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A Note on the Instability of Embeddings of Cauchy–Riemann Manifolds

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ABSTRACT. We prove that there are compact strictly pseudoconvex CR manifolds, embedded into some Euclidean space, that admit small deformations that are also embeddable but their embeddings cannot be chosen close to the original embedding.

1. Introduction

Let M be a compact differentiable manifold of odd dimension dim $M = 2n+1$. A Cauchy–Riemann structure on M is given by the choice of a (complex) subbundle $H^{1,0}M \subset \mathbb{C} \otimes TM$ of rank n satisfying the following two conditions:

(1) If $H^{0,1}M$ denotes the conjugate of $H^{1,0}M$, then $H^{0,1}M \cap H^{1,0}M = (0)$.

(2) If Z, W are local sections of $H^{1,0}M$, then so is their Lie bracket $[Z, W]$.

(In this paper all objects will be infinitely differentiable; in particular M, $H^{1,0}M$, Z, W are such, so that $[Z, W]$ makes sense.)

The pair $(M, H^{1,0}M)$ is a Cauchy–Riemann (CR) manifold. For example, compact hypersurfaces M in complex manifolds X inherit a CR structure from the ambient manifold: $H^{1,0}M$ simply consists of those $(1, 0)$ tangent vectors to X that lie in $\mathbb{C} \otimes TM$. In the past 25 years or so, much work was directed to finding out to what extent the converse is true, e.g., can any CR manifold be gotten from a complex manifold by the above or related constructions. A central question is whether a CR manifold M can be CR embedded into some Euclidean space $\mathbb{C}^k$. In other words, does there exist a (smooth) embedding $f : M \to \mathbb{C}^k$ such that for any $Z \in H^{1,0}M$,

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$f_* Z \in \mathbb{C} \otimes T\mathbb{C}^k$ is a $(1,0)$ vector. An answer to this question depends on the notion of strict pseudoconvexity. A CR manifold M is strictly pseudoconvex if for any nonvanishing local section Z of $H^{1,0}M$ $[Z, \overline{Z}]$ is never tangent to $H^{1,0}M \oplus H^{0,1}M$. If M is such, and of dimension of at least five, a theorem of Boutet de Monvel asserts that it can be CR embedded into some space $\mathbb{C}^k$ (see [1]). This is not true if $\dim M = 3$, as an example of Rossi shows; see, e.g., [2,3,9].

Along with the question of existence of CR embeddings, the problem of their stability also arises. Suppose a CR manifold $(M, H^{1,0}M)$ is embedded into some $\mathbb{C}^k$ via a mapping $f : M \to \mathbb{C}^k$, and let $(M, \widetilde{H}^{1,0}M)$ denote a small perturbation of the CR structure of $(M, H^{1,0}M)$ which again is embeddable (into some other $\mathbb{C}^\ell$). The problem is to decide whether $(M, \widetilde{H}^{1,0}M)$ can be embedded into the same $\mathbb{C}^k$ via a CR embedding $\widetilde{f} : M \to \mathbb{C}^k$ close to f. When M is strictly pseudoconvex, of dimension at least 5, an affirmative answer was given by Tanaka (see [10]), provided a certain cohomology group of M $(H^{0,1}_{\overline{\partial}_b}(M))$ vanishes. More recently, the second named author proved stability when $\dim M = 3$ and there is a CR embedding $f_0 : (M, H^{1,0}M) \to \mathbb{C}^2$ such that $f_0(M)$ is a strictly convex hypersurface [8]. (Strictly speaking, [8] proves stability only when the embedding f agrees with the convex embedding f_0, but it is easy to show that stability for f_0 implies stability for any other embedding f.)

