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# Maximizing the $L^\infty$ Norm of the Gradient of Solutions to the Poisson Equation

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ABSTRACT. We find the maximum of $\|Du_f\|_{L^\infty}$ when $u_f$ is the solution, which vanishes at infinity, of the Poisson equation $\Delta u = f$ on $\mathbb{R}^n$ in terms of the decreasing rearrangement of $f$. Hence, we derive sharp estimates for $\|Du_f\|_{L^\infty}$ in terms of suitable Lorentz or $L^p$ norms of $f$. We also solve the problem of maximizing $|Du_f^B(0)|$ when $u_f^B$ is the solution, vanishing on $\partial B$, to the Poisson equation in a ball $B$ centered at 0 and the decreasing rearrangement of $f$ is assigned.

## 1. Introduction and main results

We are concerned with a number of problems, a prototype of which can be stated in the following physical terms: Consider a distribution of electric charges such that the density is 1 and the total charge is fixed; which configuration maximizes the largest intensity of the electric field?

The problem amounts to finding

$$\max_E \left( \max_{x\in\mathbb{R}^n} |Du_{\chi_E}(x)| \right) \tag{1.1}$$

as $E$ is subject to the following conditions:

$E$ is a measurable subset of $\mathbb{R}^n$,

$m(E)$ is fixed,

and $u_{\chi_E}$ is the solution to the equation

$$\Delta u = \chi_E \tag{1.2}$$

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in $\mathbb{R}^n$, such that

$$\begin{cases} \lim_{|x|\to\infty} u(x) = 0 & \text{if } n \geq 3 \\ \limsup_{|x|\to\infty} u(x)/|x| < +\infty & \text{if } n = 2. \end{cases} \tag{1.3}$$

Here $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, vertical bars $|\ |$ denote modulus, $m$ is the Lebesgue measure, $D$ stands for gradient, and $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ for the Laplace operator. We call $\chi_E$ the characteristic function of a set $E$; namely, $\chi_E$ equals 1 in $E$ and vanishes elsewhere.

In the present paper we show that the maximum in (1.1) is attained when $E$ is a suitable egg-shaped set (a disk, in case $n = 2$) that is symmetric about an axis. This result can be precisely stated as follows.

**Theorem 1.** *Let $E$ be any measurable subset of $\mathbb{R}^n$, $n \geq 2$, such that $m(E) < +\infty$. Then*

$$\max_{x\in\mathbb{R}^n} |Du_{\chi_E}(x)| \leq |Du_{\chi_{S(E)}}(0)| \tag{1.4}$$

*where $S(E)$ is the set that has the same measure as $E$ and is defined by*

$$S(E) = \left\{x \in \mathbb{R}^n : |x|^n \leq x_1 K_n^{n-1} m(E)^{1-1/n}\right\} \tag{1.5}$$

*where*

$$K_n = \left( \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2n-1}{2n-2}\right)}{n\,\Gamma\left(\frac{n^2}{2n-2}\right)} \right)^{-1/n}$$

*Moreover,*

$$\left|Du_{\chi_{S(E)}}(0)\right| = Q_n\, m(E)^{1/n}, \tag{1.6}$$

*where*

$$Q_n = \frac{2\pi^{n-1+1/(2n)}}{n^{2-1/n}\Gamma\left(\frac{n}{2}\right)} \left( \frac{\Gamma\left(\frac{2n-1}{2n-2}\right)}{\Gamma\left(\frac{n^2}{2n-2}\right)} \right)^{1-1/n} \tag{1.7}$$

Theorem 1 enables one to solve the more general problem

$$\max_f \left( \max_{x\in\mathbb{R}^n} |Du_f(x)| \right) \tag{1.8}$$

when $f$ is constrained as follows:

$f$ is a real-valued measurable function on $\mathbb{R}^n$,

either $f_+^*$ and $f_-^*$ or $f^*$ is given,

and $u_f$ is the solution to the full Poisson equation

$$\Delta u = f \tag{1.9}$$

in $\mathbb{R}^n$, satisfying (1.3). Hereafter, $f_+$ and $f_-$ denote the positive and the negative part of $f$ respectively, i.e.,

$$f_+ = \frac{|f| + f}{2}, \quad f_- = \frac{|f| - f}{2}.$$

Above, $f^*(s) = \inf\{t \geq 0 : \mu_f(t) \leq s\}$, the decreasing rearrangement of $f$, where $\mu_f(t) = m(\{x \in \mathbb{R}^n : |f(x)| > t\})$, the distribution function of $f$ (see [HLP] for more details).

The solution to problem (1.8) is given in Theorems 2 and 3. In the statements,

$$\|f\|_{n,1} = \int_0^{+\infty} s^{1/n-1} f^*(s)\, ds, \tag{1.10}$$

an equivalent norm in the Lorentz space $L(n,1)$ (see, e.g., [Z]).

