The Journal of Geometric Analysis
Volume 6, Number 1, 1996

Spaces of Wald-Berestovskii Curvature Bounded Below

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ABSTRACT. We consider inner metric spaces of curvature bounded below in the sense of Wald, without assuming local compactness or existence of minimal curves. We first extend the Hopf-Rinow theorem by proving existence, uniqueness, and "almost extendability" of minimal curves from any point to a dense G_δ subset. An immediate consequence is that Alexandrov's comparisons are meaningful in this setting. We then prove Toponogov's theorem in this generality, and a rigidity theorem which characterizes spheres. Finally, we use our characterization to show the existence of spheres in the space of directions at points in a dense G_δ set. This allows us to define a notion of "local dimension" of the space using the dimension of such spheres. If the local dimension is finite, the space is an Alexandrov space.

1. Introduction and main results

Curvature bounded below is usually defined using Alexandrov's triangle comparisons, an approach which requires local compactness, or at least the assumption due to Berestovskii [B] that minimal curve exist locally. In this article we will use a very simple and natural definition based on one given in 1935 by Wald, [W], for which these requirements are unnecessary. Our primary result is an extension of the Hopf-Rinow theorem (Theorem 1.4) in which we use curvature in lieu of local compactness to obtain existence, uniqueness, and "almost extendability" of minimal curves, almost everywhere. We use this global theorem to extend Toponogov's theorem (Theorem 1.5), characterize spheres (Theorem 1.8), and get some control over the space of directions (Section 5). We continue the approach used in [P1] and [P2] in which assumptions involving dimension are avoided as much as possible. All of our major constructions and results are therefore valid for infinite dimensional spaces. In the finite dimensional case our conclusions (Corollaries 1.12–1.14) overlap with work done independently by Burago, Gromov and Perelman [BGP2].

Definition 1.1. An open subset U of Y is called a *region of curvature* $\geq k$ if every quadruple of points in U can be isometrically embedded in S^3_m for some $m \geq k$.

By S^3_m we mean the three-dimensional sphere ($m > 0$), Euclidean space ($m = 0$) or hyperbolic space ($m < 0$) of curvature m. The *comparison radius* on X is denoted by $c_k(x) = \sup\{r : B(p, r)$ is a region of curvature $\geq k\}$; we say X has *curvature* $\geq k$ if $c_k > 0$ on X and *curvature locally*

Math Subject Classification 53C20, 53C70.
Key Words and Phrases Wald curvature, Hopf-Rinow theorem, Toponogov's theorem, Alexandrov space.

ISSN 1050-6926

bounded below if for each $x \in X, c_k(x) > 0$ for some k (possibly depending on x). Every separable, metrizable space Y admits a metric of curvature ≥ 0 since it can be embedded in the Hilbert cube, so curvature bounded below is in itself of little topological consequence. What is necessary to turn this condition into a powerful tool is a stronger connection between the metric and the intrinsic structure of the space. The usual condition is that the metric on Y should actually measure the distance travelled from point to point within Y itself, and not within some ambient space into which Y is embedded. To express this we use the following notation. A *triple* $(b; a, c)$ in Y is an ordered set of ponts such that $a, c \neq b$. We denote the *excess* of the triple by $\varepsilon(b; a, c) = d(a, b) + d(b, c) - d(a, c)$.

Definition 1.2. The metric on Y is called *inner* (or *length* or *intrinsic*) if for every $x \neq z \in Y$ and $\delta > 0$ there exists a point $y \in Y$ such that $d(x, y) = d(x, z)/2$ and $\varepsilon(y; x, z) < \delta$.

From now on we will assume that X is a metrically complete inner metric space and all curves are parameterized proportional to arclength. We denote by $\dim(X)$ the topological dimension of X. For any $x, y \in X$ one can use Definition 1.2 repeatedly to construct a continuous mapping from the dyadic rationals of $[0, 1]$ into X whose extension to $[0, 1]$ is a curve joining x and y of length ℓ arbitrarily close to $d(x, y)$. In other words, $d(x, y)$ is the infimum of the lengths of curves joining x and y; this property is equivalent to Definition 1.2, and is the usual definition of inner metric. A curve γ from x to y is called *minimal* if $\ell(\gamma) = d(x, y)$.

If X is locally compact, Ascoli's theorem can be used to obtain the existence of minimal curves between all pairs of points, and the famous triangle comparisons of Alexandrov can be used to define bounded curvature. In this case Berestovskii's and Alexandrov's definitions coincide, and the resulting class of spaces includes all Gromov-Hausdorff limits of Riemannian manifolds of dimension $\leq n$ for some $n < \infty$, having a fixed lower bound on sectional curvature. This fact about limits has resulted in many applications in Riemannian geometry.

When one doesn't assume local compactness the situation is not so simple. Alexandrov's comparisons cannot have significant consequences if one does not know (or assume) there are enough minimal curves to make them useful. On the other hand, Wald's definition, as a price for its simplicity, lacks much immediate applicability. Our solution to this problem is to extend the Hopf-Rinow theorem using Wald's definition; the resulting minimal curves are sufficient to make Alexandrov's conditions useful.

The Hopf-Rinow theorem is the starting point for the study of Riemannian manifolds as metric spaces. In Riemannian geometry the theorem establishes the equivalence of three conditions—metric completenmess, extendability of geodesics, and the compactness of closed metric balls—and provides the existence of minimal curves between all pairs of points when the manifold is complete. The theorem can be understood in the context of inner metric spaces in the following way: A *geodesic* is a curve that is minimal on sufficiently short segments. A geodesic (minimal or otherwise) β from p to q is called *extendable* (past q) if β is the restriction to a subinterval of a geodesic γ containing q in its interior. With this terminology the Hopf-Rinow Theorem is completely generalized to topological manifolds of curvature bounded above and below [P2]. More generally, local compactness and metric completeness in an inner metric space imply existence of minimal curves and are equivalent to compactness of metric balls (cf. [C]). However, extendability fails even if X has curvature bounded below (e.g. at the apex of a cone) and existence even fails for infinite-dimensional Riemannian manifolds [A], [Gs]. Another consideration, the uniqueness of minimal curves, is usually a separate issue for Riemannian manifolds, connected with the study of the cut locus. In our theorem all three of these issues are addressed at once.

Extendability can be expressed in terms of angles in the following way. If $(b;\, a, c)$ is a triple in X and $d(a,b)+d(b,c)+d(a,c) \leq 2\pi/\sqrt{k}$ ($1/\sqrt{k} = \infty$ when $k \leq 0$) then the triple has a unique (up to isometry) *representative* $(B;\, A, C)$ in $S_k = S_k^2$ such that $d(a,b) = d(A,B)$, $d(b,c) = d(B,C)$, and $d(a,c) = d(A,C)$. We denote by $\alpha_k(b;\, a, c)$ the angle of $(B;\, A, C)$ at B. The *angle* between minimal curves α and γ is given by $\alpha(\alpha, \gamma) = \liminf_{t\to 0} \alpha_k(\alpha(0);\, \alpha(t), \gamma(t), \gamma(t))$ exists, and a curve β from p to q is extendable beyond q only if there exists a minimal curve μ starting at q such that $\alpha(\beta, \mu) = \pi$.

Definition 1.3. If p and q lie in a region U of curvature $\geq k$ then a minimal curve β from p to q is called *almost extendable (beyond q)* if for all $\varepsilon > 0$ there exists a minimal curve ζ starting at q such that $\alpha(\beta, \zeta) > \pi - \varepsilon$.

Theorem 1.4. *Let X have curvature locally bounded below and $p \in X$. There exists a dense G_δ subset J_p of X such that for all $x \in J_p$ there is a unique, almost extendable minimal curve from p to x.*

The fact that J_p is a dense G_δ (countable intersection of open sets) is very useful since, by the Baire category theorem, the intersection of countably many dense G_δ's is again a dense G_δ. Among other things this allows us to construct triangles almost everywhere, and obtain an essential equivalence of Wald's and Alexandrov's definitions (Corollary 2.10). Almost extendability allows us to use modified versions of arguments developed in [P1] for the geodesically complete case (where by definition all minimal curves are extendable); this approach is used to prove Theorem 1.8 below.

Theorem 1.5. *If X has curvature $\geq k$ then $c_k(x) = \infty$ for all $x \in X$.*

Theorem 1.5 was proved in [P1] assuming a uniform Alexandrov curvature bound and existence and extendability of minimal curves, in [BGP1] assuming local compactness and, independently, in the general case, in [BGP2]. Our proof is an extended and simplified version of the proof in [P1]. It is primarily constructive in the sense that it inductively enlarges the comparison radius at a point. Furthermore, the proof uses only Alexandrov's comparisons, and can be reduced to quite a short argument in the Riemannian special case.

Corollary 1.6. *If X has curvature $\geq k > 0$ then* $\operatorname{diam}(X) \leq \pi/\sqrt{k}$.

We generalize Toponogov's maximal diameter theorem in the following way. When geodesic completeness is relaxed, we need a stronger condition than maximal diameter.

Definition 1.7. A set of $2n$ points $x_1, y_1, \ldots, x_n, y_n$ in a metric space Y is called *spherical* if $d(x_i, y_i) = \pi$ for all i and $\det[\cos d(x_i, x_j)] > 0$.

The existence of a spherical subset in an arbitrary metric space in general does not have many useful consequences, but control over curvature yields the following theorem.

Theorem 1.8. *If X has curvature ≥ 1 and contains a spherical set Σ of $2(n+1)$ points, then there is a subset S of X isometric to S^n such that $\Sigma \subset S$.*

Theorem 1.9. *If X has curvature ≥ 1, $\dim(X) \leq n < \infty$, and X contains a spherical set of $2(n+1)$ points, then X is isometric to S^n.*

Theorem 1.8 is used in Section 5 of this article. The standard hemisphere shows that Theorem 1.9 cannot be improved by removing even a single point from the required set, and spheres of smaller dimension show that the determinant condition cannot be removed. By relaxing the criteria of Theorem 1.9 and applying the results of [P2] and [Y] we also obtain two differentiable sphere theorems (Theorems 4.7 and 4.9).

