife [44] and Gurtin [59].

First justification of the convergence to the mean curvature flow was apparently given by Rubinstein, Sternberg and Keller in 1988 [86]. By an asymptotic expansion Rubinstein, Sternberg and Keller formally justified this convergence result not only for (1.1) but also for systems of equations in any space dimension. Independently, Caginalp and Fife [25] obtained the same expansion for a two dimensional phase field model which is very similar to (1.1). Since then the formal expansion techniques have been extended to several other problems, including systems of equations for which the limit is a harmonic map [87], problems with boundary conditions and nonlocal terms [88, 87, 83].

Later deMottoni and Schatzman [76] used the asymptotic expansion technique together with hard error estimates to prove the following result:

> Suppose that the initial data u_0^ϵ is positive inside a smooth $d-1$ dimensional hypersurface Γ_0 and negative outside of Γ_0. Further assume that there is a classical solution Γ_t of the mean curvature flow on $t \leq T$. Then u^ϵ converges to +1 inside Γ_t and to -1 outside of Γ_t on $t \leq T$.

The precise result requires additional technical assumptions on the regularity and the behavior of the initial data around the initial interface Γ_0. Independently, Chen proved the same result [27]. Chen's method was to cleverly use appropriate sub and supersolutions of (1.1). In addition to the convergence result, deMottoni and Schatzman [27] and Chen [77] also analyzed the formation of the initial interface. Since in two space dimensions ($d = 2$), there is a unique classical solution of the mean curvature flow [7, 8, 51, 55], the results of deMottoni and Schatzman and Chen completely describe the asymptotics of u^ϵ. However when $d > 2$, the mean curvature flow developes singularities even if the initial surface Γ_0 is smooth [56]. Hence for $d > 2$, the results of deMottoni and Schatzman and of Chen describe only the short time behavior of u^ϵ.

It is clear that the global-in-time, asymptotic analysis of u^ϵ requires a weak notion of mean curvature flow. The first weak formulation of the mean curvature flow was given by Brakke using the theory of geometric measure theory [17]. Then DeGiorgi [37] and Bronsard and Kohn [19] proposed to use the Ginzburg-Landau equation to define a weak solution of the mean curvature flow. By using energy estimates, Bronsard and Kohn also proved a convergence result for radially symmetric u^ϵ. Their approach was influenced by the Γ-convergence results of Modica and Mortolla [75], Modica [74], Fonseca and Tartar [48], and Sternberg [95].

More recently an alternate weak formulation was proposed independently by Evans and Spruck [39] and in more generality by Chen, Giga, and Goto [30]. Their formulation which is based on an idea of Otha, Jasnow, and Kawasaki [82], Sethian [90], and Osher and Sethian [81], is to view the surface moving by mean curvature as the level set of a function defined on the whole ambient space and to derive a differential equation for this function. This level set equation is degenerate parabolic; and Evans and Spruck and Chen et al. overcame this difficulty by using the theory of viscosity solutions of nonlinear second-order partial differential equations [34, 32, 33, 67]. The level-set approach was used earlier by Barles [9] to study a first-order problem arising in flame propagation and was further developed by Evans and Spruck [40, 41, 42], Chen, Giga, and Goto [31], Giga and Goto [52], Giga et al. [53], Soner [92], Barles, Soner, and Souganidis [13], Ishii and Souganidis [66] and Ilmanen [63, 64]. In particular, an intrinsic definition that will be used in this paper was obtained in [92]. The regularity and the other properties of the solutions and the connection between the level-set solutions and Brakke's solutions were discussed in [63, 40, 41, 42]. Motions in bounded domains were studied by Sternberg and Ziemer [96], Katsoulakis, Kossioris, and Reitich [68], and Giga and Sato [54]. Katsoulakis, Kossioris, and Reitich also obtained a convergence result for of solutions of (1.1) in a bounded domain with Neumann boundary condition.

Very recently, an interesting computational algorithm for tracking the fronts moving by generalized mean curvature was proposed by Bence, Merriman, and Osher [14] and the convergence of this algorithm was proved independently by Barles and Georgelin [11] and by Evans [38]. Also

Gurtin, Soner, and Souganidis [60] and Ohnuma and Sato [80] used the level-set approach to study a class of singular anisotropic equations. Anisotropic motions with crystalline energies introduce further difficulties. We refer the reader to the excellent survey of Taylor, Cahn, and Handwerker [103] and recent articles Almgren, Taylor, and Wang [5], Almgren and Taylor [4] for more information on anisotropic motions and the use of varifolds in studying them.

Equipped with the level-set formulation for the mean curvature equation, Evans, Soner, and Souganidis [43] proved the first global in time, multidimensional convergence result for (1.1). Hence the level-set solution of the mean curvature flow and the solution proposed by DeGiorgi [37] and Bronsard and Kohn are the same. The convergence result of [43] was extended by Barles, Soner, and Souganidis [13] to include a class of equations that are more general than (1.1). Barles, Soner, and Souganidis also extended the previous work of Gärtner [50] and Barles, Bronsard, and Souganidis [10] related to a different scaling in (1.1). Recently Katsoulakis and Souganidis [69] used these results to characterize the generalized mean curvature flow as the hydrodynamic limit of an infinite particle system, generalizing a previous result of Bonaventura [16]. For more information on the derivation of the mean curvature flow from certain other spin systems, we refer the reader to a recent article of DeMasi et al. [73] and the references therein.

More precisely, Evans, Soner, and Souganidis proved the following. Let u^ϵ be the unique solution of (1.1) with initial data $u^\epsilon(0, x) = \tanh(d(x, \Gamma_0)/\epsilon))$, where Γ_0 is the boundary of a bounded region and $d(x, \Gamma_0)$ is the signed distance of x to Γ_0. Let $\varphi(t, x)$ be the solution of the level-set equation with initial data $\varphi(0, x) = d(x, \Gamma_0)$. (Recall that the zero level set $\{x\colon \varphi(t, x) = 0\}$ is defined by Evans and Spruck and Chen et al. as the level set solution of the mean curvature flow.) Then u^ϵ converges to $+1$ on $\{\varphi > 0\}$ and to -1 on $\{\varphi < 0\}$. Hence the interface is included in the zero level set of φ or equivalently in the level set solution of the mean curvature flow. Moreover the interface is equal to the zero level set when it is "thin." However, when the set $\{\varphi = 0\}$ is not "thin," the above result does not yield more information about the interface Γ or the limit of u^ϵ in the region $\{\varphi = 0\}$ (see Section 5 in [43]).

Using mainly geometric measure theory, Ilmanen [62] obtained a different convergence result for u^ϵ that does not require the level-set to be "thin." Ilmanen proved that there are a subsequence ϵ_n and a closed bounded set $\Gamma \subset (0, \infty)\times\mathcal{R}^d$ satisfying, (a) u^ϵ converges to $+1$ or -1 locally uniformly on the complement of Γ, (b) Γ is a Brakke solution of the mean curvature equation. Moreover Γ has Hausdorff dimension d. Ilmanen's elegant proof is quite different than those given in [13, 43], an important tool being his extension of the monotonicity formula of Huisken: Huisken [61] proved his formula for smooth solutions of the mean curvature flow and Ilmanen extended Huisken's formula to solutions of (1.1). A statement of Ilmanen's monotonicity formula for the solutions of (1.1) is given in Section 5, below. To further understand the asymptotic behavior of the solutions in the region $\{\varphi = 0\}$, Dang, Fife, and Peletier [36] studied the stability properties of (1.1) in the plane. They considered solutions with initial interface close to the union of two axis. Since the level set solution of the mean curvature flow starting from this initial interface is "fat," the evolution of the interface is expected to be very sensitive to perturbations of the initial data. Schatzman [89] and Dang, Fife and Peletier [36] proved this instability.

Ilmanen proved his result under the assumption that $u^\epsilon(0, x) = q(z^\epsilon(0, x)/\epsilon)$ for some function z^ϵ satisfying $|Dz^\epsilon(0, x)| < 1$. In particular this assumption implies that the initial energy is uniformly bounded in ϵ. In this paper we remove both of these assumptions. Moreover, we do not assume that the initial energy is uniformly bounded in ϵ. Our proof combines Ilmanen's monotonicity formula with weak viscosity limits of Barles and Perthame [12]. In addition to the convergence result, this combination also allows us to obtain an asymptotic expansion of u^ϵ of the form

$$u^\epsilon(t, x) = q\left(\frac{d(t, x) + O(\epsilon)}{\epsilon}\right),$$

on a subsequence of ϵ. Here $d(t, x)$ is the signed distance of x to the t-section of the interface Γ.

The analysis of a model very similar to (1.1) was carried out by Chen and Elliot [29], Blowey and Elliot [15], and Nochetto, Paolini, and Verdi [79]. The bi-stable potential W that they considered is equal to infinity outside the interval $[-1, 1]$ and it is concave, quadratic inside this interval: the Euler equation related to this energy functional is the "double obstacle problem." Solutions of this problem take on the values ± 1 on two different regions and in the interface they solve a linear equation. Sharp error estimates for this model and numerical approximations of the mean curvature flow were obtained in [15, 29, 79]. Also Caginalp and Socolovsky [26] used (1.1) to numerically approximate the mean curvature flow.

In this paper, we will not survey the literature on systems of equations generalizing (1.1). A brief discussion of the connection between these equations and the harmonic maps is given in [62]. For information on problems with more than two phases and "triple junctions," we refer the reader to Taylor [101, 102], Bronsard and Reitich [20], Sternberg and Ziemer [97], and the references therein. Reader interested in "slow motion" or in the Cahn-Hilliard equation should consult Alikakos, Bates, and Fusco [3], Bronsard and Kohn [18], and Pego [84].

We complete our historical remarks with a very brief survey of convergence results for the phase field model for solid–liquid phase transitions in a pure material. This model was proposed by Langer [70], Fix [46], Caginalp [21, 22], and Collins and Levine [35] and more recently modified versions of the phase field equations have been derived by Penrose and Fife [85] and Fried and Gurtin [49]. Mathematically, the phase field model consists of two equations. One of these equations is very similar to (1.1) and the other is a heat equation with a source term. A rigorous asymptotic analysis of the phase field model has proved to be difficult. Formal expansions were obtained by Caginalp and Fife [25] and Caginalp [23]. More recently Stoth [98, 99] carried out an analysis of the one-dimensional and the radially symmetric problems and Caginalp and Chen [24] studied a version of the phase field model in an annular domain, with radial symmetry and special boundary conditions. For a generalized Stefan model of solidification with melting temperature proportional to curvature, Luckhaus [72] and Almgren and Wang [6] proved the convergence of a "time-step energy mininmization" method. Also, computational studies of the limiting equations were carried out by Strain [100] and Sethian and Strain [91]. However the convergence analysis of the multidimensional phase field model still remains open and further understanding of the Ginzburg-Landau equation (1.1) may prove to be useful in this direction.

After the completion of this work, Soner [93] proved the convergence of the phase field model to the mean curvature equation coupled with a heat equation, without assuming the existence of a smooth solution. For the Hele-Shaw model, a similar result was proved by Alikakos, Bates, and Chen

[2] by using a recent spectral estimate of Chen [28]. They assume the existence of a smooth solution of the Hele-Shaw problem.

Outline of our proof. We always assume that $|u^\epsilon(0,x)| \leq 1$. Then $|u^\epsilon(t,x)| < 1$ for every $(t,x) \in (0,\infty)\times\mathcal{R}^d$. Let $q = \tanh$. We now introduce a new function z^ϵ by

$$u^\epsilon(t,x) = q\left(\frac{z^\epsilon(t,x)}{\epsilon}\right).$$

Using (1.1) we obtain

$$z^\epsilon_t - \Delta z^\epsilon + \frac{2u^\epsilon}{\epsilon}[|Dz^\epsilon|^2 - 1] = 0. \tag{1.2}$$

Formally the above equation suggests that in the limit $|Dz^\epsilon| = 1$. However the only statement one can prove is the following. Let

$$\begin{aligned} z^*(t,x) &= \limsup_{\epsilon\to 0,\ (s,y)\to(t,x)} z^\epsilon(s,y), \\ z_*(t,x) &= \liminf_{\epsilon\to 0,\ (s,y)\to(t,x)} z^\epsilon(s,y) \\ \mathcal{P}_u &= \{(t,x)\colon \liminf u^\epsilon(t,x) > 0\}, \\ \mathcal{N}_u &= \{(t,x)\colon \limsup u^\epsilon(t,x) < 0\}, \end{aligned}$$

and

$$T_{ext} = \inf\{T \in [0,\infty]\colon |z^*(t,x)|, |z_*(t,x)| < \infty, \forall (t,x) \in (0,T)\times\mathcal{R}^d\}.$$

Then z^*, z_* are Lipschitz continuous in the x-variable with a Lipschitz constant one and satisfy the following in the viscosity sense (see Lemma 4.1, below):

$$-|Dz^*| + 1 \leq 0, \quad \text{in } \mathcal{N}_u \cap (0, T_{ext}) \times \mathcal{R}^d, \tag{1.3}$$

$$|Dz_*| - 1 \geq 0, \quad \text{in } \mathcal{P}_u \cap (0, T_{ext}) \times \mathcal{R}^d, \tag{1.4}$$

In general z^* and z_* are not continuous in the t-variable and therefore we do not expect them to be equal to each other. However, if z^* and z_* are upper and lower semicontinuous envelopes of the same function z, respectively, then by (3), (4) and the fact that z^*, z_* are Lipschitz continuous with Lipschitz constant one, we can show that z is equal to the signed distance function to the interface. So it is clear that to establish this connection between z^* and z_* is an important step in the convergence analysis of u^ϵ. But since we have not made any assumption on the initial data, the behavior of u^ϵ on different subsequences may be quite different and consequently z^* and z_* may not be related in any way. To establish a connection between z^* and z_*, we introduce the measure

$$\mu^\epsilon(A;t) = \int_A \left[\frac{\epsilon}{2}|Du^\epsilon(t,x)|^2 + \frac{1}{\epsilon}W(u^\epsilon(t,x))\right]dx.$$

Then following Ilmanen [62], we construct a subsequence ϵ_n and a measure μ such that $\mu^{\epsilon_n}(\cdot;t)$ converges to $\mu(\cdot;t)$ for each t. This construction assumes (6). The precise statement of this result and a sketch of its proof are given in Section 2, below.