**Theorem 2.** *Let $f$ be any measurable function on $\mathbb{R}^n$ such that $\|f\|_{n,1} < +\infty$. Then*

$$\max_{x \in \mathbb{R}^n} |Du_f(x)| \leq |Du_{S(f)}(0)|, \tag{1.11}$$

*where $S(f)$ is the function having the following properties:*

*$S(f)_+$ and $S(f)_-$ are equidistributed with $f_+$ and $f_-$, respectively.*

*the level sets of $S(f)_+$ are the maximizing sets described in Theorem 1; the level sets of $S(f)_-$ are the symmetrical about the hyperplane $\{x : x_1 = 0\}$ of such maximizing sets.*

*In formulas,*

$$S(f)_\pm(x) = \begin{cases} f_\pm^*\left((\pm x_1)^{-n/(n-1)} |x|^{n^2/(n-1)} K_n^{-n}\right) & \text{if } x_1 \gtrless 0 \\ 0 & \text{if } x_1 \lessgtr 0. \end{cases} \tag{1.12}$$

**Theorem 3.** *Assume the same hypotheses as in Theorem 2. Then*

$$\max_{x \in \mathbb{R}^n} |Du_f(x)| \le |Du_{\hat{S}(f)}(0)|, \tag{1.13}$$

*where $\hat{S}(f)$ is the function equidistributed with $f$, such that $\hat{S}(f)^*_+ = \hat{S}(f)^*_-$ and whose level sets are defined analogously to the level sets of $S(f)$ (Theorem 2).*

*In formulas,*

$$\hat{S}(f) = sign(x_1) f^* \left( 2|x_1|^{-n/(n-1)} |x|^{n^2/(n-1)} K_n^{-n} \right). \tag{1.14}$$

Theorem 3 makes it possible to derive sharp upper bounds for $\max_{x \in \mathbb{R}^n} |Du_f(x)|$ in terms of the $L(n,1)$ norm or the $L^1$ and $L^\infty$ norms of $f$ (Corollary, Section 2).

Now we want to emphasize that problem (1.8) can be regarded as a particular case $(p = \infty)$ of

$$\max_f \|Du_f\|_{L^p} \tag{1.15}$$

as $f^*$ is assigned. Recall that, when $p = 2$, the norm of $Du_f$ in (1.15) has a precise physical meaning: it represents the (square root of the) energy of the electrostatic system described above.

Problem (1.15) can be solved if $p \le 2$. In fact, Theorem 1 of [T] tells us that

$$\|Du_f\|_{L^p} \le \|Du_{f^\star}\|_{L^p}$$

for $p \le 2$, where $f^\star$, the spherically symmetric rearrangement of $f$, is the function whose level sets $\{x \in \mathbb{R}^n : f^\star(x) > t\} = \{x \in \mathbb{R}^n : |f(x)| > t\}^\star$. We recall that, given any measurable subset $\Omega$ of $\mathbb{R}^n$, $\Omega^\star$ denotes the Schwarz symmetrized of $\Omega$, i.e. the ball, centered at the origin, which has the same measure as $\Omega$.

As far as we know, problem (1.15) is open for $2 < p < \infty$. Theorem 3 solves the problem in case $p = \infty$. The result suggests that when $2 < p < \infty$, a loss of the spherical symmetry should be expected. Indeed, we stress that the rearrangement $\hat{S}(f)$, which maximizes in (1.15) if $p = \infty$, does not agree with $f^\star$.

A further question arises quite naturally: What can one say if, in (1.15), $u_f$ is replaced by the solution to equation (1.9) in a domain $\Omega$, vanishing on $\partial\Omega$?

For $0 < p \le 2$, the solution to the above question follows again from Theorem 1 of [T], whereas it is not known if $2 < p < \infty$. When $p = \infty$, the case considered in this paper, we present a partial answer in this direction.

We deal with the solutions $u^B_{\chi_E}$ and $u^B_f$ to (1.2) and (1.9), respectively, in the unit ball $B^n$ of $\mathbb{R}^n$, which satisfy the condition

$$u = 0 \quad \text{on} \quad \partial B^n \tag{1.16}$$

and replace max $|Du|$ by $|Du(0)|$, the modulus of the gradient at the center of $B^n$; namely, we consider the problems

$$\max_E |Du^B_{\chi_E}(0)|, \tag{1.17}$$

and

$$\max_f |Du^B_f(0)| \tag{1.18}$$

as $m(E)$ and either $f^*_\pm$ or $f^*$ are fixed, respectively. In Theorem 4 the corresponding maximizers $S^B(E)$, $S^B(f)$, and $\hat{S}^B(f)$ are exhibited.

**Theorem 4.** Part I. *Let $E$ be any measurable subset of $B^n$, $n \ge 2$. Then*

$$|Du^B_{\chi_E}(0)| \le |Du^B_{\chi_{S^B(E)}}(0)|, \tag{1.19}$$

*where $S^B(E) \subseteq B^n$ is the set, having the same measure as $E$, defined by*

$$S^B(E) = \begin{cases} \{x \in \mathbb{R}^n : 0 \le x_1 \le t_{m(E)}, x_2^2 + \cdots + x_n^2 \le W(x_1)\} & \text{if } m(E) \le C_n/2 \\ B^n \setminus \{x \in \mathbb{R}^n : (-x_1, x_2, \ldots, x_n) \in S^B(B^n \setminus E)\} & \text{otherwise.} \end{cases} \tag{1.20}$$

*Here:*

$t_{m(E)}$ *is the solution in* $[0,1]$ *to the equation* $t^n + (n-1)C_n\lambda(m(E))t^{n-1} - 1 = 0$,

$W_{m(E)}(t) = t^{2/n}((n-1)C_n\lambda(m(E)) + t)^{-2/n} - t^2$,

$\lambda(\cdot)$ *is the function defined by* $(3.6)$ *(Section 3),*

$C_n = \pi^{n/2}/\Gamma(1 + n/2)$, *the measure of* $B^n$.

