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Domains in $\mathbf{C}^{n+1}$ with Noncompact Automorphism Group

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ABSTRACT. We consider certain pseudoconvex domains in $\mathbf{C}^{n+1}$ and show that if the automorphism group is noncompact, then the domain is equivalent to $E_m = \{|w|^2 + |z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2 < 1\}$ for some integer $m \geq 1$.

Introduction

We consider relatively compact domains $\Omega \subset \mathbf{C}^{n+1}$ with smooth boundary, and we ask when it is possible for $\operatorname{Aut}(\Omega)$ to be noncompact. It is a result of Wong [W], Klembeck [Kl], and Rosay [R] that if Ω is strongly pseudoconvex and if $\operatorname{Aut}(\Omega)$ is noncompact, then Ω is biholomorphically equivalent to the unit ball B^{n+1}. Ω is said to be of *finite type* if there is a number κ such that for any $p \in \partial\Omega$, a germ of a holomorphic variety V_p containing p cannot be tangent to $\partial\Omega$ to order higher than κ. A bounded domain with real analytic boundary is always of finite type (see [DF1]). Our main result is the following:

Theorem. *Let $\Omega \subset \mathbf{C}^{n+1}$ be a bounded pseudoconvex domain of finite type whose boundary is smooth of class C^∞, and suppose that the Levi form has rank at least $n-1$ at each point of the boundary. If* $\operatorname{Aut}(\Omega)$ *is noncompact, then Ω is biholomorphically equivalent to the domain*

$$E_m = \{(w, z_1, \ldots, z_n) \in \mathbf{C}^{n+1} : |w|^2 + |z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2 < 1\} \qquad (1)$$

for some integer $m \geq 1$.

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In proving this theorem we develop three variants of a scaling argument. Scaling and related methods have been used in the convex case by [F], [GK1], [Ki1–2], and [Ko1–4]. In the papers by Greene and Krantz, Kodama, and Kim, it is shown that if $\mathrm{Aut}(\Omega)$ is noncompact and if Ω is locally equivalent near p_∞ to a special domain E of ellipsoid type, then Ω is globally equivalent to E. The scaling method also works well on strongly pseudoconvex domains, which are locally equivalent to strongly convex domains. In fact, the method can be used to obtain many of the basic results about holomorphic mappings between strongly pseudoconvex domains (see [P]). In particular, this gives the easiest proof of the result that if $\mathrm{Aut}(\Omega)$ is noncompact and p_∞ is a strongly pseudoconvex point, then Ω is equivalent to the unit ball.

For the characterization of domains Ω for which $\mathrm{Aut}(\Omega)$ is noncompact, it is necessary to consider the case where p_∞ is not strongly pseudoconvex. It is well known that (weakly) pseudoconvex domains are not locally equivalent to convex domains. Thus, since we must rescale with a more general (nonlinear) holomorphic mapping, the scaling techniques that were used in the convex and strictly pseudoconvex cases are not applicable here. The proof of convergence is more difficult in this case and can be reduced to estimates of an invariant metric on a family of domains D_ν, which are independent of ν. In the case $n = 1$, this was achieved in [BP] by reducing this to estimates on the asymptotic behavior of the Kobayashi metric and then using the precise estimates on the Kobayashi metric of a domain in $\mathbf{C}^2$ given by Catlin [C]. Such estimates are not known in higher dimension. However, it is known that Ω has a plurisubharmonic (psh) exhaustion, and we show that is possible to use another intrinsic metric, introduced by Sibony [S], that is defined in terms of psh functions.

With this technique, we apply a scaling in Section 1 to obtain a biholomorphic mapping $g : D = \{v + P(z_1) + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n < 0\} \to \Omega$, where $P = P(z_1)$ is a real polynomial that is only known to be subharmonic. Any domain of the form D contains the translations $T_t(w, z) = (w+t, z)$ and thus has a noncompact automorphism group. The mapping g extends to a homeomorphism between $\bar{D}\cup\{\infty\}$ and $\bar{\Omega}$. Thus Ω has the 1-parameter family of automorphisms $g \circ T_t \circ g^{-1}$, which are parabolic with fixed point $\bar{p} = g(\infty)$, i.e., $\lim_{t\to\pm\infty} g\circ T_t \circ g^{-1}(p) = \bar{p}$ for all $p \in \bar{\Omega}$. The translations T_t are generated by the vector field $Re\, 2(\partial/\partial w)$, so the parabolic subgroup gT_tg^{-1} is generated by the holomorphic vector field $H = g_*(2(\partial/\partial w))$.

The rest of the proof will be concerned with a local analysis of H and Ω near $\bar{p}$. We may choose coordinates such that $\bar{p} = 0$ and near the origin

$$\Omega = \left\{ v + \psi(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n} z_\alpha \bar{z}_\alpha + \cdots < 0 \right\},$$

where ψ is a homogeneous polynomial of some degree $2m$ and the dots denote smaller terms. We make ψ unique up to scalar multiple by the assumption that it contains no harmonic terms. We assign weight $1/2m$ to the variable z_1 and weight $-1/2m$ to $\partial/\partial z_1$; weight $1/2$ to z_α and weight $-1/2$ to $\partial/\partial z_\alpha$, $2 \le \alpha \le n$; and weight 1 to w, weight -1 to $\partial/\partial w$. Thus the domain

$$\Omega_{\text{hom}} = \left\{ v + \psi(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n} z_\alpha \bar{z}_\alpha < 0 \right\}$$

is homogeneous of weight 1 and is the homogeneous model of Ω at $\bar{p}$. In Section 2 we determine the holomorphic vector fields Q such that $Re\, Q$ is tangent to $\partial\Omega_{\text{hom}}$. The tangent holomorphic vector fields of weight 0, $\mathcal{A}^{(0)}$, are of the form (19). Thus the dimension of $\mathcal{A}^{(0)}$ is $2+(n-1)^2$ if $\psi = c|z_1|^{2m}$ and $1+(n-1)^2$ otherwise. In Section 2 we show that there are no tangent vector fields to $\partial\Omega_{\text{hom}}$ of weight > 0 unless $\psi = c|z_1|^{2m}$. In this case the only possible weights are $1/2$ and 1, and the corresponding vector fields are given by (21) and (24).

