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Gaussian Curvature on Singular Surfaces

By *Wenxiong Chen and Congming Li*

ABSTRACT. We consider prescribing Gaussian curvature on surfaces with conical singularities in both critical and supercritical cases. First we prove a variant of Kazdan–Warner type necessary conditions. Then we obtain sufficient conditions for a function to be the Gaussian curvature of some pointwise conformal singular metric. We only require that the values of the function are not too large at singular points of the metric with the smallest angle, say, less or equal to 0, or less than its average value. To prove the results, we apply some new ideas and techniques. One of them is to estimate the total curvature along a certain minimizing sequence by using the "Distribution of Mass Principle" and the behavior of the critical points at infinity.

1. Introduction

Given a function $R(x)$ on a compact surface S, can it be realized as the Gaussian curvature of some pointwise conformal metric? This is an interesting problem in geometry. A usual way to solve this problem is to pick up a pointwise conformal metric g_o, the so-called basic metric, then try to conformally deform it to a metric g with the desired curvature. If we let $g = e^{2u}g_o$, then it is equivalent to solve the following nonlinear elliptic equation:

$$-\Delta u + R_o(x) = R(x)e^{2u} \tag{$*$}$$

where Δ and $R_o(x)$ are the Laplacian operator and the Gaussian curvature of g_o respectively.

In the last few years, a lot of work has been done to understand this on smooth surfaces (see the surveying article of Kazdan [3] and [1,2,4,7,8,9,10,11]).

Let χ be the Euler characteristic of the surface S, for $\chi < 0$, people usually solve the problem by using a method of sub- and supersolutions (cf. Kazdan and Warner [1]) and for $\chi \geq 0$ by variational method.

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The case $0 \le \chi < 2$ is the so-called subcritical case where one relies on the inequality

$$\int_S \exp u dA \le C \exp\left\{\frac{1}{16\pi}\int_S |\nabla u|^2 dA + \frac{1}{\text{Area}(S)}\int_S u dA\right\}, \tag{1}$$

a direct consequence of the Moser–Trudinger inequality (cf. [5]).

The case $\chi = 2$ is called the critical case. In this case, the corresponding variational functional loses its compactness. There are obstructions found by Kazdan and Warner [1] and then by Bourguignon and Ezin [13], which show that a few functions cannot be realized as such Gaussian curvature. Then for which $R(x)$, can one solve $(*)$? What are the necessary and sufficient conditions for $(*)$ to have a solution? These are challenging problems in geometry. Since the analysis here is more delicate, one needs the best value of the constant C in the above inequality. It was shown by Onofri [6] and Hong [7] that on the standard sphere S^2, the best constant C in (1) is 4π (we would like to remind the readers that there is another well-known best constant in the inequality (1), $1/16\pi$, which is obtained by Moser in [5]). Based on this, many results on prescribing Gaussian curvature on S^2 were obtained by using various techniques (cf. [8,9,10,11]). One of the powerful tool used is called the "Center of Mass" analysis.

Then comes a natural question: Can one generalize these results to surfaces with singularities? Under what conditions can a function be the Gaussian curvature of some pointwise conformal singular metrics?

To start with, one may consider a compact surface with conical singularities. Roughly speaking, a compact surface with conical singularities is a compact Riemannian surface with finitely many points $p_1, p_2, \ldots, p_k$ removed. Locally, near the singular point p_i, the surface is diffeomorphic to a cone with an angle $\theta_i > 0$ and the metric can be written as $ds^2 = \rho(x)|x|^{2\beta_i}|dx|^2$ in a local coordinate centered at p_i, where $\beta_i = \frac{\theta_i}{2\pi} - 1$ and $\rho(x)$ is a smooth function (cf. [17] for more details). These kinds of singularities appear in many situations, such as orbifolds, branched coverings etc. They also describe the ends of complete Riemannian surfaces with finite total curvature.

In his recent paper, Troyanov [17] systematically studied surfaces S with conical singularities of angle θ_i at point p_i, $i = 1, 2, \ldots, k$. First, he generalized a series of inequalities on smooth surfaces to surfaces with conical singularities, such as Poincaré, Sobolev, and Trudinger inequalities. Then he considered prescribing Gaussian curvature. He pointed out that in this process, one can use the number

$$\chi(S, \theta) = \chi(S) + \sum_{i=1}^{k}\left(\frac{\theta_i}{2\pi} - 1\right)$$

instead of $\chi(S)$ to characterize it in four cases:

(i) negative case : $\chi(S, \theta) < 0$,

(ii) subcritical case : $0 \le \chi(S, \theta) < \min_i \left\{2, \frac{\theta_i}{\pi}\right\}$,

(iii) critical case : $\chi(S,\theta) = \min_i \left\{2, \frac{\theta_i}{\pi}\right\}$ and

(iv) supercritical case : $\chi(S,\theta) > \min_i \left\{2, \frac{\theta_i}{\pi}\right\}$.

In the negative and subcritical cases, he obtained conditions for a function to be the Gaussian curvature of some pointwise conformal metric. His results are parallel to those on smooth surfaces. One of his key ingredient is the inequality

$$\int_S e^u dA \le C_1 \exp\left\{C_2 \int_S |\nabla u|^2 dA + \frac{1}{\text{area}(S)} \int_S u dA\right\}. \tag{2}$$

It guarantees the compactness of the functional in the subcritical case. He posed the critical case as an open problem.