If $p \in X$ lies in a region of curvature $\geq k$, the *space of directions* $\bar{S}_p$ at a point p is defined to be the metric completion of the *space of geodesic directions* S_p. S_p, in turn, consists of all arclength parameterized geodesics (of maximal domain of definition) starting at p, with the angle metric. Prior to Theorem 1.4 it was not known whether S_p, defined in this way, was even *nonempty* without the assumption of local compactness. It is still unknown whether, in general, $\bar{S}_p$ is an inner metric space unless it is known to be compact or the space is geodesically complete [P1]. We let $\bar{T}_p$ denote the metric cone on $\bar{S}_p$, and call it the *tangent space* at p. One central difficulty that occurs when one removes either the upper curvature bound or geodesic completeness from the traditional assumptions of synthetic differential geometry is that compactness of $\bar{S}_p$ and finite dimension no longer follow from local compactness of X. In general, only after assuming finiteness of dimension and showing $\bar{S}_p$ is then a compact inner metric space of bounded curvature does $\bar{T}_p$ become a useful approximation of X near p. This strategy was first carried out by us in 1989 for spaces of curvature bounded both above and below (cf. [P2]). In Section 5 we show that $\bar{S}_p$ can be understood to a certain extent without any assumptions on dimension. Using the almost extendability part of Theorem 1.4 we find a spherical set in the space of directions at a dense set of points. We then construct an inner metric space contained in the space of directions and containing the spherical set. Theorem 1.8 can then be applied to produce an actual sphere (Theorem 5.6). This construction motivates the following definition.

Definition 1.10. If X has curvature locally bounded below we let

$$N^n(X) = \{p \in X \colon \bar{S}_p \text{ has a subset isometric to } S^{n-1}\},$$

and define the *local dimension* of X by $\operatorname{ldim}(X) = \sup\{n\colon\ N^n(X) \neq \emptyset\}$.

Theorem 1.11. *If X has curvature bounded below and* $\operatorname{ldim}(X) \geq n$ *then* $N^n(X)$ *contains a dense* G_δ.

Corollary 1.12. *If X has curvature bounded below and* $\operatorname{ldim}(X) = n$, *then for any* $p \in N^n(X)$, $\bar{S}_p$ *is isometric to* S^{n-1}. *X is locally compact and for each* $p \in X$, $\bar{S}_p$ *is a compact inner metric space of curvature* ≥ 1. *Every compact subset of X has finite n-dimensional Hausdorff measure.*

Corollary 1.13. *If X has curvature bounded below, then* $\dim(X) \leq \operatorname{ldim}(X)$.

Corollary 1.14. *If X has curvature bounded below and for some* $p \in X$, $\bar{S}_p$ *is isometric to* S^n, *then* $\operatorname{ldim}(X) = n$.

Corollary 1.15. *If X has curvature bounded below and* $\dim(X) = \infty$, *then there is a dense* G_δ *subset* N^∞ *of X such that for all* $p \in N^\infty$ *and* $n \geq 0$, $\bar{S}_p$ *has a subset isometric to the unit sphere in* ℓ^2.

Note that by the results of [P3], if $\operatorname{ldim}(X_i) \leq n$ and the Gromov-Hausdorff limit X of the X_i's has curvature bounded below, then $\operatorname{ldim}(X) \leq n$.

In light of Theorem 1.8, the notion of local dimension is essentially the same as that of "strain number" [BGP2]. It follows from Theorem 1.12 and results of [BGP2] that if $\operatorname{ldim}(X) < \infty$ then $\operatorname{ldim}(X) = \operatorname{hdim}(X) = \dim(X)$. It remains an interesting open question whether $\dim(X) < \infty$ implies $\operatorname{ldim}(X) < \infty$. In fact this seems to be unknown even if X is a topological manifold. However, this implication is true in the following situation. The exponential map $\exp_p$ is defined on a subset of $\bar{T}_p$ as the unique continuous extension of the map $t\gamma \mapsto \gamma(t)$, $\gamma \in S_p$ to the closure of its domain of definition. Proposition 2.10, [P1], can be restated in the following way.

Proposition 1.16. *If X has curvature* $\geq k$, *then at any point p,* $\exp_p$ *does not decrease topological dimension and does not increase Hausdorff dimension.*

The following corollary is immediate.

Corollary 1.17. *If X has curvature bounded below,* $\dim(X) = n$ *and there exists a* $p \in N^n(X)$ *such that the domain of definition of* $\exp_p$ *contains an open set, then* $\operatorname{ldim}(X) = n$.

A point p is called a *geodesic terminal* [P1] if there are a point q and a minimal curve γ from q to p which cannot be extended as a geodesic beyond p. If there are no geodesic terminals in $B(p, r)$ then $\exp_p$ is defined on $B(0, r) \subset \bar{T}_p$.

Corollary 1.18. *If X has curvature bounded below and the geodesic terminals in X are not dense, then* $\dim(X) = \mathrm{ldim}(X)$.

An example of Otsu-Shioya [OS] shows that the hypothesis of Corollary 1.18 need not be satisfied even for a topological sphere. However, this example is arbitrarily Lipschitz close to spaces having nowhere dense terminals. It would be interesting to know whether approximations by spaces with nowhere dense terminals exist more generally.

Corollary 1.18 and Theorem 1.4 imply that results of [P1], [P2], [P3], which require both finite topological dimension and either local compactness or existence of minimal curves, are true without either of the last two assumptions. In connection with this improvement in [P2], it is interesting to observe that even curvature bounded above *and* below has no significant consequences if the metric is not inner. Simply possessing an inner metric without curvature bounds also has few topological consequences. On the other hand, a finite-dimensional space which is both inner and has curvature bounded above and below, is, by [P2], a smooth manifold with boundary.

We conclude this section with a few background results. The next proposition is essentially proved in [B], and gives a more useful curvature tool than Definition 1.1. By S_m we mean the two-dimensional simply connected space form (as opposed to S_m^3).

Proposition 1.19. *Any four points a, b, c, d in a region U of curvature $\geq k$ can be isometrically embedded in S_m for some $m \geq k$. Furthermore, $\alpha_k(a; b, c) + \alpha_k(a; b, d) + \alpha_k(a; c, d) \leq 2\pi$ and any minimal curve triangle in U satisfies Alexandrov's comparison conditions for curvature $\geq k$.*

We denote the three equivalent Alexandrov curvature conditions in the following way. Let T be a triangle of minimal curves in a region U of curvature $\geq k$. Condition A0 states that there is a representative T' in S_k (same side length) and the distance from any corner of T to a point on the opposite side is $\geq$ the corresponding distance on T'. Condition A1 also states that T' exists, and the angles of T are all $\geq$ the corresponding angles on T'. If W is a wedge in U (two minimal curves starting at a common point), then Condition A2 states that there is a representative W' in S_k (same angle and side lengths), and the distance between the endpoints opposite the angle in W is $\leq$ the corresponding distance in W'. We say A0, A1, or A2 hold *with equality* if the above inequalities can be replaced by equalities. As an immediate consequence of the definition of the angle between minimal curves and Proposition 1.19, we have the following extension of the theorem of complementary angles in [R].

Corollary 1.20. *If $\pi - \alpha(\beta, \gamma) < \varepsilon$ then for any minimal curve ζ with the same initial point, $\alpha(\beta, \zeta) + \alpha(\zeta, \gamma) < \pi + \varepsilon$.*

The *strong excess* of a triple is $\sigma(b; a, c) = \varepsilon(b; a, c)/\min\{d(a, b), d(c, b)\}$. From the elementary geometry of S_k it is not hard to prove that $\sigma(b, a, c)$ is small if and only if $\pi - \alpha_k(b; a, c)$ is small, with the exact relationship depending on k. (For $k > 0$ we require that $d(a, b) + d(b, c) < \pi/\sqrt{k}$). We will use these two quantities interchangably.

2. The Extended Hopf-Rinow Theorem

Lemma 2.1. *Let p, q be distinct points in an inner metric space X. For every positive σ and $0 < \delta < d(p,q)$, there exists a point $q' \in X$ such that $d(q,q') = \delta$ and $\sigma(q'; p, q) < \sigma$.*

Proof. Without loss of generality we can assume $\delta < d(p,q)/2$. Let γ be a curve from p to q such that $\ell(\gamma) < d(p,q) + \sigma\delta$. By the definition of the length of a curve and the triangle inequality, for any point x on γ, $\varepsilon(x; p, q) < \sigma\delta$. By the intermediate value theorem there is a point q' on γ such that $d(q,q') = \delta$, which meets the requirements of the lemma. □

Definition 2.2. Let p be a point in X, a complete inner metric space. We define J_p to be the G_δ set

$$\bigcap_{i=1}^{\infty}\{y \in X \colon \exists\, z \in X \text{ such that } \sigma(y; p, z) < 2^{-i}, d(y,z) < 2^{-i}\}.$$

Proposition 2.3.. *For any $p \in X$, J_p is dense.*

Proof. Fix $q_0 \in X$ and positive $\eta_0 < 1$, and suppose we wish to find a point $q \in J_p$ such that $d(q_0, q) \leq \eta_0$. We will construct q as a limit of an inductively defined Cauchy sequence $\{q_j\}$. Using Lemma 2.1, let q_1 be a point such that $d(q_1, q_0) = \eta_0/2$ and $\sigma(q_1; p, q_0) < 1/4$. By continuity of the distance function there exists an $\eta_1 < \eta_0/2$ such that if $d(x, q_1) < \eta_1$, then $\sigma(x; p, q_0) < 1/2$. Now suppose we have found the following for all $i < j$: points q_i and numbers η_i, where $d(q_i, q_{i-1}) = \eta_{i-1}/2$, and $\eta_i < \eta_{i-1}/2$ is such that if $d(x, q_i) < \eta_i$ then $\sigma(x; p, q_{i-1}) < 2^{-i-1}$. We can now pick q_j such that $d(q_j, q_{j-1}) = \eta_{j-1}/2$ and $\sigma(q_j; p, q_{j-1}) < 2^{-j}$, and η_j satisfying the necessary property by continuity of the distance function. Since $d(q_k, q_m) \leq \sum_{i=k}^{m-1} d(q_i, q_{i+1}) < \sum_{i=k}^{m-1} 2^{-i}\eta_0$, $\{q_j\}$ is Cauchy, and $q = \lim q_j \in B(q_0, \eta_0)$. Furthermore, for any j, $d(q, q_j) < \eta_j < 2^{-j}$ and $\sigma(q; p, q_j) < 2^{-j}$. Therefore $q \in J_p$. □