Now let Q denote the homogeneous part of $H = g_*(2(\partial/\partial w))$ of smallest weight. By Lemma 6, $Q \neq 0$. The vector fields of weight zero are linear combinations of dilations and rotations, so if Q has weight zero, it must correspond to a rotation. (In the case of dimension 2 we can conclude at this point that $\psi = c|z_1|^{2m}$ and thus Ω is locally convex at $\bar{p}$; with these facts, we can proceed directly to the final rescaling in Section 5.) Unfortunately, if $n > 1$, a nontrivial rotation term might involve only the variables z_α, $2 \leq \alpha \leq n$, which would not allow us to conclude that $\psi = c|z_1|^{2m}$. Further, if $n > 1$, the fact that $\psi = c|z_1|^{2m}$ does not guarantee that Ω is locally convex at the origin, and we must examine Q more closely.

A parabolic vector field remains parabolic after the addition of a (weight 0) rotation term, so there is no "geometric" reason why Q should not have weight 0. But under the change of coordinates $w \mapsto -1/w$ (which is merely "formal" since the domain is not known to be convex), $\partial/\partial w$ (with fixed point at ∞) is taken to $w^2(\partial/\partial w)$, which has weight 1. So, formally, if g were "smooth" at infinity, the weight of $g_*(\partial/\partial w)$ would be 1. In Lemma 7, we use a second scaling argument, based on homogeneous dilations of the mapping g, to make this formal argument rigorous. Thus it follows that Q has weight > 0, and so by the results of Section 2 we conclude that $\psi = c|z_1|^{2m}$.

Our third scaling argument will be carried out on the parabolic subgroup $\{g \circ T_t \circ g^{-1}\}$ generated by the vector field $H = Q + \ldots$. In order to obtain more information from the new rescaling than we obtained from the original one, we need to know the asymptotic behavior of the orbit $\{g \circ T_t \circ g^{-1}(p_0)\}$ of a point $p_0 \in \Omega$ as $t \longrightarrow \pm\infty$. By Sections 2 and 3 we know that Q has either the form (22), corresponding to weight $1/2$, or (24), corresponding to weight 1. These cases involve different geometric behaviors, and we analyze them separately. In Section 4 we show that in the case of weight $1/2$ the orbit has the asymptotic behavior given in Lemma 8, and the w-coordinate satisfies $u(t) = o(v(t))$. The case of weight 1 is more delicate; the asymptotic behavior of the orbit is given in Lemmas 9 and 10, and the w-coordinate satisfies $v(t) = O(u(t)^2)$.

We perform our final rescaling in Section 5. We have shown that the orbit is well behaved as $t \longrightarrow \pm\infty$. With this information we can use very specific linear functions S_t for rescaling, and we can control the behavior of the domains $S_t(\Omega)$ well enough to conclude that the limit domain D is given by the homogeneous model Ω_{hom}. The proof is now complete, since this is biholomorphically equivalent to (1).

We note that in the case $n = 1$ the theorem was proved in [BP] for the case where Ω has real analytic boundary.[1] Bell and Catlin [BC] showed us a version of Lemma 6 that allows the

[1] F. Berteloot brought to our attention a gap in the proof of Proposition 2.3 of [BP]; Lemma 2 both generalizes Proposition 2.3 and fills this gap. We have recently received the preprint [BCo] in which Berteloot and Cœuré have found an independent proof of this proposition.

arguments of [BP] to carry over to boundaries of finite type. A natural direction for generalization of the theorem above would be to weaken the hypothesis on the rank of the Levi form. This case will be more difficult because a general pseudoconvex point does not have a nondegenerate homogeneous model, peak functions are not known to exist, and there are many domains (not of type (1)) with noncompact automorphism groups. For instance, let us choose positive weights $\delta_1, \ldots, \delta_n$ of the form $\delta_j = 1/2m_j$, m_j a positive integer, and let $p(z)$ be a real polynomial of the form

$$p(z) = \sum_{\mathrm{wt}(J)=\mathrm{wt}(K)=\frac{1}{2}} a_{J,K} z^J \bar{z}^K,$$

where we set $\mathrm{wt}(J) = \mathrm{wt}(j_1, \ldots, j_n) = \delta_1 j_1 + \cdots + \delta_n j_n$. The condition on the weights of the multiindices is equivalent to saying that $\Omega = \{|w|^2 + p(z) < 1\}$ is invariant under the 2-torus action $(\varphi, \theta) \mapsto (e^{i\varphi} w, e^{i\delta_1\theta} z_1, \ldots, e^{i\delta_n\theta} z_n)$. For rather general choices of the numbers $a_{J,K}$ Ω is bounded and has smooth boundary (which may be pseudoconvex or not). But the mapping $(w, z) \mapsto ((w/4 + i)(-w/4 + i)^{-1}, z_1(-w/4 + i)^{-2\delta_1}, \ldots, z_n(-w/4 + i)^{-2\delta_n})$ maps the domain $D = \{v + p(z) < 0\}$ (which has noncompact automorphism group) onto Ω.