Now let Γ be the support of μ and redefine z_* and z^* by using the subsequence ϵ_n instead of the whole sequence ϵ. Then the monotonicity formula of Ilmanen and a gradient estimate (Proposition 4.1, below) imply that on the complement of Γ, u^ϵ converges to $+1$ or -1 locally uniformly. It turns out that this result and (3), (4) are enough to make the connection between z_* and z^*. Indeed in Section 6 we show that z^* is equal to the upper semicontinuous envelope of d and z_* is equal to the lower semicontinuous envelope of d , where as before $d(t,x)$ is the signed distance of x to the t-section of Γ. Once this result is established , then it is easy to show that $\mathcal{P}_u$, $\mathcal{N}_u$ and Γ are disjoint subsets of $(0,\infty)\times\mathcal{R}^d$ and their union is the whole space. Moreover u^{ϵ_n} converges to $+1$ uniformly on compact subsets of $\mathcal{P}_u$ and to -1 uniformly on compact subsets of $\mathcal{N}_u$.

The above convergence result enables us to prove two important properties of the interface Γ. First we observe that the monotonicity result yields the "clearing-out" lemma (Theorem 5.1, below) and the "clearing-out" lemma implies that the Hausdorff dimension of the interface Γ is d. Recall that Γ is a subset of $(0,\infty)\times\mathcal{R}^d$ and therefore the interface Γ is a "sharp." Moreover Γ is a weak solution of the mean curvature flow. This fact follows from (1.2) and the techniques developed by Barles, Souganidis and the author in [13]. To give the basic idea let us assume that $|Dz^\epsilon(0,x)| \le 1$. Then by maximum principle and (1.2), $|Dz^\epsilon(t,x)| < 1$, for every (t,x) in $(0,\infty)\times\mathcal{R}^d$. We let ϵ_n go to zero in (1.2) to obtain

$$\begin{aligned} d_t - \Delta d &\ge 0 \text{ on } \mathcal{P}_u, \\ d_t - \Delta d &\le 0 \text{ on } \mathcal{N}_u, \end{aligned}$$

of course in the viscosity sense. Since d is a distance function the above inequalities immediately imply that Γ is a weak solution of the mean curvature in the sense defined in [92]. The general initial condition case requires more analysis. In particular, we use the gradient estimate (Proposition 4.1, below) and the techniques developed in [13].

Organization of the paper is as follows. In section 2, we recall several known results about (1.1). The main results of this paper are stated in Section 3. In Section 4, we define and study the functions z^* and z_*. In particular we state a gradient estimate for z^ϵ. The proof of this gradient estimate is given in the Appendix. Section 5 recalls the monotonicity result of Ilmanen and a corollary to this monotonicity result is also proved in this section. Finally proofs of the main results are completed in Section 6. In the Appendix 1, we state a result of deMottoni and Schatzman and of Chen and then prove a corollary that is used in the earlier sections. The gradient estimate for z^ϵ is proved in Appendix 2.

I would like to acknowledge several very helpful discussions with Guy Barles, Tom Ilmanen and Takis Souganidis and

2. Preliminaries

Let

$$f(u) = 2u(u^2 - 1) = W'(u), \qquad W(u) = (u^2 - 1)^2/2. \tag{2.1}$$

Consider the scalar Ginzburg-Landau equation,

$$u_t^\epsilon(t,x) - \Delta u^\epsilon(t,x) + \frac{1}{\epsilon^2} f(u^\epsilon(t,x)) = 0, \qquad (t,x) \in (0,\infty)\times\mathcal{R}^d, \tag{2.2}$$

with initial data

$$u^\epsilon(0,x) = u_0^\epsilon(x), \qquad x \in \mathcal{R}^d. \tag{2.3}$$

We will always assume that $u_0^\epsilon \leq 1$ and continuous on $\mathcal{R}^d$. Then the standard parabolic theory implies that there is a unique, bounded, real-valued

$$u^\epsilon \in C^\infty((0,\infty)\times\mathcal{R}^d) \cap C([0,\infty)\times\mathcal{R}^d)$$

satisfying (2.2) and (2.3).

Equation (2.2) is the gradient flow of the energy functional

$$\mathcal{E}^\epsilon(\psi) = \int_{\mathcal{R}^d} \left[\frac{\epsilon}{2}|D\psi(x)|^2 + \frac{1}{\epsilon}W(\psi(x))\right] dx. \tag{2.4}$$

Indeed by simple integration by parts and approximation arguments we can show that

$$\mathcal{E}^\epsilon(u^\epsilon(t,\cdot)) = \mathcal{E}^\epsilon(u_0^\epsilon) - \epsilon \int_{\mathcal{R}^d} \int_0^t [u_t^\epsilon(s,x)]^2 \, ds \, dx.$$

Since the above identity will not be used in this paper, we leave its derivation to the reader. For $t > 0$ define a measure on Borel subsets of $\mathcal{R}^d$ by

$$\mu^\epsilon(A;t) = \int_A \left[\frac{\epsilon}{2}|Du^\epsilon(t,x)|^2 + \frac{1}{\epsilon}W(u^\epsilon(t,x))\right] dx. \tag{2.5}$$

In view of the energy identity, if $\mathcal{E}^\epsilon(u_0^\epsilon)$ is uniformly bounded in ϵ, then $\mu^\epsilon(\mathcal{R}^d;t)$ is uniformly bounded in ϵ and t. Then we can use well known compactness of Radon measures to extract a weak* convergent subsequence. In this paper however, we do not assume that initial energy is uniformly bounded. Instead we assume that for every $\delta > 0$ there are positive constants K_δ and η satisfying

$$\sup\left\{\int |\psi(x)|\mu^\epsilon(dx;t) : \quad \epsilon \in (0,1), t \in [\delta, 1/\delta]\right\} \\ \leq K_\delta \sup\{|\psi(x)|e^{\eta|x|} : x \in \mathcal{R}^d\}, \tag{2.6}$$

I wish to thank Tom Ilmanen for sending me a preliminary copy of his manuscript [62] which influenced this work greatly. I also thank Mort Gurtin for bringing several references to my attention.

for every continuous function ψ. The above condition is satisfied if $\mathcal{E}^\epsilon(u_0^\epsilon)$ is uniformly bounded in ϵ, but also holds under more general hypotheses. For example, (6) is satisfied under the hypotheses of [27, 77], i.e., if the initial condition is three times continuously differentiable, has nonzero gradient on its zero set and its zero level set is bounded, then (6) holds. A proof of this fact and other sufficient conditions for (6) is the subject of the sequel of this paper [94].

Let $\psi(x)$ be a compactly supported, smooth, nonnegative real-valued function. Then following Ilmanen [62] we obtain

$$\begin{aligned}
&\frac{d}{dt}\int \psi(x)\mu^\epsilon(dx;t)\\
&\quad= -\epsilon\int \psi(x)\left(-\Delta u^\epsilon(t,x)+\frac{1}{\epsilon^2}f(u^\epsilon(t,x))\right)^2 dx\\
&\qquad+\epsilon\int D\psi(x)\cdot Du^\epsilon(t,x)\left(-\Delta u^\epsilon(t,x)+\frac{1}{\epsilon^2}f(u^\epsilon(t,x))\right) dx\\
&\quad\le -\epsilon\int \psi(x)\left(-\Delta u^\epsilon(x,t)+\frac{1}{\epsilon^2}f(u^\epsilon(t,x))-\frac{D\psi(x)\cdot Du^\epsilon(t,x)}{2\psi(x)}\right)^2 dx\\
&\qquad+\epsilon\int |Du^\epsilon(t,x)|^2\frac{|D\psi(x)|^2}{4\psi(x)}\,dx\\
&\quad\le C_1(\psi)\mu^\epsilon(\{\psi>0\};t),
\end{aligned}\tag{2.7}$$

for some constant $C_1(\psi)$ depending only on ψ. Since ψ is compactly supported, (6) implies that for $t>\delta$ $\mu^\epsilon(\{\psi>0\};t)$ is bounded by some constant $C_{2,\delta}(\psi)$ depending on ψ but not on ϵ. Therefore the map

$$t \to \int \psi(x)\mu^\epsilon(dx;t) - C_1(\psi)C_{2,\delta}(\psi)t$$

is nondecreasing on $t>\delta$.

Now using the weak* compactness of Radon measures, (6) and the above monotonicity property in a diagonal argument we construct a subsequence $\epsilon_n \to 0$ and a Radon measure μ satisfying

$$\lim_{n\to\infty}\int_{\mathcal{R}^d}\psi(x)\mu^{\epsilon_n}(dx;t) = \int_{\mathcal{R}^d}\psi(x)\mu(dx;t), \tag{2.8}$$

for every $t>0$ and a compactly supported smooth function ψ. The above argument originates in Brakke [17] and for the details of this argument we refer to Section 5.4 in Ilmanen [62]. The density arguments together with (6) show that (2.8) actually holds for all continuous $\psi(x)$ that decay faster than $e^{\eta|x|}$ as $|x|$ tends to infinity (here the constant η is as in (6)). Finally, let

$$\Gamma = \text{support } \mu. \tag{2.9}$$

We will show in addition to several other properties of Γ that it has Hausdorff dimension d and that it is the "sharp interface" separating the two regions on which u^{ϵ_n} converge -1 and 1, see Section 3 below.

Using (6) together with (7) and the techniques of Bronsard and Kohn [19], we show that there are a further subsequence, denoted by ϵ_n again, and a function u satisfying, $|u| = 1$ and

$$u^{\epsilon_n} \to u, \qquad n \to \infty, \tag{2.10}$$

locally in $L^1((0, \infty) \times \mathcal{R}^d)$. Under the assumption that the initial energy is uniformly bounded in ϵ, the above argument is given in detail in [19] (also see section 5 in [43]). Since in this paper we assume only (6), we need to localize the argument of Bronsard and Kohn by using (7). Details of this routine localization argument is left to the reader. In the remainder of this paper, we only study the properties of the sequence u^{ϵ_n}. So we introduce the notation

$$u^n = u^{\epsilon_n}, \qquad \mu^n = \mu^{\epsilon_n}.$$

Another important object in our analysis is the travelling wave associated to the cubic nonlinearity f. Using the explicit form of f, we can easily show that

$$q(r) = \tanh(r), \qquad r \in \mathcal{R}$$

is the unique solution of the equation

$$q''(r) = f(q(r)), \qquad \forall r \in \mathcal{R}, \tag{2.11}$$

with boundary conditions $q(\pm\infty) = \pm 1$ and $q(0) = 0$. Clearly the map

$$q \colon \mathcal{R} \to (-1, 1)$$

is one-to-one and onto. Therefore if $|u^n(t, x)| < 1$, then we can define a real-valued function z^n satisfying

$$u^n(t, x) = q\left(\frac{z^n(t, x)}{\epsilon_n}\right). \tag{2.12}$$

3. Main results

Let u^ϵ be a smooth, bounded solution of (2.1), (2.2), the sequence ϵ_n be as in Section 2 satisfying (2.8), (2.10) and $u^n = u^{\epsilon_n}$. In addition to (6), we will always assume that

$$|u_0^\epsilon(x)| \leq 1, \quad \forall x \in \mathcal{R}^d, \tag{3.1}$$

then by maximum principle $|u^n(t, x)| < 1$ for every $(t, x) \in (0, \infty) \times \mathcal{R}^d$ and therefore $z^n(t, x)$ is defined everywhere. Let us recall that sufficient conditions for (6) are obtained in [94]. In particular (6) holds if $\mathcal{E}^\epsilon(u_0^\epsilon)$ is uniformly bounded in ϵ.

Now we are ready to state the main results of this paper. Proofs of these results will be given in the subsequent sections.

Theorem 3.1. *Assume* (6) *and* (3.1). *Let* Γ *be as in* (2.9). *Then there are open disjoint subsets* $\mathcal{P}$ *and* $\mathcal{N}$ *of* $(0,\infty)\times\mathcal{R}^d$ *satisfying,*

$$\Gamma \cup \mathcal{P} \cup \mathcal{N} = (0,\infty)\times\mathcal{R}^d, \qquad \Gamma \cap \mathcal{P} = \Gamma \cap \mathcal{N} = \emptyset. \tag{3.2}$$

Moreover,

$$u^n \to +1 \ \ \textit{uniformly in}\ \mathcal{P}, \tag{3.3}$$

$$u^n \to -1 \ \ \textit{uniformly in}\ \mathcal{N}. \tag{3.4}$$

Let Γ_t be the t-section of Γ and $d(t,x)$ be the signed-distance between x and Γ_t, i.e.,

$$d(t,x) = \begin{cases} \operatorname{dist}(x,\Gamma_t), & (t,x)\in\mathcal{P} \\ -\operatorname{dist}(x,\Gamma_t), & (t,x)\in\mathcal{N} \\ 0, & (t,x)\in\Gamma, \end{cases} \tag{3.5}$$

if Γ_t is empty, we define $\operatorname{dist}(x,\Gamma_t)=\infty$ for all x. Set

$$t_{ext} = \inf\{t\in[0,\infty] \colon \Gamma\subset[0,t]\times\mathcal{R}^d\}.$$

Let $\psi(t,x)$ be a real-valued function. Then the *upper semicontinuous envelope* of ψ is the smallest upper semi continuous function that is greater than or equal to ψ and it is denoted by $\psi^*(t,x)$. Similarly the *lower semi continuous envelope* of ψ is the largest lower semi continuous function that is less than or equal to ψ and it is denoted by $\psi_*(t,x)$.