Proposition 2.4. *Let $p, q \in X$ with $q \in J_p$. If $c_k(q) \geq r$ then there exists a unique point x such that $d(x,q) = r$, $\varepsilon(x; p, q) = 0$. Furthermore, there is a unique minimal curve joining q and x.*

Proof. Let x_i be such that $\varepsilon(x_i; p, q) < 2^{-i}$ and $d(q, x_i) = r$. We claim that $\{x_i\}$ is Cauchy. Note that the definition of J_p implies that for any $\delta > 0$ there exists an $M > 0$ and a point $q' \in B(p, r)$ such that for all $i > M$, $\sigma(q; x_i, q') < \delta$. Now if $\sigma(q; x_i, q')$ and $\sigma(q;' x_j, q')$ are small, $\alpha_k(q; x_i, q')$ and $\alpha_k(q; x_j, q')$ are close to π and by Proposition 1.18, $\alpha_k(q; x_i, x_j)$ is small. Since $d(q, x_i) = d(q, x_j)$, $d(x_i, x_j)$ is also small, $\{x_i\}$ is Cauchy. Now $x := \lim x_i$ satisfies $\varepsilon(x; p, q) = 0$, and x is unique because the sequence $\{x_i\}$ was arbitrarily chosen. This proves the first part of the theorem. To finish the proof of the theorem, we can apply the first part to x and q to construct a unique "midpoint" $m \in X$ between them. Proceeding with this standard construction we can produce an isometry of the dyatic rationals in $[0, d(x, q)]$, which can be extended to a minimal curve joining them. The minimal curve is unique because m is unique. $\square$

Proof of Theorem 1.4. We will first prove that if $q \in J_p$ then there is a unique minimal curve γ between p and q. The curve will be constructed starting a q, and extended toward p. Since $c_k(q) \geq r > 0$ for some r and k, by Proposition 2.4 there is a unique point x_1 such that $d(q, x_1) = r$, $\varepsilon(x_1; p, q) = 0$, and there is a unique minimal curve from q to x_1. Let Ω denote the supremum of all ω such that there exists an $x \in X$ such that (1) $d(x, q) = \omega$, (2) $\varepsilon(x; p, q) = 0$ and (3) there is a unique minimal curve joining q and x. Then $\Omega \geq r$. Let ω_i approach Ω from below and choose corresponding points x_i and minimal curves γ_i. Since the minimal curves are unique, each γ_i is an extension of γ_{i-1}; hence the sequence x_i is Cauchy with a limit point x. By continuity of the distance function, x satisfies (1), (2), and (3), except possibly for the uniqueness of the minimal curve $\gamma = \lim \gamma_i$ joining q and x. Let β be any minimal curve from q to p and y be on β such that $d(q, y) = r$. Then $\varepsilon(y; x, q) = 0$ and so

$$0 \leq \varepsilon(y; p, q) \leq d(p, x) + d(x, y) + d(y, q) - d(p, q) = \varepsilon(x; p, q) = 0.$$

But by uniqueness of x_1, $y = x_1$, and $\beta = \gamma$.

We have reduced the proof to showing that $\Omega \geq d(p, q)$. Suppose otherwise. By the above argument we have a point x satisfying (1), (2), (3) for $\omega = \Omega$; denote by γ the unique minimal curve from q to x. Then $c_K(x) \geq R > 0$ for some $R < d(x, q)$ and K. We need to show $x \in J_p$. Choose points w_i on γ such that $d(x, w_i) = 2^{-i}$. Then $\varepsilon(x; p, q) = \varepsilon(w_i; x, q) = 0$ implies

$$\begin{aligned} 0 \ &\leq \ \varepsilon(x; w_i, p) = d(p, q) - d(x, q) + d(x, w_i) - d(p, w_i) \\ &= d(p, q) - d(x, w_i) - d(w_i, q) + d(x, w_i) - d(p, w_i) \leq 0, \end{aligned}$$

i.e., $x \in J_p$. Now we can apply Proposition 2.4 to find a unique point x' such that $d(x', x) = R$ and $\varepsilon(x'; p, x) = 0$ and a minimal curve γ' from x to x'. Now $\varepsilon(x; p, q)$ implies

$$0 \leq \varepsilon(x; x', q) = d(p, x) - d(p, x') + d(p, q) - d(p, x) - d(x', q) \leq 0.$$

In other words, γ and γ' together form a minimal curve from q to x'. The curve is unique, again because x_1 is unique.

To complete the proof of Theorem 1.4, note that we can find a $z \in J_q$ such that $\sigma(q; p, z)$ is small, and hence $\pi - \alpha_k(q; p, z)$, is arbitrarily small. By what we have just proved, we can find a minimal curve α from q to z. By A1, $\pi - \alpha(\gamma, \alpha)$ is small, and this proves that γ is almost extendable. $\square$

Corollary 2.5. *If $B = B(p, r)$ is a region of curvature $\geq k$ in X, $q, x \in B$ are such that $\varepsilon(x; p, q) = 0$ then there is a minimal curve from p to q.*

Proof. Since $\sigma(x; p, q) = \varepsilon(x; p, q) = 0$, Proposition 2.1 implies that there are minimal curves β from p to x and γ from x to q. But β and γ together form a minimal curve from p to q. $\square$

Remark 2.6. Note that what we have in fact proved is that J_p consists of points q which can be connected to p via a strictly minimal curve which is *almost strictly extendable* past q in the sense that for any $\varepsilon > 0$ there exists a point q' such that $\pi - \alpha_k(q; p, q') < \varepsilon$. This is strictly stronger than almost extendable (even in the Riemannian case) because it forces uniqueness of the joint between p and q. This result can be considered as a strengthened extension of the fact that, for Riemannian manifolds, the non-conjugate cut locus is contained in an F_σ. In the Riemannian case the cut locus is also nowhere dense; in our case, the F_σ is at most first category. $\square$

Example 2.7. In a geodesically complete space of curvature $\geq k$, every minimal curve is extendable and hence almost extendable. On the other hand, let X be the space obtained by gluing a half circle of diameter d to a side S of a square of area d^2. The two sides of the square perpendicular to S are minimal curves which cannot be extended past their intersection with the boundary of the circle, but are clearly *almost* extendable. By gluing two copies of X along their boundaries, we obtain the same situation in a manifold without boundary.

Example 2.8. Let E be the infinite-dimensional ellipsoid given in [A] and let p, q be the two points in E which cannot be joined by a minimal curve. Take the one point union E' of E with another inner metric space X at q. Then no point in X can be joined to p via a minimal curve, and the set of points joined to p is not dense. Of course q is a branch point in E' so E' has no lower curvature bound. We do not know of an example of an inner metric space *without branch points* in which the set of points joined to a given point via a minimal curve is not dense. If $\mathcal{E}$ denotes an infinite product of copies of E then the point $P = (p, p, \ldots)$ cannot be joined via a minimal curve to a dense subset $\mathcal{Z}$ of $\mathcal{E}$. In other words, J_P, which is contained in the complement of $\mathcal{Z}$, cannot contain an open set.

Corollary 2.9. *Suppose $B(p, r)$ is a region of curvature $\geq k$. For any set $\{x, x_1, \ldots, x_n\} \subset B(p, r/3)$, there exist points $x_1', \ldots, x_n'$ arbitrarily close to $x_1, \ldots, x_n$, respectively, such that each pair of points in $\{x, x_1', \ldots, x_n'\}$ is joined by a minimal curve.*

Proof. If $z \in B(p, r/3)$ then $B(z, 2r/3)$ is a region of curvature $\geq k$ which includes $B(p, r/3)$. The proof is by induction on n; the case $n = 1$ is immediate from Theorem 1.4. Suppose we have chosen points $x_1', \ldots, x_k'$ arbitrarily close to $x_1, \ldots, x_k$, respectively, so that all pairs in $\{x, x_1', \ldots, x_k'\}$ can be joined by a minimal curve. The sets J_x, $J_{x_i'}$ for $i \leq k$ are dense G_δ's, and so is their intersection I by the Baire category theorem. To complete the induction step, we simply choose $x_{k+1}' \in I$ close to x_{k+1}. $\square$

Following Berestovskii's recent work we can state the next corollary; given Theorem 1.4, the proof involves only minor modifications of the results in [B]. From this corollary, all of the traditional theory involving Alexandrov's comparisons, such as existence of angles, can be built up in the present more general setting.

Corollary 2.10. *If $x \in X$, $c_k(x) > 0$ if and only if (1) x lies in a region of curvature bounded below in the original sense of Alexandrov and (2) there is an open set U about x such that every $y \in U$ is joied by a minimal curve to each point in a dense G_δ subset J_y of U.*

3. The Global Comparison Theorem

If X has curvature locally bounded below and there are at most two directions at some point, then using Theorem 1.4 it is easy to verify that X is isometric to a circle or an interval. In this case the global comparison theorem has no meaning, and some of the constructions given in the remainder of this article fail. *From now on we will always assume that if $c_k(x) > 0$, then there are at least three directions at x.*

Given a triple $(b; a, c)$ such that a and c are joined to b by minimal curves, we can make sense of the Conditions A1 and A2 in terms of the angle between minimal curves γ_{bc} and γ_{ba}. We say that a triple $(b; a, c)$ is A1 if A1 and A2 hold for every pair of minimal curves γ_{bc} and γ_{ba}. By continuity of the distance function, in order to prove Theorem 1.5 it suffices to prove that all quadruples of points from some dense set of X can be embedded in S_m, $m \geq k$. In light of [B] and Theorem 1.4, we need only show that every triple $(b; a, c)$, such that a and c are joined to b by a unique minimal curve, is A1. *From now on, we will require that any triple $(b; a, c)$ be joined in this way. We will take the comparison radius to mean the maximal radius of a ball in which every triple is A1.* We will denote $\alpha(\gamma_{bc}, \gamma_{ba})$ by $\alpha(b; a, c)$. In proving a triple $(b; a, c)$ is A1, it will sometimes be necessary to introduce new points. Because the new points are sometimes required to lie on some given minimal curve, we cannot assume by Theorem 1.4 *alone* that they are joined to a and c by unique minimal curves. However, there exist points a_i, c_i arbitrarily close to a and c that are uniquely joined to any given finite set of points. By the definition of the angle, $\limsup_{i\to\infty} \alpha(b; a_i, c_i) \leq \alpha(b; a, c)$; since the distance function is continuous it suffices to prove only that $(b; a_i, c_i)$ is A1 for all i. In other words, when all the requirements imposed on a and c are *open* conditions, we can, in fact, always assume that the points we introduce can be joined to a and c by unique minimal curves. We will use this assumption without further comment (this includes the statement of Lemma 3.1).