In the critical case, one encounters the same kinds of difficulties as on S^2. Here, the corresponding variational functional loses its compactness. A minimizing or minimax sequence may blow up. To compensate, one first needs to obtain the best values of the constants in inequality (2). Fortunately, our previous result [18] on the best constant of the Trudinger inequality on singular surfaces makes this possible. It implies immediately that the largest possible value of C_2 is $1/4b_o$ with $b_o = 2\pi \min_i\{2, \frac{\theta_i}{\pi}\}$.

In [15], we start from a special critical case, a sphere S with two singularities of equal angle, $0 < \theta_1 = \theta_2 < 2\pi$. It looks like a football.

First, we proved a variant of Aubin's [12] inequality. It implies that every minimizing or minimax sequence with distributed mass possesses a convergent subsequence. We call this the "Distribution of Mass" Principle. One knows that the Center of Mass analysis once played an important role in prescribing Gaussian curvature on S^2. However, it relies on the coordinates in R^3 and hence cannot be applied to other kinds of surfaces. Our Distribution of Mass Principle is much more general and can be applied to any compact surface, smooth or singular. It can also be applied to a subdomain of a surface.

Then, with the help of the Distribution of Mass Principle, we proved that the best value of the constant C_1 in inequality (2) is the area of S.

Finally, using our previous techniques on S^2, we obtained sufficient conditions for a symmetric function (invariant under the action of some isometry group) to be the Gaussian curvature on the football. The conditions are parallel to those we obtained for S^2 (cf. [8]).

In the supercritical case, the variational functional is in general neither bounded nor compact. This makes the variational approach even more difficult. In [16], McOwen considered a special case, a sphere with one singularity. Assuming that $R(x)$ approaches 0 in some order near the singularity, he was able to regain the boundedness and compactness for the functional and hence arrived at a solution of $(*)$.

In this paper, we consider general critical and supercritical cases. First, we prove a variant of the Kazdan–Warner obstruction in some special critical and supercritical cases. Then, we obtained a series of sufficient conditions for prescribing Gaussian curvature on general surfaces with conical singularities in critical and supercritical cases. Because of the use of some brand new ideas and techniques, we are able to greatly generalize our previous results [15] in the critical case. In the supercritical case, we obtained some sufficient conditions disjoint to that of McOwen's, and our conditions are somewhat more relaxed.

In Section 2, inspired by an idea of Kazdan and Warner [1], we obtain necessary conditions for a function $R(x)$ to be the Gaussian curvature of some pointwise conformal metric on a sphere with two singularities of equal angle (a critical case) and a sphere with one singularity (a supercritical case) (cf. Theorem 2.1). One knows that, by Kazdan–Warner's necessary condition

$$\int_{S^2} \nabla R \cdot \nabla \psi \exp u dA = 0$$

a monotone function can never be realized as the Gaussian curvature of some pointwise conformal metric on a smooth standard sphere. We show that this is the case on the sphere with two singularities of equal angle. While on the sphere with one singularity, it is interesting that the above identity is replaced by strict inequalities, which seem to suggest that monotone functions are now the right candidates for Gaussian curvature. To partially justify this, we provide some examples (cf. Example 1 and 2 in Section 2).

In Section 3, we obtain sufficient conditions for a function $R(x)$ to be the Gaussian curvature of some pointwise conformal metric in a special critical case, the sphere with two singularities of equal angle (cf. Theorem 3.1). These conditions are very weak. Besides the obvious necessary condition that $R(x)$ be positive somewhere, we only require that the value of R is not too large at the singular points of the metric, say less than or equal to 0, or less than the average value of R on the surface, while no other restrictions are imposed on $R(x)$ elsewhere. Surprisingly, our sufficient conditions here are much weaker than those on S^2, while we obtain some similar obstructions as Kazdan–Warner did for S^2.

Some brand new ideas are employed to prove the above results. Instead of estimating the value of the functional, as people did traditionally, we estimate the total curvature along a certain minimizing sequence. Our Distribution of Mass Principle implies that the minimizing sequence can only blow up at one point. Then using blowing up and rescaling technique and our knowledge on the behavior of the critical points at infinity [14], we showed that the sequence can only blow up at one of the singularities. This enabled us to drop the symmetry assumption on $R(x)$ in [15]. Finally, imposing some conditions on $R(x)$ just at such a singularity, we are able to control the convergence of the minimizing sequence and hence arrive at a solution.

In Section 4, we consider general critical cases and obtain similar sufficient conditions (cf. Theorem 4.1). We only require the values of $R(x)$ are not too large at singular points of the metric with the smallest angle. Our results apply to any conical singular surface satisfying

$$\chi(s, \theta) = \min_i \left\{2, \frac{\theta_i}{\pi}\right\}.$$

However, because we are not able to obtain the best value of the constant C_1 in inequality (2) for a general surface, some of our conditions here are not as accurate as in Section 3. To obtain this results, one can use the idea in Section 3. However, we present an alternate approach, which is also interesting in its own right.

In Section 5, we study some supercritical cases. Our sufficient conditions are that $R(x)$ must be positive somewhere and $R(x)$ must be less than 0 at the singular points of the metric with the smallest angle (cf. Theorem 5.1). Our proof is based on the idea introduced in Section 4. We first show by contradiction that, under our assumptions, the corresponding variational functional is bounded from below. Then we can treat a minimizing sequence as in Sections 3 or 4 and arrive at a solution.

2. Necessary conditions and examples

In this section, we derive some necessary conditions for prescribing Gaussian curvature on the sphere with two singularities of equal angles (a critical case) and on the sphere with one singularity (a supercritical case). We approach from open manifolds.