Theorem 3.2. *Assume* (6) *and* (3.1). *Then the Hausdorff dimension of* Γ *is equal to* d. *Moreover in* $(0,t_{ext})\times\mathcal{R}^d$, Γ *is a weak solution of the mean curvature equation in the sense defined in* [92], *i.e.,*

$$\frac{\partial}{\partial t}[d\wedge 0] - F^*(D[d\wedge 0], D^2[d\wedge 0]) \le 0, \tag{3.6}$$

$$\frac{\partial}{\partial t}[d\vee 0] - F_*(D[d\vee 0], D^2[d\vee 0]) \ge 0, \tag{3.7}$$

where the above inequalities are understood in the viscosity sense in $(0, t_{ext}) \times \mathcal{R}^d$, *and* $d \wedge 0 = \min\{d, 0\}$, $d \vee 0 = \max\{d, 0\}$ *and for a symmetric matrix* H *and a nonzero vector* $p \in \mathcal{R}^d$,

$$F(p, H) = \text{trace}\left[I - \frac{p \otimes p}{|p|^2}\right] H.$$

Our final result is a refinement of (3) and (4).

Theorem 3.3. *Assume* (6) *and* (3.1). *Then, for every* $(t, x) \in (0, t_{ext}) \times \mathcal{R}^d$ *we have*

$$\limsup_{n \to \infty, (s,y) \to (t,x)} z^n(s, y) = d^*(t, x), \tag{3.8}$$

$$\liminf_{n \to \infty, (s,y) \to (t,x)} z^n(s, y) = d_*(t, x), \tag{3.9}$$

where z^n *is as in* (2.12), d^* *is the upper semicontinuous envelope of* d, *and* d_* *is the lower semicontinuous envelope of* d. *On* $(t_{ext}, \infty) \times \mathcal{R}^d$ *we have either one of the following:*

$$(t_{ext}, \infty) \times \mathcal{R}^d \subset \mathcal{P}, \quad z^* = z_* = +\infty,$$

or

$$(t_{ext}, \infty) \times \mathcal{R}^d \subset \mathcal{N}, \quad z^* = z_* = -\infty.$$

In general, the distance function d is not continuous in the t-variable. However, when d is continuous at a point (t_0, x_0), then (3.8) and (3.9) imply that z^n converges to d uniformly in a bounded neighborhood of (t_0, x_0). In this neighborhood we have the following expansion:

$$u^n(t, x) = q\left(\frac{d(t, x) + O(\epsilon_n)}{\epsilon_n}\right).$$

4. Weak viscosity limits

Recall that since $|u_0^\epsilon| < 1$, $|u^\epsilon(t, x)| < 1$ for all (t, x) and therefore $z^\epsilon(t, x)$ is defined everywhere and solves the following parabolic equation in $(0, \infty) \times \mathcal{R}^d$:

$$z_t^\epsilon - \Delta z^\epsilon + \frac{2u^\epsilon}{\epsilon}[|Dz^\epsilon|^2 - 1] = 0. \tag{4.1}$$

Set

$$w^\epsilon = |Dz^\epsilon(t, x)|^2, \qquad w^n = w^{\epsilon_n} = |Dz^n(t, x)|^2,$$

where ϵ_n is the sequence chosen in Section 2 and D denote the differentiation with respect to the spatial variable x alone. By differentiating (4.1) we obtain

$$w_t^\epsilon + \mathcal{L}_t^\epsilon w^\epsilon + R^\epsilon(t, x, w^n) = -2||D^2 z^\epsilon||^2 \text{ in } (0, \infty)\times\mathcal{R}^d, \tag{4.2}$$

where for a real number r, $(t, x) \in [0, \infty)\times\mathcal{R}^d$ and a smooth function $\varphi \in C^2(\mathcal{R}^d)$,

$$\begin{aligned} \mathcal{L}_t^\epsilon \varphi(x) &= -\Delta \varphi(x) + \frac{4u^\epsilon(t, x)}{\epsilon} Dz^\epsilon(t, x)\cdot D\varphi(x), \\ R^\epsilon(t, x, r) &= \frac{4}{\epsilon^2} q'\left(\frac{z^\epsilon(t, x)}{\epsilon}\right) r(r-1), \end{aligned}$$

where as before $q = \tanh$. Observe that if $|Dz^\epsilon(0, x)| \leq 1$ for every x, then by maximum principle $|Dz^\epsilon(t, x)| \leq 1$ for every (t, x). This fact was used in an essential way in [62] (see Section 4.1 in [62]). In this paper we make no such assumption on the initial data. Instead we have the following gradient estimate.

Proposition 4.1. *Assume* (3.1). *Then there exists* $0 < \epsilon_0 < 1$ *such that for every* $\epsilon \leq \epsilon_0$ *we have,*

$$w^\epsilon(t, x) = |Dz^\epsilon(t, x)|^2 \leq 1 + \frac{2}{\ln(1/\epsilon)} \frac{(z^\epsilon(t, x))^2 + 1}{t}, \tag{4.3}$$

for every $(t, x) \in (0, \infty) \times \mathcal{R}^d$.

We prove (3.5) by maximum principle and appropriate of supersolutions. Since this proof is somehow tangential to the main trust of this paper, we postpone it to Appendix 2. We should also note that a "scale-invariant" version of (4.3) is also discussed in Appendix 2, Remark 8.1.

Although the proof of (4.3) is not important in the subsequent sections, (4.3) itself is an essential tool in our analysis. Note that after letting ϵ go to zero in (4.3), we see that the right hand side of tends to one provided that t is positive and z^ϵ is uniformly bounded in ϵ. Hence at least formally, in the limit as ϵ tends to zero we recover the estimate "$|Dz^\epsilon| \leq 1$."

By a simple application of the Gronwall's inequality, we obtain the following corollary:

Corollary 4.1. *Assume* (3.1). *Let* ϵ_0 *be as in Proposition 4.1. Then for every positive* t, *there exists a constant* $K^\epsilon(t)$ *such that*

$$|z^\epsilon(t, x) - z^\epsilon(t, y)| \leq [e^{K^\epsilon(t)|x-y|} - 1]\left[\min\{|z^\epsilon(t, x)|, |z^\epsilon(t, y)|\} + \frac{1 + K^\epsilon(t)}{K^\epsilon(t)}\right], \tag{4.4}$$

where

$$K^\epsilon(t) = \sqrt{2}[t \ln(1/\epsilon)]^{-1/2}.$$

Now following Barles and Perthame [12] (also see Chapter 7 in [47]), we define two possibly extended-valued functions z^* and z_* by

$$\begin{aligned} z^*(t,x) &= \limsup_{n\to\infty,\ (s,y)\to(t,x)} z^n(s,y), \\ z_*(t,x) &= \liminf_{n\to\infty,\ (s,y)\to(t,x)} z^n(s,y). \end{aligned} \tag{4.5}$$

We also define

$$\begin{aligned} \mathcal{P} &= \{(t,x) \in (0,\infty)\times\mathcal{R}^d\colon\ z_*(t,x) > 0\}, \\ \hat{\mathcal{P}} &= \{(t,x) \in (0,\infty)\times\mathcal{R}^d\colon\ z_*(t,x) \geq 0\}, \\ \mathcal{N} &= \{(t,x) \in (0,\infty)\times\mathcal{R}^d\colon\ z^*(t,x) < 0\}, \\ \hat{\mathcal{N}} &= \{(t,x) \in (0,\infty)\times\mathcal{R}^d\colon\ z^*(t,x) \leq 0\}, \\ \mathcal{P}_u &= \{(t,x) \in (0,\infty)\times\mathcal{R}^d\colon\ \liminf u^n(t,x) > 0\}, \\ \mathcal{N}_u &= \{(t,x) \in (0,\infty)\times\mathcal{R}^d\colon\ \limsup u^n(t,x) < 0\}, \end{aligned} \tag{4.6}$$

and

$$T_{ext} = \inf\{T \in [0,\infty]\colon\ |z^*(t,x)|, |z_*(t,x)| < \infty, \forall (t,x) \in (0,T)\times\mathcal{R}^d\}.$$

Clearly u^n satisfies (3) and (4), and in particular $\mathcal{P} \subset \mathcal{P}_u, \mathcal{N} \subsetneq \mathcal{N}_u$. One of the main objects of the rest of this paper is to show that the complement of $\mathcal{P} \cup \mathcal{N}$ is equal to Γ. We will prove this fact by carefully analyzing the properties of z^* and z_*. First observe that, by passing to the limit $\epsilon_n \to 0$ in (4.4) we obtain

$$\begin{aligned} |z^*(t,x) - z^*(t,y)| &\leq |x-y|, \quad t \in (0,T_{ext}), x, y \in \mathcal{R}^d, \\ |z_*(t,x) - z_*(t,y)| &\leq |x-y|, \quad t \in (0,T_{ext}), x, y \in \mathcal{R}^d. \end{aligned} \tag{4.7}$$

However z^* and z_* may fail to be continuous in the t-variable. But z^* is upper semi continuous and z_* is lower semi continuous. Next we multiply (4.1) by ϵ and then pass to the limit in the viscosity sense to obtain the following result.

Lemma 4.1. *Assume* (3.1). *Then z_* and z^* satisfy the following differential inequalities in the viscosity sense,*

$$-|Dz^*| + 1 \leq 0, \quad in\, \mathcal{N}_u, \tag{4.8}$$

$$|Dz_*| - 1 \geq 0, \quad in\, \mathcal{P}_u. \tag{4.9}$$

Proof. Let ψ be a smooth function and $(t,x) \in \mathcal{N}_u$ be a strict local maximizer of the difference $z^* - \psi$ on $(0,\infty) \times \mathcal{R}^d$. Since $(t,x) \in \mathcal{N}_u$, $z^*(t,x) \le 0$ and since $z^* - \psi$ has a local maximum at (t,x), $z^*(t,x) > \infty$. Hence $z^*(t,x)$ is finite and there are a subsequence n_k and local maximizers (t_k, x_k) of the difference $z^{n_k} - \psi$ converging to (t,x) as k tends to ∞. By calculus at (t_k, x_k) we have

$$D\psi = Dz^{n_k}, \quad \psi_t = (z^{n_k})_t, \quad D^2\psi \le D^2 z^{n_k}.$$

We use the above in (4.1) to obtain the following at (t_k, x_k),

$$\psi_t - \Delta\psi + \frac{2u^{n_k}}{\epsilon_{n_k}}[|D\psi|^2 - 1] \le 0.$$

Now if $(t,x) \in \mathcal{N}_u$, then $u^{n_k}(t_k, x_k) < 0$ for sufficiently large k. Then the above inequality implies that at (t_k, x_k),

$$-|D\psi|^2 + 1 \le \frac{\epsilon_{n_k}}{2u^{n_k}}[\psi_t - \Delta\psi].$$

Let k go to infinity in the above inequality to obtain

$$-|D\psi(t,x)|^2 + 1 \le 0.$$

This completes the proof of (8). The other inequality (9) is proved exactly the same way. □

Recall that $\mathcal{P} \subset \mathcal{P}_u$, $\mathcal{N} \subset \mathcal{N}_u$. Also the distance function is a unique viscosity solution of the Eikonal equation $|Dd| = 1$. These observations and comparison results for viscosity sub and supersolutions of the Eikonial equation [65], yield the following result.

Proposition 4.2. *Assume* (3.1). *Then*

$$\mathcal{N}_u = \mathcal{N}, \quad \mathcal{P}_u = \mathcal{P}. \tag{4.10}$$

Moreover,

$$\begin{aligned} z^*(t,x) &\le -\mathrm{dist}(x, \mathcal{N}_t^c), \quad (t,x) \in \mathcal{N}, \\ z^*(t,x) &= -\mathrm{dist}(x, \mathcal{N}_t^c), \quad (t,x) \in \mathcal{N} \cap (0, T_{ext}) \times \mathcal{R}^d, \\ z^*(t,x) &\le \mathrm{dist}(x, \hat{\mathcal{N}}_t), \quad (t,x) \in \mathcal{N}^c \cap (0, T_{ext}) \times \mathcal{R}^d, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} z_*(t,x) &\ge \mathrm{dist}(x, \mathcal{P}_t^c), \quad (t,x) \in \mathcal{P}, \\ z_*(t,x) &= \mathrm{dist}(x, \mathcal{P}_t^c), \quad (t,x) \in \mathcal{P} \cap (0, T_{ext}) \times \mathcal{R}^d, \\ z_*(t,x) &\ge -\mathrm{dist}(x, \hat{\mathcal{P}}_t), \quad (t,x) \in \mathcal{P}^c \cap (0, T_{ext}) \times \mathcal{R}^d, \end{aligned} \tag{4.12}$$

where A^c denote the complement of A and $A_t \subset \mathcal{R}^d$ denote the t-section of $A \subset [0,\infty) \times \mathcal{R}^d$.

Proof. The only subtle point in the proof of (11) and (12) is the fact that (8) and (9) hold only in open subsets of $(0, \infty)\times\mathcal{R}^d$ but not necessarily at every t. Set

$$\hat{d}(t,x) = \mathrm{dist}(x, (\mathcal{P}_u)_t^c).$$

Let (t_0, x_0) be in $\mathcal{P}_u$ with $t_0 < T_{ext}$.

Case a: $\hat{d}(t_0, x_0) < \infty$. Then $\hat{d}(t,x) > 0$ in a neighborhood of (t_0, x_0) and for every $\delta > 0$, there is $r(\delta) > 0$ satisfying

$$Q_\delta = \{(t,x)\colon |x - x_0| < \hat{d}(t_0, x_0) - \delta, |t - t_0| < r(\delta)\} \subset \mathcal{P}_u.$$

For $\gamma > 0$, set

$$v(t,x) = \hat{d}(t_0, x_0) - \delta - |x - x_0| - \gamma[r(\delta) - |t - t_0|]^{-2}, \quad (t,x) \in Q_\delta.$$

Then by (9)

$$|Dz_*| \geq 1, \text{ and } |Dv| = 1 \text{ in } Q_\delta.$$

Recall that D denote the differentiation with respect to x-variable only and the above inequalities hold in the viscosity sense. We also have $z_* \geq v$ on the boundary of Q_δ and v is continuous in Q_δ. Hence by a comparison result for viscosity sub and supersolutions (see for example [12, 65, 33]) we have $z_* \geq v$ in Q_δ for every positive δ and γ. Let δ and γ go to zero to conclude that $z_*(t_0, x_0) \geq \hat{d}(t_0, x_0)$.