Part II. *Let $f$ be any measurable function on $B^n$, $n \ge 2$, such that $\|f\|_{n,1} < \infty$. Then:*

$$|Du^B_f(0)| \le \left|Du^B_{S^B(f)}(0)\right|, \tag{i}$$

*where $S^B(f)$ is the function having the following properties:*

*$S^B(f)_+$ and $S^B(f)_-$ are equidistributed with $f_+$ and $f_-$, respectively;*

*the level sets of $S^B(f)_+$ are the maximizing sets described in Part I; the level sets of $S^B(f)_-$ are the symmetrical about the hyperplane $\{x : x_1 = 0\}$ of such maximizing sets;*

$$\left|Du_f^B(0)\right| \leq \left|Du_{\hat{S}^B(f)}^B(0)\right| \tag{ii}$$

*where $\hat{S}^B(f)$ is the function equidistributed with $f$, such that $\hat{S}^B(f)_+^* = \hat{S}^B(f)_-^*$ and whose level sets are defined analogously to the level sets of $S^B(f)$ above.*

## 2. The problem in $\mathbb{R}^n$

Our proof of Theorems 1 through 3 is based on a representation formula for the solution $u_f$ to problem (1.3)–(1.9) in $\mathbb{R}^n$. As is well known, $u_f$ agrees with the Newtonian (logarithmic, in case $n = 2$) potential of $f$; namely,

$$u_f(x) = \frac{1}{(2-n)nC_n} \int_{\mathbb{R}^n} f(y)|x-y|^{2-n}\, dy \tag{2.1a}$$

if $n \geq 3$, or

$$u_f(x) = \frac{1}{2\pi} \int_{\mathbb{R}^n} f(y) \ln|x-y|\, dy \tag{2.1b}$$

if $n = 2$. Thus, we have

$$Du_f(x) = \frac{1}{nC_n} \int_{\mathbb{R}^n} f(y)(x-y)|x-y|^{-n}\, dy. \tag{2.2}$$

Observe that $|Du_f|$ is bounded in $\mathbb{R}^n$, provided $f \in L(n,1)$. In fact,

$$|Du_f(x)| \leq \frac{1}{nC_n} \int_{\mathbb{R}^n} |f(y)|\, |x-y|^{1-n}\, dy. \tag{2.3}$$

On the other hand, the Hardy–Littlewood inequality says that

$$\int_{\mathbb{R}^n} g(x)h(x)\, dx \leq \int_{\mathbb{R}^n} g^\star(x)h^\star(x)\, dx \tag{2.4}$$

for any couple of measurable functions $g, h : \mathbb{R}^n \to \mathbb{R}$ (see [HLP]). Therefore, from (2.3) and (2.4) we obtain

$$|Du_f(x)| \le \frac{1}{nC_n} \int_{\mathbb{R}^n} f^\star(y)|y|^{1-n}\,dy.$$

Consequently,

$$|Du_f(x)| \le \frac{1}{nC_n^{1/n}} \int_0^{+\infty} s^{1/n-1} f^*(s)\,ds,$$

as $f^\star(y) = f^*(C_n|x|^n)$.

Clearly, $f \in L(n,1)$ is a sharp condition for the boundedness of $|Du_f|$. A sufficient condition for $f$ to belong to $L(n,1)$ is $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for some $n < p \le \infty$ and $1 \le q < n$. This fact follows from the Hölder inequality, since

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)| = \sup_{s\in\mathbb{R}} f^*(s) = \sup_{x\in\mathbb{R}^n} f^\star(x) \tag{2.5}$$

and

$$\int_{\mathbb{R}^n} |f(x)|^p\,dx = \int_0^{+\infty} f^*(s)^p\,ds = \int_{\mathbb{R}^n} f^\star(x)^p\,dx. \tag{2.6}$$

Now, let us fix a few notations. If $\Omega$ is any measurable subset of $\mathbb{R}^n$, we define the "cylindrically symmetrized" $\Omega^{\star,1}$ of $\Omega$ along the $x_1$-axis by

$$\Omega^{\star,1} = \{x \in \mathbb{R}^n : (x_2, \dots, x_n) \in (\Omega \cap \{y \in \mathbb{R}^n : y_1 = x_1\})^\star\}, \tag{2.7}$$

where, on the right-hand side, $\star$ stands, with abuse of notation, for $(n-1)$-dimensional Schwarz symmetrization on the hyperplane $\{y \in \mathbb{R}^n : y_1 = x_1\}$.

Analogously, if $g$ is any measurable function on $\Omega$, we call $g^{\star,1} : \Omega^{\star,1} \to \mathbb{R}$ the cylindrically symmetric rearrangement of $g$ if $g^{\star,1}$ depends only on $x_1$ and $(x_2^2 + \cdots + x_n^2)^{1/2}$ and

$$g^{\star,1}(x_1, \cdot) = (g(x_1, \cdot))^\star. \tag{2.8}$$

By the Cavalieri principle,

$$m(\Omega^{\star,1}) = m(\Omega). \tag{2.9}$$

Moreover, by the Fubini theorem and equation (2.4),

$$\int_\Omega g(x)h(x)\,dx \le \int_{\Omega^{\star,1}} g^{\star,1}(x)h^{\star,1}(x)\,dx \tag{2.10}$$

for any pair of measurable functions $g$ and $h$ on $\Omega$.