Let S^2 be the sphere with standard metric g, and let (θ, ϕ) be the spherical coordinate with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. Let p be the north pole with $\theta = 0$ and q be the south pole with $\theta = \pi$. Write $S_1 = S^2 \setminus \{p\}$ and $S_0 = S^2 \setminus \{p, q\}$.

Given a smooth function $R(x)$ on S^2, we study the solvability of the semilinear elliptic problems

$$\begin{cases} -\triangle u + 2 = R(x) \exp u & \text{on } S_1 \\ u(x) = \gamma \ln \theta + v(x) & \text{near } p \end{cases} \tag{3}$$

and

$$\begin{cases} -\triangle u + 2 = R(x) \exp u & \text{on } S_0 \\ u(x) = \gamma \ln \theta + v(x) & \text{near } p \\ u(x) = \gamma \ln(\pi - \theta) + v(x) & \text{near } q, \end{cases} \tag{4}$$

where $\gamma > -2$, $v(x)$ is a smooth function and $\triangle$ is the Laplacian of the standard metric g. It is well known that if u is a solution of (4) then $R(x)$ is the scalar curvature (twice of the Gaussian curvature) of the metric $\exp u \, g$.

Let $\psi = \cos\theta$ be the spherical harmonic function. We have the following Kazdan–Warner type conditions.

Theorem 2.1. *(i) If u is a solution of (3), then*

$$\int_{S_1} \nabla R \cdot \nabla \psi \exp u dA > 0 \qquad \textit{for} \qquad \gamma > 0$$

and

$$\int_{S_1} \nabla R \cdot \nabla \psi \exp u dA < 0 \qquad \textit{for} \qquad \gamma < 0.$$

(ii) If u is a solution of (4), then

$$\int_{S_1} \nabla R \cdot \nabla \psi \exp u dA = 0 \;\; \forall \, \gamma > -2.$$

Proof of Theorem 2.1. The proof is similar to that of Kazdan and Warner's [1]. The difference here is that we need to take care of the boundary terms. Let $B_\epsilon(p)$ be the ball of radius ϵ centered at p. Let B be the set $S^2 \setminus B_\epsilon(p)$ in problem (3) and $S^2 \setminus \{B_\epsilon(p) \cup B_\epsilon(q)\}$ in problem (4). Multiply both side of the equation in (3) or in (4) by $\nabla u \cdot \nabla \psi$ and integrate on B. Taking into account of the fact that

$$\psi_{,ij} = g_{ij}\psi$$

we obtain

$$\int_{\partial B} \left\{ \frac{1}{2}|\nabla u|^2 \frac{\partial \psi}{\partial n} - \frac{\partial u}{\partial n}\nabla u \cdot \nabla \psi + 2\frac{\partial u}{\partial n}\psi \right\} ds = -\int_B \nabla R \cdot \nabla \psi \exp u dA \tag{5}$$

where $\partial/\partial n$ is the outward normal derivative on ∂B.

To show the first part of Theorem 2.1, we take into account of the following facts:

$$|\nabla u| \sim \left|\frac{\partial u}{\partial n}\right|, \qquad \frac{\partial u}{\partial n} \sim -\frac{\gamma}{\theta}, \qquad \psi \sim 1, \qquad \text{and} \qquad \frac{\partial \psi}{\partial n} \sim \theta, \text{ for } \theta \text{ small.}$$

Letting $\epsilon \to 0$, (5) becomes

$$\pi\gamma^2 + 4\pi\gamma = \int_B \nabla R \cdot \nabla \phi \exp u dA, \tag{$*$}$$

and the conclusion of (i) follows.

To verify the second part of Theorem 2.1, one simply notice that on the two boundary parts of B, the integrands on the left-hand side of (5) have the same asymptotic growth with opposite sign. Thus the left-hand side of (5) approaches 0 as $\epsilon \to 0$.

This completes the proof of the theorem. □

Remark 2.1. Due to technical limitation, we can only prove the Kazdan–Warner type necessary condition in the two special cases. It is known that in the smooth case both Bourguignon and Ezin [13] and Schoen showed that the same necessary condition (appropriately stated) was present for all compact surfaces, not just the sphere. However, the only case the necessary condition gave an obstruction to solvability was on the sphere. Then for general surfaces with conical singularities two natural questions arose:

(a) Can one obtain an appropriate necessary conditions?

(b) Are there obstructions for singular surfaces other than the punctured sphere?

It is well known that by Kazdan Warner's condition a monotone function cannot be realized as the Gaussian curvature of some pointwise conformal metric on smooth standard S^2, while our Theorem 2.1 seems to suggest that monotone functions are now the right candidates for Gaussian curvature on the sphere with one singularity. At this stage, we are not able to prove this fact; we will try in the near future. However, we can provide some examples of $R(x)$, which satisfy our necessary conditions in Theorem 2.1, and for such $R(x)$, we can find explicit solutions for (3).

Example 1. Let $\gamma > -2$. Consider a family of functions

$$R_\gamma(\theta) = \left(\frac{\gamma}{2} + 2\right)\left(\sin\frac{\theta}{2}\right)^{-\gamma/2}$$

The corresponding solution of (3) are

$$u_\gamma(\theta) = \gamma \ln\left(\sin\frac{\theta}{2}\right).$$

It can be seen easily that for $\phi = \cos\theta$

(i) for $\gamma > 0$, $R_\gamma'(\theta) < 0$, hence $\int_{S_1} \nabla R_\gamma \cdot \nabla\phi \exp u_\gamma dA > 0$;

(ii) for $\gamma < 0$, $R_\gamma'(\theta) > 0$, hence $\int_{S_1} \nabla R_\gamma \cdot \nabla\phi \exp u_\gamma dA < 0$;

(iii) for $\gamma = 0$, $u_\gamma = 0$ and $R_\gamma = 2$, this corresponds to the smooth sphere.