1. Rescaling of domains

If $\mathrm{Aut}(\Omega)$ is noncompact, then there exists $p_\infty \in \partial\Omega$ and a sequence $\{f_\nu\} \subset \mathrm{Aut}(\Omega)$ that converges uniformly on compact subsets to p_∞, since there are no germs of complex varieties in $\partial\Omega$ (cf. [BP, Lemma 2.2]). We may assume that the rank of the Levi form at p_∞ is exactly $n-1$. Let r be a smooth defining function for $\Omega = \{r < 0\}$. Writing $w = u + iv$, we may assume that $p_\infty = 0$, and r is given near 0 as

$$r = v + u\varphi(u, z) + |z_2|^2 + \cdots + |z_n|^2 + o(|z|^2) \tag{2}$$

where $\varphi(0,0) = 0$. Since Ω is pseudoconvex and finite type, there exists a homogeneous polynomial $\psi(z_1, \bar{z}_1)$ of degree $2m$ such that $\partial\Omega$ is given by

$$\begin{aligned} r &= v + u\varphi(u, z) + \psi(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n} (Re(B_\alpha(z_1, \bar{z}_1)\, z_\alpha) + z_\alpha \bar{z}_\alpha) \\ &\quad + o(|z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2) = 0. \end{aligned} \tag{3}$$

It is standard to perform the change of coordinates

$$\tilde{w} = w + \sum (b_J + c_J w) z^J, \qquad \tilde{z} = z, \tag{4}$$

with $|J| \le 2m$, which serves to remove the "pure" terms from (3), i.e., it removes z^J and $\bar{z}^J$ terms from ψ and $\varphi(0, z)$ as well as $z_1^j z_\alpha$ from the summation.

Thus φ contains no linear terms in z_α by (4), so the only linear term is of the form cu. Since Ω is pseudoconvex, we may apply [BP, Lemma 2.1] to obtain

$$\varphi(u,z) - cu = O\left(|u|^2 + |u||z| + |z_1|^{m+1} + \sum_{\alpha=2}^{n}(|z_1 z_\alpha| + |z_\alpha|^2)\right). \tag{5}$$

We may also perform a change of coordinates

$$\tilde{w} = w, \qquad \tilde{z}_1 = z_1, \qquad \tilde{z}_\alpha = z_\alpha + \sum_j c_{\alpha j} z_1^j \tag{6}$$

to remove terms of the form $\bar{z}_1^j z_\alpha$ from the summation in (3).

Before we begin the scaling procedure, we make preliminary changes of coordinates of the form (4) and (6) to remove the corresponding terms from (3). To start the scaling procedure, we choose a point $p_0 \in \Omega$ and set $p_\nu = f_\nu(p_0)$ so that for ν large, we may write $p_\nu = (w^{(\nu)}, z^{(\nu)})$ in the local coordinates at $p_\infty = 0$. We introduce new coordinates

$$\hat{z} = z - z^{(\nu)}, \qquad \hat{w} = e^{i\theta_\nu} w - w_*^{(\nu)} - \sum a_j \left(z_j - z_j^{(\nu)}\right) \tag{7}$$

where $w_*^{(\nu)} \in \mathbf{C}$, $\theta_\nu \in \mathbf{R}$, and $a_j \in \mathbf{C}$ are chosen so that in the coordinates $(\hat{w}, \hat{z})$ we have (i) the point $(0,\ldots,0) \in \partial\Omega$, (ii) p_ν is given as $(-i\epsilon_\nu, 0, \ldots, 0)$ for some $\epsilon_\nu > 0$, and (iii) the tangent to $\partial\Omega$ at $(0,\ldots,0)$ is $\{Im\,\hat{w} = 0\}$. The boundary of Ω is given in the $(\hat{w}, \hat{z})$ coordinate system by

$$\begin{aligned}\hat{\rho}^{(\nu)} &= \hat{c}^{(\nu)}\hat{v} + \sum_{k=2}^{2m} \hat{\psi}_k^{(\nu)}(\hat{z}_1, \bar{\hat{z}}_1) + \hat{u}\hat{\varphi}^{(\nu)}(\hat{u}, \hat{z}) + \sum_{k=2}^{m}\sum_{\alpha=2}^{n} Re\left(\hat{B}_{\alpha,k}^{(\nu)}(\hat{z}_1, \bar{\hat{z}}_1)\, z_\alpha\right) \\ &\quad + \hat{Q}^{(\nu)}(\hat{z}_2, \ldots, \hat{z}_n) + \hat{E}^{(\nu)}(\hat{z}, \bar{\hat{z}}) = 0 \end{aligned} \tag{8}$$

where $\hat{\psi}_k^{(\nu)}$ and $\hat{B}_{\alpha,k}^{(\nu)}$ are homogeneous of degree k, $\hat{Q}^{(\nu)}$ is a quadratic form, and $\hat{E}^{(\nu)} = o(|z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2)$.

Now we make coordinate changes of the form (4) and (6) to remove the terms of the form z_1^j, $2 \le j \le 2m$, $z_1^j z_\alpha$, $\bar{z}_1^j z_\alpha$, $2 \le j \le m$, and their conjugates from the expansion of $\hat{\rho}^{(\nu)}$. We observe that these changes of coordinates, which we will denote by $(4^{(\nu)})$ and $(6^{(\nu)})$, are biholomorphic mappings defined in a fixed neighborhood U of p_∞ in $\mathbf{C}^{n+1}$, which is independent of ν. Further, as $\nu \to \infty$, the composition of coordinate changes $(4^{(\nu)})$, $(6^{(\nu)})$, and $(7^{(\nu)})$, associated to the point p_ν, approaches the identity transformation. Thus there exists $c > 0$ such that $\sum_k \hat{\psi}_k^{(\nu)}$, $\hat{\varphi}^{(\nu)}$, and $\sum_k \hat{B}_{\alpha,k}^{(\nu)}$ approach ψ, φ, and B_α in $C^\infty(|u| + |z| < c)$ as

$\nu \longrightarrow \infty$, and $\hat{Q}^\nu$ approaches $|z_2|^2 + \cdots + |z_n|^2$. Further, there exist large $N, K < \infty$ such that

$$\begin{aligned} |\hat{\varphi}^{(\nu)}| &\leq K(|u| + |z_1|^m + |z_2| + \cdots + |z_n|) \\ |\hat{E}^{(\nu)}| &\leq K\,|z|(|z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2) \end{aligned} \tag{9}$$

hold for $\nu \geq N$ and $|u| + |z| < c$.