The next lemma is a standard trick in proofs of Toponogov theorem (cf. [CE]). The proof is essentially the same as in the Riemannian case.

Lemma 3.1. *Let $(b; a, c)$ be a triple. Suppose there is a partition $x_0 = b, \ldots, x_n = c$ of γ_{bc} such that $(x_i; a, x_{i+1})$ for $0 \leq i < n$, and $(x_i; a, x_{i-1})$ for $0 < i < n$, are A1, then $(b; a, c)$ is A1.*

Proposition 3.2. *Suppose $c_k \geq \kappa$ for some $\kappa > 0$ on $B = B(a, D) \subset X$. Then every triple $(b; a, c)$ in B such that $d(a, b)$, $d(a, c) < D - \kappa$ and $d(b, c) < 2\kappa$, is A1.*

Proof. Suppose first that $D < \pi/\sqrt{k}$. Choose positive $\chi < \min\{\kappa, \pi\}/8$ such that the following two conditions hold.

(a) for any $R < D$ if $A, B, C \in S_k$ are such that $d(A, B) < R$, $d(A, C) < R - 2\chi$ and $d(B, C) < 4\chi$ then $d(A, F) < R$ for all F on Γ_{BC}.

If $k \leq 0$ any $\chi < \kappa/8$ will work. If $k > 0$, note that by monotonicity we can assume $d(A, B) > d(A, C)$; by convexity we can suppose $d(A, B) > \pi/2\sqrt{k}$. If $\alpha(\chi) = \sup\{\alpha(B; A, C)\}$ over all possible triangles for a given χ, then by the spherical cosine law,

$$\lim_{\chi \to 0} \cos\alpha(\chi) > \lim_{\chi \to 0} \frac{\cos\sqrt{k}(R - 2\chi) - \cos\sqrt{k}R\cos\sqrt{k}4\chi}{\sin\sqrt{k}4\chi} = \frac{\sin\sqrt{k}R}{2} > \frac{\sin\sqrt{k}D}{2}.$$

In other words, for small χ, $\alpha(\chi) < \pi/2$ is bounded away from $\pi/2$ by a positive number depending only on D, which completes the argument.

(b) if $A, B, C \in S_k$ are such that $d(A, B) \geq \kappa$, $d(A, B) - d(A, C) > \chi$ and $d(B, C) := a < 3\chi$, then $\alpha(B; A, C)$ is an increasing function of a.

We prove only the spherical case $k = 1$; the other cases are similar. Let $R = d(A, B)$ and $R' = d(A, C)$. By the spherical cosine law we need to show that the derivative of

$$f(a) = \frac{\cos(R') - \cos(a)\cos(R)}{\sin(a)\sin(R)}$$

is negative for small enough χ and $a < 3\chi$. This reduces to showing $\cos(R) < \cos(R')\cos(a)$ or $\cos(R) < g(\chi) := \cos(R - \chi)\cos(3\chi)$ for suitably small χ. But this is immediate from $g'(0) = \sin(R) > \min\{\sin(\kappa), \sin(D)\}$.

We will show by induction that the following statement holds for every $n \leq (D - \chi)/\chi$:

$S(n)$. If $(b; a, c)$ is a triple such that $d(a, b), d(a, c) < n \cdot \chi$ and $d(b, c) < \chi$, then $(b; a, c)$ is A1.

An immediate consequence of $S(n)$ and Lemma 3.1 is that if $d(a, b)$, $d(a, c) < n \cdot \chi$ and any minimal curve γ from b to c lies in $B(a, n\chi)$ then $(b; a, c)$ is A1 (subdivide γ into intervals of length $< \chi$). By the way χ was chosen, $S(n)$ is true for $n \leq 8$; suppose $S(n)$ is valid for some $n \geq 8$ and let b, c be such that $d(a, b)$, $d(a, c) < (n + 1) \cdot \chi$ and $d(b, c) < \chi$. Let d be on γ_{ab} such that $d(d, b) = 3\chi$. Since $d(d, c) < 4\chi$, $(d; b, c)$ and $(b; c, d)$ are both A1. Lemma 3.1 reduces the proof to showing: If $(d; a, c)$ is a triple such that $d(a, d) < (n - 2)\chi$ and $d(d, c) < 4\chi$ then $(d; a, c)$ is A1. Let $A, D, C \in S_k$ be such that $\alpha(D; A, C) = \alpha(d; a, c)$, $d(a, d) = d(A, D)$, and $d(d, c) = d(D, C)$.

Suppose first that $d(A, C) < n\chi$. Compactness and (a) imply that there is $\zeta > 0$ such that for any F on Γ_{DC}, $d(A, F) < n\chi - \zeta$. To show $(d; a, c)$ is A1, it suffices to prove γ_{dc} lies in $B(a, n\chi)$. If γ_{dc} doesn't lie in $B(a, n\chi)$ there is a point f on γ_{dc} closest to d such that $d(a, f) = n\chi$. Let f' be on γ_{dc} between f and d arbitrarily close to f. Then $(a; f', d)$ is A1 so $d(a, f') \leq n\chi - \zeta$ and, hence, $d(a, f) \leq n\chi - \zeta$, a contradiction.

We can now assume that $d(A, C) \geq n\chi > d(A, D)$ and also that $d(a, c) > n\chi$ since otherwise $(d; a, c)$ is A1 by definition. We will show that for all $\omega, \delta > 0$ there exist points $d' \in X$ and $D' \in S_k$ such that the following properties hold.

(1) $\alpha(d'; a, c) \leq \alpha(D'; A, C) + \delta$,

(2) $d(a, d') = d(A, D') = d(a, d)$,

(3) $d(d', c) \leq d(D', C) + \delta$,

(4) $\varepsilon(D'; A, C) < \omega$.

This will complete the inductive argument, since then $d(a, c) \leq d(a, d') + d(d', c) \leq d(A, D') + d(D', C) + \delta \leq d(A, C) + \delta + \omega$.

Certainly d and D satisfy these properties for possibly large ω and any δ. Let Ω be the infimum of all ω such that for all $\delta > 0$ there exist d', D' satisfying (1)–(4), and d' is joined by minimal curves to a and c; suppose $\Omega > 0$. For $\omega_i \to \Omega$ and $\delta_i \to 0$, choose d_i', D_i' satisfying (1)–(4) for ω_i and δ_i: we can assume $D_i' \to D' \in S_k$. Let E_i be the point on $\Gamma_{AD_i'}$ such that $d(E_i, D_i') = \chi$, and D_i'' be the point on Γ_{E_iC} such that $d(A, D_i'') = d(A, D)$. If D'' denotes the corresponding point constructed for D' then $D_i'' \to D''$. Since $d(D', C) - d(D'', C) > 0$, $\varepsilon(D''; A, C) < \Omega$ and $\varepsilon(D_i''; A, C) < \Omega$ for large i. We will obtain a contradiction by constructing d_i'' such that conditions (1)–(3) are satisfied by D_i'' and d_i'' and numbers $\delta_i' \to 0$. For simplicity, we will use δ_i' several (but finitely many) times to denote numbers tending to 0; the final choice of δ_i' will be the maximum of these various values.

Since $d(a, d_i') \geq d(a, c) - d(d_i', c) > (n-4)\chi \geq 4\chi$, we can choose e_i on the minimal curve from a to d_i' such that $d(e_i, d_i') = \chi$. Then e_i, d_i', c all lie in a region of curvature $\geq k$. By continuity of the distance function, we can choose d_i'' on γ_{e_ic} such that $d(d_i'', a) = d(d, a)$ and d_i'' can be joined to a by a unique minimal curve. Now $d(a, c) > n\chi$ implies $d(d_i', c) > 2\chi = 2d(D_i', E_i)$. Since $d(D_i', C) < 4\chi < \pi/2$, if we were to move C along the minimal curve from C to D_i' until $d(C, D_i') = d(c, d_i') - \delta_i$ we would shorten $d(E_i, C)$. By A2 and (1) (for d_i' and D_i'), $d(e_i, c) \leq d(E_i, C) + \delta_i'$.

If we were to decrease the angle at D_i', reducing $d(E_i, C)$ to $d(e_i, c) - \delta_i'$ we would increase $\alpha(E_i; D_i', C)$; it now follows from A1 that $\alpha(e_i; d_i', c) \geq \alpha(E_i; D_i', C) - \delta_i'$ and hence that $\alpha(e_i; a, d_i'') \leq \alpha(E_i; A, D_i'') + \delta_i'$. Note that $d(d_i'', e_i) = d(e_i, c) - d(d_i'', c) < 5\chi - (n\chi - (n-2)\chi) = 3\chi$ so that the minimal curve from d_i'' to e_i lies in $B(a, n\chi)$. It follows that both $(e_i; d_i'', a)$ and $(d_i''; e_i, a)$ are A1. We now claim $d(e_i, d_i'') \geq d(E_i, D_i'') - \delta_i'$. Otherwise, there exists an $\varepsilon > 0$ such that $d(e_i, d_i'') < d(E_i, D_i'') - \varepsilon$ for infinitely many i. For such i we could decrease $d(E_i, D_i'')$ to $d(e_i, d_i'')$ by an amount greater than ε. Since $d(E_i, D_i'') > \chi$, we would increase $\alpha(E_i; A, D_i'')$ by at least an amount $\varepsilon' > 0$ independent of i. A1 then implies

$\alpha(e_i; d_i'', a) > \alpha(E_i; D_i'', A) + \varepsilon$, a contradiction. Thus $d(e_i, d_i'') \geq d(E_i, D_i'') - \delta_i'$ and $d(d_i'', c) \leq d(D_i'', C) + \delta_i'$, proving (3).