Case b: $\hat{d}(t_0, x_0) = \infty$. Then for every N, there is $r_N > 0$ such that

$$Q_N = \{(t,x)\colon |x - x_0| < N, |t - t_0| < r_N\} \subset \mathcal{P}_u.$$

Set

$$v_N(t,x) = N - |x - x_0| - \gamma[r_N - |t - t_0|]^{-2}, \quad (t,x) \in Q_N.$$

Now proceed as in the previous case to show that $z_*(t_0, x_0) \geq v_N(t_0, x_0) = N$ for every N. Hence

$$z_*(t_0, x_0) = \hat{d}(t_0, x_0) = \infty.$$

Hence we have proved that

$$z_*(t,x) \geq \hat{d}(t,x) = \mathrm{dist}(x, (\mathcal{P}_u)_t^c), \quad \forall (t,x) \in \mathcal{P}_u.$$

Now suppose that $(t,x) \in \mathcal{P}_u$. Then $\hat{d}(t,x) > 0$. Therefore $z_*(t,x) > 0$, and $(t,x) \in \mathcal{P}$. Since by construction $\mathcal{P}$ is a subset of $\mathcal{P}_u$, we conclude that they are equal to each other. In summary, we have proved that

$$\mathcal{P} = \mathcal{P}_u, \quad z_*(t,x) \geq \operatorname{dist}(x, \mathcal{P}_t^c), \ \forall (t,x) \in \mathcal{P}.$$

Moreover on $(0, T_{ext}) \times \mathcal{R}^d$, by (7) we have

$$z_*(t,x) \ \leq \ \operatorname{dist}(x, \mathcal{P}_t^c), \ \forall (t,x) \in \mathcal{P} \cap (0, T_{ext}) \times \mathcal{R}^d,$$
$$z_*(t,x) \ \geq \ -\operatorname{dist}(x, \hat{\mathcal{P}}_t), \ \forall (t,x) \in \mathcal{P}^c \cap (0, T_{ext}) \times \mathcal{R}^d.$$

Combining above inequalities to obtain (12). The second part of (4.10) and (11) are proved similarly. □

Remark 4.1. Suppose that there are a subsequence n_k and an open subset Q of $(0,\infty) \times \mathcal{R}^d$ such that as k tends to infinity, $u^{n_k} \to +1$ uniformly on compact subsets of Q. Then by restricting the arguments of Lemma 4.1 and Proposition 4.2 to the subsequence n_k, we can show that

$$\liminf_{k\to\infty,\ (s,y)\to(t,x)} z^{n_k}(s,y) \geq \operatorname{dist}(x, Q_t),$$

for all $(s,x) \in Q$. Clearly a similar result holds if $u^{n_k} \to -1$ uniformly on compact subsets of an open set Q.

5. Monotonicity Formula

In this section we recall a remarkable monotonicity formula derived by Ilmanen [62]. Ilmanen's formula is an extension of the Huisken's monotonicity formula for smooth manifolds moving by their mean curvature [61]. In our analysis the monotonicity formula is essential in connecting the subsequences on which $z_*(t,x)$ and $z^*(t,x)$ are achived. Following the notation and the terminology of Section 3 in [62], we fix a 'blow-up' point (y,s) and set

$$\rho(t,x;y,s) = \frac{1}{(4\pi(s-t))^{(d-1)/2}} \exp\left(-\frac{|x-y|^2}{4(s-t)}\right), \quad x \in \mathcal{R}^d, t < s. \tag{5.1}$$

For $t < s$, Ilmanen proved the following (see Section 3.3 in [62]).

$$\frac{d}{dt}\int \rho(t,x;y,s)\mu^\epsilon(dx;t) \leq \frac{1}{2(s-t)}\int \rho(x,t;y,s)\xi^\epsilon(dx;t), \tag{5.2}$$

where

$$\xi^\epsilon(A;t) = \int_A \left[\frac{\epsilon}{2}|Du^\epsilon(t,x)|^2 - \frac{1}{\epsilon}W(u^\epsilon(t,x))\right] dx,$$
$$= \int_A \left[\frac{1}{2\epsilon}\left(q'\left(\frac{z^\epsilon(t,x)}{\epsilon}\right)\right)^2 [|Dz^\epsilon(t,x)|^2 - 1\right] dx. \tag{5.3}$$

In the above derivation we used the identity $(q')^2 = 2W(q)$. Set

$$\alpha^\epsilon(t;s,y) = \int \rho(t,x;y,s)\mu^\epsilon(dx;t).$$

Observe that if $|Dz^\epsilon(t,x)|^2 \le 1$, then ξ is negative and therefore $\alpha^\epsilon(t;s,y)$ is nonincreasing in t. In general the gradient estimate (4.3) yields the following analogue of this monotonicity result. Let ϵ_0 be as in Proposition 4.2.

Corollary 5.1. *Assume* (3.1). *Then for all* $(s,y) \in (0,\infty)\times\mathcal{R}^d$, $0 < \epsilon \le \epsilon_0$ *and* $0 < \delta \le t \le r < s$, *we have,*

$$\alpha^\epsilon(r;s,y) \le \alpha^\epsilon(t;s,y)\left(\frac{s-t}{s-r}\right)^{K(\epsilon,\delta)}$$
$$+ 4\epsilon\sqrt{\pi}K(\epsilon,\delta)\int_t^r \left(\frac{s-\tau}{s-r}\right)^{K(\epsilon,\delta)} \frac{d\tau}{\sqrt{s-\tau}}, \tag{5.4}$$

where

$$K(\epsilon,\delta) = [\delta \ln(1/\epsilon)]^{-1}.$$

In particular, for every $0 < t \le r < s$, *we have*

$$\int \rho(r,x;y,s)\mu(dx;r) \le \int \rho(t,x;y,s)\mu(dx;t),$$

where μ *is the limit of* μ^n (*see* (2.8)).

Proof. Ilmanen's monotonicity formula (5.2) , (3) and the gradient estimate (4.3) imply that,

$$\frac{d}{dt}\alpha^\epsilon(t;s,y) \le \frac{1}{2(s-t)}\int \rho(t,x;s,y)\frac{1}{2\epsilon}\left(q'\left(\frac{z^\epsilon(t,x)}{\epsilon}\right)\right)^2 [|Dz^\epsilon(t,x)|^2 - 1]dx,$$
$$\le [\ln(1/\epsilon)t(s-t)]^{-1}$$
$$\times \int \rho(t,x;s,y)\frac{1}{2\epsilon}\left(q'\left(\frac{z^\epsilon(t,x)}{\epsilon}\right)\right)^2 [(z^\epsilon(t,x))^2 + 1]dx.$$

Observe that

$$\left[\frac{1}{2\epsilon}\left(q'\left(\frac{z^\epsilon(t,x)}{\epsilon}\right)\right)^2\right]dx \le \mu^\epsilon(t,dx),$$

and

$$|q'(r)r| \le 4 \;\forall r \;\Rightarrow\; \left(q'\left(\frac{z^\epsilon}{\epsilon}\right)\right)^2 (z^\epsilon)^2 \le 4\epsilon^2.$$

Hence we have

$$\frac{d}{dt}\alpha^\epsilon(t;s,y) \le [\ln(1/\epsilon)t(s-t)]^{-1}\int \rho(t,x;s,y)[\mu^\epsilon(dx,t) + 2\epsilon dx].$$

Since $\int \rho(t,x;s,y)dx = \sqrt{4\pi(s-t)}$ and $t \ge \delta$,

$$\frac{d}{dt}\alpha^\epsilon(t;s,y) \le [\ln(1/\epsilon)\delta(s-t)]^{-1}[\alpha^\epsilon(t;s,y) + 4\epsilon\sqrt{\pi(s-t)}].$$

Finally an application of Gronwald's inequality yields (4). ☐

We are ready to prove a slight extension of the "clearing-out" lemma proved by Ilmanen [62]. Our proof follows very closely Section 6.1 in [62].

Theorem 5.1 (Ilmanen). *Assume* (3.1) *and* (6). *Let* μ *be the limit of* μ^n (*cf.* (2.8)). *Then for every* $\delta > 0$ *there exists* $\eta(\delta) > 0$ *such that if*

$$\int \rho(t,x;s_0,y_0)\mu(dx;t) \le \eta(\delta), \tag{5.5}$$

for some (s_0, y_0) *and* t *satisfying* $\delta \le t < s_0 \le t+1 < 1/\delta$, *then there is a neighborhood* O *of* (s_0, y_0) *such that*

$$z_*(s,y) > 0, \quad \forall (s,y) \in O, \quad or \quad z^*(s,y) < 0, \quad \forall (s,y) \in O, \tag{5.6}$$

and $u^n \to \pm 1$ *uniformly in this neighborhood. In particular, if* (5.5) *holds then*

$$(s_0, y_0) \in \mathcal{P} \cap \mathcal{N},$$

and

$$(s_0, y_0) \notin \overline{\{(t,x)\colon x \in \text{support } \mu(\cdot;t)\}}.$$

Proof. Assume $t \in [\delta, 1/\delta]$. All the constants in this proof depend on δ, but we suppress this dependence.

1. Suppose that

$$\alpha(t; s_0, y_0) = \int \rho(t, x; s_0, y_0)\mu(dx; t) \leq \eta,$$

for some η that will be chosen later in this proof. Set $\alpha^n = \alpha^{\epsilon_n}$. Then by the continuity of ρ, assumption (6) and the convergence of $\mu^n(\cdot; t)$ for every t (cf. (2.8)), there are an integer n_0 and a neighborhood U of (s_0, y_0) satisfying

$$\alpha^n(t; s, y) \leq 2\eta, \quad (s, y) \in U, n > n_0.$$

Here U and n_0 may depend on η and t, s_0, y_0.

2. Use (4) with $r = s - \epsilon_n^2$ to construct a constant k_1 (independent of η and n) and $n(\eta) \geq n_0$ satisfying

$$\alpha^n(s - \epsilon_n^2; s, y) \leq k_1\eta, \quad \forall (s, y) \in U, n > n(\eta).$$

Let $B_{\epsilon_n}(y)$ be the sphere centered at y with radius ϵ_n. Then, we have

$$\begin{aligned} \mu^n(B_{\epsilon_n}(y); s - \epsilon_n^2) &\leq [\min_{x \in B_{\epsilon_n}(y)} \{\rho(s - \epsilon_n^2, x; s, y)\}]^{-1}\alpha^n(s - \epsilon_n^2; s, y) \\ &\leq k_2\eta{\epsilon_n}^{d-1}, \quad \forall (s, y) \in U, n > n(\eta). \end{aligned} \tag{5.7}$$

Observe that the constant k_2 is independent of η.

Now define

$$\beta = \liminf_{n \to \infty} \inf_{(s,y) \in U} |u^n(s - \epsilon_n^2, y)|$$

Let c be any number sufficiently close to one, say (7/8). In the next step we will show that for a carefully chosen value of η, $\beta \geq c = (7/8)$.

3. Suppose that $\beta < (7/8)$. Then there are a subsequence n_k and $(s_k, y_k) \in U$ satisfying

$$\begin{aligned} &|u^{n_k}(s_k - \epsilon_{n_k}^2, y_k)| < (7/8) \\ &\quad \Rightarrow |z^{n_k}(s_k - \epsilon_{n_k}^2, y_k)| < \epsilon_{n_k} q^{-1}(7/8), \end{aligned}$$

for every k. Using (4.4) we conclude that there is k_0, independent of η, such that for $k \geq k_0$ we have

$$|z^{n_k}(s_k - \epsilon_{n_k}^2, x)| < \epsilon_{n_k}[q^{-1}(7/8) + 2], \quad \forall x \in B_{\epsilon_{n_k}}(y_k).$$

Consequently $W(u^{n_k}(s_k - \epsilon_{n_k}^2, x)) > W(q[q^{-1}(7/8) + 2])$, and

$$\mu^{n_k}(B_{\epsilon_{n_k}}(y_k); s - \epsilon_{n_k}^2) > \int_{B_{\epsilon_n}(y_k)} \frac{W(u^{n_k}(s_k - \epsilon_{n_k}^2, x))}{\epsilon_{n_k}} dx$$
$$> \omega_d W(q[q^{-1}(7/8) + 2])(\epsilon_{n_k})^{d-1}, \qquad (5.8)$$

where ω_d is the volume of the d dimensional unit sphere. Now choose

$$\eta = \omega_d W(q[q^{-1}(7/8) + 2])/k_2,$$

where k_2 is as in Step 2. With this choice of η, (8) contradicts (7) for sufficiently large k. Hence $\beta \geq (7/8)$.

4. Since $\beta \geq (7/8)$ and u^n is continuous, for a sufficiently large n we have either $u^n(s - \epsilon_n^2, y) \geq (3/4)$ for all $(s, y) \in U$ or $u^n(s - \epsilon_n^2, y) \leq -(3/4)$ for all $(s, y) \in U$. We also know that the sequence u^n is convergent (cf. (2.10)). Hence we conclude that, we have either $u^n(s - \epsilon_n^2, y) \geq (3/4)$ for all sufficiently large n and $(s, y) \in U$ or $u^n(s - \epsilon_n^2, y) \leq -(3/4)$ for all sufficiently large n and $(s, y) \in U$. Without loss of generality suppose that we have the first case. Then by a result of deMottoni and Schaztman [77] or Chen [27] (see Corollary 7.1, below), we conclude that u^n converges to one uniformly on U. Hence

$$U \subset \mathcal{P}_u = \mathcal{P},$$

and by (11),

$$z_*(s, y) > 0, \quad \forall (s, y) \in U.$$

The other conclusions of the theorem easily follows from (5.6). ☐

6. Conclusion

In this section we complete the proofs of Theorems 3.1, 3.2 and 3.3.

Proof of Theorem 3.1. Let $\mathcal{P}$ and $\mathcal{N}$ be as in (6). Then clearly they are disjoint.