**Proof of Theorem 1.** Since $\Delta$ is translation- and rotation-invariant, we can restrict ourselves to maximizing $-(\partial/\partial x_1)u_{\chi_E}(0)$.

As

$$-\frac{\partial}{\partial x_1}u_{\chi_E}(0)=\frac{1}{nC_n}\int_{\mathbb{R}^n}\chi_E(y)y_1|y|^{-n}\,dy$$

(see formula (2.2)) and $y_1|y|^{-n}\geq 0$ for $y_1\geq 0$, $-(\partial/\partial x_1)u_{\chi_E}(0)$ increases is we replace $E$ (if necessary) by a new set $\tilde{E}$ that has the same measure as $E$ and is contained in the half-space $\mathbb{R}^n_+=\{x\in\mathbb{R}^n : x_1\geq 0\}$. We put

$$\tilde{E}=\begin{cases} E & \text{if } E\subset\mathbb{R}^n_+ \\ (E\cap\mathbb{R}^n_+)\cup F & \text{otherwise,}\end{cases}$$

where $F$ is any set, contained in $\mathbb{R}^n_+$, such that $m(F)=m(E\setminus\mathbb{R}^n_+)$ and $F\cap E\cap\mathbb{R}^n_+=\emptyset$ (such a set $F$ exists, for $m(E)<+\infty$). So doing, we have

$$-\frac{\partial}{\partial x_1}u_{\chi_E}(0)\leq\frac{1}{nC_n}\int_{\mathbb{R}^n}\chi_{\tilde{E}}(y)y_1|y|^{-n}\,dy. \tag{2.11}$$

Now, since the kernel $y_1|y|^{-n}$ is cylindrically symmetric on $\mathbb{R}^n_+$ (see (2.8)) and $(\chi_{\tilde{E}})^{\star,1}=\chi_{\tilde{E}^{\star,1}}$, inequality (2.10) implies

$$\frac{1}{nC_n}\int_{\mathbb{R}^n}\chi_{\tilde{E}}(y)y_1|y|^{-n}\,dy\leq\frac{1}{nC_n}\int_{\mathbb{R}^n}\chi_{\tilde{E}^{\star,1}}(y)y_1|y|^{-n}\,dy. \tag{2.12}$$

Combining (2.11) and (2.12) yields

$$-\frac{\partial}{\partial x_1}u_{\chi_E}(0)\leq\frac{1}{nC_n}\int_{\mathbb{R}^n}\chi_{\tilde{E}^{\star,1}}(y)y_1|y|^{-n}\,dy. \tag{2.13}$$

Notice that the right-hand side of (2.13) is nothing but $-(\partial/\partial x_1)u_{\chi_{\tilde{E}^{\star,1}}}(0)$. Consequently, due to (2.9), inequality (2.13) enables us to infer

$$-\frac{\partial}{\partial x_1}u_{\chi_E}(0)\leq\max_{m(E(v))=m(E)}\left(-\frac{\partial}{\partial x_1}u_{\chi_{E(v)}}(0)\right). \tag{2.14}$$

Here $E(v)$ denotes any symmetric set about the $x_1$-axis of the form

$$E(v)=\{x\in\mathbb{R}^n : x_1\geq 0,\quad (x_2^2+\cdots+x_n^2)^{1/2}\leq v(x_1)\},$$

where $v$, the meridian of $E(v)$, fulfills

$$v \in L^{n-1}[0,+\infty), \quad v(t) \geq 0. \tag{2.15}$$

Therefore, the last step of our proof consists of maximizing

$$\frac{(n-1)C_{n-1}}{nC_n} \int_0^{+\infty} \left( \int_0^{v(t)/t} s^{n-2}(1+s^2)^{-n/2}\, ds \right) dt,$$

under the constraint

$$C_{n-1} \int_0^{+\infty} v^{n-1}(t)\, dt \quad \text{given,}$$

among those functions $v$ that satisfy (2.15).

To this purpose, let us set

$$\begin{aligned} J(v) &= \int_0^{+\infty} \left( \int_0^{v(t)/t} s^{n-2}(1+s^2)^{-n/2}\, ds \right) dt, \\ G(v) &= C_{n-1} \int_0^{+\infty} v^{n-1}(t)\, dt, \end{aligned} \tag{2.16}$$

and consider, for fixed $\lambda > 0$, the functional $J - \lambda G$. Setting $a_\lambda(t,v) = \int_0^{v/t} s^{n-2}(1+s^2)^{-n/2}\, ds - \lambda C_{n-1} v^{n-1}$, we have $(J - \lambda G)(v) = \int_0^{+\infty} a_\lambda(t, v(t))\, dt$.

It is easily seen that, for every $v \geq 0$, $a_\lambda(t,v) \leq a_\lambda(t, v_\lambda(t))$, where $v_\lambda(t)$ equals $(t^{2/n}((n-1)C_{n-1}\lambda)^{-2/n} - t^2)^{1/2}$ if $0 \leq t \leq ((n-1)C_{n-1}\lambda)^{-1/(n-1)}$ and vanishes otherwise. Therefore, $v_\lambda$ maximizes $J - \lambda G$ in the class of functions $v$ fulfilling (2.15) and maximizes $J$ among those functions $v$ that, in addition, satisfy $G(v) = G(v_\lambda)$.