Now let us drop the hats (^) from the coordinates and make the following rescaling of coordinates:

$$\begin{aligned} \epsilon_\nu \tilde{w} &= w \\ \delta_\nu \tilde{z}_1 &= z_1 \\ \sqrt{\epsilon_\nu}\,\tilde{z}_\alpha &= z_\alpha, \qquad 2 \leq \alpha \leq n. \end{aligned} \tag{10}$$

The defining function for the corresponding domain is given by

$$\begin{aligned} \rho^{(\nu)} &= \frac{1}{\epsilon_\nu}\hat{\rho}^{(\nu)} \\ &= v + \epsilon_\nu^{-1}\left(\sum_{k=2}^{2m} \delta_\nu^k \psi_k^{(\nu)}(z_1) + \sum_{k=2}^{m}\sum_{\alpha=2}^{n} \sqrt{\epsilon_\nu}\,\delta_\nu^k\, Re\left(B_{\alpha,k}^{(\nu)}(z_1,\bar{z}_1)\, z_\alpha\right)\right) \\ &\quad + Q^{(\nu)}(z_2,\ldots,z_n) + \ u\varphi^{(\nu)}(\epsilon_\nu u, \delta_\nu z_1, \sqrt{\epsilon_\nu} z_2, \ldots, \sqrt{\epsilon_\nu} z_n) \\ &\quad + \epsilon_\nu^{-1} E^{(\nu)}(\delta_\nu z_1, \sqrt{\epsilon_\nu} z_2, \ldots, \sqrt{\epsilon_\nu} z_n), \end{aligned} \tag{11}$$

where we have dropped the tildes (~) from all the coordinates. Now we choose δ_ν such that the coefficient of the largest term in parentheses in (11) has modulus 1. Since the function $\psi_{2m}^{(\nu)}$ converges to ψ as $\nu \longrightarrow \infty$, it follows that

$$\sup_\nu \epsilon_\nu^{-1}\delta_\nu^{2m} < \infty. \tag{12}$$

We may pass to a subsequence as $\nu \longrightarrow \infty$ so that the polynomials inside the large parentheses in (11) converge to a limit. By (9), the $\varphi^{(\nu)}$ and $E^{(\nu)}$ terms disappear as $\nu \longrightarrow \infty$, and thus the functions $\rho^{(\nu)}$ converge to a function of the form

$$\rho = v + P(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n}(Re(C_\alpha(z_1,\bar{z}_1)\, z_\alpha) + z_\alpha \bar{z}_\alpha),$$

where P and C_α are polynomials with $\deg P \leq 2m$ and $\deg C_\alpha \leq m$. Since (11) holds on the set $\{|w| < c\epsilon_\nu^{-1}, |z_1| < c\delta_\nu^{-1}, |z_\alpha| < c\epsilon_\nu^{-1/2}\}$, it follows that the convergence takes place in C^∞ of compact subsets of $\mathbf{C}^{n+1}$.

Let $D^{(\nu)}$ denote the domain that is the image of $\Omega \cap U$ under the original coordinate changes (4) and (6), followed by ($4^{(\nu)}$), ($6^{(\nu)}$), ($7^{(\nu)}$), and ($10^{(\nu)}$). It follows that the domains $D^{(\nu)}$ converge to the domain $D = \{\rho < 0\}$.

Lemma 1. *We have*

$$D = \{v + P(z_1, \bar{z}_1) + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n < 0\}$$

where P is a subharmonic polynomial whose Laplacian does not vanish identically. Further, there is a holomorphic function φ on D such that $|\varphi(w, z)| < 1$ for $(w, z) \in D$ and

$$\lim_{D \ni (w,z) \to \infty} \varphi(w, z) = 1.$$

Proof. Since D is the smooth limit of the pseudoconvex domains $D^{(\nu)}$, it is pseudoconvex. Thus the function ρ in (12) is plurisubharmonic, and so we must have

$$\frac{\partial^2 P(z_1, \bar{z}_1)}{\partial z_1 \partial \bar{z}_1} + \sum_{\alpha=2}^{n} Re \frac{\partial^2 C_\alpha(z_1, \bar{z}_1)}{\partial z_1 \partial \bar{z}_1} z_\alpha \geq 0.$$

Since this must hold for all values of z_α, it follows that the terms in the summation must vanish. On the other hand, since we removed all of the pure harmonic terms from C_α, it follows that the terms C_α themselves vanish. Since we have also removed all the pure harmonic terms from P, it follows that the Laplacian cannot vanish identically.

Let P be a polynomial of degree $2k$, and let P_{2k} denote the homogeneous part of P of degree $2k$. Thus for $\epsilon > 0$ we may choose a large C such that

$$D \subset \tilde{D} := \{v + P_{2k}(z_1, \bar{z}_1) + |z_2|^2 + \cdots + |z_n|^2 < \epsilon(|w| + |z_1|)^{2k} + C\}.$$

Let us recall the construction of a peak function for the point $0 \in \partial\tilde{D}$ as given in [BF]. This involved constructing a function that was linear along certain sectors and vanished at the origin. The reciprocal of this function vanishes at infinity and may be used to construct a function that peaks at the point at infinity of $\tilde{D}$. This proves the lemma. $\square$

Now let $h_\nu : D^{(\nu)} \longrightarrow \Omega \cap U$ denote the biholomorphic mapping defined by taking the inverse of the coordinate changes ($4^{(\nu)}$), ($6^{(\nu)}$), ($7^{(\nu)}$), and ($10^{(\nu)}$), followed by the inverses of the original (4) and (6). Any compact subset of Ω is mapped to $U \cap \Omega$ under f_ν for large ν, and so $h_\nu^{-1} \circ f_\nu$ is defined on any compact subset of Ω for ν large enough. Furthermore, for any compact $K \subset D$, we have $K \subset D^{(\nu)}$ for ν sufficiently large, and thus $g_\nu := f_\nu^{-1} \circ h_\nu$ is defined on K. Since Ω is bounded, we may extract a convergent subsequence $\{g_{\nu_j}\}$ which converges to a mapping $g : D \longrightarrow \bar{\Omega}$. By the choice of ϵ_ν, it follows that $g_\nu(-i, 0) = p_0$ holds for $\nu = 1, 2, 3, \ldots$. Thus $g(-i, 0) = p_0 \in \Omega$.