1. Suppose that $(s_0, y_0) \notin \Gamma$ and $s_0 > 0$. Since $\rho(t, x; s_0, y_0)$ decays exponentially as $|x|$ tends to infinity, (2.8) holds with $\psi(x) = \rho(t, x; s_0, y_0)$ for every $t < s_0$. Moreover $\rho(t, x; s_0, y_0)$ tends to zero exponentially fast as t tends to s_0, for all $x \neq y_0$. Using these facts and (6), it is easy to show that (5.5) is satisfied at every t sufficiently close to s_0. Hence by Theorem 5.1, (5.6) holds and consequently

$$(s_0, y_0) \in \mathcal{P} \cup \mathcal{N}.$$

2. Suppose that $(s_0, y_0) \in \mathcal{P}$ and $s_0 > 0$. Then there are $\delta > 0, \alpha > 0$ and $n_0 > 0$ satisfying

$$z^n(s, y) > \alpha, \quad \forall |s - s_0|, |y - y_0| \leq \delta, \ n > n_0. \tag{6.1}$$

Definition of μ^n and the gradient estimate (4.3) imply that for sufficiently large n and $|s - s_0| \leq \delta$,

$$\begin{aligned} \mu^n(B_\delta(y_0); s) &= \int_{B_\delta(y_0)} \frac{1}{2\epsilon_n} \left(q'\left(\frac{z^n(t,x)}{\epsilon_n}\right)\right)^2 [|Dz^n(t,x)|^2 + 1]\,dx \\ &\leq \int_{B_\delta(y_0)} \frac{1}{2\epsilon_n} \left(q'\left(\frac{z^n(t,x)}{\epsilon_n}\right)\right)^2 [|z^n(t,x)|^2 + 3]\,dx. \end{aligned}$$

Let α be as in (6.1). Then for sufficiently small ϵ, the function

$$z \to \left(q'\left(\frac{z}{\epsilon}\right)\right)^2 [z^2 + 3]$$

is decreasing on $z > \alpha$. Therefore, for $|s - s_0| < \delta$,

$$\lim_{n\to\infty} \mu^n(B_\delta(y_0); s) \leq \lim_{n\to\infty} \int_{B_\delta(y_0)} \frac{1}{2\epsilon_n} \left(q'\left(\frac{\alpha}{\epsilon_n}\right)\right)^2 [\alpha^2 + 3]\,dx = 0.$$

Hence $(s_0, y_0) \notin \Gamma$.

3. Suppose that $(s_0, y_0) \in \mathcal{P}$ and $s_0 > 0$. Then the same argument as in the previous step yields that $(s_0, y_0) \notin \Gamma$. □

We need the following result in the proof of Theorem 3.3. Recall that $B_\delta(x)$ is the d dimensional sphere centered at x with radius δ.

Lemma 6.1. *Assume* (3.1) *and* (6).

a) *Suppose that* $z^*(t,x) > 0$ *at some* $(t,x) \in (0,\infty)\times\mathcal{R}^d$. *Then there is a positive constant* δ *satisfying*

$$z_*(s, y) \geq \delta, \quad \forall (s, y) \in (t, t + \delta] \times B_\delta(x).$$

b) *Suppose that* $z_*(t,x) < 0$ *at some* $(t,x) \in (0,\infty)\times\mathcal{R}^d$. *Then there is a positive constant* δ *satisfying*

$$z^*(s, y) \leq -\delta, \quad \forall (s, y) \in (t, t + \delta] \times B_\delta(x).$$

Proof. We will prove only part a. Proof of part b is similar.

1. Since $z^*(t,x) = \gamma > 0$, the definition z^* implies that there are a subsequence n_k and $(t_k, x_k) \to (s, x)$ satisfying

$$z^{n_k}(t_k, x_k) \geq \gamma/2.$$

Since $x_k \to x$, in view of (4.4) there is a neighborhood U of x such that for sufficiently large k,

$$z^{n_k}(t_k, y) \geq \gamma/4 \quad \forall y \in U.$$

Hence for sufficiently large k,

$$u^{n_k}(t_k, y) \geq q(\gamma/4\epsilon) \quad \forall y \in U.$$

2. By a result of deMottoni and Schatzman [77] and Chen [27] (see Corollary 7.1, below), there is $\delta > 0$ such that $u^{n_k} \to +1$ uniformly on every compact subset of $(t, t+\delta] \times B_{3\delta}(x)$. Since this convergence is only on a subsequence, we can not yet use (12). But by Remark 4.1 we have,

$$\liminf_{k\to\infty,\ (s',y')\to(s,y)} z^{n_k}(s', y') \geq \delta,$$

for all $(s, y) \in (t, t+\delta] \times B_{2\delta}(x)$.

3. Arguing as in Step 2 of the previous proof, for $s \in (t, t+\delta]$ we obtain

$$\lim_{n\to\infty} \mu^{n_k}(B_{2\delta}(x); s) \leq \lim_{n\to\infty} \int_{B_{2\delta}(x)} \frac{1}{2\epsilon_{n_k}} \left(q'\left(\frac{\delta}{\epsilon_{n_k}}\right)\right)^2 [\delta^2 + 3] dx = 0.$$

Recall that μ is the limit of μ^n (cf. (2.8)). Hence $\mu(B_{2\delta}(x); s) = 0$ for all $s \in (t, t+\delta]$ or equivalently

$$(t, t+\delta] \times B_{2\delta}(x) \cap \Gamma = \emptyset.$$

Then by Theorem 3.1,

$$(t, t+\delta] \times B_{2\delta}(x) \subset \mathcal{P} \cup \mathcal{N}.$$

Since u^n is convergent in L^1 (cf. (2.10)) and u^{n_k} converges to one uniformly in $(t, t+\delta] \times B_{2\delta}(x)$, we conclude that

$$(t, t+\delta] \times B_{2\delta}(x) \subset \mathcal{P}.$$

We complete the proof of part a by using (12) in $(t, t+\delta] \times B_\delta(x)$. ☐

We continue with the proof of Theorem 3.3. The above lemma essentially shows that the boundaries of the sets $\mathcal{P} = \{z_* > 0\}$ and $\mathcal{N} = \{z^* < 0\}$ in $(0, \infty)\times\mathcal{R}^d$ are equal. This observation and Proposition 4.2 will be used to complete the proof of Theorem 3.3.

Proof of Theorem 3.3.

1. Suppose that $z_*(t, x) > 0$ and $t < T_{ext}$. Then by Proposition 4.2, (12),

$$z_*(t, x) = \text{dist}(x, \mathcal{P}_t^c).$$

Moreover by (3.2) we have $\mathcal{P}_t^c = \Gamma_t \cup \mathcal{N}_t$. Hence $\text{dist}(x, \mathcal{P}_t^c) \leq d(t, x)$. Let $y \in \mathcal{P}_t^c$ be such that, $\text{dist}(x, \mathcal{P}_t^c) = |x - y|$. Since z_* is Lipschitz continuous in the x-variable, $z_*(t, y) = 0$. Recall that $y \in \mathcal{P}_t^c = \Gamma_t \cup \mathcal{N}_t$ and $z^*(t, x) \geq z_*(t, x) = 0$. Therefore $y \in \Gamma_t$ and consequently,

$$d(t, x) \leq |x - y| = \text{dist}(x, \mathcal{P}_t^c) \leq d(t, x)$$

$$\Rightarrow z_*(t, x) = d(t, x).$$

Also $(t, x) \in \mathcal{P}$ and therefore in a neighborhood of (t, x) we have $d = g$, where $g(t, x)$ is the distance between x and the t-section of the closed set $\Gamma \cup \mathcal{N}$. Then it is elementary to show that g is lower semi continuous and since $d = g$ in a neigborhood of (t, x) we have,

$$d_*(t, x) = g_*(t, x) = g(t, x) = d(t, x).$$

Hence we have proved (3.9) when $z_*(t, x) > 0$ and $t < T_{ext}$. Similarly we can show that

$$d^*(t, x) = d(t, x) = z^*(t, x),$$

provided that $z^*(t, x) < 0$ and $t < T_{ext}$.

2. Suppose that $z_*(t, x) = 0$ and $t < T_{ext}$. Since $0 = z_*(t, x) \leq z^*(t, x)$, (t, x) is not in $\mathcal{P} \cup \mathcal{N}$. Therefore by (3.2) $x \in \Gamma_t$ and $d(t, x) = 0$. We now claim that $d(t, x) = d_*(t, x)$. Indeed if there is a sequence $(t_k, x_k) \to (t, x)$ satisfying

$$\liminf_{k\to\infty} d(t_k, x_k) < 0.$$

Then by the previous step and (11), $z^*(t_k, x_k) = d(t_k, x_k)$ for all k. Hence

$$z_*(t, x) \leq \liminf_{k\to\infty} z^*(t_k, x_k) = \liminf_{k\to\infty} d(t_k, x_k) < 0.$$

But this contradicts with the hypothesses of this step; $z_*(t, x) = 0$. Therefore $d(t, x) = d_*(t, x) = 0$, whenever $z_*(t, x) = 0$. Combining with the first step we conclude that

$$z_*(t, x) = d(t, x) = d_*(t, x), \quad \forall (t, x) \text{ satisfying } z_*(t, x) \geq 0. \tag{6.2}$$

3. We proceed as in the previous step to obtain

$$z^*(t,x) = d(t,x) = d^*(t,x), \quad \forall (t,x) \text{ satisfying } z^*(t,x) \le 0. \tag{6.3}$$

4. Suppose that $z_*(t,x) < 0$ and $t < T_{ext}$. Set

$$\hat{d}(t,x) = \liminf_{(s,y)\to(t,x),\ s>t} d(s,y), \quad \gamma = \operatorname{dist}(x, \hat{\mathcal{P}}_t),$$

where $\hat{\mathcal{P}}$ is as in (6). For $\epsilon > 0$, we use Lemma 6.1b together with a compactness argument to construct $\delta > 0$ satisfying

$$(t, t+\delta] \times B_{\gamma-\epsilon}(x) \subset \mathcal{N}. \tag{6.4}$$

Hence,

$$\liminf_{(s,y)\to(t,x),\ s>t} \operatorname{dist}(y, \Gamma_s) \ge \gamma,$$

and therefore

$$\operatorname{dist}(x, \hat{\mathcal{P}}_t) = \gamma \le \liminf_{(s,y)\to(t,x),\ s>t} \operatorname{dist}(y, \Gamma_s) = -\hat{d}(t,x).$$

Since by (12) $z_*(t,x) \ge -\gamma$,

$$z_*(t,x) \ge \hat{d}(t,x). \tag{6.5}$$

We also have

$$z_*(t,x) \le (z^*)_*(t,x) \le \liminf_{(s,y)\to(t,x),\ s>t} z^*(s,y).$$

Step 3 and (6.4) imply that for every s sufficiently close to t and $s > t$ we have, $z^*(s,x) = d(s,x)$. We use this in the above inequality and then recall (6.5) to obtain

$$z_*(t,x) = \hat{d}(t,x), \quad \forall (t,x) \text{ satisfying } z_*(t,x) < 0.$$

5. In this step we will show that $\hat{d}(t,x)$ defined in the previous step is equal to $d_*(t,x)$ whenever $z_*(t,x) < 0$. Indeed by construction $\hat{d} \ge d_*$. We already know that $\hat{d}(t,x) = z_*(t,x) < 0$. Set $\varrho = d_*(t,x)$. Then $\varrho < 0$. By the definition of d_* and the Lipschitz continuity of d, there is a sequence $t_k \to t$ satisfying, $d(t_k,x) \to \varrho$. Since $d(t_k,x) < 0$ for sufficiently large k, $(t_k,x) \in \mathcal{N}$ and step 3 implies that $z^*(t_k,x) = d(t_k,x)$. In summary,

$$\begin{aligned} z_*(t,x) &= \hat{d}(t,x) \ge d_*(t,x) = \lim_{k\to\infty} d(t_k,x) \\ &= \lim_{k\to\infty} z^*(t_k,x) \ge (z^*)_*(t,x) \ge z_*(t,x). \end{aligned}$$

Hence $\hat{d}(t,x) = d_*(t,x)$ and this step together with steps 2 and 4 complete the proof of (3.9) on $t < T_{ext}$.

6. Suppose that $z^*(t,x) > 0$ and $t < T_{ext}$. We proceed as in step 4 and then step 5 to prove that

$$z^*(t,x) = \limsup_{(s,y)\to(t,x),\ s>t} d(t,x) = d^*(t,x).$$

This identity and step 3 complete the proof of (3.8) on $t < T_{ext}$.

7. An application of Lemma 6.1 and (4.4) yield that on $(T_{ext}, \infty) \times \mathcal{R}^d$ we have either $z_* = z^* = -\infty$ or $z_* = z^* = +\infty$. Therefore

$$(T_{ext}, \infty) \times \mathcal{R}^d \cap \Gamma = \emptyset. \tag{6.6}$$

Suppose that $\mathcal{P}_t$ is empty for some t. Then (3.8) implies that if $t < T_{ext}$ then $z^*(t,x) = +\infty$ for every x. Therefore $t \geq T_{ext}$. A similar argument shows that if $\mathcal{N}_t$ is empty, then $t \geq T_{ext}$. Hence both $\mathcal{P}_t$ and $\mathcal{N}_t$ are nonempty for all $t < T_{ext}$. So by (3.2) Γ_t is nonempty for all $t < T_{ext}$. This and (6.6) imply that $T_{ext} = t_{ext}$ (recall that t_{ext} is defined in Section 3).

Combinning all the steps we conclude that (3.8) and (3.9) hold with $t_{ext} = T_{ext}$. ☐

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. The $\epsilon \to 0$ limit of (4), assumption (6), the "clearing-out," Theorem 5.1, and Ilmanen's argument in Section 6.3 in [62] yields a local upper bound on the d-dimensional Hausdorff measure of $\Gamma \cap U$ for any compact subset U of $(0,\infty)\times\mathcal{R}^d$. The second assertion of the theorem is a consequence of Theorem 3.3, equation (4.1), gradient estimate(4.3) and Section 10.2 in [13]. Since the notation and the assumptions used in [13] are quite different than the ones used in this paper, we will now briefly sketch this argument.

1. In this step we will show that

$$z_{*t} - \Delta z_* \geq 0,$$

on $\mathcal{P} \cap (0, T_{ext}) \times \mathcal{R}^d$. In fact when $|Dz^\epsilon| \leq 1$, this inequality follows immediately from (4.1). In general we will prove the above inequality by using a transformation, the gradient estimate (4.3) and (3.9).