Finally, if we were to increase $d(E_i, D_i'')$ to $d(e_i; d_i'') + \delta_i'$ then by (b) we would increase $\alpha(D_i''; E_i, A)$; i.e., A1 implies that $\alpha(d_i''; e_i, a) \geq \alpha(D_i''; E_i, A) - \delta_i'$. This shows $\alpha(d_i''; c, a) \leq \alpha(D_i''; C, A) + \delta_i'$ and proves (1).

We now complete the proof of the proposition. If γ_{bc} lies in $B(a, \pi/\sqrt{k})$ then γ_{bc} lies in some $B(a, D)$, $D < \pi/\sqrt{k}$, we can subdivide into segments of length χ and apply Lemma 3.1. This completes the proof for $k \leq 0$. Suppose $k > 0$ and suppose $p \in X$ such that $d(p, a) = \pi$. Let $q \in J_a \cap J_p$. Then we have shown that for $p_i \to p$ along γ_{qp}, $(q; a, p_i)$ is A1. By continuity it follows that γ_{aq} and γ_{qp} together form a minimal curve from a to p through q. If we choose $w \in J_a \cap J_p$ such that $\alpha(a; q, w) > 0$ (we can do this because there are at least three directions at (a), then we have a second distinct minimal curve from a to p. Thus no minimal curve passes strictly through a point of distance $\pi/\sqrt{k}$ from a, and we have shown that the diameter of X is at most $\pi/\sqrt{k}$. For any point p' such that $d(p', a) = \pi/\sqrt{k}$ and $q' \in J_a \cap J_p \cap J_{p'}$, from the above argument we have minimal curves from a to p and p' through q'. But these would have to coincide between a and q', which means $p = p'$. The only remaining case is that the unique "antipodal" point $p \neq c$ lies on γ_{bc}. If $b = p$, we can consider triples $(p_i; a, c)$ with $p_i \to p$ and $p_i \in J_a \cap J_c$, and apply semicontinuity of the angle to complete the proof. If $p \neq b$ then $\alpha(b; a, c) = \pi$ and we need only show a representative exists. But we can choose $c_i \in J_p \cap J_a$ with $c_i \to c$. Continuity and the fact that c_i lies on a minimal curve from a to p imply $d(a, c) = \pi - d(p, c)$, which is the required distance. □

Corollary 3.3. *If $c_k \geq \kappa$ for some $\kappa > 0$ on $B(p, D) \subset X$ then $c_k(p) \geq D/3$.*

Proof. If $(b; a, c)$ is a triple in $B(p, D/3)$ then $c_k \geq \kappa$ on $B(a, 2D/3)$. Since γ_{bc} lies inside $B(a, 2D/3)$, we can subdivide it into segments of length $\leq 2\kappa$ and apply Lemma 3.1 and Proposition 3.2 to complete the proof. □

Corollary 3.4. *If $c_k(p) < \rho$ then for every $\varepsilon > 0$ there exists a point $q \in B(p, 3\rho)$ such that $c_k(q) < \varepsilon$.*

Proof of Theorem 1.5. Suppose there is a triple $(b; a, c)$ in X which is not A1. Then if $\rho = \max\{d(a, c), d(a, b)\}$, $c_k(a) < \rho$. By Corollary 3.4 there exists a point $x_1 \in B(a, 3\rho)$ such that $c_k(x_1) < \rho/6$. But then there exists a point $x_2 \in B(x_2, \rho/2)$ such that $c_k(x_2) < \rho/12$. Continuing in this way we can construct a Cauchy sequence $\{x_i\}$ such that $c_k(x_i) \to 0$. But if $\lim x_i = x$, $c_k(x) = \delta$ for some $\delta > 0$. Then for all large enough i, $d(x_i, x) < \delta/2$ and $c_k(x_i) \geq \delta/2$, a contradiction. □

4. The Sphere Theorems

The next lemma was proved in [GP2] assuming existence of minimal curves.

Lemma 4.1. *Suppose X has curvature ≥ 1 and $x, y \in X$ satisfy $d(x, y) = \pi$. For any point $z \neq x, y$ in X, there is a minimal curve from x to y through z. In consequence, any minimal curve starting at x can be extended as a minimal curve from x to y.*

Proof. Let $z_i \to z$ be such that there exist unit minimal curves α_i and β_i from z_i to x and y, respectively. Then A1 implies $\alpha(\alpha_i, \beta_i) = \pi$; that is, α_i and β_i form a unit minimal curve γ_i from x to y through z_i. We claim that the sequence $\{\gamma_i\}$ is uniformly convergent. In fact, A0 and the monotonicity of distance as a function of angle implies that if $d(z_i, z_j)$ is small then $d(\gamma_i(t), \gamma_j(t))$ is uniformly small for all $t \in [0, \pi]$. Now $\gamma := \lim \gamma_i$ is the desired minimal curve. □

Definition 4.2. A subset Y of X is said to be *metrically embedded* if the induced metric on Y is an inner metric.

If Y is locally compact, the following are trivially equivalent: (1) Y is metrically embedded in X, (2) the induced metric and induced inner metric ([P1]) on Y are equal, and (3) every pair of points in Y is joined by a curve which is minimal in the metric of X and lies in Y. The term "metrically embedded" therefore has the same meaning as "convex" in [P1] and [R]; but we have decided to abandon the latter term due to possible confusion with other usages. If Y (with the induced metric) is isometric to an inner metric space (e.g. the sphere) then Y is automatically metrically embedded. Note that if Y is a submanifold of a Riemannian manifold then the distance derived from the induced Riemannian metric on Y is the induced *inner* metric on Y. If X has no branch points (in particular if X has curvature bounded below) and Y is geodesically complete and metrically embedded then Y is totally geodesic in the sense that any geodesic of X which lies in Y for some positive length lies entirely in Y.

Lemma 3.3. *Suppose X has curvature ≥ 1 and γ is a metrically embedded geodesic loop in X of length 2π. For any geodesic β starting at $\gamma(0)$, the wedge $(\beta|_{[0,s]}, \gamma|_{[0,t]})$ is A2 with equality for all $s, t \in (0, \pi)$.*

Proof. Since γ is metrically embedded, every segment of γ of length $\leq \pi$ is minimal. We need to show that if γ', β' are minimal in S_k such that $\alpha(\gamma', \beta') = \alpha(\gamma, \beta)$ then for all $s, t \in (0, \pi)$, $d(\gamma(s), \beta(t)) = d(\gamma'(s), \beta'(t))$. By Lemma 4.1, $d(\gamma(s), \beta(t)) = \pi - d(\gamma(s + \pi), \beta(t))$, and A2 implies

$$\begin{aligned}\pi - d(\gamma(s+\pi), \beta(t)) &\geq \pi - d(\gamma'(s+\pi), \beta'(t))\\ &= d(\gamma'(s), \beta'(t))\\ &\geq d(\gamma(s), \beta(t)).\end{aligned}$$ □

Lemma 4.4. *If X has curvature ≥ 1 and γ is a closed geodesic in X having a minimal segment of length π then γ is metrically embedded of length 2π.*

Proof. By reparameterizing if necessary, we can assume that γ is minimal from $p = \gamma(0)$ to $q = \gamma(\pi)$. For sufficiently small $\varepsilon > 0$, $\beta = \gamma|_{[\pi-\varepsilon,\pi+\varepsilon]}$ and $\eta = \gamma|_{[0,\pi-\varepsilon]}$ are also minimal. Letting $\zeta = \gamma|_{[\pi+\varepsilon,2\pi]}$ and applying A0 to the triangle (ζ, β, η) we obtain that $d(\gamma(s), \gamma(s+\pi)) = \pi$ for all $s \leq \varepsilon$. If $S = \sup\{s\colon d(\gamma(s), \gamma(s+\pi)) = \pi\}$ and we suppose $S < \pi$, we can apply the same argument to $p' = \gamma(S)$ and $q' = \gamma(S+\pi)$ to obtain a contradiction. Therefore $d(\gamma(s), \gamma(s+\pi)) = \pi$ for all s and so the induced metric on γ is intrinsic. □

Lemma 4.5. *Suppose X has curvature ≥ 1 and contains a subset S isometric to S^n, for some n, with the induced metric. If there exist points $x, y \in X\backslash S$ such that $d(x, y) = \pi$, then X contains a subset S' isometric to S^{n+1} containing x, y, and S.*

Proof. Let $Z = \{\gamma \in S_x\colon \gamma(s) \in S \text{ for some } s < \pi\}$ and $S' = \{p = \gamma(t)\colon \gamma \in Z$ and $t \leq \pi\}$. By Lemma 4.1 if $\gamma \in Z$ then $\gamma|_{[0,\pi]}$ is minimal from x to y and $\gamma|_{[0,s]}$ is the unique minimal curve from x to $\gamma(s) \in S$. Furthermore, since S is totally geodesic, $\gamma|_{[0,\pi]}$ intersects S in exactly one point. By Lemma 4.1 there is a minimal curve γ_w from x to each point in $w \in S$; the map $w \mapsto \gamma_w$ is a homeomorphism from $S = S^n$ to Z. Using this map we can topologically identify Z with the unit sphere in the tangent space at a point on S^{n+1}. With this identification we can define a homeomorphism $\varphi\colon S' \to S^{n+1}$ which carries a geodesic starting at x to the corresponding geodesic in S^{n+1}. We will show that φ is actually an isometry, and for this it suffices to prove that if $\gamma, \beta \in Z$ and γ', β' are minimal in S^{n+1} such that $\alpha(\gamma', \beta') = \alpha(\gamma, \beta)$ then for all $s, t \in (0, \pi)$, $d(\gamma(s), \beta(t)) = d(\gamma'(s), \beta'(t))$, or equivalently, the wedge $W = (\beta|_{[0,s]}, \gamma|_{[0,t]})$ is A2 with equality.