In order to control the convergence of the sequence $\{g_\nu\}$, we will use the invariant metric introduced by Sibony [S]. This assigns a length $F(p, \xi; D)$ to a point $p \in D$ and a tangent vector ξ at p. F decreases under holomorphic mappings, i.e., if $g : D \to \Omega$ is holomorphic then

$$F(g(p), g_*(\xi); \Omega) \leq F(p, \xi; D).$$

By the distance decreasing property it is easy to show that $F \leq$ Kobayashi metric. The property of this metric that is most useful for our work is that $F(p, \xi; D)$ is defined in terms of certain bounded psh functions on D. In particular, Sibony [S] shows that if there is a smooth, psh function ρ defined on all of D with $\rho \leq 1$, and if there is a $\delta > 0$ such that $dd^c\rho(\xi, \bar{\xi}) \geq \delta|\xi|^2$ holds at all p such that the Euclidean distance satisfies $\text{dist}(p, p_0) < \delta$, then there exists $\epsilon = \epsilon(\delta)$ such that

$$F(p_0, \xi; D) \geq \epsilon|\xi|. \tag{13}$$

We will find (13) useful for metric estimates that hold uniformly for a family of domains D. Equivalently, we could use the Kobayashi metric in the proofs below and appeal to the Sibony metric merely as a device for estimating the Kobayashi metric from below.

Lemma 2. *The mapping $g : D \to \Omega$ is a biholomorphism.*

Proof. By [DF2] there is a $\delta > 0$ and a smooth function α with $\alpha(0) = 0$ such that $\tilde{r} := -e^{\alpha}(-r)^{\delta}$ is a plurisubharmonic exhaustion function for Ω. Then

$$\tilde{\rho}^{(\nu)} = -e^{\alpha o h_\nu}(-r \circ h_\nu)^{\delta}\epsilon_\nu^{-\delta} = -e^{\alpha o h_\nu}(-\rho^{(\nu)})^{\delta}$$

is plurisubharmonic, and $\tilde{\rho}^{(\nu)}$ converges to $\tilde{\rho} := -(-\rho)^{\delta}$ as $\nu \to \infty$. Let us choose a point $q_0 \in D$ near $(-i, 0)$ such that $g(q_0) \in \Omega$ and $e^{\tilde{\rho}}$ is strictly plurisubharmonic at q_0. For ν sufficiently large, the functions $e^{\tilde{\rho}^{(\nu)}}$ will be uniformly strictly plurisubharmonic at the point q_0. Thus for a tangent vector ξ there will be a uniform lower bound for the Sibony metric

$$F(q_0, \xi; D^{(\nu)}) \geq \epsilon|\xi|.$$

Since this metric is nonincreasing under the functions g_ν, and since F_Ω is strictly positive, it follows that the differential $Dg_\nu(q_0)$ is bounded below. It follows then by the theory of H. Cartan (see [N]) that $g : D \to g(D)$ is a biholomorphic mapping.

It remains to show that g is onto. First we will show that g maps the boundary of D to the boundary of Ω. Let us choose small neighborhoods B_0 of q_0 and B_1 of $g(q_0)$ such that $B_1 \subset g_\nu(B_0)$ for ν sufficiently large. We choose $c > 0$ such that $\tilde{\rho} < -c$ on $\overline{B_0}$, and thus $\tilde{\rho}^{(\nu)} < -c$ on $\overline{B_0}$ for ν large. Now let h denote the harmonic function on $\Omega - B_1$ such that $h = -c$ on ∂B_1 and $h = 0$ on $\partial\Omega$. By the Hopf lemma, there is a constant $\epsilon > 0$ such that $-\epsilon\,\text{dist}(p, \partial\Omega) > h(p)$ for $p \in \Omega$. For any $\epsilon' > 0$, we may choose a compact K such that

$h > -\epsilon'$ outside K, and thus for ν sufficiently large, we have

$$-\epsilon \operatorname{dist}(p, \partial\Omega) > h \geq \tilde{\rho}^{(\nu)}((g^{(\nu)})^{-1}(p)) - \epsilon'$$

for $p \in K$. If $p \in g(D)$, we may pass to the limit as $\nu \to \infty$ and obtain the same inequality for $\tilde{\rho}$ with $\epsilon' = 0$. Finally, we note that for $R > 0$ there is a constant $\eta > 0$ such that $-\eta \operatorname{dist}(q, \partial D) < \rho(q)$ holds for $q \in D$ such that $|q| < R$. We conclude, then, that for $q \in D$ with $|q| < R$ we have

$$\epsilon^{1/\delta}\eta \operatorname{dist}(g(q), \partial\Omega)^{1/\delta} \leq \operatorname{dist}(q, \partial D). \tag{14}$$

Now we suppose that g is not onto. By (14) we must have $\lim_{p\to p_0} g^{-1}(p) = \infty$ for any $p_0 \in \Omega \cap \partial(g(D))$. Let φ be the peak function given in Lemma 1. It follows that

$$\lim_{p\to\Omega\cap\partial(g(D))} \varphi(g^{-1}(p)) = 1.$$

Thus, by Rado's theorem, $\varphi \circ g^{-1}$ extends analytically to Ω if we set it equal to 1 on $\Omega - g(D)$. However, $|\varphi \circ g^{-1}| < 1$ on $g(D)$, and so the extended function is bounded in modulus by 1 on Ω. Thus $\Omega - g(D)$ is empty by the Maximum Principle. $\square$

2. Tangent vector fields

In this section we will analyze, from the algebraic point of view, the tangent vector fields for the homogeneous model of Ω at the point $\bar{p} = 0$. We may assume that $\partial\Omega$ has the form (3) at 0. (Note, however, that the point $\bar{p}$ is in general different from p_∞ and the degree $2m$ of the homogeneous polynomial ψ at $\bar{p}$ may differ from the value encountered earlier.) We attach the following weights to the coordinates: w has weight 1, z_1 has weight $1/2m$, and z_α has weight $1/2$ for $2 \leq \alpha \leq n$.