For a small positive constant β, let η be a smooth, increasing function satisfying, $\eta(0) = 0$, $0 < \eta'' < \beta$, $\eta' > 1$ in $(0,\infty)$, $\eta' < 1$ in $(-\infty, 0)$, and $\eta \geq -\beta$. Following [13], we define

$$\hat{z}^\epsilon(t,x) = \inf_{\mathcal{R}^d}(\eta(z^\epsilon(t,y)) + |x-y|). \tag{6.7}$$

This transformation is very similar to the inf-convolution used extensively in the theory of viscosity solutions [67],[71]. We refer the reader to [33] and Chapter 5 in [47] , for the properties of the inf-convolution. Since η is bounded from below by β, the above minimum is achieved, say at $y(x)$ ($y(x)$ also depends on t and ϵ, but we suppress this dependence in our notation.) By calculus, (see [13] for details), we obtain

$$q(z^\epsilon(t, y(x))/\epsilon)[1 - |Dz^\epsilon(t, y(x))|^2] \geq 0, \quad \text{on } \{\hat{z}^\epsilon > 0\}.$$

Now by using equation (4.1) and the properties of viscosity solutions and the inf-convolution we obtain

$$\begin{aligned} &\hat{z}^\epsilon_t(t, x) - \Delta\hat{z}^\epsilon(t, x) + \eta''(z^\epsilon(t, y(x)))|Dz^\epsilon(t, y(x))|^2 \\ &\quad \geq \eta'(z^\epsilon(t, y(x)))[z^\epsilon_t(t, y(x)) - \Delta z^\epsilon(t, y(x))] \\ &\quad = \frac{2}{\epsilon}\eta'(z^\epsilon(t, y(x)))q(z^\epsilon(t, y(x))/\epsilon)[1 - |Dz^\epsilon(t, y(x))|^2], \end{aligned}$$

in the viscosity sense. Since $\eta'' < \beta$, we have

$$\hat{z}^\epsilon_t(t, x) - \Delta\hat{z}^\epsilon(t, x) + \beta|Dz^\epsilon(t, y(x))|^2 \geq 0, \quad \text{on } \{\hat{z}^\epsilon > 0\}. \tag{6.8}$$

Let

$$\hat{z}_*(t, x) = \liminf_{(s,y)\to(t,x),\ n\to\infty} \hat{z}^{\epsilon_n}(s, y).$$

Then the properties of η, (6.7) and (3.9) imply that $\hat{z}_* = z_*$ on $\mathcal{P} \cap (0, T_{ext})$. Also by carefully letting ϵ go to zero in (6.8) we obtain

$$z_{*t}(t, x) - \Delta z_*(t, x) + \beta|Dz_*(t, x)|^2 \geq 0,$$

on $\mathcal{P} \cap (0, T_{ext}) \times \mathcal{R}^d$. Now let β go to zero and use the gradient estimate (4.3) to conclude that on $\mathcal{P} \cap (0, T_{ext}) \times \mathcal{R}^d$,

$$z_{*t}(t, x) - \Delta z_*(t, x) \geq 0. \tag{6.9}$$

2. Lemma 2 in [13] and (6.9) imply (3.7). The intuitive idea behind this is simple. First we observe that since z_* is a distance function, $|Dz_*|^2 = 1$ in the viscosity sense. We then formally differentiate this identity to obtain $D^2z_*Dz_* = 0$. Hence formally $\Delta z_* = F(Dz_*, D^2z_*)$, where F is as in the statement of Theorem 3.2. This identity and (6.9) imply

$$z_{*t}(t, x) - F(Dz_*(t, x), D^2z_*(t, x)) \geq 0,$$

on $\mathcal{P} \cap (0, T_{ext}) \times \mathcal{R}^d$. See [13] for the details of this argument. □

Remark 6.1. Following the ideas of Ilmanen, it may be possible to prove that Γ is a solution of the mean curvature flow in the sense of Brakke [17]. Since Brakke's solutions satisfy the distance function property of [13, 92], this would be a stronger result than Theorem 3.2. Indeed this result was proved by Ilmanen under the additional assumption that $u_0 = q(z_0/\epsilon)$ and $|Dz_0| \leq 1$ [62]. □

7. Appendix: A result of Chen and deMottoni and Schatzman

In this section we state a result of deMottoni and Schatzman and Chen. Then we prove a corollary which was used in earlier sections.

The following is a special case of a result proved by deMottoni and Schatzman [77] and Chen [27]. The particular result stated below follows immediately from Theorem 3 in [27].

Theorem 7.1 (Chen and deMottoni-Schatzman). *Let u^ϵ be a solution of* (2.2) *and* (2.3). *Suppose that there are $t_0 > 0$ and a one parameter family of bounded closed subsets Ω_t of $\mathcal{R}^d$ satisfying*

(a) *boundary of Ω_t is a classical solution of the mean curvature flow on $(0, t_0)$,*

(b) *signed distance function $d(t, x)$ is three times continuously differentiable on $[0, t_0] \times \mathcal{R}^d$, (where $d(t, x)$ is the signed distance between x and the boundary of Ω_t),*

(c) *$u_0^\epsilon > 0$ in the interior of Ω_0 and $u_0^\epsilon < 0$ in the complement of Ω_0, and there are positive constants C, h, independent of ϵ, such that*

$$|u_0^\epsilon(x)| \geq C|d(0, x)|, \quad |Du_0^\epsilon(x)| \geq C, \quad \forall |d(0, x)| < h.$$

Then u^ϵ converges to $+1$ uniformly on compact subsets of $\{(t, x)\colon\ t \in (0, t_0), x \in int(\Omega_t)\}$, and u^ϵ converges to -1 uniformly on compact subsets of $\{(t, x)\colon\ t \in (0, t_0), x \notin \Omega_t\}$.

Corollary 7.1. *Suppose that there are subsequences $\epsilon_n \to 0$ and $t_n \to t_0$ and an open set O of $\mathcal{R}^d$ satisfying*

$$\liminf_{n\to\infty} \inf_{x\in O} u^{\epsilon_n}(t_n, x) = \alpha > 0,$$

or

$$\limsup_{n\to\infty} \sup_{x\in O} u^{\epsilon_n}(t_n, x) = -\alpha < 0.$$

Then for every $x \in O$ there is $\delta > 0$ such that u^{ϵ_n} converges to $+1$ or -1, uniformly on every compact subset of $(t, t + \delta] \times B_\delta(x)$.

Proof. Without loss of generality we may assume that $u^{\epsilon_n}(t_n, x) > 0$ on O. Then we have

$$u^{\epsilon_n}(t_n, x) = \alpha/2 > 0, \quad \forall x \in O, \tag{7.1}$$

and sufficently large n. Fix $x_0 \in O$ and let v^ϵ be the solution of (2.2) with initial data

$$v^\epsilon(0, x) = \frac{\alpha}{2} - \frac{|x - x_0|^2}{\delta_0}\left[1 + \frac{\alpha}{2}\right],$$

where

$$\delta_0 = [\inf\{|y - x_0|: y \notin O\}]^2.$$

Then for every $x \notin O$, $|x - x_0| \geq \delta_0$, and

$$v^\epsilon(0, x) \leq \frac{\alpha}{2} - \left[1 + \frac{\alpha}{2}\right] \leq -1, \quad \forall x \notin O.$$

Also for every x, $v^\epsilon(0, x) \leq \frac{\alpha}{2}$. Therefore by (7.1),

$$v^{\epsilon_n}(0, x) \leq u^{\epsilon_n}(t_n, x), \quad \forall x \in \mathcal{R}^d, \tag{7.2}$$

and by maximum principle

$$v^{\epsilon_n}(t, x) \leq u^{\epsilon_n}(t_n + t, x), \quad \forall (t, x) \in [0, \infty)\times\mathcal{R}^d.$$

Observe that the zero level set of $v^\epsilon(0, \cdot)$ is the boundary of a ball centered at x_0 with radius

$$r_0 = \left[\frac{\alpha\delta_0}{2 + \alpha}\right]^{1/2}.$$

Hence the previous theorem holds with

$$\Omega_t = \{x \in \mathcal{R}^d: |x - x_0| \leq \sqrt{(r_0)^2 - 2(d - 1)t}\}.$$

Now the conclusion of the corollary follows from (7.2) and Theorem 7.1. □

8. Appendix: A gradient estimate.

In this section we prove the gradient estimate (4.3). As in Section 4, we define

$$w^\epsilon = |Dz^\epsilon(t, x)|^2.$$

Recall that

$$w_t^\epsilon + \mathcal{L}_t^\epsilon w^\epsilon + R^\epsilon(t, x, w^n) \leq 0, \quad \text{in } (0, \infty)\times\mathcal{R}^d, \tag{8.1}$$

where for a real number r, $(t, x) \in [0, \infty)\times\mathcal{R}^d$ and a smooth function $\varphi \in C^2(\mathcal{R}^d)$,

$$\begin{aligned}\mathcal{L}_t^\epsilon\varphi(x) &= -\Delta\varphi(x) + \frac{4u^\epsilon(t,x)}{\epsilon}Dz^\epsilon(t,x)\cdot D\varphi(x),\\ R^\epsilon(t,x,r) &= \frac{4}{\epsilon^2}q'\left(\frac{z^\epsilon(t,x)}{\epsilon}\right)r(r-1),\end{aligned}$$

and as before $q(r) = \tanh(r)$. Set

$$W(t,x) = 1 + \frac{2}{\ln(1/\epsilon)}\frac{(z^\epsilon(t,x))^2+1}{t}.$$

Recall that the gradient estimate (4.3) states that for sufficiently small ϵ, $w^\epsilon \leq W$. We will first show that W is a supersolution of an equation very similar to (8.1). Then we complete the proof of (4.3) by an application of the maximum principle.

Lemma 8.1. *There are a constant* ϵ_0 *and a function* $\lambda^\epsilon(t)$ *such that for every* $\epsilon \leq \epsilon_0$,

$$W_t + \mathcal{L}_t^\epsilon W + R^\epsilon(t,x,W) \geq \frac{\lambda^\epsilon(t)}{t}[W - w^\epsilon] \quad \text{in } (0,\infty)\times\mathcal{R}^d. \tag{8.2}$$

Moreover $\lambda^\epsilon(t)$ *is uniformly bounded in* ϵ, *i.e.*,

$$\lambda_0 = \sup\{|\lambda^\epsilon(t)|:\ t \geq,\ \epsilon \leq \epsilon_0\} < \infty. \tag{8.3}$$

Proof. Set

$$\psi = (z^\epsilon)^2 + 1, \quad K(\epsilon) = 2[\ln(1/\epsilon)]^{-1}.$$

1. We directly calculate that

$$W_t + \mathcal{L}_t^\epsilon W = -\frac{K(\epsilon)\psi}{t^2} + \frac{K(\epsilon)}{t}\left[\frac{4u^\epsilon}{\epsilon}z^\epsilon(w^\epsilon+1) - 2w^\epsilon\right],$$

and

$$W_t + \mathcal{L}_t^\epsilon W + R^\epsilon(t,x,W) = I + J, \tag{8.4}$$

where

$$\begin{aligned} I &= \frac{1}{t}\left[-\frac{K(\epsilon)\psi}{t} + \frac{2u^\epsilon z^\epsilon K(\epsilon)}{\epsilon}(w^\epsilon + 1)\right] + \frac{1}{2}R^\epsilon(t,x,W), \\ J &= \frac{K(\epsilon)}{t}\left[\frac{2u^\epsilon z^\epsilon}{\epsilon}(w^\epsilon + 1) - 2w^\epsilon\right] + \frac{1}{2}R^\epsilon(t,x,W). \end{aligned}$$

In the following steps we will estimate I and J separately.

2. We split the estimate of I into two cases and start with the case

$$K(\epsilon)|z^\epsilon| \geq \epsilon. \tag{8.5}$$

(The other case will be analyzed in the next step.) The above inequality yields

$$|u^\epsilon| = q\left(\frac{|z^\epsilon|}{\epsilon}\right) \geq q\left(\frac{1}{K(\epsilon)}\right) = \frac{1-\epsilon}{1+\epsilon} \geq \frac{1}{2},$$

provided that $\epsilon \leq 1/3$. Since $u^\epsilon z^\epsilon = |u^\epsilon||z^\epsilon|$,

$$2u^\epsilon z^\epsilon K(\epsilon)/\epsilon = 2|u^\epsilon||z^\epsilon|K(\epsilon)/\epsilon \geq 1.$$

Using the above estimate and the positivity of $R^\epsilon(t,x,W)$ we obtain

$$I \geq \frac{1}{t}\left[-\frac{K(\epsilon)\psi}{t} + (w^\epsilon + 1)\right] \geq -\frac{1}{t}[W - w^\epsilon].$$

3. Suppose that (8.5) does not hold, i.e.,

$$K(\epsilon)|z^\epsilon| < \epsilon \;\Rightarrow q'\left(\frac{z^\epsilon}{\epsilon}\right) \geq q'\left(\frac{1}{K(\epsilon)}\right).$$

Since $q'(r) = 4e^{2r}(e^{2r}+1)^{-2}$,

$$q'\left(\frac{z^\epsilon}{\epsilon}\right) \geq q'\left(\frac{1}{K(\epsilon)}\right) = 4\epsilon(\epsilon+1)^{-2} \geq \epsilon,$$

provided that $\epsilon \leq 1$. We use this inequality in the definition of R^ϵ to obtain

$$\begin{aligned} R^\epsilon(t,x,W) &= \frac{4}{\epsilon^2}q'\left(\frac{z^\epsilon}{\epsilon}\right)(W-1)W, \\ &\geq \frac{4}{\epsilon}\frac{K(\epsilon)\psi}{t}\left[\frac{K(\epsilon)\psi}{t} + 1\right], \\ &\geq \frac{4}{\epsilon}\frac{(K(\epsilon))^2\psi}{t^2}. \end{aligned}$$

In the third inequality we used the fact that $\psi \geq 1$. Since $2K(\epsilon) \geq \epsilon$ for all $\epsilon \leq 1$ and the product $u^\epsilon z^\epsilon$ is always positive, the above inequality yields

$$\begin{aligned} I &\geq -\frac{K(\epsilon)\psi}{t^2} + \frac{1}{2}R^\epsilon(t, x, W), \\ &\geq \frac{K(\epsilon)\psi}{t^2}\left[-1 + \frac{2K(\epsilon)}{\epsilon}\right] \geq 0. \end{aligned}$$