Since in the latter case, the relevant cohomology group is infinite dimensional, one may be led to believe that stability of embeddings of strictly pseudoconvex CR manifolds always holds, without regard to the vanishing of some cohomology groups. The purpose of this note is to show that this is not so, and unstable CR embeddings do exist (see Theorem 2.1). The CR manifolds with unstable embeddings will arise as unit circle bundles in hermitian line bundles over projective algebraic manifolds. The instability of CR embeddings will be a consequence of the instability of an algebraic property: very ampleness of line bundles.

2. Formulation of the main theorem

By a smooth family of CR manifolds we shall mean a family $H_t^{1,0}M$ of CR structures on a fixed compact manifold M, $-\varepsilon < t < \varepsilon$, such that the bundles $H_t^{1,0}M \subset \mathbb{C} \otimes TM$ depend smoothly (C^∞) on t. If $(M, H_0^{1,0}M)$ is strictly pseudoconvex, so will $(M, H_t^{1,0}M)$ for sufficiently small t.

Theorem 2.1. *For any $n = 1, 2, \ldots$, there are a smooth family $(M, H_t^{1,0}M)$ of strictly pseudoconvex CR manifolds of dimension $2n+1$, each CR embeddable into some Euclidean space; a CR embedding $f : (M, H_0^{1,0}M) \to \mathbb{C}^k$; and a positive number δ with the following property. If $t \neq 0$, there is no CR embedding $g : (M, H_t^{1,0}M) \to \mathbb{C}^k$ such that $|g(p) - f(p)| < \delta$ for every $p \in M$.*

3. Very ample line bundles

Let L denote a line bundle over a compact complex manifold N. The space $H^0(L)$ of holomorphic sections of L is finite dimensional. If $\sigma_0, \ldots, \sigma_k \in H^0(L)$ is a basis, we obtain a mapping $\sigma = (\sigma_0 : \ldots : \sigma_k)$ of N into projective space $\mathbb{P}_k$ (which may not be defined everywhere on N).

Definition 3.1. *L is called very ample if the above mapping σ biholomorphically embeds N into $\mathbb{P}_k$.*

For example, if $N \subset \mathbb{P}_k$, and H is the hyperplane section bundle on $\mathbb{P}_k$, the restriction of H to N is very ample. Indeed, the homogeneous coordinates $z_0, \ldots, z_k$ on $\mathbb{P}_k$ are sections of H and their restrictions to N biholomorphically embed N into $\mathbb{P}_k$.

A divisor D on N determines a line bundle $[D]$ (see, e.g., [5]). D is called very ample if $[D]$ is. This is not a stable concept (in contrast to ampleness = positivity): arbitrarily small perturbations of very ample divisors or line bundles may not be very ample.

Example 3.2. Let $F \subset \mathbb{P}_2$ denote the Fermat curve $z_0^m = z_1^m + z_2^m$. A simple computation involving the Riemann–Hurwitz formula shows that the genus of F is $g = (m-1)(m-2)/2$ (see, e.g., [5]). Let $p \in F$ have homogeneous coordinates $(1:1:0)$. Then the divisor mp is very ample but for nearby points $\widetilde{p} \in F$, $m\widetilde{p}$ is not, provided $m \geq 5$. Indeed, $\dim H^0([m\widetilde{p}]) = 1$.

Proof. (i) The line $\Lambda \subset \mathbb{P}_2$ given by $z_0 = z_1$ has one single point on F, p, and the order of contact between F and Λ is m. Hence the restriction of the hyperplane section bundle $[\Lambda]$ to F is $[mp]$, so that by our previous remark $[mp]$ is indeed very ample. We also see that sections of $[mp]$ can embed F into $\mathbb{P}_2$.