That W is A2 with equality in turn follows from Lemmas 4.3 and 4.4 if we show that γ can be extended as a geodesic loop of length 2π lying in S'. But if $\gamma(r) = v \in S$ and w is antipodal to v in S, the minimal curve ζ from y to x through w extends γ as a geodesic, by A1; γ, in turn, extends ζ. □

Proof of Theorem 1.8. We proceed by induction. The proof is trivial for $n = 0$. Now suppose the theorem holds for $2n$ points, $n \geq 0$ and there is a spherical set $\{x_1, y_1, \ldots, x_{n+1}, y_{n+1}\}$ in X. By the induction hypothesis there is a subset S' of X isometric to S^{n-1} containing x_i and y_i for all $i \leq n$. Lemma 4.5 implies that we need only now show that $x_{n+1}, y_{n+1} \notin S'$, but this is immediate from the definition of spherical set and the fact that S' is metrically embedded. □

Proof of Theorem 1.9. By Theorem 1.8, X contains a subset S isometric to S^n. Suppose there exists some $x \in X\backslash S$. By Lemma 4.1, for each point $y \in S$ there is a unique minimal curve from y to x. In addition, if y_1 and y_2 are close in S then A1 applied to a triangle formed by y_1, x, and a point antipodal to y_2 implies that the angle between minimal curves from x to y_1 and y_2, respectively, is small. Thus the subset Z of S_x corresponding to minimal curves from x to a point in S is homeomorphic to S to S^n. That is, the tangent space T_x contains a subset R homeomorphic to $\mathbb{R}^{n+1}$, and the exponential map $\exp_x$ is a homeomorphism on $B(0, \pi/2) \cap R$. Thus $B(x, \pi) \subset X$ contains a closed subset of dimension $n + 1$, a contradiction. $\square$

Before proving our pinching theorems we give a little background on Gromov-Hausdorff convergence and curvature. For more details, see [P2]. Suppose $\{X_i\}$ is Gromov-Hausdorff convergent to X and each X_i is an inner metric space. Then there is a compact metric space into which all the spaces can be embedded, so that we can use the classic Hausdorff convergence and make sense of uniform convergence ([G]). By Ascoli's theorem, if γ_i is unit (arclength parameterized) minimal in X_i then $\{\gamma_i\}$ has a subsequence which is uniformly convergent to a minimal curve in X. On the other hand, in [P2] the following is proved.

Proposition 4.6. *If each X_i has curvature $\geq k$ for some fixed k, then for any minimal curve γ starting at a point p in X and points $p_i \in X_i$ with $p_i \to p$, there exist minimal curves γ_i in X_i starting at p_i such that $\{\gamma_i\}$ converges uniformly to γ. Furthermore, if γ can be extended as a geodesic beyond p then for any minimal curves β starting at p and β_i starting at p_i, if β_i converges uniformly to β then* $\lim \alpha(\beta_i, \gamma_i) = \alpha(\beta, \gamma)$.

For simplicity, in what follows we will often work with the limit of curves without mentioning each time that we might have to take a subsequence in order for that limit to exist. The reader can verify in each case that this does not affect the validity of the proof.

Theorem 4.7. *For any n, $\delta > 0$ there exists an $\varepsilon > 0$ such that if M is a Riemannian manifold of dimension $\leq n$ and sectional curvture ≥ 1, and M has $2(n + 1)$ points $x_1, y_1, \ldots, x_{n+1}, y_{n+1}$ such that*

(1) $d(x_i, y_i) \geq \pi - \varepsilon$ *for all i and*

(2) $\det[\cos d(x_i, x_j)] \geq \delta$,

then M is diffeomorphic and almost isometric to S^n.

Proof. By the definition of the Gromov-Hausdorff metric, if $\{M_i\}$ is a convergent sequence of Riemannian manifolds of sectional curvature ≥ 1 having "almost spherical" sets satisfying the hypothesis of Theorem 4.7 for $\varepsilon = 1/i$, then $X := \lim M_i$ contains a spherical set of $2(n + 1)$ points. By Theorem A of [GP], $\dim(X) \leq n$. Theorem 8, [P2] (cf. also [GP]) implies that X has curvature ≥ 1 and now Theorem 1.9 proves that X is in fact a sphere. The proof of the theorem is now complete by Yamaguchi's pinching theorem ([Y]). $\square$

Lemma 4.8. *Let $\{X_i\}$ be a sequence of inner metric spaces of curvature ≥ 1 converging to a locally compact limit space X. Suppose there exist points $p_i, q_i \in X_i$ such that* $\lim d(p_i, q_i) = \pi$.

If γ_i is a geodesic in X_i starting at p_i and $\lim \ell(\gamma_i) = \pi$*, then a subsequence of $\{\gamma_i\}$ converges uniformly to a minimal curve in X from $p = \lim p_i$ to $q = \lim q_i$.*

Proof. By choosing a subsequence and reparameterizing, if necessary, we can assume that γ_i is unit and $\{\gamma_i\}$ converges uniformly to a curve γ in X of length π. Let $z_i = \gamma_i(\ell(\gamma_i))$. Then by A2 (applied to $\gamma_i|_{[0,\pi]}$ if $\ell(\gamma_i) > \pi$), $d(z_i, q_i) \to 0$ and so $z_i \to q$. Since $d(p,q) = \pi$, the proof is complete. □

Theorem 4.9. *For any n, $\delta > 0$ there exists an $\varepsilon > 0$ such that if M is a Riemannian manifold of dimension $\leq n$ and sectional curvature ≥ 1, and for some points $p, q \in M$ and $v_1, \ldots, v_n \in S_pM$ such that*

(1) $\det[\langle v_i, v_j\rangle] \geq \delta$,
(2) $d(\exp_p((\pi/2)v_i), \exp_p((-\pi/2)v_i)) \geq \pi - \varepsilon$, *for all i, and*
(3) $d(p,q) \geq \pi - \varepsilon$,

then M is diffeomorphic and almost isometric to S^n.

Proof. Suppose $\{M_i\}$ is a convergent sequence of Riemannian manifolds of curvature ≥ 1 such that there exist $p_i, q_i \in M_i$ and $v_{i1}, \ldots, v_{in} \in S_{p_i}M_i$ such that

(1) $\det[\langle v_{ik}, v_{ij}\rangle] \geq \delta$,
(2) $d(\exp_p((\pi/2)v_{ij}), \exp_p((-\pi/2)v_{ij})) \geq \pi - 1/i$, for all j, and
(3) $d(p,q) \geq \pi - 1/i$.

As in the proof of Theorem 4.7, we need only show that $X := \lim M_i$ contains a spherical subset having $2(n+1)$ elements.

Let γ_{ij} be the unit geodesic corresponding to v_{ij}; i.e., $\gamma_{ij}(t) = \exp_{p_i}(tv_{ij})$. Choosing a subsequence if necessary we can find $p = \lim p_i, q = \lim q_i$. Let $\gamma_j = \lim \gamma_{ij}|_{[0,\pi]}$ and $-\gamma_j = \gamma_{ij}|_{[0,-\pi]}$. By Lemma 4.8, γ_j and $-\gamma_j$ are minimal from p to q. Furthermore, $d(\gamma_j(\pi/2), -\gamma_j(\pi/2)) = \pi$, so A2 implies $\alpha(\gamma_j, -\gamma_j) = \pi$. In fact, Lemma 4.4 implies γ_j and $-\gamma_j$ together form a metrically embedded closed geodesic.

Let β_{ij} be minimal from p_i to $\gamma_{ij}(\pi/2)$ and $-\beta_{ij}$ be minimal from p_i to $\gamma_{ij}(-\pi/2)$. Then A1 implies that $\lim \alpha(\beta_{ij}, \gamma_{ij}|_{[0,-\pi]}) = \pi$, and so $\lim \alpha(\beta_{ij}, \gamma_{ij}|_{[0,\pi]}) = 0$. Likewise, $\alpha(-\beta_{ij}, \gamma_{ij}|_{[0,-\pi]}) = 0$. Furthermore, since γ_j is the unique minimal curve from p to $\gamma_j(\pi/2)$, $\lim \beta_{ij} = \gamma_j$ and $\lim -\beta_{ij} = -\gamma_j$ for all j. By Proposition 4.6, $\alpha(\gamma_k, \gamma_j) = \lim \alpha(\beta_{ik}, \beta_{ij}) = \lim \alpha(\gamma_{ik}, \gamma_{ij}) = \lim \cos\langle v_k, v_j\rangle$. In addition, Lemma 4.3 implies that if $x_j = \gamma_j(\pi/2)$ and $y_j = -\gamma_j(-\pi/2)$ for $j = 1, \ldots, n$ then $d(x_j, x_k) = \alpha(\gamma_j, \gamma_k)$ and these $2n$ points form a spherical set with $\det[\cos d(x_i, x_j)] = \det[\cos \alpha(\gamma_i, \gamma_j)] \geq \delta > 0$. Letting $x_{n+1} = p$ and $y_{n+1} = q$ completes the proof. □

5. Dimension

We define the *cut radius* of $\gamma \in S_p$ to be $C(\gamma) = \sup\{t\colon \gamma|_{[0,t]}$ is defined and minimal$\}$, and extend it to be 0 on $\bar{S}_p \backslash S_p$. The cut radius map is upper semicontinuous but not in general continuous (Example 2.13, [P1]) on $\bar{S}_p$.

Lemma 5.1. *Let $B(p, r)$ be a region of curvature $\geq k$ and $\{\alpha_i\}$, $\{\beta_i\}$ be Cauchy in S_p. Suppose there exist $\delta_i < C(\alpha_i)$ and $\varepsilon_i < C(\beta_i)$ such that $\delta_i, \varepsilon_i \to 0$, $c_1 \leq \varepsilon_i/\delta_i \leq c_2$ for positive (finite) constants c_1 and c_2, and there is a minimal curve η_i from $a_i = \alpha_i(\delta_i)$ to $b_i = \beta_i(\varepsilon_i)$. If τ_i denotes the segment of α_i from a_i to p then $\lim_{i\to\infty} \alpha(\tau_i, \eta_i) = \lim_{i\to\infty} \alpha_k(a_i; p, b_i)$.*