We perform the scaling of coordinates at the point $\bar{p} = 0$ given by $w = \eta\tilde{w}$, $z_1 = \eta^{1/2m}\tilde{z}_1$, $z_\alpha = \eta^{1/2}\tilde{z}_\alpha$, $2 \leq \alpha \leq n$. As $\eta \to 0$, we obtain the weighted homogeneous domain

$$v + \psi(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n} Re\,(B_{\alpha,m}(z_1, \bar{z}_1)z_\alpha) + z_\alpha\bar{z}_\alpha < 0. \tag{15}$$

Since Ω is pseudoconvex, (15) must be pseudoconvex, too, so if we argue as in the proof of Lemma 1 we conclude that the B_α terms vanish. Thus we have

$$r = v + \psi(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n} z_\alpha\bar{z}_\alpha + \cdots \tag{16}$$

where the dots denote terms of weight greater than 1.

We assign weight -1 to $\partial/\partial w$, $-1/2$ to $\partial/\partial z_\alpha$, $2 \le \alpha \le n$, and weight $-1/2m$ to $\partial/\partial z_1$, and we let $\mathcal{A}^{(\mu)}$ denote the holomorphic vector fields Q that are homogeneous of weight μ and such that $Re\, Q$ is tangential to the homogeneous model $\{v + \psi(z_1, \bar z_1) + \sum_{\alpha=2}^n z_\alpha \bar z_\alpha = 0\}$. Thus $\oplus \mathcal{A}^{(\mu)}$ is a graded Lie algebra, and $[\mathcal{A}^{(\mu)}, \mathcal{A}^{(\nu)}] \subset \mathcal{A}^{(\mu+\nu)}$.

If we write $Q = Q_0(\partial/\partial w) + \sum_{j=1}^n Q_j(\partial/\partial z_j)$, then the tangency condition is equivalent to

$$Re\left[-\frac{i}{2}Q_0 + \psi_1 Q_1 + \sum_{\alpha=2}^n \bar z_\alpha Q_\alpha\right] = 0, \tag{17a}$$

where we write $\psi_1 = \partial\psi/\partial z_1$, and we substitute

$$w = u - i\left(\psi + \sum z_\alpha \bar z_\alpha\right) \tag{17b}$$

with z and u arbitrary. It is evident that -1 is the lowest possible weight, and $\mathcal{A}^{(-1)}$ consists of real multiples of $\partial/\partial w$. Since $m > 1$, every vector field of $\mathcal{A}^{(-1/2)}$ is easily seen to have the form

$$Q^{(-\frac{1}{2})} = \sum_{\alpha=2}^n c_\alpha L_\alpha, \quad c_\alpha \in \mathbf{R}, \tag{18}$$

where

$$L_\alpha = z_\alpha \frac{\partial}{\partial w} - \frac{i}{2}\frac{\partial}{\partial z_\alpha}\,. \tag{19}$$

$\mathcal{A}^{(-1/2)}$ may also be described by changing coordinates so that the homogeneous model becomes

$$v + \psi(az_1) + Re\,(cz_1^{2m}) + y_2^2 + \cdots + y_n^2 = 0.$$

In these coordinates the vector field L_α corresponds to $\partial/\partial z_\alpha$, which generates a translation in the x_α-direction. It will be seen in Lemma 4 that $\mathcal{A}^{(-1/2m)} \ne 0$ if and only if $a, c \in \mathbf{C}$ can be chosen so that $\psi(az_1) + Re\,(cz_1^{2m}) = y_1^{2m}$, and in this case $\mathcal{A}^{(-1/2m)}$ is generated by $\partial/\partial z_1$.

Let us make the useful observation that if $\mathrm{wt}(Q) \ge 0$, then $Q_j(0) = 0$ for $0 \le j \le n$. Thus there are no pure holomorphic or antiholomorphic terms in $\psi_1 Q_1$ or $\bar z_\alpha Q_\alpha$. So if (17a) holds, then there can be no pure holomorphic terms in Q_0, under the condition (17b). This means:

$$\text{If } \mathrm{wt}(Q) \ge 0, \text{ then } Q_0 \text{ is divisible by } w. \tag{20}$$

It is an elementary calculation to check that a solution of (17) with weight 0 has the form

$$Q^{(0)} = \mu\left(2mw\frac{\partial}{\partial w} + z_1\frac{\partial}{\partial z_1} + m\sum_{\alpha=2}^{n} z_\alpha\frac{\partial}{\partial_\alpha}\right) + i\gamma z_1\frac{\partial}{\partial z_1} + \sum_{\alpha,\beta=2}^{n}\gamma_{\alpha,\beta}z_\alpha\frac{\partial}{\partial z_\beta}, \quad (21)$$

with $\mu, \gamma \in R$ and $\gamma_{\alpha,\beta} = -\bar{\gamma}_{\beta,\alpha}$. Conversely, if (21) is a solution of (17) and if $\gamma \neq 0$, then $\psi = c|z_1|^{2m}$.