Here as $\theta \to 0$,

$$R_\gamma(\theta) \to \begin{cases} +\infty \text{ for } \gamma > 0 \\ +0 \text{ for } \gamma < 0. \end{cases}$$

In the following, we present a family of bounded monotone increasing or decreasing functions $R(x)$ with singular metric.

Example 2. Let

$$R_\gamma(\theta) = \frac{2(4\gamma\cos\theta + \gamma + 4)}{\gamma\cos\theta + \gamma + 4}.$$

One can verify that $R_\gamma(\theta)$ satisfy the above (i), (ii), and (iii) in Example 1. Also one can see that the corresponding metrics are

$$d\theta^2 + \left[\left(\frac{\gamma}{4}\cos\theta + 1 + \frac{\gamma}{4}\right)\sin\theta\right]^2 d\phi^2$$

which possess the desired conical singularity at $\theta = 0$.

3. Sufficient Conditions for a Special Critical Case

In this section, we find sufficient conditions for prescribing Gaussian curvature in a special critical case, the sphere with two singularities of equal angles. We approach from a singular surface.

Again let S_0 be the sphere S^2 with north and south poles removed. Equip S_0 with the metric

$$g_0 = d\theta^2 + (\alpha\sin\theta)^2 d\phi^2$$

where $0 < \alpha < 1$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. One can see that g_0 has two singularities of equal angle $2\pi\alpha$ at the north pole and south pole. The corresponding Laplacian is

$$\triangle = \frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{(\alpha\sin\theta)^2}\frac{\partial^2}{\partial\phi^2}$$

and the area element $dA = \alpha\sin\theta d\theta d\phi$ with total area $4\pi\alpha$. A straightforward calculation shows that the scalar curvature of g_0 is equal to 2.

Given a smooth function $R(x)$ on S^2, one would like to know whether it can be the scalar curvature of some metric g pointwise conformal to g_0. If we let $g = \exp(u)\, g_0$, then it is equivalent to solve the following elliptic equation:

$$-\triangle u + 2 = R(x)\exp u(x) \quad x \in S_0. \tag{6}$$

To find a solution of the equation, we use variational method. Consider the functional

$$J(u) = \frac{1}{2}\int_{S_0}|\nabla u|^2 dA + 2\int_{S_0} u dA - 8\pi\alpha\ln\int_{S_0} R\exp u dA$$

defined on $H = \{u \in H^1(S_0) | \int_{S_0} R \exp u dA > 0\}$, where $H^1(S_0)$ is the Hilbert space with norm (cf. [17]):

$$||u|| = \left(\int_{S_0} \left(|\nabla u|^2 + |u|^2 \right) dA \right)^{1/2}$$

We prove

Theorem 3.1. *Let p and q be the north pole and south pole of S^2, respectively. Assume that* $\max_{S^2} R(x) > 0$. *Let $m = \max\{R(p), R(q)\}$. Then problem (6) has a solution provided* ***either one*** *of the following conditions hold:*

(i) $m \leq 0$ *or*

(ii) $m > 0$ *and* $\inf_H J < -8\pi\alpha \ln(4\pi\alpha m)$.

Remark 3.1. Condition (ii) in Theorem 3.1 is satisfied if *either one* of the following holds:

(a) $0 < m < \frac{1}{\text{Area}(S_0)} \int_{S_0} R(x) dA$, or

(b) $R(q) < R(p)$ and $\triangle R(p) > 0$.

To see that (a) implies (ii), we simply consider $J(0)$. To see that (b) implies (ii), we estimate $J(\phi_\lambda - \tilde{\phi}_\lambda)$, where

$$\phi_\lambda(\theta) = \ln \frac{1 - \lambda^2}{(1 - \lambda \cos\theta)^2} \qquad \text{for} \qquad \lambda \in [0, 1),$$

and

$$\tilde{\phi}_\lambda = \frac{1}{\text{Area}(S_0)} \int_{S_0} \phi_\lambda \, dA \qquad \text{for} \qquad \lambda \in [0, 1).$$

Similar to the proof of Theorem 2.2 in [8], using the second-order Taylor expansion of $R(x)$ near the point p, one can verify that

$$J(\phi_\lambda - \tilde{\phi}_\lambda) < -8\pi\alpha \ln 4\pi\alpha m$$

for λ sufficiently close to 1.

Remark 3.2. One can easily see that $R(x)$ is positive somewhere is a necessary condition. Besides this, we only require that the values of $R(x)$ are not too large at two poles. No other restrictions are imposed on $R(x)$ elsewhere.

The outline of the proof of Theorem 3.1. First we show that for each $\lambda < 2$, the equation

$$-\triangle u + \lambda = R(x) \exp u(x) \quad x \in S_0 \tag{7}$$

has a solution.

Then we consider a sequence $\lambda_k \to 2$ and the solution u_k of (7) associated to it.

If $\{u_k\}$ is bounded, then it converges to a solution of (6). We are done.

If $\{u_k\}$ blows up, by our Distribution of Mass Principle, it can blow up at only one point, say y_0. We show that y_0 has to be one of the poles by estimating the total curvature $\int_{S_0} R(x) \exp u_k dA$ along the blowing-up sequence. To estimate the total curvature, we apply the rescaling technique to get a limiting equation in R^2 and then use our knowledge on the solutions of the limiting equation, the so-called "the behavior of the critical points at infinity" (cf. [14,15]). Finally, we show that $\{u_k\}$ cannot blow up at the poles under the assumptions of Theorem 3.1, and we arrive at our desired solution.