Proof. By A1, $\lim_{i\to\infty} \alpha(\tau_i, \eta_i) \geq \lim_{i\to\infty} \alpha_k(a_i; p, b_i)$. Let $A = \lim \alpha(\alpha_i, \beta_i)$. Fix $\delta > 0$ and some m such that $\alpha(\alpha_i, \alpha_j) < \delta$ and $\alpha(\beta_i, \beta_j) < \delta$ whenever $i, j > m$. By Lemma 1.3, [P1] and the "strong existence" of angles [R], $|\alpha_k(p; a_m, b_i) - A| < 3\delta$ for large i. Lemma 1.3, [P1] also implies that $\lim d(a_i, b_i)^2/\varepsilon_i^2 = 1 + (\delta_i/\varepsilon_i)^2 - 2\delta_i \cos A/\varepsilon_i$ (i.e., the Euclidean cosine law). If Γ_1, Γ_2 are minimal in S_k starting at a point P such that $\alpha(\Gamma_1, \Gamma_2) = A$, $A_i = \Gamma_i(\varepsilon_i)$, and $B_i = \Gamma_2(\delta_i)$ then $\lim_{i\to\infty} |d(A_m, B_i) - d(a_m, b_i)|/\varepsilon_i < \zeta$, where $\zeta \to 0$ as $\delta \to 0$. In other words, if κ_i is the segment of α_m from $\alpha_m(\varepsilon_i)$ to a_m, K_i is the corresponding segment on Γ_1, Λ_i is minimal from A_i to B_i, and T_i is the segment of Γ_1 from P to A_i, then $\lim_{i\to\infty} \pi - \alpha(T_i, \Lambda_i) - \alpha_k(a_i; a_m, b_i) = \lim_{i\to\infty} \alpha(K_i, \Lambda_i) - \alpha_k(a_i; a_m, b_i) < \zeta'$, where $\zeta' \to 0$ as $\delta \to 0$. Since $\lim_{i\to\infty} \pi - \alpha_k(a_i; a_m, b_i) - \alpha_k(a_i; p, b_i)$ tends to 0 as $\delta \to 0$ (and hence m becomes large), the proof is complete. □

Lemma 5.2. *Let numbers $c_1, c_2 > 0$ and k be fixed; for every $\varepsilon > 0$ there exists a $\delta > 0$ such that the following holds. Suppose U is a region of curvature $\geq k$, $p_1, p_2, p_3, p_4 \in U$ and γ_{34} is minimal in U from p_3 to p_4 with the property that (1) $\alpha_k(p_2; p_1, p_3) \geq \pi - \delta$ and (2) $c_1 \leq d(p_2, p_3)/d(p_3, p_4) \leq c_2$. Assume further that Γ_{13}, Γ_{34}, Γ_{41} are minimal in S_k forming a triangle with corresponding corners P_1, P_3, P_4 such that $\ell(\Gamma_{13}) = d(p_1, p_2) + d(p_2, p_3)$, $\ell(\Gamma_{34}) = d(p_3, p_4)$, and $\ell(\Gamma_{41}) = d(p_4, p_1)$ and q, Q are the midpoints of γ_{34}, Γ_{34}. Then $(d(p_1, q) - d(P_1, Q))/d(p_2, p_3) + \varepsilon > 0$.*

Proof. By A0 and Corollary 2.9 it suffices to show the following. Let P_3' be the point in S_k closest to P_3 such that $(P_1; P_3', P_4)$ represents $(p_1; p_3, p_4)$ and Q' be midway between P_3' and P_4. Then for $\delta > 0$ small enough, $d(Q, Q')/d(p_2, p_3)$ is small. If δ is small then by Lemma 1.2, [P1], $[d(P_1, P_2) + d(P_2, P_3) - d(P_1, P_3)]/d(P_2, P_3)$ is small. Therefore if P_3'' is the point on Γ_{13} such that $d(P_3'', P_1) = d(P_1, P_3)$, $d(P_3, P_3'')/d(P_3, P_4) \leq c_2 d(P_3, P_3'')/d(P_2, P_3)$ is also small. This implies that if Q'' is midway between P_3'' and P_4 then $d(Q, Q'')/d(P_2, P_3)$ is also small. Furthermore, $|d(P_3, P_4) - d(P_3'', P_4)| \leq d(P_3, P_3'')$ implies that $d(P_3', P_3'')/d(P_3, P_4)$ is small, and so $d(Q'', Q')/d(P_3, P_4)$ is small, again by Lemma 1.2, [P1]. The lemma now follows from the triangle inequality and the fact that $d(P_3, P_4) \leq d(P_2, P_3)/c_1$. □

The next Lemma extends Lemma 1.4, [P1]. The case when $\alpha(\bar\alpha, \bar\beta) = \pi$ is treated in Theorem 5.6, from which it also follows that $\bar\zeta$ is unique.

Lemma 5.3. *Suppose $p \in X$ lies in a region of curvature $\geq k$ and $\bar\alpha, \bar\beta, \bar\alpha', \bar\beta' \in \bar S_p$ are such that $\alpha(\bar\alpha, \bar\beta) < \pi$ and $\alpha(\bar\alpha, \bar\alpha') = \alpha(\bar\beta, \bar\beta') = \pi$. Then there exists a $\bar\zeta \in \bar S_p$ such that $\alpha(\bar\zeta, \bar\alpha) = \alpha(\bar\zeta, \bar\beta) = \alpha(\bar\alpha, \bar\beta)/2$.*

Proof. Let $A = \alpha(\bar\alpha, \bar\beta)$. Choose $\alpha_i, \beta_i, \alpha_i', \beta_i' \in S_p$ such that $\alpha_i \to \bar\alpha$, $\beta_i \to \bar\beta$, $\alpha_i' \to \bar\alpha'$, $\beta_i' \to \bar\beta'$. By Corollary 2.9 we can assume that α_i, β_i, α_i', β_i' were chosen so that there exist $\delta_i, \varepsilon_i, \delta_i', \varepsilon_i' \to 0$ such that $\delta_i \leq C(\alpha_i)$, $\varepsilon_i \leq C(\beta_i)$, $\delta_i' \leq C(\alpha_i')$, $\varepsilon_i' \leq C(\beta_i')$, $\lim \delta_i/\varepsilon_i = \lim \delta_i'/\varepsilon_i' = \lim \delta_i/\delta_i' = \lim \varepsilon_i/\varepsilon_i' = 1$, and there are minimal curves η_i from $a_i = \alpha_i(\delta_i)$ to $b_i = \beta_i(\varepsilon_i)$. Let p_i denote the midpoint of η_i. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be minimal in S_k starting at a point P such that $\alpha(\Gamma_1, \Gamma_2) = A$, and $\alpha(\Gamma_1, \Gamma_3) = \pi$. Denote by P_i the point midway between $\Gamma_1(\delta_i)$ and $\Gamma_3(\varepsilon_i)$. By Lemma 5.1 and A2, $\lim d(p_i, p)/\varepsilon_i \leq \lim d(P_i, P)/\varepsilon_i$, but since A0 implies $d(p_i, p) \geq d(P_i, P)$ we in fact have $\lim d(p_i, p)/\varepsilon_i = \lim d(P_i, P)/\varepsilon_i$. By Lemma 1.3, [P1], for sufficiently large i, $\alpha_k(p; \alpha_i(\delta_i), \alpha_i'(\delta_i'))$ is close to π, $\alpha_k(p; \alpha_i(\delta_i), \beta_i(\varepsilon_i))$ is close to A, and $\alpha_k(p; \beta_i(\varepsilon_i), \alpha_i'(\delta_i'))$ is close to $\pi - A$. We now have immediately that $\lim \alpha_k(p; \alpha_i(\varepsilon_i), p_i)) = A/2$. On the other hand, Lemma 5.2 now implies that $\lim \alpha_k(p; p_i, \alpha_i'(\delta_i')) = \pi - A/2$. If we choose a minimal curve γ_i from p to a point p_i' sufficiently close to p_i then $\lim \alpha_k(p; \alpha_i(\varepsilon_i), p_i')) = A/2$ and $\lim \alpha_k(p; p_i', \alpha_i'(\delta_i')) = \pi - A/2$; the first implies that $\lim \alpha(\alpha_i, \gamma_i) \geq A/2$ and the second implies that $\lim \alpha(\gamma_i, \alpha_i') \geq \pi - A/2$. Since $\lim \alpha(\alpha_i, \gamma_i) + \alpha(\gamma_i, \alpha_i') = \pi$, $\alpha(\alpha_i, \gamma_i) = A/2$. A similar argument shows that $\alpha(\beta_i, \gamma_i) = A/2$. □

Lemma 5.4. *If $B(p, r)$ is a region of curvature $\geq k$ and $\beta, \gamma \in S_p$ such that $\pi - \alpha(\beta, \gamma) \leq \delta$ then for any $\zeta \in S_p$, $\alpha(\beta, \zeta) + \alpha(\zeta, \gamma) \leq \pi + \delta$. In particular, if $\bar\beta, \bar\gamma \in \bar S_p$ such that $\alpha(\bar\beta, \bar\gamma) = \pi$ then for any $\bar\zeta \in \bar S_p$, $\alpha(\bar\beta, \bar\zeta) + \alpha(\bar\zeta, \bar\gamma) = \pi$.*

The above lemma is an immediate consequence of Proposition 1.19, the triangle inequality for angles, and the definition of the angle. Lemma 5.4 extends Lemma 2.3, [P1], and using it we can extend Proposition 2.4, [P1] to obtain the following.

Proposition 5.5. *If $c_k(p) > 0$ and a subset S of $\bar S_p$ is a convex inner metric space, then S has curvature uniformly ≥ 1.*

Theorem 5.6. *If $c_k(p) > 0$ and $\bar S_p$ contains a spherical set Σ of $2(n+1)$ points then there is a subset S of $\bar S_p$ isometric to S^n containing Σ.*

Proof. We will prove that there exists a subset $\mathcal{L}$ of S_p containing Σ such that $\mathcal{L}$ (with the induced metric) is a convex inner metric space. Proposition 5.5 then shows that $\mathcal{L}$ has curvature uniformly ≥ 1, and Theorem 1.8 then gives the existence of S.