Imposing $G(v_\lambda) = m(E)$ yields

$$\lambda = \frac{1}{(n-1)C_{n-1}} \left( \frac{\pi^{(n-1)/2}\Gamma\left(\frac{2n-1}{2n-2}\right)}{n\,\Gamma\left(\frac{n^2}{2n-2}\right)} \right)^{1-1/n} m(E)^{1/n-1}.$$

Thus, by virtue of (2.14), we get (1.4), as $-(\partial/\partial x_1) u_{\chi_{S(E)}}(0) = |Du_{S(E)}(0)|$. Finally, (1.6) follows through straightforward computation of $|Du_{S(E)}(0)|$. □

**Proof of Theorem 2.** We have

$$Du_f(x) = \int_0^{+\infty} Du_{\chi_{\{f_+ > t\}}}(x)\, dt - \int_0^{+\infty} Du_{\chi_{\{f_- > t\}}}(x)\, dt. \tag{2.17}$$

In fact, splitting $f$ into its positive and negative parts and making use of the formula $f_\pm(x) = \int_0^{+\infty} \chi_{\{f_\pm>t\}}(x)\,dt$ in (2.2) give

$$Du_f(x) = \frac{1}{nC_n}\int_{\mathbb{R}^n} \frac{(x-y)}{|x-y|^n} \times \left(\int_0^{+\infty} \chi_{\{f_+>t\}}(y)\,dt - \int_0^{+\infty} \chi_{\{f_->t\}}(y)\,dt\right) dy. \tag{2.18}$$

Hence, (2.17) follows via the Fubini theorem.

From (2.17) one gets

$$|Du_f(x)| \le \int_0^{+\infty} \left(\left|Du_{\chi_{\{f_+>t\}}}(x)\right| + \left|Du_{\chi_{\{f_->t\}}}(x)\right|\right) dt. \tag{2.19}$$

By (1.4),

$$\left|Du_{\chi_{\{f_+>t\}}}(x)\right| \le \left|Du_{\chi_{S(\{f_\pm>t\})}}(0)\right| = -\frac{\partial}{\partial x_1} u_{\chi_{S(\{f_\pm>t\})}}(0) \qquad \text{for } t>0. \tag{2.20}$$

for $t > 0$. Clearly,

$$-\frac{\partial}{\partial x_1} u_{\chi_{S(\{f_->t\})}}(0) = \frac{\partial}{\partial x_1} u_{\chi_{S(\{f_->t\})_-}}(0), \tag{2.21}$$

where, for any set $F \subseteq \mathbb{R}^n$, the notation

$$F_- = \{x \in \mathbb{R}^n : (-x_1, x_2, \dots, x_n) \in F\} \tag{2.22}$$

is used.

Now, thanks to formula (2.2) and Fubini's theorem, we have

$$\int_0^{+\infty} \left(-\frac{\partial}{\partial x_1} u_{\chi_{S(\{f_+>t\})}}(0) + \frac{\partial}{\partial x_1} u_{\chi_{S(\{f_->t\})_-}}(0)\right) dt = \frac{1}{nC_n}\int_{\mathbb{R}^n} \frac{y_1}{|y|^n}\left(\int_0^{+\infty} \chi_{S(\{f_+>t\})}(y)\,dt - \int_0^{+\infty} \chi_{S(\{f_->t\})_-}(y)\,dt\right) dy. \tag{2.23}$$

Observe that the right-hand side of (2.23) equals $-(\partial/\partial x_1)u_{S(f)}(0)$, for the term in brackets in the integrand agrees with $S(f)$. Furthermore, $-(\partial/\partial x_1)u_{S(f)}(0) = |Du_{S(f)}(0)|$, since the level sets of $S(f)$ are symmetric about the $x_1$-axis and $S(f)(x) \ge 0$ if $x_1 \ge 0$, $S(f)(x) \le 0$ if $x_1 \le 0$.

Thus, (2.23) yields

$$\int_0^{+\infty}\left(-\frac{\partial}{\partial x_1}u_{\chi_{S(\{f_+>t\})}}(0)+\frac{\partial}{\partial x_1}u_{\chi_{S(\{f_->t\})_-}}(0)\right)dt=\left|Du_{S(f)}(0)\right|. \tag{2.24}$$

Combining (2.19), (2.20), (2.21), and (2.24) completes the proof.

We notice that the maximizing function $S(f)$ can be obtained by rearranging $f$ on the level sets of the harmonic function $x_1/|x|^n$. □

In the proof of Theorem 3 we shall make use of the following lemma.