(ii) Holomorphic sections of $[m\widetilde{p}]$ are in one-to-one correspondence with meromorphic functions on F with a single pole at $\widetilde{p}$ of order at most m. If for a point $\widetilde{p} \in F$ there exist nonconstant meromorphic functions with a single pole at $\widetilde{p}$ of order at most g $(\geq m)$, the point is called a Weierstrass point, and it is known that there are only finitely many such points on F (see, e.g., [4]). Hence for $\widetilde{p}$ close to but different from p, the only meromorphic functions of the above type are the constants, so $\dim H^0([m\widetilde{p}]) = 1$. $\square$

Example 3.3. With F and p as above, and n a positive integer, let N be the n–fold product of F with itself. With $\widetilde{p} \in F$ define $D_{\widetilde{p}} \subset N$ by $D_{\widetilde{p}} = \{(p_1, \ldots, p_n) \in F \times \ldots \times F : p_i = \widetilde{p}$ for some $i\}$. Then the divisor mD_p is very ample but for $\widetilde{p} \neq p$ close to p, $mD_{\widetilde{p}}$ is not, provided $m \geq 5$; again, $\dim H^0([mD_{\widetilde{p}}]) = 1$.

This is a straightforward consequence of Example 3.2.

4. Deformations of line bundles

Let F be the Fermat curve of degree m as before (or any compact Riemann surface), k an integer, and $p(t)$ $(-\varepsilon < t < \varepsilon)$ a small smooth curve on F. The line bundles $L_t = [kp(t)]$ are smoothly equivalent to one another, so we can think of L_t as having the same underlying smooth line bundle as L_0, but of course for $t \neq 0$ the holomorphic (= complex) structures will differ. However, the identifications $L_t \leftrightarrow L_0$ can be chosen so that the complex structure of L_t depends smoothly on t, by which we mean that the bundle of $(1,0)$ vectors $T^{1,0}L_t \subset \mathbb{C}\otimes TL_t = \mathbb{C}\otimes TL_0$ depends smoothly on t.

To see this, choose a coordinate neighborhood $U \subset F$ that contains the closure of the curve $p(t)$. Let $z : U \to \mathbb{C}$ be a local coordinate on U. We can assume that $z(U)$ is the unit disc and $z(p(0)) = 0$. Let $V \subset F$ denote a neighborhood of $F\backslash U$ such that $\overline{V}$ is disjoint from the closure of the curve $p(t)$. The bundle L_t can be defined as

$$L_t = (U \times \mathbb{C}) \cup (V \times \mathbb{C}) / \sim \tag{4.1}$$

where $(q_1, \xi_1) \in U \times \mathbb{C}$ and $(q_2, \xi_2) \in V \times \mathbb{C}$ are identified if $q_1 = q_2$ and

$$\left(z(q_1) - z(p(t))\right)^k \xi_2 = \xi_1. \tag{4.2}$$

Next choose a smooth function $f : U \times (-\varepsilon, \varepsilon) \to \mathbb{C}\backslash\{0\}$ such that $f(q,0) = 1$ $(q \in U)$ and

$$f(q,t) = z(q)^k / (z(q) - z(p(t)))^k \qquad (q \in U \cap V). \tag{4.3}$$

With this f, for every $t \in (-\varepsilon, \varepsilon)$ construct a smooth self-diffeomorphism φ_t of the disjoint union $(U \times \mathbb{C}) \cup (V \times \mathbb{C})$ by putting for $(q_1, \xi_1) \in U \times \mathbb{C}$, $(q_2, \xi_2) \in V \times \mathbb{C}$

$$\varphi_t(q_1, \xi_1) = (q_1, f(q_1, t)\xi_1), \qquad \varphi_t(q_2, \xi_2) = (q_2, \xi_2).$$

Because of (4.2), (4.3) φ_t descends to a smooth bundle isomorphism $\Phi_t : L_t \to L_0$. Then Φ_t provides the required identification $L_t \leftrightarrow L_0$.