Lemma 3. *If $Q \in \mathcal{A}^{(\mu)}$, $\mu > 0$, and if $[\partial/\partial w, Q] = 0$ (i.e., if the coefficients of Q are independent of the variable w), then $Q = 0$.*

Proof. By (20) we have $Q_0 = 0$. And since $\mathrm{wt}(Q) > 0$, it follows that $\partial Q_j(0)/\partial z_\alpha = 0$ for $2 \leq j, \alpha \leq n$. Differentiating (17a) with respect to $\bar{z}_\alpha$, $2 \leq \alpha \leq n$, we obtain

$$Q_\alpha + \psi_1\overline{\left(\frac{\partial Q_1}{\partial z_\alpha}\right)} + \sum_{\beta=2}^{n} z_\beta\overline{\left(\frac{\partial Q_\beta}{\partial z_\alpha}\right)} = 0.$$

Thus $Q_j = 0$ for $j \neq 1$, and so $Q = 0$. $\square$

Lemma 4. *If $Q \in \mathcal{A}^{(\mu)}$ and $Q \neq 0$, then either $\mu = j/2$ for some integer $j \geq -2$, or $\mu = -1/2m$, in which case the following hold:*

(i) After a change of variable $z_1 \mapsto az_1$,

$$\psi(z_1) = \left(\left(\frac{z_1 - \bar{z}_1}{2i}\right)^{2m} - \left(\frac{z_1}{2i}\right)^{2m} - \left(\frac{\bar{z}_1}{2i}\right)^{2m}\right).$$

(ii) $Q = \lambda\left(iz_1^{2m-1}(\partial/\partial w) + c(\partial/\partial z_1)\right)$ for some $\lambda \in \mathbf{R}$ and $c = 4^{m-1}(-1)^m/m$.

Proof. In the case $-1 < \mu < -1/2$, we can only have $Q = Q_0\partial/\partial w$, and it is obvious that $Q = 0$. If $-1/2 < \mu < 0$, then $Q = Q_0\partial/\partial w + \sum_{j=1}^{n} Q_j\partial/\partial z_j$, and no coefficient Q_α, $2 \leq \alpha \leq n$ is a nonzero constant. Either $Q_1 = 0$ or we are in the case of weight $-1/2m$, and Q_1 is constant. $[\partial/\partial w, Q] = 0$ since it has weight < -1, so the coefficients Q_j are independent of w for $0 \leq j \leq n$. Further, $[L_\alpha, Q]$ has weight strictly between -1 and $-1/2$, so it, too, vanishes. Thus the coefficients Q_β, $2 \leq \beta \leq n$ are also independent of the variables

z_α, $2 \le \alpha \le n$, i.e., $Q_\beta = Q_\beta(z_1)$. Condition (17a) is now given as

$$Re\left[-\frac{i}{2}Q_0 + Q_1\psi_1(z_1) + \sum \bar{z}_\beta Q_\beta(z_1)\right] = 0,$$

and it follows that $Q_\beta = 0$ for $2 \le \beta \le n$. In the case of weight $< -1/2m$ we must have $Q_1 = 0$, so $Q = 0$. If Q has weight $-1/2m$, then Q_1 is constant, and we consider the new coordinates $\tilde{z}_1 = Q_0^{-1}z_1$. Dropping the $\tilde{}$ from our notation, we have $-(i/2)Q_0 = bz_1^{2m-1}$, so the previous equation becomes

$$Re\ (bz_1^{2m-1} + \psi_1) = Re\left(bz_1^{2m-1} + \frac{1}{2}\psi_{x_1}\right) = 0.$$

It follows that, modulo harmonic terms, ψ is independent of x_1. Thus ψ has the form (i), and Q can easily be shown to have the form (ii).

Next we show that $\mathcal{A}^{(1-(1/2m))} = 0$. If $Q \in \mathcal{A}^{(1-(1/2m))}$, then $[\partial/\partial w, Q] \in \mathcal{A}^{(-1/2m)}$. By Lemma 3, we have $Q = 0$ unless $\mathcal{A}^{(-1/2m)} \neq 0$, and in this case ψ must have the form (i) above and $[\partial/\partial w, Q]$ must have the form (ii). Thus by (20) we have $Q_0 = \lambda iwz_1^{2m-1}$. And by Lemma 3 again, we have $Q_1 = \lambda cw + R_1$, $Q_\alpha = R_\alpha$, $2 \le \alpha \le n$, where $R_j(z)$ depends on the variables z alone. Now we set $u = 0$ and $z_\alpha = 0$ for $2 \le \alpha \le n$, so (17a) becomes

$$Re\ [\lambda cvmy_1^{2m-1} + R_1(z_1, 0, \ldots, 0)\psi_1(z_1)] = 0.$$

Now we note that $v = -\psi$, and by homogeniety we must have $R_1(z_1, 0, \ldots, 0) = Az_1^{2m}$ for some constant A, so this equation becomes

$$Re\left[\lambda acy_1^{4y-1} + Az_1^{2m}\frac{m}{i}y_1^{2m-1}\right] = 0.$$

Expanding this out into powers of x_1 and y_1 we see that we must have $\lambda = A = 0$, and thus $Q = 0$.

For $\mu > 0$ with $\mu \neq 1 - (1/2m)$, we have $[\partial/\partial w, Q] \in \mathcal{A}^{(\mu-1)}$, and the lemma follows from Lemma 3 by induction. $\square$

In case $\psi = c|z_1|^{2m}$, it is easily seen that $\mathcal{A}^{(1/2)}$ contains

$$Q^{(\frac{1}{2})} = \sum_{\alpha=2}^{n} c_\alpha P_\alpha, \qquad \text{with } c_\alpha \in \mathbf{R}, \tag{22}$$

and

$$P_\alpha = mwz_\alpha\frac{\partial}{\partial w} + z_1 z_\alpha\frac{\partial}{\partial z_1} - \frac{im}{2}w\frac{\partial}{\partial z_\alpha} + m\sum_{\beta=2}^{n} z_\alpha z_\beta\frac{\partial}{\partial z_\beta}, \tag{23}$$

and $\mathcal{A}^{(1)}$ contains

$$Q^{(1)} = \lambda w \left(mw \frac{\partial}{\partial w} + z_1 \frac{\partial}{\partial z_1} + m \sum_{\alpha=2}^{n} z_\alpha \frac{\partial}{\partial z_\alpha} \right). \tag{24}$$

Lemma 5. *If $\mathcal{A}^{(\mu)} \neq 0$ for some $\mu \neq -1, -1/2, -1/2m, 0$, then the following hold:*

(i) $\mu = 1/2$ *or* 1.