To prove the theorem, we need the following three useful tools developed in our previous papers [14,15].

The first is the so-called Distribution of Mass Principle, which implies that every minimizing sequence of $J(u)$ possesses a subsequence that can only blow up at one point. This principle is true for general compact Riemaniann surfaces, smooth or with singularities (cf. [15]). For simplicity, here we state it for our football S_0 only.

Proposition 3.1. *Let Ω_1, $\Omega_2 \subset S_0$ be two sets that* $\text{dist}(\Omega_1, \Omega_2) \geq \epsilon_0 > 0$. *Let α_0 be a number $0 < \alpha_0 \leq 1/2$. Then for any $\epsilon > 0$, there exists a constant $C = C(\alpha_0, \epsilon_0, \epsilon)$ such that if $u \in H^1(S_0)$ satisfies*

$$\frac{\int_{\Omega_i} \exp u dA}{\int_{S_0} \exp u dA} \geq \alpha_0, \qquad i = 1, 2,$$

then

$$\int_{S_0} \exp u dA \leq C \exp \left\{ \left(\frac{1}{32\pi\alpha} + \epsilon \right) \int_{S_0} |\nabla u|^2 dA + \frac{1}{4\pi\alpha} \int_{S_0} u dA \right\}.$$

The second tool provides information on the behavior of the critical points at infinity of the functional J. Here we only state the part that we need in this paper (cf. [14] for more details).

Proposition 3.2. *Let $u(x)$ be a solution of*

$$-\triangle u = C \exp u \; x \in R^2$$

with $\int_{R^2} \exp u dx < \infty$. *Then* $\int_{R^2} C \exp u dx = 8\pi$.

The third tool gives the best constant in a key inequality.

Proposition 3.1 *The inequality*

$$\int_{S_0} \exp u dA \leq 4\pi\alpha \exp\left\{ \frac{1}{16\pi\alpha} \int_{S_0} |\nabla u|^2 dA + \frac{1}{4\pi\alpha} \int_{S_0} u dA \right\} \tag{8}$$

holds for all $u \in H^1(S_0)$.

The proof of Theorem 3.1.

Step 1. To show that a solution of (7) exists for each fixed $\lambda < 2$, we simply minimize the functional

$$J_\lambda(u) = \frac{1}{2} \int_{S_0} |\nabla u|^2 dA + \lambda \int_{S_0} u dA - 4\pi\alpha\lambda \ln \int_{S_0} R \exp u dA$$

on H. By (8), we have

$$J_\lambda(u) \geq \left(\frac{1}{2} - \frac{\lambda}{4} \right) \int_{S_0} |\nabla u|^2 dA - 4\pi\alpha\lambda \ln \left(4\pi\alpha \max_{S^2} R \right).$$

It follows that for any minimizing sequence $\{v_k\}$ of J_λ, $u_k = v_k + c_k$ is bounded in H with a suitable choice of constants c_k. Thus $\{u_k\}$ possesses a weakly convergent subsequence. Note that the functional $J_\lambda(\cdot)$ is weakly lower semicontinuous and satisfies

$$J_\lambda(u + c) = J_\lambda(u) \text{ for any constant } c,$$

the weak limit $\bar{u}_\lambda$ of the subsequence is a minimizer of J_λ. It is easy to see that $u_\lambda = \bar{u}_\lambda + c_\lambda$ is a solution of (7) and a minimizer of J_λ for a suitable choice of constant c_λ.

Step 2. Let $\{\lambda_k\}$ be a sequence approaching 2. For each λ_k, let u_k be the solution of

$$-\triangle u + \lambda_k = R(x) \exp u(x) \; x \in S_0.$$

Let $m_k = \max_{S^2} u_k(x)$.

If $\{m_k\}$ is bounded from above, then one can show by a standard argument that $\{u_k\}$ possesses a convergent subsequence in H; hence the limit u_0 is a solution of (6). We are done.

Now suppose that $m_k \to +\infty$ as $k \to \infty$. We are going to apply the Distribution of Mass Principle to show that $\{u_k\}$ can only blow up at one point. To this end, we need the boundedness of $J_{\lambda_k}(u_k)$. In fact, we can show that

$$\limsup_{\lambda \to 2} \inf_H J_\lambda \leq \inf_H J. \tag{9}$$

To see this, for any $\epsilon > 0$, choose $u \in H$, s.t.

$$J(u) < \inf_H J + \epsilon.$$

Then let λ be sufficiently close to 2, s.t.

$$|J_\lambda(u) - J(u)| \leq (2-\lambda)\left(\left|\int_{S_0} u dA\right| + 4\pi\alpha \left|\ln \textstyle\int_{S_0} \exp u dA\right|\right) < \epsilon.$$

Now (9) follows easily.

Let y_k be one of the maximum points of u_k on S^2, i.e., $u_k(y_k) = \max_{S^2} u_k \equiv m_k$. Passing to a subsequence, one may assume that $y_k \to y_0 \in S^2$.

Applying inequality (9) and Proposition 3.1, we claim that the masses of the sequence $\{\exp u_k\}$ have to concentrate at y_0. More precisely, for any $\epsilon > 0$, we have

$$\frac{\int_{S_0 \setminus B_\epsilon(y_0)} \exp u_k dA}{\int_{S_0} \exp u_k dA} \to 0 \qquad \text{as} \qquad k \to \infty. \tag{10}$$

Consequently, by (9), we derive that (cf. [15])

$$R(y_0) > 0. \tag{11}$$

Step 3. At this stage, we apply the blowing-up technique to estimate the total curvature $\int_{S_0} R(x) \exp u_k dA$ and arrive at a contradiction. Hence conclude that y_0 has to be one of the poles. Or in other words, $\{u_k\}$ can only blow up at one of the singular points of the metric.