4. Combining the two previous steps we conclude that for all $\epsilon \leq 1/3$,

$$I \geq -\frac{1}{t}\mathbf{1}_A[W - w^\epsilon],$$

where $\mathbf{1}_A$ is the characteristic function of the set A and

$$A = \{(t, x)\colon\ K(\epsilon)|z^\epsilon(t, x)| \geq \epsilon\}.$$

5. We continue with an estimate of J. First suppose that

$$|z^\epsilon| \geq 2\epsilon. \tag{8.6}$$

(The other case will be analyzed in the next step.) Then we have

$$\frac{2}{\epsilon}u^\epsilon z^\epsilon = 2q\left(\frac{|z^\epsilon|}{\epsilon}\right)\frac{|z^\epsilon|}{\epsilon} \geq 4q(2) \geq 2.$$

Hence,

$$J \geq \frac{K(\epsilon)}{t}[2(w^\epsilon + 1) - 2w^\epsilon] \geq 0.$$

6. Now suppose that (8.6) does not hold, i.e., $|z^\epsilon| < 2\epsilon$. Then,

$$R^\epsilon(t, x, W) = \frac{4}{\epsilon^2}q'\left(\frac{z^\epsilon(t, x)}{\epsilon}\right)(W - 1)W \geq \frac{4}{\epsilon^2}q'(2)\left[\frac{K(\epsilon)\psi}{t}\right]W.$$

Recall that $u^\epsilon z^\epsilon \geq 0$. Therefore for $\epsilon \leq \sqrt{q'(2)}$,

$$\begin{aligned} J &\geq -\frac{2K(\epsilon)w^\epsilon}{t} + \frac{1}{2}R^\epsilon(t, x, W), \\ &\geq \frac{2K(\epsilon)}{t}\left[-w^\epsilon + \frac{q'(2)W}{\epsilon^2}\right], \\ &\geq \frac{2K(\epsilon)}{t}[W - w^\epsilon]. \end{aligned}$$

In the second inequality, we used the fact that $\psi \geq 1$.

7. Combining the two previous steps we conclude that for every $\epsilon \leq \sqrt{q'(2)}$,

$$J \geq -\frac{2K(\epsilon)}{t}\mathbf{1}_B[W - w^\epsilon],$$

where

$$B = \{(t,x) \colon |z^\epsilon(t,x)| \leq 2\epsilon\}.$$

Now (8.2) follows from this step, Step 4 and (8.4) with $\epsilon_0 = \sqrt{q'(2)}$ and

$$\lambda^\epsilon(t,x) = -\mathbf{1}_A + 2K(\epsilon)\mathbf{1}_B.$$

Finally, we prove (8.3) after observing that $K(\epsilon) \to 0$ as $\epsilon \to 0$. ☐

Proof of Proposition 4.1. Fix $\epsilon \leq \epsilon_0$ and $T > 0$. We first assume that there exists a constant $C(\epsilon) \geq 1$ satisfying

$$w^\epsilon(0,x) \leq C(\epsilon), \ \forall x \in \mathcal{R}^d. \tag{8.7}$$

We will remove this restriction in Step 6, below. Since $C(\epsilon) \geq 1$, $w \equiv C(\epsilon)$ is a supersolution of (8.1). Therefore by (8.7) and the maximum principle we have

$$w^\epsilon(t,x) \leq C(\epsilon), \ \forall (t,x) \in [0,\infty)\times\mathcal{R}^d. \tag{8.8}$$

To prove (4.3) we first assume that

$$\inf_{[0,T]\times\mathcal{R}^d} [W - w^\epsilon] < 0,$$

and then obtain a contradiction in the next five steps.

1. Set

$$\varrho = \frac{4}{\epsilon^2}\|q'\|_\infty + \frac{\lambda_0 C(\epsilon)}{K(\epsilon)} + 1,$$

where λ_0 and $K(\epsilon)$ are as in the previous lemma. This technical choice of ϱ will be clear in the next several steps. Then we have

$$\inf\{e^{-\varrho t}[W(t,x) - w^\epsilon(t,x)] \colon (t,x) \in [0,T] \times \mathcal{R}^d\} = -2b < 0.$$

For $\delta \in (0,b]$, choose $(t_\delta, x_\delta) \in [0,T] \times \mathcal{R}^d$ such that

$$e^{-\varrho t_\delta}(W - w^\epsilon)(t_\delta, x_\delta) < -2b + \delta.$$

2. Let η be a smooth function satisfying

$$0 \le \eta \le 1, \ \ \eta(0) = 1, \ \ \eta(x) = 0, \forall |x| \ge 1.$$

Consider the auxiliary function,

$$\Phi(t, x) = e^{-\varrho t}[W(t, x) - w^\epsilon(t, x)] - \delta\eta(x - x_\delta).$$

Then Φ achieves its maximum over $[0, T] \times \mathcal{R}^d$, say at $(\hat{t}, \hat{x})$. Moreover,

$$\Phi(\hat{t}, \hat{x}) \le \Phi(t_\delta, x_\delta) = e^{-\varrho t_\delta}(W - w^\epsilon)(t_\delta, x_\delta) \le -2b + \delta. \tag{8.9}$$

Since $W(t, x) \ge K(\epsilon)/t$, (8.8) implies that for all $\delta \le \min\{b, 1\}$,

$$0 \ge b + \Phi(\hat{t}, \hat{x}) \ge e^{-\varrho\hat{t}}(W - w^\epsilon)(\hat{t}, \hat{x}) \ge e^{-\varrho\hat{t}}\left(\frac{K(\epsilon)}{\hat{t}} - C(\epsilon)\right).$$

Therefore,

$$\hat{t} \ge K(\epsilon)/C(\epsilon). \tag{8.10}$$

3. Set

$$\bar{w}(t, x) = e^{-\varrho t}w^\epsilon(t, x), \quad \bar{W}(t, x) = e^{-\varrho t}W(t, x).$$

Then

$$\begin{aligned} \bar{w}_t - \mathcal{L}_t^\epsilon\bar{w} + \bar{R}(t, x, \bar{w}) &\le 0, & (8.11)\\ \bar{W}_t - \mathcal{L}_t^\epsilon\bar{W} + \bar{R}(t, x, \bar{W}) &\ge \frac{\lambda^\epsilon}{t}[\bar{w} - \bar{W}], & (8.12) \end{aligned}$$

where for $r > 0$, $(t, x) \in [0, \infty)\times\mathcal{R}^d$,

$$\bar{R}(t, x, r) = \varrho r + e^{-\varrho t}R^\epsilon(t, x, e^{\varrho t}r).$$

Using the definitions of R^ϵ and ϱ, we directly estimate that

$$\begin{aligned} \bar{R}_r(t, x, r) &= \varrho + R_r^\epsilon(t, x, e^{\varrho t}r),\\ &\ge \varrho - \frac{4}{\epsilon^2}q'\left(\frac{z^\epsilon(t, x)}{\epsilon}\right),\\ &\ge 1 + \frac{\lambda_0 C(\epsilon)}{K(\epsilon)}. \end{aligned} \tag{8.13}$$

4. Let

$$\varphi = \bar{W} - \bar{w}.$$

Subtract (11) from (12) to obtain

$$\varphi_t - \mathcal{L}^\epsilon_t \varphi \geq \bar{R}(t, x, \bar{w}) - \bar{R}(t, x, \bar{W}) - \frac{\lambda^\epsilon}{t}\varphi.$$

Since $\varphi(\hat{t}, \hat{x}) \leq 0$, (13) and (8.10) yield

$$\bar{R}(\hat{t}, \hat{x}, \bar{w}) - \bar{R}(\hat{t}, \hat{x}, \bar{W}) - \frac{\lambda^\epsilon(\hat{t})}{\hat{t}}\varphi \geq -\varphi.$$

Hence at $(\hat{t}, \hat{x})$ we have

$$\varphi_t - \mathcal{L}^\epsilon_t \varphi \geq -\varphi. \tag{8.14}$$

5. Recall that the auxiliary function $\Phi = \varphi - \delta\eta$ defined in step 2 attains its maximum at $(\hat{t}, \hat{x}) \in (0, T] \times \mathcal{R}^d$. Therefore at $(\hat{t}, \hat{x})$,

$$\Phi_t - \mathcal{L}^\epsilon_t \Phi \leq 0.$$

Now we use (8.14) and (8.9) to obtain

$$2b - \delta[1 + \eta(\hat{x} - x_\delta)] \leq -\varphi(\hat{t}, \hat{x}) \leq \varphi_t - \mathcal{L}^\epsilon_t \varphi = \Phi_t - \mathcal{L}^\epsilon_t \Phi - \delta \mathcal{L}^\epsilon_t \eta \leq \delta L(\epsilon),$$

where $L(\epsilon)$ is a constant depending on the function η and the operator $\mathcal{L}^\epsilon$. Since this constant is independent of δ, we obtain a contradiction by letting δ go to zero in the above string of inequalities. Hence $W \geq w^\epsilon$ on $[0, T] \times \mathcal{R}^d$ for every $T > 0$. So we have completed the proof of (4.3) under the additional assumption (8.7).

6. In this step we remove the restriction (8.7). First observe that

$$|u^\epsilon_t(t, x) - \Delta u^\epsilon(t, x)| \leq \frac{2}{\epsilon^2}, \quad |u^\epsilon(0, x)| \leq 1.$$

By well known properties of the heat kernel, we have

$$|Du^\epsilon(t, x)| \leq C_d \sqrt{t}\left[\frac{2}{\epsilon^2} + \frac{1}{t}\right],$$

where C_d is an appropriate constant, depending only on the dimension d. Fix ϵ, and for a positive integer k, let u^k be the solution of (2.2) with initial data,

$$u^k(0, x) = \min\{\max\{u^\epsilon(k^{-1}, x), 1 - k^{-1}\}, -1 + k^{-1}\}.$$

Define z^k by

$$u^k = q\left(\frac{z^k}{\epsilon}\right).$$

Clearly as k tends to infinity, u^k converges to u^ϵ in $C^m_{loc}((0,\infty)\times\mathcal{R}^d)$ for any m. Moreover,

$$|Dz^k(0,x)| = \epsilon|Du^k(0,x)|\left[q'\left(\frac{z^k(0,x)}{\epsilon}\right)\right]^{-1} \le \frac{\epsilon C_d}{\sqrt{k}}\left[\frac{2}{\epsilon^2}+k\right]\left[q'\left(\frac{1-k^{-1}}{\epsilon}\right)\right]^{-1}.$$

Therefore by the previous steps, $|Dz^k|^2 \le W$. Now we let k to go to infinity to complete the proof of (4.3). □

The following remark was pointed out to us by Ilmanen.

Remark 8.1. Observe that if $u^\epsilon(t,x)$ is a solution (1.1), then for any $\lambda > 0$,

$$v^\lambda(t,x) = u^\epsilon(\lambda^2 t, \lambda x),$$

is again a solution of (1.1) with ϵ replaced with ϵ/λ. Then by the gradient estimate (4.3),

$$\hat{z}^\lambda(t,x) = \frac{\epsilon}{\lambda}q^{-1}(v^\lambda(t,x)) = \frac{z^\epsilon(\lambda^2 t, \lambda x)}{\lambda},$$

satisfies

$$|D\hat{z}^\lambda(t,x)|^2 \le 1 + \frac{1}{2\ln(\lambda/\epsilon)}\frac{(\hat{z}^\lambda(t,x))^2+1}{t}.$$

The above estimate holds for all λ satisfying, $\epsilon/\lambda \le \epsilon_0$, where ϵ_0 is as in the statement of Propsition 4.1. Since

$$z^\epsilon(t,x) = \lambda\hat{z}^\lambda\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right),$$

the above estimate yields that,

$$|Dz^\epsilon(t,x)|^2 \le 1 + \frac{1}{2\ln(\lambda/\epsilon)}\frac{(z^\epsilon(t,x))^2+\lambda^2}{t}.$$

Now by minimizing the right-hand side over $\lambda \ge \epsilon/\epsilon_0$, we obtain a scale-invariant version of the estimate (4.3). □

References

[1] Allen, S., and Cahn, J. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metall.* **27**, 1084–1095 (1979).

[2] Alikakos, N. D., Bates, P. W., and Chen, X. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Arch. Rat. Mech. Anal.* **128**, 165–205 (1994).

[3] Alikakos, N. D., Bates, P. W., and Fusco, G. Slow motion for the Cahn-Hilliard equation in one space dimension. *J. Diff. Equations* **90**, 81–135 (1991).

[4] Almgren, F., and Taylor, J. E. Flat flow is motion by crystalline curvature for curves with crystalline energies. *J. Differential Geom.* **42**, 1–22 (1995).

[5] Almgren, F., Taylor, J. E., and Wang, L. Curvature driven flows: a variational approach. *SIAM J. Cont. and Opt.* **31**, 387–438 (March 1993). Issue dedicated to W. H. Fleming.

[6] Almgren, F., and Wang, L. Mathematical existence of crystal growth with Gibbs-thompson curvature effects. Forthcoming.

[7] Angenent, S. Parabolic equations for curves on surfaces. I. Curves with p-integrable curvature. *Ann. Math.* **132**, 451-483 (1990).

[8] Angenent, S. Parabolic equations for curves on surfaces. II. Intersections, blow-up and generalized solutions. *Ann. Math.* **133**, 171–217 (1991).

[9] Barles, G. Remark on a flame propogation model. Rapport INRIA # 464 (1985).

[10] Barles, G., Bronsard, L., and Souganidis, P. E. Front propogation for reaction-diffusion equations of bi-stable type. *Ann. I.H.P. Anal. Nonlin.* **9**, 479–496 (1992).

[11] Barles, G., and Georgelin, C. A simple proof of convergence for an approximation scheme for computing motions by mean curvature. *SIAM J. Num. Anal.* **32**, 484–500 (1995).