**Lemma.** *Let $E$, $F$, and $G$ be any subsets of $\mathbb{R}^n$ such that $2m(E)=m(F)+m(G)$. Then*

$$-\frac{\partial}{\partial x_1}u_{\chi_{S(F)}}(0)-\frac{\partial}{\partial x_1}u_{\chi_{S(G)}}(0)\le -2\frac{\partial}{\partial x_1}u_{\chi_{S(E)}}(0). \tag{2.25}$$

**Proof.** Suppose, by contradiction

$$-\frac{\partial}{\partial x_1}u_{\chi_{S(F)}}(0)-\frac{\partial}{\partial x_1}u_{\chi_{S(G)}}(0)> -2\frac{\partial}{\partial x_1}u_{\chi_{S(E)}}(0). \tag{2.26}$$

Assume, for instance, $m(F)<m(G)$. By adding $(\partial/\partial x_1)u_{\chi_{S(E)}}(0)$ to both sides of (2.26) and observing that, under our assumptions, $S(F)\subset S(E)\subset S(G)$, we obtain

$$-\frac{\partial}{\partial x_1}u_{\chi_{S(G)\setminus(S(E)\setminus S(F))}}(0)> -\frac{\partial}{\partial x_1}u_{\chi_{S(E)}}(0). \tag{2.27}$$

But $m(S(G)\setminus(S(E)\setminus S(F)))=m(E)$. Thus, inequality (2.27) contradicts (1.4), as $-(\partial/\partial x_1)u_{\chi_{S(E)}}(0)=|Du_{\chi_{S(E)}}(0)|$ and $S(S(G)\setminus(S(E)\setminus S(F)))=S(E)$. □

**Proof of Theorem 3.** Let us set

$$L(t)=\{x\in\mathbb{R}^n: f_+(2^{1/n}x)>t\}\cup\{x\in\mathbb{R}^n: f_-(2^{1/n}x)>t\},\quad t>0. \tag{2.28}$$

Obviously,

$$m(L(T))=\frac{1}{2}\mu_f(t). \tag{2.29}$$

Therefore, owing to the lemma above and to equality (2.21),

$$-\frac{\partial}{\partial x_1}u_{\chi_{S(\{f_+>t\})}}(0)+\frac{\partial}{\partial x_1}u_{\chi_{S(\{f_->t\})_-}}(0)$$
$$\le -\frac{\partial}{\partial x_1}u_{\chi_{S(L(t))}}(0)+\frac{\partial}{\partial x_1}u_{\chi_{S(L(t))_-}}(0). \tag{2.30}$$

On the other hand,

$$\int_0^{+\infty}\left(-\frac{\partial}{\partial x_1}u_{\chi_{S(L(t))}}(0)+\frac{\partial}{\partial x_1}u_{\chi_{S(L(t))_-}}(0)\right)dt$$
$$=\frac{1}{nC_n}\int_{\mathbb{R}^n}\frac{y_1}{|y|^n}\left(\int_0^{+\infty}\chi_{S(L(t))}(y)\,dt-\int_0^{+\infty}\chi_{S(L(t))_-}(y)\,dt\right)dy. \tag{2.31}$$

Note that the expression in brackets in the latter integrand agrees with $\hat{S}(f)$. Therefore, (2.31) implies

$$\int_0^{+\infty}\left(-\frac{\partial}{\partial x_1}u_{\chi_{S(L(t))}}(0)+\frac{\partial}{\partial x_1}u_{\chi_{S(L(t))_-}}(0)\right)dt=\left|Du_{\hat{S}(f)}(0)\right|, \tag{2.32}$$

since $-(\partial/\partial x_1)u_{\hat{S}(f)}(0)=|Du_{\hat{S}(f)}(0)|$.

Thus, (1.13) follows through (1.4), (2.24), (2.30), and (2.32). □

**Corollary.** *Assume the hypotheses of Theorem 2. Then*

$$\max_{x\in\mathbb{R}^n}|Du_f(x)|\le\frac{2^{1-1/n}}{n}Q_n\|f\|_{n,1} \tag{i}$$

*and equality holds if* $f=\hat{S}(f)$;

$$\max_{x\in\mathbb{R}^n}|Du_f(x)|\le 2^{1-1/n}Q_n\|f\|_{L^\infty}^{1-1/n}\|f\|_{L^1}^{1/n} \tag{ii}$$

*and equality holds if* $f=c(\chi_{S(E)}-\chi_{S(E)_-})$, *where* $E$ *is any set (recall (2.22)) and* $c$ *is any real number.*

*The constant* $Q_n$ *is given by* (1.7).

**Proof.** Taking into account (1.13) and using (2.32) to compute $|Du_{\hat{S}(f)}(0)|$ yield

$$\max_{x\in\mathbb{R}^n}|Du_f(x)|\le\int_0^{+\infty}\left(-\frac{\partial}{\partial x_1}u_{\chi_{S(L(t))}}(0)+\frac{\partial}{\partial x_1}u_{\chi_{S(L(t))_-}}(0)\right)dt.$$

Hence, by (1.6) and (2.29), we get

$$\max_{x\in\mathbb{R}^n} |Du_f(x)| \le 2^{1-1/n} Q_n \int_0^{+\infty} \mu_f(t)^{1/n}\, dt. \tag{2.33}$$

Now, one has

$$\int_0^{+\infty} \mu_f(t)^{1/n}\, dt = \frac{1}{n}\int_0^{+\infty} s^{1/n-1} f^*(s)\, ds. \tag{2.34}$$

This is shown by an integration by parts, which uses the formula $f^*(s) = \int_0^{+\infty} \chi_{[0,\mu(t)]}(s)\, dt$. Thus, (i) is a consequence of (2.33) and (2.34).