5. Unstable embeddings of CR manifolds

With $m \geq 5$, F, p as in section 3, construct a smooth arc $p(t)$ $(-\varepsilon < t < \varepsilon)$ in F so that $p(0) = p$, but for $t \neq 0$, $mp(t)$ is not very ample. Let L_t denote the line bundle $[-mp(t)]$. As smooth line bundles the L_t's will be identified, as explained above. Choose holomorphic sections $\sigma_0, \sigma_1, \sigma_2$ of $[mp] = L_0^*$ that embed F into $\mathbb{P}_2$, and define a hermitian metric h on L_0 by $h(v) = \Sigma|\sigma_j(v)|^2$, $v \in L_0$. h is a strictly plurisubharmonic function on $L_0\backslash F$, so that the circle bundle $M = \{v \in L_0 : h(v) = 1\}$ is a strictly pseudoconvex hypersurface, and inherits a CR structure $H_0^{1,0}M = (\mathbb{C} \otimes TM) \cap T^{1,0}L_0|_M$. M can be CR embedded into $\mathbb{C}^3$ by the mapping

$$M \ni v \mapsto f(v) = (\sigma_0(v), \sigma_1(v), \sigma_2(v)) \in \mathbb{C}^3. \tag{5.1}$$

We can also endow M with a CR structure $H_t^{1,0}M$ inherited from L_t, and thus (after possibly shrinking the interval $(-\varepsilon, \varepsilon)$) we obtain a smooth family of compact strictly pseudoconvex CR manifolds.

Each of these CR manifolds bounds a compact strictly pseudoconvex complex manifold, namely $\{v \in L_t : h(v) \leq 1\}$; hence it follows from [6] that $(M, H_t^{1,0}M)$ embeds into some Euclidean space.

There exists a $\delta > 0$ such that no complex line in $\mathbb{C}^3$ contains $f(M)$ in its δ neighborhood. With this δ we have the following Proposition, which implies Theorem 2.1 for $n = 1$:

Proposition 5.1. *For $t \neq 0$ there is no CR embedding g of $(M, H_t^{1,0}M)$ into $\mathbb{C}^3$ such that $|g(v) - f(v)| < \delta$ for every $v \in M$.*

Proof. Suppose there is, and define a CR mapping $k : (M, H_t^{1,0}M) \to \mathbb{C}^3$ by

$$k(v) = \frac{1}{2\pi} \int_0^{2\pi} g(ve^{i\theta})e^{-i\theta} d\theta, \qquad v \in M. \tag{5.2}$$

By a result due to Kohn and Rossi, any CR function on the boundary of a compact strictly pseudoconvex complex manifold extends holomorphically to the manifold [7]. In our case this means that k extends holomorphically to $\Omega_t = \{v \in L_t : h(v) < 1\}$. Equation (5.2) implies that $k(e^{i\theta}v) = e^{i\theta}k(v)$ for $\theta \in \mathbb{R}$, $v \in M$, hence also for $v \in \Omega_t$, so that k is in fact linear on the fibers of L_t. In other words, the components k_0, k_1, k_2 of k are holomorphic sections of L_t^*. Since $\dim H^0(L_t^*) = 1$ according to Section 3, $k = (k_0, k_1, k_2)$ maps L_t, and so M, into some complex line. From (5.2),

$$|f(v) - k(v)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(ve^{i\theta}) - g(ve^{i\theta}) \right| d\theta < \delta$$

for $v \in M$, which now contradicts the choice of δ. $\square$

Remark 5.2. If in the above construction we choose σ_0, σ_1, σ_2 to be the homogeneous coordinates z_0, z_1, z_2 on $F \subset \mathbb{P}_2$, we find that via (5.1) $(M, H_0^{1,0}M)$ embeds into $\mathbb{C}^3$ as

$$\{z \in \mathbb{C}^3 : z_0^m = z_1^m + z_2^m, \quad |z_0|^2 + |z_1|^2 + |z_2|^2 = 1\}. \qquad \square$$

The proof of Theorem 2.1 in case $n > 1$ follows the same path as above, except that F will be replaced by $N = F \times \ldots \times F$ and, with notation as in Example 3.3, L_t will be defined $[-mD_{p(t)}]$. The details will have to be omitted.

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