[12] Barles, G., and Perthame, B. Discontinuous solutions of deterministic optimal stopping problems. *Math. Modelling Numerical Analysis* **21**, 557–579 (1987).

[13] Barles, G., Soner, H. M., and Souganidis, P. E. Front propagation and phase field theory. *SIAM. J. Cont. Opt.* (March 1993). Issue dedicated to W. H. Fleming.

[14] Bence, J., Merriman, B., and Osher, S. Diffusion generated motion by mean curvature. Preprint (1992).

[15] Blowey, J. F., and Elliot, C. M. Curvature dependent phase boundary motion and parabolic double obstacle problems. *IMA* **47**, 19–60. Springer, New York, 1993.

[16] Bonaventura, L. Motion by curvature in an interacting spin system. Preprint (1992).

[17] Brakke, K.A. *The Motion of a Surface by Its Mean Curvature.* Princeton University Press, Princeton, NJ, 1978.

[18] Bronsard, L., and Kohn, R. On the slowness of the phase boundary motion in one space dimension. *Comm. Pure Appl. Math.* **43**, 983–998 (1990).

[19] Bronsard, L., and Kohn, R. Motion by mean curvature as the singular limit of Ginzburg-Landau model. *Jour. Diff. Equations* **90**, 211–237 (1991).

[20] Bronsard, L., and Reitich, F. On the three-phase boundary motion and the singular limit of a vector-valued Ginzburgh-Landau equation. *Arch. Rat. Mech. An.* **124**, 355–379 (1993).

[21] Caginalp, G. Surface tension and supercooling in solidification theory. *Lecture Notes in Physics* **216**, 216–226 (1984).

[22] Caginalp, G. An analysis of a phase field model of a free boundary. *Arch. Rat. Mech. An.* **92**, 205–245 (1986).

[23] Caginalp, G. Stefan and Hele Shaw type models as asymptotic limits of the phase-field equations. *Physical Review A* **39/11**, 5887–5896 (1989).

[24] Caginalp, G., and Chen, X. Phase field equations in the singular limit of sharp interface equations. Preprint (1992).

[25] Caginalp, G., and Fife, P. Dynamics of layered interfaces arising from phase boundaries. *SIAM J. Appl. Math.* **48**, 506–518 (1988).

[26] Caginalp, G., and Socolovsky, E. A. Efficient computation of a sharp interface by spreading via phase field methods. *Appl. Math. Let.* **2/2**, 117–120 (1989).

[27] Chen, X. Generation and propagation of the interface for reaction-diffusion equations. *Jour. Diff. Equations* **96**, 116–141 (1992).

[28] Chen, X. Spectrums for the Allen-Cahn, Cahn-Hilliard and phase-field equations for generic interfaces. Preprint (1993).

[29] Chen, X., and Elliot, C. M. Asymptotics for a parabolic double obstacle problem. Preprint (1991).

[30] Chen, Y.-G., Giga, Y., and Goto, S. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geometry* **33**, 749–786 (1991). (Announcement: *Proc. Japan Academy Ser. A* **67/10**, 323–328 (1991).)

[31] Chen, Y.-G., Giga, Y., and Goto, S. Analysis toward snow crystal growth. *Proc. Func. Analysis and Rel. Topics* (S. Koshi, ed.) Sapporo, 1990. To appear.

[32] Crandall, M. G., Evans, L. C., and Lions, P.-L. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. AMS* **282**, 487–502 (1984).

[33] Crandall, M. G., Ishii, H., and Lions, P.-L. User's guide to viscosity solutions of second order partial diffrential equations. *Bull. AMS* **27/1**, 1–67 (1992).

[34] Crandall, M. G., and Lions, P.-L. Viscosity solutions of Hamilton-Jacobi equations. *Trans. AMS* **277**, 1–43 (1983).

[35] Collins, J. B., and Levine, H. Diffuse interface model of diffusion-limited crystal growth. *Phys. Rev. B.* **31**, 6119–6122 (1985).

[36] Dang, H., Fife, P. C., and Peletier, L. A. Saddle solutions of the bi-stable diffusion equation. Preprint (1992).

[37] DeGiorgi, E. Some conjectures on flow by mean curvature. *Proceedings of Capri Workshop* (1990).

[38] Evans, L. C. Convergence of an algorithm for mean curvature motion. Preprint (1993).

[39] Evans, L. C., and Spruck, J. Motion of level sets by mean curvature. *J. Differential Geometry* **33**, 635–681 (1991).

[40] Evans, L. C., and Spruck, J. Motion of level sets by mean curvature II. *Trans. AMS* **330**, 635–681 (1992).

[41] Evans, L. C., and Spruck, J. Motion of level sets by mean curvature III. *J. Geom. Analysis* **2**, 121–150 (1992).

[42] Evans, L. C., and Spruck, J. Motion of level sets by mean curvature IV. *J. Geom. Analysis* **5**, 77–114 (1995).

[43] Evans, L. C., Soner, H. M., and Souganidis, P. E. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.* **45**, 1097–1123 (1992).

[44] Fife, P.C. Dynamics of internal layers and diffusive interfaces. *CBMS-NSF Regional Conference Series in Applied Math.* **53** (1988), SIAM, Philadelphia.

[45] Fife, P. C., and McLeod, B. The appoarch of solutions of nonlinear diffusion equation to travelling front solutions. *Arc. Rat. Mech. An.* **65**, 335–361 (1977).

[46] Fix, G. Phase field methods for free boundary problems. In *Free Boundary Problems: Theory and Applications* (B. Fasano and M. Primicerio, eds.). Pitman, London 1983, pp. 580–589.

[47] Fleming, W. H., and Soner, H. M. *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag, New York, 1993.

[48] Fonseca, I., and Tartar, L. The gradient theory of phase transitions for systems with two potential wells. *Proc. Royal Soc. Edinburgh Sect. A* **111**, 89–102 (1989).

[49] Fried, E., and Gurtin, M. Continuum theory of thermally induced phase transitions based on an order parameter. *Physica D* **68**, 326–343 (1993).

[50] Gärtner, J. Bistable reaction-diffusion equations and excitable media. *Math. Nachr.* **112**, 125–152 (1983).

[51] Gage, M., and Hamilton, R. S. The heat equation shrinking convex plane curves. *J. Differential Geometry* **23**, 69–95 (1986).

[52] Giga, Y., and Goto, S. Motion of hypersurfaces and geometric equations. *J. Math. Soc. Japan* **44/1**, 99–111 (1992).

[53] Giga, Y., Goto, S., Ishii, H., and Sato, M. H. Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains. *Indiana Math. J.* **40/2**, 443–470 (1991).

[54] Giga, Y., and Sato, M. H. Generalized interface condition with the Neumann boundary condition. *Proc. Japan Acad. Ser. A Math* **67**, 263–266 (1991).

[55] Grayson, M. A. The heat equation shrinks embedded plane curves to round points. *J. Differential Geometry* **26**, 285–314 (1987).

[56] Grayson, M. A. A short note on the evolution of surfaces via mean curvature. *Duke Math. J.* **58**, 555–558 (1989).

[57] Gurtin, M. E. Multiphase thermomechanics with interfacial structure 1. Heat conduction and the capillary balance law. *Arch. Rat. Mech. An.* **104**, 195–221 (1988).

[58] Gurtin, M. E. Multiphase thermomechanics with interfacial structure. Towards a nonequilibrium thermomechanics of two phase materials. *Arch. Rat. Mech. An.* **104**, 275–312 (1988).

[59] Gurtin, M. E. *Thermodynamics of Evolving Phase Boundaries in the Plane.* Oxford University Press, 1993.

[60] Gurtin, M. E., Soner, H. M., and Souganidis, P. E. Anisotropic motion of an interface relaxed by the formation of infinitesimal wrinkles. *Journal of Differential Equations* **119**, 54–108 (1995).

[61] Huisken, G. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geometry* **31**, 285–299 (1990).

[62] Ilmanen, T. Convergence of the Allen-Cahn equation to the Brakke's motion by mean curvature. *J. Differential Geom.* **38**, 417–461 (1993).

[63] Ilmanen, T. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. AMS* **108** (1994).

[64] Ilmanen, T. Generalized motion of sets by mean curvature on a manifold. *Univ. Indiana Math. J.* **41**, 671–705 (1992).

[65] Ishii, H. A simple direct proof for uniqueness for solutions of the Hamilton-Jacobi equations of the Eikonial type. *Proc. AMS.* **100**, 247–251 (1987).

[66] Ishii, H., and Souganidis, P. E. Forthcoming.

[67] Jensen, R. The maximum principle for viscosity solutions of second order fully nonlinear partial differential equations. *Arch. Rat. Mech. An.* **101**, 1–27 (1988).

[68] Katsoulakis, M., Kossioris, G., and Reitrich, F. Generalized motion by mean curvature with Neumann condition and the Allen-Cahn model for phase transitions. Preprint (1992).

[69] Katsoulakis, M., and Souganidis, P. E. Interacting particle systems and generalized mean curvature evolution. Preprint (1992).

[70] Langer, J. S. Unpublished notes (1978).

[71] Lasry, J. M., and Lions, P.-L. A remark on regularization in Hilbert spaces. *Israel J. Math.* **551**, 257–266 (1988).

[72] Luckhaus, S. Solutions of the two phase Stefan problem with the Gibbs-Thompson law for the melting temperature. *European J. Appl. Math.* **1**, 101–111 (1990).

[73] DeMasi, A., Orlandi, E., Presutti, E., and Triolo, L. Motion by curvature by scaling non local evolution equations. Preprint (1993).

[74] Modica, L. Gradient theory of phase transitions and the minimal interface criteria. *Arch. Rat. Mech. An.* **98**, 123–142 (1987).

[75] Modica, L., and Mortola, S. Il limite nella Γ-convergenza di una famiglia di funzionali ellittici. *Boll. Un. Math. Ital.* **A 14** (1977).

[76] deMottoni, P., and Schatzman, M. Geometrical evolution of developed interfaces. *Trans. AMS.* To appear. (Announcement: Evolution géométric d'interfaces. *C.R. Acad. Sci. Sér. I. Math.* **309**, 453–458 (1989).)

[77] deMottoni, P., and Schatzman, M. Development of surfaces in $\mathcal{R}^d$. *Proc. Royal Edinburgh Sect. A* **116**, 207–220 (1990).

[78] Mullins, W. Two dimensional motion of idealized grain boundaries. *J. Applied Physics* **27**, 900–904 (1956).

[79] Nochetto, R. H., Paolini, M., and Verdi, C. Optimal interface error estimates for the mean curvature flow. *Ann. Scuola Nor. Pisa* **21**, 193–212 (1994).

[80] Ohnuma, M., and Sato, M. Singular degenerate parabolic equations with applications to geometric evolutions. Preprint (1992).

[81] Osher, S., and Sethian, J. Fronts propogating with curvature dependent speed. *J. Comp. Phys.* **79**, 12–49 (1988).

[82] Otha, T., Jasnow, D., and Kawasaki, K. Universal scaling in the motion of a random interface. *Physics Review Letters* **49**, 1223–1226 (1982).

[83] Owen, N., Rubinstein, J., and Sternberg, P. Minimizers and gradient flow for singularly perturbed bi-stable potentials with a Dirichlet condition. *Proc. Royal Soc. London, A*, **429**, 505–532 (1990).

[84] Pego, R. L. Front migration in the nonlinear Cahn-Hilliard equation. *Proc. Roy. Soc. London A* **422**, 261–278 (1989).

[85] Penrose, O., and Fife, P. Theormodynamically consistent models for the kinetics of phase transitions. *Physica D* **43**, 44–62 (1990).

[86] Rubinstein, J., Sternberg, P., and Keller, J. B. Fast reaction, slow diffusion and curve shortening. *SIAM J. Appl. Math.* **49**, 116–133 (1989).

[87] Rubinstein, J., Sternberg, P., and Keller, J. B. Reaction diffusion processes and evolution to harmonic maps. *SIAM J. Appl. Math.* **49**, 1722–1733 (1989).

[88] Rubinstein, J., and Sternberg, P. Nonlocal reaction diffusion equations and nucleation. *J. IMA.* To appear.

[89] Schatzman, M. On the stability of the saddle solution of Allen-Cahn's equation. Preprint (1993).

[90] Sethian, J. Curvature and evolution of fronts. *Comm. Math. Physics* **101**, 487–495 (1985).

[91] Sethian, J., and Strain, J. Crystal growth and dentritic solidification. *Jour. Comp. Physics* **2**, 231–253 (1992).

[92] Soner, H. M. Motion of a set by the curvature of its boundary. *Jour. Diff. Equations* **101/2**, 313–372 (1993).

[93] Soner, H. M. Convergence of the phase-field equations to the Mullins-Sekerka problem with kinetic undercooling. **131**, 139–197 (1995).

[94] Soner, H. M. Ginzburg-Landau equation and motion by mean curvature, II: Development of the initial interface. *J. Geom. Anal.* **7**, 000–000 (1997).

[95] Sternberg, P. The effect of a singular perturbation on nonconvex variational problems. *Arch. Rat. Mech. An.* **101**, 209–260 (1988).

[96] Sternberg, P., and Ziemer, W. Generalized motion by curvature with a Dirichlet condition. Preprint (1992).

[97] Sternberg, P., and Ziemer, W. Local minimizers of a three phase partition problem with triple junctions. Preprint (1993).

[98] Stoth, B. A model with sharp interface as limit of phase-field equations in one space dimension. *European J. Appl. Math.* (1992). To appear.

[99] Stoth, B. The Stefan problem with the Gibbs-Thompson law as singular limit of phase-field equations in the radial case. *European J. Appl. Math.* (1992). To appear.

[100] Strain, J. A boundary integral approach to unstable solidification. *Jour. Comp. Physics* **85**, 342–389 (1989).

[101] Taylor, J. E. Motion of curves by crystalline curvature, including triple junctions and boundary points. *Proc. Symp. Pure Math.* To appear.

[102] Taylor, J. E. The motion of multiple phase junctions under prescribed phase-boundary velocities. Preprint (1992).

[103] Taylor, J. E., Cahn, J. W., and Handwerker, A. C. Geometric models of crystal growth. *Acta Met.* **40**, 1443–1474 (1992).

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