As far as (ii) is concerned, we obtain from (2.33), via Hölder inequality,

$$\max_{x\in\mathbb{R}^n} |Du_f(x)| \le 2^{1-1/n} Q_n (\sup f^*)^{1-1/n} \left(\int_0^{+\infty} \mu_f(t)\, dt\right)^{1/n}.$$

Since $\int_0^{+\infty} \mu_f(t)\, dt = \|f\|_{L^1}$ (see, e.g., [Z]) and (2.5) holds, (ii) is proved. □

**Remark 1.** Theorems 1 and 2 make it possible, via reflection arguments, to solve maximum problems analogous to (1.1) and (1.8) for the gradient of solutions to equations (1.2) and (1.9), respectively, in the half-space $\mathbb{R}^n_+$.

Let $E$ be any subset of $\mathbb{R}^n_+$ having finite measure and let $u^+_{\chi_E}$ be the solution to (1.2) in $\mathbb{R}^n_+$, which satisfies (1.3) and the boundary condition

$$u(0, x_2, \dots, x_n) = 0. \tag{2.35}$$

Then,

$$\max_{x\in\mathbb{R}^n_+} |Du^+_{\chi_E}(x)| \le |Du^+_{\chi_{S(E)}}(0)|. \tag{2.36}$$

In fact, it is easily verified that

$$u^+_{\chi_E} = u_{\chi_E} + u_{\chi_{E_-}} \tag{2.37}$$

(recall (2.21)). Therefore, $|Du^+_{\chi_E}(x)| \le |Du_{\chi_E}(x)| + |Du_{\chi_{E_-}}(x)|$ for every $x \in \mathbb{R}^n_+$. Applying Theorem 1 yields $|Du^+_{\chi_E}(x)| \le |Du_{\chi_{S(E)}}(x)| + |Du_{\chi_{S(E)_-}}(x)|$. By (2.37), the right-hand side of the latter inequality agrees with $|Du^+_{\chi_{S(E)}}(x)|$, inasmuch as $S(E)$ and $S(E)_-$ are symmetric sets about the $x_1$-axis. Hence, (2.36) follows.

More generally, consider any nonnegative function $f \in L(n,1)$ on $\mathbb{R}^n_+$ and call $u^+_f$ the solution to the equation (2.9) on $\mathbb{R}^n_+$ satisfying (1.3) and (2.35). Then, arguing as in the proof of Theorem 2 and making use of (2.36) instead of (1.4) shows that

$$\max_{x \in \mathbb{R}^n_+} |Du^+_f(x)| \leq |Du^+_{S(f)}(0)|.$$

An analogous result holds for nonpositive $f$. The details are omitted for brevity. □

## 3. The problem in a ball

The same role played by (2.1) in Section 2 is performed here by the following formula, which makes use of the Green function, for the solution $u^B_f$ to problems (1.9)–(1.16) in $B^n$:

$$u^B_f(x) = \frac{1}{(2-n)nC_n} \int_{B^n} f(y)\left(|x-y|^{2-n} - |y|^{2-n}|x-\tilde{y}|^{2-n}\right) dy \tag{3.1a}$$

if $n \geq 3$, or

$$u^B_f(x) = \frac{1}{2\pi} \int_{B^2} f(y)\left(\ln|x-y| - \ln(|y|\,|x-\tilde{y}|)\right) dy \tag{3.1b}$$

if $n = 2$, where $\tilde{y} = y/|y|^2$.

Hence,

$$Du^B_f(x) = \frac{1}{nC_n} \int_{B^n} f(y)\left((x-y)|x-y|^{-n} - |y|^{2-n}(x-y)|x-\tilde{y}|^{-n}\right) dy. \tag{3.2}$$

Arguing as in Section 2 shows that $|Du^B_f|$ is bounded in $B^n$ provided $f \in L(n,1)$. Since $B^n$ has finite measure, $f$ belongs to $L(n,1)$ if $f \in L^p(B^n)$ for some $n < p \leq \infty$.

**Proof of Theorem 4, Part I.** To begin with, let us consider the case where $m(E) \leq C_n/2$.

Since $\Delta$ and $B^n$ are invariant with respect to rotations about 0, we may assume without loss of generality that $|Du^B_{\chi_E}(0)| = -(\partial/\partial x_1)u^B_{\chi_E}(0)$. By (3.2),

$$-\frac{\partial}{\partial x_1}u^B_{\chi_E}(0) = \frac{1}{nC_n}\int_{B^n}\chi_E(y)y_1(|y|^{-n}-1)\,dy.$$

Moreover, $y_1(|y|^{-n}-1) \geq 0$ if $y \in B^n_+$; henceforth, $B^n_+ = \{x \in B^n : x_1 \geq 0\}$. Thus, in order to increase $-(\partial/\partial x_1)u^B_{\chi_E}(0)$, we modify (if necessary) the set $E$ by putting

$$\tilde{E} = \begin{cases} E & \text{if } E \subseteq B^n_+ \\ (E \cup F) \cap B^n_+ & \text{otherwise,} \end{cases}$$

where $F$ is any measurable subset of $B^n_+ \setminus E$ such that $m(F) = m(E \setminus B^n_+)$. This set $F$ exists because we are assuming $m(E) \leq C_n/2$.