Suppose y_0 is not one of the poles. Fix an $\epsilon > 0$, such that on the ball $B_\epsilon(y_0)$,

$$R > 0. \tag{12}$$

This is possible due to (11).

Choose a local coordinate system centered at y_0, such that the metric of S_0 can be written as

$$g_0(y) = \rho(y)|dy|^2$$

where $\rho(y)$ is a positive smooth function with $\rho(y_0) = 1$.

Set $v_k(x) = u_k(\epsilon_k x + y_k) - m_k$ with $\epsilon_k = \exp(-m_k/2)$. Denote the Euclidean Laplacian in R^2 by $\triangle_0$. Then

$$-\frac{1}{\rho(\epsilon_k x + y_k)}\triangle_0 v_k(x) + \epsilon_k^2\lambda_k = R(\epsilon_k x + y_k)\exp v_k, \qquad x \in B_{\frac{\epsilon}{\epsilon_k}}(0) \tag{13}$$

Since $\{v_k(x)\}$ is bounded from above and $v_k(0) = 0$, by (13) and a standard approach, one can see that in any bounded region $\Omega \subset R^2$, $\{v_k(x)\}$ converges weakly to a function $v_0(x)$ in $W^{2,2}(\Omega)$ and the limiting function satisfies

$$-\triangle_0 v_0(x) = R(y_0)\exp v_0(x), \qquad x \in R^2.$$

One can obtain $\int_{S_0}\exp u_k dA < C < \infty$ from (9) and (10). Consequently one sees that the integral $\int_{R^2}\exp v_0\, dx$ is finite. Then by Proposition 3.2, we have

$$\int_{R^2} R(y_0)\exp v_0(x)dx = 8\pi. \tag{14}$$

On the other hand, integrating both sides of (7) with $\lambda = \lambda_k$ and taking into account of (10), we obtain

$$\begin{aligned} 4\pi\alpha\lambda_k &= \textstyle\int_{S_0} R(x)\exp u_k dA \\ &= [R(y_0) + \circ(1)]\int_{B_\epsilon(y_0)} \exp u_k dA \\ &= [R(y_0) + \circ(1)]\int_{B_{\epsilon/\epsilon_k}(0)} \exp v_k dx \\ &\geq \circ(1) + \int_{R^2} R(y_0)\exp v_0(x)dx \\ &= \circ(1) + 8\pi. \end{aligned}$$

This is impossible because $\alpha < 1$ and $\lambda_k \leq 2$. Therefore y_0 has to be one of the poles.

Step 4. Now suppose the sequence $\{u_k\}$ blows up at one of the poles, say p. We want to show that both assumptions (i) and (ii) in the theorem are violated.

First, by (11), $R(p) > 0$. This contradicts (i).

To derive a contradiction with (ii), we apply (8) and (10) and obtain the estimate

$$J_{\lambda_k}(u_k) \geq -4\pi\alpha\lambda_k \ln\left[4\pi\alpha(R(p) + o(1))\right].$$

Hence

$$\liminf_{k\longrightarrow\infty} J_{\lambda_k}(u_k) \geq -8\pi\alpha \ln(4\pi\alpha m).$$

Consequently, by (9),

$$\inf_H J \geq -8\pi\alpha \ln(4\pi\alpha m),$$

which contradicts assumption (ii). This completes the proof. □

4. Sufficient conditions for general critical cases

In this section, we consider general critical cases. Let S be a compact Riemannian surface with metric g_o having conical singularities of angle θ_i at point p_i, $i = 1, 2, \ldots, m$. Assume that

$$\theta_1, \theta_2, \ldots, \theta_s = \min_i \theta_i \equiv \theta_o < 2\pi \quad \text{and} \quad \chi(S, \theta) = \min_i\{2, \theta_i/\pi\}. \tag{15}$$

Let $\triangle_o$, $R_o(x)$, and dA_o be the Laplacian, the scalar curvature, and the area element of the metric g_o. To find the solution for the equation

$$-\triangle_o u + R_o(x) = R(x)e^u, \qquad x \in S \tag{16}$$

we minimize the corresponding variational functional

$$J(u) = \frac{1}{2}\int_S |\nabla_o u|^2 dA_o - 4\theta_o \ln \int_S R(x)e^u dA_o$$

in

$$H_o = \left\{u \in H^1(S) \mid \int_S R(x)e^u dA_o > 0 \quad \text{and} \quad \int_S R_o(x)u dA_o = 0\right\}.$$

It is easy to see that a critical point of $J(u)$ in H_o plus a suitable constant is a solution of (16).

To estimate the minimizing sequence of the functional, we need the following inequalities.

The first is our Moser–Trudinger inequality on surfaces with conical singularities (cf. [14]). Its version for the surface S is that

$$\int_S e^{2\theta_o u^2} dA_o \leq C_1 \tag{17}$$

holds for all $u \in H^1(S)$ with $\int_S |\nabla_o u|^2 dA_o \leq 1$ and $\int_S u dA_o = 0$; where θ_o is the smallest angle of the singularities.