Define $\mathcal{L}_0 = \Sigma$ and $\mathcal{L}_m$ inductively as follows. Suppose $\mathcal{L}_{m-1}$ has been defined, and has the property that for each $\bar{\alpha} \in \mathcal{L}_{m-1}$ there exists an $\bar{\alpha}' \in \mathcal{L}_{m-1}$ such that $\alpha(\bar{\alpha}, \bar{\alpha}') = \pi$. Then for every $\bar{\alpha}, \bar{\beta} \in \mathcal{L}_{m-1}$ such that $\alpha(\bar{\alpha}, \bar{\beta}) < \pi$ there exists, by Lemma 5.3, an element $\bar{\gamma} \in \bar{S}_p$ such that $\alpha(\bar{\alpha}, \bar{\gamma}) = \alpha(\bar{\gamma}, \bar{\beta}) = \alpha(\bar{\alpha}, \bar{\beta})/2$. We define $\mathcal{L}_m$ to be the union of $\mathcal{L}_{m-1}$ with set of all such elements $\bar{\gamma}$. For any $\bar{\zeta} \in \mathcal{L}_m$ either $\bar{\zeta} \in \mathcal{L}_{m-1}$, in which case there is a $\bar{\zeta}' \in \mathcal{L}_{m-1} \subset \mathcal{L}_m$ such that $\alpha(\bar{\zeta}, \bar{\zeta}') = \pi$, or $\bar{\zeta} \notin \mathcal{L}_{m-1}$. In the latter case we can find $\bar{\alpha}, \bar{\beta} \in \mathcal{L}_{m-1}$ such that $\alpha(\bar{\alpha}, \bar{\gamma}) = \alpha(\bar{\gamma}, \bar{\beta}) = \alpha(\bar{\alpha}, \bar{\beta})/2$ and $\bar{\alpha}', \bar{\beta}' \in \mathcal{L}_{m-1}$ such that $\alpha(\bar{\alpha}, \bar{\alpha}') = \pi$ and $\alpha(\bar{\beta}, \bar{\beta}') = \pi$. But there is an element $\bar{\gamma}' \in \mathcal{L}_m$ such that $\alpha(\bar{\alpha}', \bar{\gamma}') = \alpha(\bar{\gamma}', \bar{\beta}') = \alpha(\bar{\alpha}', \bar{\beta}')/2$; i.e., such that $\alpha(\bar{\gamma}, \bar{\gamma}') = \pi$. Thus $\mathcal{L}_m$ has the desired "betweenness" property. The set $\mathcal{L} = \cup \mathcal{L}_m$ has the property that for each $\bar{\alpha}, \bar{\beta} \in \mathcal{L}$ such that $\alpha(\bar{\alpha}, \bar{\beta}) < \pi$ there exists an element $\bar{\gamma} \in \mathcal{L}$ such that $\alpha(\bar{\alpha}, \bar{\gamma}) = \alpha(\bar{\gamma}, \bar{\beta}) = \alpha(\bar{\alpha}, \bar{\beta})/2$. By successive approximation it is therefore possible to find, for any $\lambda \in [0, 1]$ and element $\bar{\eta} \in \mathcal{L}$ such that $\alpha(\bar{\alpha}, \bar{\eta}) = \lambda \cdot \alpha(\bar{\alpha}, \bar{\beta})$ and $\varepsilon(\bar{\eta}; \bar{\alpha}, \bar{\beta}) = 0$. If $\alpha(\bar{\alpha}, \bar{\beta}) = \pi$ then we can choose any $\bar{\zeta} \in \mathcal{L}$ different from $\bar{\alpha}$ and $\bar{\beta}$ and, using the fact that $\alpha(\bar{\alpha}, \bar{\zeta}) + \alpha(\bar{\zeta}, \bar{\beta}) = \pi$, find a "midpoint" between $\bar{\alpha}$ and $\bar{\beta}$. Corollary 2.5 now shows that $\mathcal{L}$ is a convex inner metric space. □

The next lemma can be considered an extension of Lemma 2.11, [P1].

Lemma 5.7. *Suppose points p, a, b, c and minimal curves α, β, from p to a, b, lie in a region of curvature $\geq k$ and let $\omega = \max\{\pi - \alpha_k(p; a, c), \pi - [\alpha_k(p; a, b) + \alpha_k(p; c, b)]\}$. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d(q, p) < \delta$ and α', β' are minimal from q to a, b, then $|\alpha(\alpha, \beta) - \alpha(\alpha', \beta')| < \varepsilon = 3\omega$.*

Proof. Let $q_i \to p$ and α_i, β_i be minimal from q_i to a, b respectively. By Corollary 2.9 (choosing a "new" c, if necessary) we can assume there exist minimal curves χ from p to c and χ_i from q_i to c. A1 and Lemma 5.4 imply $\liminf \alpha(\alpha_i, \beta_i) \geq \alpha_k(p; a, b) \geq \pi - \alpha_k(p; c, b) - \omega \geq \pi - \alpha(\chi, \beta) - \omega \geq \alpha(\alpha, \beta) - 2\omega$. On the other hand, Lemma 5.4 and A1 imply that $\limsup \alpha(\chi_i, \alpha_i) \geq \pi - \omega$ so $\limsup \alpha(\alpha_i, \beta_i) \leq \pi - \limsup \alpha(\chi_i, \beta_i) + \omega \leq \alpha(\chi, \beta) + 3\omega$ by the above argument. This completes the proof. □

Proposition 5.8. *Suppose $c_k(p) \geqslant 0$ and $\bar{S}_p$ contains a subset S isometric to S^n. Then if $\bar{S}_p \neq S^n$ then $N^{n+1} \cap B(p, \delta)$ contains a dense G_δ for all small enough $\delta > 0$.*

Proof. Since S_p is dense in $\bar{S}_p$ we can find a spherical subset $\Sigma = \{\gamma_1, \bar{\gamma}_1, \ldots, \gamma_{n+1}, \bar{\gamma}_{n+1}\}$ of $\bar{S}_p$, where $\gamma_i \in S_p$. We now choose points a_i, $\bar{a}_i$ such that $\alpha_k(p; a_i, \bar{a}_j)$, $\alpha_k(p; a_i, a_j)$, and $\alpha_k(p; \bar{a}_i, \bar{a}_j)$ are all arbitrarily close to $\alpha(\gamma_i, \bar{\gamma}_j)$, $\alpha(\gamma_i, \gamma_j)$, and $\alpha(\bar{\gamma}_i, \bar{\gamma}_j)$, respectively. If $\bar{S}_p \neq S^n$ then we can find some $\beta \in S_p \backslash S$ and a point b such that $\alpha_k(p; b, a_i)$ and $\alpha_k(p; b, \bar{a}_i)$ are close to $\alpha(\beta, \gamma_i)$ and $\alpha(\beta, \bar{\gamma}_i)$, respectively. We can assume all of these points lie in $B(p, c_k(p)/2)$. By Corollary 2.9 we can find a dense G_δ subset A of B such that if $q \in A$ then there are almost extendable minimal curves α_i from a_i to q and ζ from b to q. By Lemma 5.7 and the "openness" of the second condition in Definition 1.7, if q is close enough to p, the curves α_i, together with their complementary curves $\bar{\alpha}_i \in \bar{S}_q$, form a spherical set Σ' of $2(n + 1)$ elements; adding ζ and its complement in $\bar{S}_p$ to Σ' we form a spherical set of $2(n+2)$ elements. The proof is complete by Theorem 5.6. $\square$

Proof of Theorem 1.11. It is immediate from Proposition 5.8 that if $\mathrm{ldim}(X) \geq n$ then $N^n(X)$ is dense in X. Furthermore, for each $p \in X$ there exists a $B_p = B(p, r)$ such that $B_p \cap N^n(X)$ contains a G_δ. That is, there exist open sets U_i^p such that $\bigcap_{i=1}^\infty U_i^p \subset B_p \cap N^n(X)$. Let $\mathcal{U}_i = \bigcup_{p\in X} U_i^p$. It is easy to verify that $\bigcap_{i=1}^\infty \mathcal{U}_i$ is a dense G_δ in X contained in $N^n(X)$. $\square$

Proof of Corollary 1.12. By Proposition 5.8 there is a point p such that $\bar{S}_p = S^{n-1}$, then in particular $\bar{S}_p$ is compact and so every sequence of minimal curves of bounded length has a uniformity convergent sequence. It follows from Theorem 1.4 that every point in $B = B(p, c_k(p))$ is joined to p by a minimal curve. In fact, choose $q_i \to q$ and unit minimal curves γ_i from p to q. Choosing a subsequence, if necessary, we can assume $\alpha(\gamma_i, \gamma_j) \to 0$ as i, j become large. From A2 we deduce that $\{\gamma_i\}$ converges uniformly to a minimal curve from p to q. Now $\exp_p$ is defined on a closed (hence compact) set containing the origin, whose image is $\bar{B}$. Therefore X is locally compact, and every pair of poits in X is joined by a minimal curve. Since X is locally compact, for any $r > 0$, $c_k(p) \geq r$ for some k, and so every compact set lies in a region $B(p, k)$ of curvature $\geq k$ for some k. If one puts the metric of constant curvature k on $B(0, r) \subseteq \bar{T}_p$ then $\exp_p$ is distance decreasing on its domain of definition, and hence decreases Hausdorff measure. Finally, let p be arbitrary, and suppose there exist $\delta > 0$ and $\gamma_i \in S_p$ such that for all $i, j, \alpha(\gamma_i, \gamma_j) > \delta$. By the semi-continuity of the angle (cf. Lemma 2.2, [P1]), for any $m > 0$ there exists an $\varepsilon > 0$ such that if $d(p, q) < \varepsilon$ then $\bar{S}_q$ contains a sequence $B_1, \ldots, B_n$ such that $\alpha(\beta_i, \beta_j) > \delta/2$. But q can be chosen so that $\bar{S}_q = S^{n-1}$, and for arbitrarily large m such a sequence cannot exist, a contradiction. It follows that $\bar{S}_p$ is compact, and Proposition 2.4, [P1] implies it is an inner metric space of curvature ≥ 1. $\square$

Proofs of Corollaries 1.13–1.15. Corollary 1.13 is immediate from that fact that $\dim(X) \leq \mathrm{hdim}(X)$ and Corollary 1.12. Corollary 1.14 follows from Proposition 5.8 as in the proof of Theorem 1.11. To prove the last corollary, let $N^\infty = \bigcap_{n=1}^\infty N^n$. Note the union in $\bar{S}_p$ of any isometric copies S, S' of S^n and S^{n+1}, respectively (where S may not be a great sphere of S') must contain a copy S'' of S^{n+1} such that S' *is* a great sphere of S''. Therefore we can construct a sequence $\{S_n\}$, where S_n is a copy of S^n, such that S_n is a great sphere of S_{n+1}. The metric completion of the limit of S_n is isometric to S^∞. $\square$

Acknowledgment

I thank the referee for valuable comments, Y. Burago and G. Perelman for pointing out an unsubstantiated claim in an early version of the paper, and Karsten Grove for suggesting a simplification of the proof of Theorem 1.5.

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Received June 11, 1993

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