Therefore, we have

$$-\frac{\partial}{\partial x_1}u^B_{\chi_E}(0) \leq \frac{1}{nC_n}\int_{B^n}\chi_{\tilde{E}}(y)y_1(|y|^{-n}-1)\,dy.$$

Hence, as the kernel $y_1(|y|^{-n}-1)$ is cylindrically symmetric on $B^n_+$ (recall (1.8)), one can repeat the argument used in the proof of Theorem 1 and infer

$$-\frac{\partial}{\partial x_1}u^B_{\chi_E}(0) \leq \max_{m(E(v))=m(E)}\left(-\frac{\partial}{\partial x_1}u^B_{\chi_{E(v)}}(0)\right). \tag{3.3}$$

Here $E(v)$ is any subset of $B^n_+$, symmetric about the $x_1$-axis, having the form

$$E(v) = \{x \in \mathbb{R}^n : 0 \leq x_1 \leq 1, (x_2^2 + \cdots + x_n^2)^{1/2} \leq v(x_1)\},$$

where

$$0 \leq v(t) \leq (1-t^2)^{1/2}, \quad t \in [0,1]. \tag{3.4}$$

Thus, our task is to maximize

$$\frac{(n-1)C_{n-1}}{nC_n}\int_0^1\left(\int_0^{v(t)/t}\left(s^{n-2}(s^2+1)^{-n/2}-t^2\right)ds\right)dt,$$

subject to the constraint

$$G(v) = m(E) \tag{3.5}$$

(see (2.16)), among those functions $v$ that satisfy (3.4).

Let $\lambda \geq 0$ be fixed. Setting

$$J^B(v) = \int_0^1 \left( \int_0^{v(t)/t} \left( s^{n-2}(s^2+1)^{-n/2} - t^2 \right) ds \right) dt$$

and arguing as in the proof of Theorem 1 shows that the function $v_\lambda^B$, which equals $(t^{2/n}((n-1)C_{n-1}\lambda + t)^{-2/n} - t^2)^{1/2}$ if $0 \leq t \leq t(\lambda)$ and vanishes otherwise, maximizes the functional $J^B - \lambda G$ in the class of those functions fulfilling (3.4). Here, $t(\lambda)$ denotes the inverse of $(t^{1-n} - t)/((n-1)C_{n-1})$ for $t \in (0, 1]$ (clearly, $t(\lambda)^n + (n-1)C_{n-1}\lambda t(\lambda)^{n-1} - 1 = 0$).

Now, consider the function $\phi$ defined as $\phi(\lambda) = G(v_\lambda^B)$. It is easily verified, by direct computation, that $d\phi/d\lambda < 0$ if $\lambda \geq 0$; moreover, we have $\phi(0) = C_n/2$ and $\lim_{\lambda \to +\infty} \phi(\lambda) = 0$. Thus, $\phi$ is bijective and decreasing from $[0, +\infty)$ into $(0, C_n/2]$.

By setting

$$\lambda(\cdot) = \phi^{-1}(\cdot), \tag{3.6}$$

we have $G(v^B_{\lambda(m(E))}) = m(E)$. Consequently, $v^B_{\lambda(m(E))}$ minimizes the functional $J^B$ under the constraints (3.4), (3.5). Hence, by (3.3), we get (1.19).

Note that, if $m(E) = C_n/2$, then the maximizing set $S(E) = B^n_+$.

Finally, if $m(E) > C_n/2$, then inequality (1.19) easily follows from the case proved above. Indeed, for any $E \subset B^n$, $|Du^B_{\chi_E}(0)| = |Du^B_{\chi_{B^n \setminus E}}(0)|$, because

$$\begin{aligned} |Du^B_{\chi_E}(0)| &= \frac{1}{nC_n} \left| \int_{B^n} \chi_E(y) y_1(|y|^{-n} - 1)\, dy \right|, \\ |Du^B_{\chi_{B^n \setminus E}}(0)| &= \frac{1}{nC_n} \left| \int_{B^n} \chi_{B^n \setminus E}(y) y_1(|y|^{-n} - 1)\, dy \right| \end{aligned}$$

and $\int_{B^n} y_1(|y|^{-n} - 1)\, dy = 0$. □

**Proof of Theorem 4, Part II, sketched.** A proof of inequalities (i) and (ii) starts from formula (3.2) and proceeds through the same steps as in the proof of Theorems 2 and 3. One has to make use of Theorem 4, Part I, in the place of Theorem 1 and of $S^B(f)$ and $\hat{S}^B(f)$ instead of $S(f)$ and $\hat{S}(f)$.

Notice that, in order to prove (ii), a new version (with identical proof) of the lemma in Section 2 is needed, where $S(\cdot)$ and $u$ in (2.25) are replaced by $S^B(\cdot)$ and $u^B$, respectively.

We remark that the maximizing function $S^B(f)$ is nothing but the rearrangement of $f$ on the level sets of the harmonic function $x_1(|x|^{-n} - 1)$. □

**Remark 2.** Obviously, Theorem 4 still holds, with simple suitable changes in the definitions of $S^B(E)$, $S^B(f)$, and $\hat{S}^B(f)$, if $B^n$ is replaced by a ball $B^n(R)$ of any radius $R$.

It is quite easy to verify that the maximizing set for the problem in $B^n(R)$ converges to the maximizer for the problem in the whole space $\mathbb{R}^n$, as $R$ goes to $+\infty$. This is what one reasonably expects, since, heuristically speaking, the former problem approaches the latter one when $R$ tends to $+\infty$. $\square$

**Remark 3.** We point out that, by arguing as in Remark 1, the results of Theorem 4 can be used to solve analogous questions for the gradient of the solutions of Poisson's equation with zero boundary data, in the half-ball $B^n_+$. $\square$

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