As a direct consequence of (17), we have for all $u \in H^1(S)$

$$\int_S e^u dA_o \leq C_2 \exp\left\{\frac{1}{8\theta_o}\int_S |\nabla_o u|^2 dA_o + \frac{1}{A_o}\int_S u dA_o\right\} \tag{18}$$

Based on (18), we can prove

Lemma 4.1. *For all $u \in H^1(S)$,*

$$\int_S e^u dA_o \leq C_o \exp\left\{\frac{1}{8\theta_o}\int_S |\nabla_o u|^2 dA_o + \frac{1}{4\theta_o}\int_S R_o(x) u dA_o\right\} \tag{19}$$

holds where the constant C_o depends on the metric g_o.

Proof. Let $R_o = \frac{1}{A_o}\int_S R_o(x) dA_o$ be the average of $R_o(x)$ on S. Then by the generalized Gauss–Bonnet theorem (cf. [17]), we have

$$R_o = \frac{1}{A_o} \cdot 4\pi\chi(S,\theta) = \frac{4\theta_o}{A_o}\,. \tag{20}$$

Let $v(x)$ be the solution of

$$\triangle v = R_o(x) - R_o \tag{21}$$

and let $u = w + v$. Then by (16), (18), and (19), we have

$$\begin{aligned}
\textstyle\int_S e^u dA_o &\leq C_3 \textstyle\int_S e^w dA_o \leq C_4 \exp\left\{\frac{1}{8\theta_o}\textstyle\int_S w(-\triangle_o w) dA_o + \frac{1}{A_o}\textstyle\int_S w dA_o\right\} \\
&= C_4 \exp\Big\{\tfrac{1}{8\theta_o}\big(\textstyle\int_S |\nabla_o u|^2 dA_o + 2\textstyle\int_S u\triangle_o v dA_o + \textstyle\int_S |\nabla_o v|^2 dA_o\big) \\
&\quad + \frac{1}{A_o}\textstyle\int_S (u - v) dA_o\Big\} \\
&\leq C_o \exp\left\{\tfrac{1}{8\theta_o}\textstyle\int_S |\nabla_o u|^2 dA_o + \tfrac{1}{4\theta_o}\textstyle\int_S R_o(x) u dA_o\right\}
\end{aligned}$$

This completes the proof. □

Now, we are ready to prove the existence theorem.

Theorem 4.1. *Assume that $R(x)$ is positive somewhere and C_o be defined in* (17). *Let $m = \max_{i=1,\ldots,s} R(p_i)$. Then problem* (16) *has a solution if either one of the following conditions hold:*

(i) $m \leq 0$, *or*

(ii) $m > 0$ *and* $\inf_H J(u) < -4\theta_o \ln(C_o m)$.

Remark 4.1. Condition (ii) is satisfied if $m < \frac{1}{C_o}\int_S R(x) dA_o$.

Proof. Let $\{u_k\}$ be a minimizing sequence of $J(u)$ in H_o.

If $\{u_k\}$ is bounded in $H^1(S)$, we are done.

Now suppose $\{u_k\}$ blows up. Then by the Distribution of Mass Principle, passing to a subsequence of $\{u_k\}$, there exists a point $x_o \in S$, such that for any $\epsilon > 0$

$$\frac{\int_{B_\epsilon(x_o)} e^{u_k} dA_o}{\int_S e^{u_k} dA_o} \rightarrow 1 \tag{22}$$

as $k \rightarrow \infty$.

We are going to show that x_o has to be one of the singularities $p_1, \ldots, p_s$ with the smallest angle. To this end, one could estimate the total curvature as we did in the proof of Theorem 3.1. However, here we would rather present an alternate approach, the idea of which is interesting and can be applied to investigate some other problems, as will be seen in Section 5.

We prove by contradiction. Suppose x_o is a smooth point or is one of the other singular points $p_{s+1}, \ldots, p_m$. Let $\theta = \min\{\theta_{s+1}, \ldots, \theta_m, 2\pi\}$. Then by the definition of θ_o, we have

$$\theta > \theta_o. \tag{23}$$

Choose a smooth positive function $v(x)$ on S, such that $v(x) \equiv 1$ in a small neighborhood of x_o and $v(x) \sim [\text{dist}(x, p_i)]^{2-\frac{\theta_o}{\pi}}$ near the singularities p_i, for $i = 1, 2, \ldots, s$. Then the metric $g = v g_o$ has conical singularities of angle θ_i at points p_i for $i = s+1, \ldots, m$. Here we use $v(x)$ to smooth out the singularities $p_1, \ldots, p_s$ of the original metric g_o. Denote the corresponding gradient and the area element of g by $\bigtriangledown$ and dA. Then our Moser–Trudinger inequality becomes

$$\int_S e^{2\theta u^2} dA \leq C$$

for any u satisfying $\int_S |\nabla u|^2 dA \leq 1$ and $\int_S u dA = 0$. Consequently, for $u \in H^1(S)$, the following holds:

$$\int_S e^u dA \leq C \exp\left\{\tfrac{1}{8\theta}\int_S |\nabla u|^2 dA + \frac{1}{A}\int_S u dA\right\}. \tag{24}$$

Choose $\epsilon > 0$ so small that $v(x) \equiv 1$ in $B_\epsilon(x_o)$. Then for sufficiently large k, by (22) and (24), we have

$$\begin{aligned} \int_S e^{u_k} dA_o &\leq 2\int_{B_\epsilon(x_o)} e^{u_k} dA_o = 2\int_{B_\epsilon(x_o)} e^{u_k} dA \\ &\leq 2\int_S e^{u_k} dA \leq C \exp\left\{\tfrac{1}{8\theta}\int_S |\nabla u_k|^2 dA + \frac{1}{A}\int_S u_k dA\right\} \\ &= C \exp\left\{\tfrac{1}{8\theta}\int_S |\nabla_o u_k|^2 dA_o + \frac{1}{A}\int_S u_k v dA_o\right\}. \end{aligned} \tag{25}$$

Since $\int_S R_o(x) u_k dA_o = 0$, the generalized Poincaré inequality implies

$$\frac{1}{A}\int_S u_k^2 dA_o \leq C\int_S |\nabla_o u_k|^2 dA_o.$$

Consequently, by Hölder inequality, we have

$$\begin{aligned} \frac{1}{A}\int_S u_k v dA_o &\leq \frac{1}{A}\left\{\int_S u_k^2 dA_o\right\}^{\frac{1}{2}}\left\{\int_S v^2 dA_o\right\}^{\frac{1}{2}} \leq C\left\{\int_S |\nabla_o u_k|^2 dA_o\right\}^{\frac{1}{2}} \\ &\leq \mu\int_S |\nabla_o u_k|^2 dA_o + C_\mu \end{aligned} \tag{26}$$

for any $\mu > 0$. Choose $\mu = \frac{1}{2}(\frac{1}{8\theta_o} - \frac{1}{8\theta})$. Then by (25) and (26),

$$\int_S e^{u_k} dA_o \leq C_\mu \exp\left\{\left(\tfrac{1}{8\theta_o} - \mu\right)\int_S |\nabla_o u_k|^2 dA_o\right\}.$$

It follows that

$$J(u_k) \geq \mu\int_S |\nabla_o u_k|^2 dA_o - C.$$

Consequently $\{u_k\}$ is bounded in $H^1(S)$. This is a contradiction with our assumption that $\{u_k\}$ blows up.

The above contradiction shows that the minimizing sequence $\{u_k\}$ can only blow up at one of the singularities $p_1, \ldots, p_s$ with the smallest angle. Now similar to the proof of Theorem 3.1, we can show that under assumption (i) or (ii) of Theorem 4.1, the sequence $\{u_k\}$ can blow up nowhere and hence has to converge weakly to a minimizer of J in H_o.

This completes the proof of the theorem. □

5. Sufficient conditions for supercritical cases

In this section, we provide sufficient conditions for prescribing Gaussian curvature in supercritical cases. To better illustrate our idea, we only state our results and present the proof in a special case, the sphere S with one conical singularity at point p of angle $\theta_o < 2\pi$. We start from a metric g_o on S having such a singularity, and then pointwise conformally deform it to a metric with the same singularity and having the prescribed scalar curvature $R(x)$. Since $\chi(S,\theta) = 1 + \frac{\theta_o}{2\pi} > \frac{\theta_o}{\pi}$, we are in a supercritical case.

As we did in Section 4, let $\triangle_o$, $R_o(x)$, and dA_o be the Laplacian, the scalar curvature and the area element of the metric g_o. To find the solution for the equation

$$- \triangle_o u + R_o(x) = R(x)e^u, \qquad x \in S \tag{27}$$

we minimize the corresponding variational functional

$$J(u) = \frac{1}{2}\int_S |\nabla_o u|^2 dA_o - 4\pi\chi(S,\theta)\ln\int_S R(x)e^u dA_o$$

in

$$H_o = \left\{u \in H^1(S) \mid \int_S R(x)e^u dA_o > 0 \quad \text{and} \quad \int_S R_o(x)u dA_o = 0\right\}.$$

Unfortunately, in supercritical cases, the functional $J(u)$ is no longer bounded from below in general. Hence we first need to regain some boundedness for J by imposing some conditions on $R(x)$.

Theorem 5.1. *Assume that $R(x)$ is positive somewhere and $R(p) < 0$. Then problem (27) has a solution.*

Proof. *Step 1.* We show by contradiction that under the assumption $R(p) < 0$, the functional $J(u)$ is in fact bounded from below in H_o.

Otherwise, there exists a sequence $\{u_k\}$ in H_o, such that $J(u_k) \to -\infty$.

(i) If $\{u_k\}$ is bounded in H_o, then passing to a subsequence $\{u_k\}$ converges weakly to an element u_o in H_o. Since $J(\cdot)$ is weakly lower-semicontinuous, we have

$$-\infty = \liminf J(u_k) \geq J(u_o).$$

This is obviously impossible.

(ii) Now suppose $\{u_k\}$ blows up. Since $J(u_k)$ is bounded from above, by our Distribution of Mass Principle, $\{u_k\}$, or passing to a subsequence, can only blow up at one point, say x_o.

By an argument similar to the proof of Theorem 4.1, we can show that $\{u_k\}$ can only blow up at singularity p.

Now taking into account the fact that $R(p) < 0$ and

$$J(u_k) = \frac{1}{2}\int_S |\nabla_o u_k|^2 dA_o - 4\pi\chi(S,\theta)\ln\{[R(p) + o(1)]\int_S e^{u_k} dA_o\}.$$

We again arrive at a contradiction. Therefore, the functional J has to be bounded from below in H_o.

Step 2. Now we consider a minimizing sequence $\{u_k\}$ of J in H_o. Applying the similar argument as in the proof of Theorem 4.1, we can show that $\{u_k\}$ can blow up nowhere and hence arrive at a solution. This completes the proof. □

Remark. One can easily generalized the results in Theorem 5.1 to some other supercritical cases. □

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Department of Mathematics, Southwest Missouri State University, Springfield MO 65804
School of Mathematics, Institute for Advanced Study
(C. Li is now in the Program in Applied Mathematics, University of Colorado, Boulder CO 80309)