(ii) $\psi = c|z_1|^{2m}$ *for some* $c > 0$.

(iii) $\mathcal{A}^{(1/2)}$ *is spanned by the vector fields* $Q^{(1/2)}$ *in (22).*

(iv) $\mathcal{A}^{(1)}$ *is spanned by the vector fields* $Q^{(1)}$ *in (24).*

Proof. By Lemma 4, μ must be a positive half-integer. By Lemma 3, $[\partial/\partial w, \mathcal{A}^{(\mu)}] \neq 0$, and so we have $\mathcal{A}^{(\mu-1)} \neq 0$. Continuing in this way, we have either $\mathcal{A}^{(1/2)} \neq 0$ or $\mathcal{A}^{(1)} \neq 0$. Let us suppose first that there exists a nonzero $Q \in \mathcal{A}^{(1/2)}$. If $n = 1$, then $Q = 0$ by Lemma 3. Thus we may suppose that $n > 1$, and $[\partial/\partial w, Q]$ has the form (18), so in this case $\partial Q_0/\partial w = \sum_{\alpha=2}^{n} c_\alpha z_\alpha$. By (20) Q_0 is divisible by w, so we must have $Q_0 = \sum_{\alpha=2}^{n} c_\alpha z_\alpha w$. By Lemma 3, there exist functions $R_1, \ldots, R_n$ that are independent of w such that $Q_1 = R_1$, and $Q_\alpha = -(i/2)c_\alpha w + R_\alpha$, for $2 \leq \alpha \leq n$.

The commutator $[L_\gamma, Q]$ has weight 0, and the coefficient of $\partial/\partial w$ is given by $z_\gamma \sum c_\alpha z_\alpha - \frac{i}{2} c_\gamma w - R_\gamma$. By (20) this is divisible by w, so we must have

$$R_\gamma = z_\gamma \sum_{\alpha=2}^{n} c_\alpha z_\alpha$$

for $2 \leq \alpha \leq n$. Now we have determined all of the Q_j except Q_1. Substituting into (17a), we obtain

$$Re \left(Q_1 \psi_1(z_1) - \sum_{\alpha=2}^{n} c_\alpha z_\alpha \psi(z_1) \right) = 0,$$

so we conclude that $\psi = c|z_1|^{2m}$ and $Q_1 = (1/m) z_1 \sum_{\alpha=2}^{n} c_\alpha z_\alpha$. This proves (ii) and (iii) in case $\mathcal{A}^{(1/2)} \neq 0$.

The other case is that there exists a nonzero $Q \in \mathcal{A}^{(1)}$. The commutator $[\partial/\partial w, Q]$ has the form (21), so by (20) we have $Q_0 = \mu m w^2$. By Lemma 3, we see that $Q_1 = \mu w z_1 + i\gamma w z_1 + R_1(z)$ and $Q_\alpha = \mu w z_\alpha + \sum_\beta \gamma_{\beta,\alpha} z_\beta w + R_\alpha(z)$ for $2 \leq \alpha \leq n$. Now we consider the coefficient of $\partial/\partial w$ in $[L_\alpha, Q] \in \mathcal{A}^{(1/2)}$. A simple calculation shows that this is equal to R_α plus terms divisible by w. So by (20) it follows that $R_\alpha(z) = 0$ for $2 \leq \alpha \leq n$. Our vector field now has the form

$$Q = Q^{(1)} + (i\gamma w z_1 + R_1(z))\frac{\partial}{\partial z_1} + \sum \gamma_{\beta,\alpha} z_\beta w \frac{\partial}{\partial z_\alpha} , \tag{25}$$

where $Q^{(1)}$ is as in (24) with $\lambda = \mu$. Setting now $u = z_1 = 0$, we see that (17a) takes the form $Re \sum \gamma_{\beta,\alpha} z_\beta iv \bar{z}_\alpha = 0$. By the condition $\gamma_{\alpha,\beta} = -\bar{\gamma}_{\beta,\alpha}$, we see that $\gamma_{\alpha,\beta} = 0$ for $2 \leq \alpha, \beta \leq n$.

We distinguish now two subcases: $\gamma = 0$ and $\gamma \neq 0$. If $\gamma = 0$, then we note that on $v + \psi + \sum z_\alpha \bar{z}_\alpha = 0$ we have $Re\, Q^{(1)}(v + \psi + \sum z_\alpha \bar{z}_\alpha) = \mu(Re\, w z_1 \psi_1) - \mu m u \psi$. Thus for the vector field Q, (17a) becomes

$$Re\,(\mu i v z_1 \psi_1 + R_1(z)\psi_1) = 0, \tag{26}$$

where we have set $u = 0$. If $n > 1$, then there is a $z_\alpha \bar{z}_\alpha$ in v, and the coefficient of $z_\alpha \bar{z}_\alpha$ (26) must vanish, so

$$\mu Re\,(i z_1 \psi_1) = 0, \tag{27}$$

and from this follows that $R_1 = 0$. If $n = 1$, then we must have $R_1(z) = c_1 z_1^{2m+1}$, and in this case also we must have (27) and $c_1 = 0$. In both cases it is evident that (iv) holds. Further, since $Q \neq 0$, we must have $\mu \neq 0$. Thus (27) gives $Im\, z_1 \psi_1 = 0$, so it follows that $\psi = c|z_1|^{2m}$ for some $c > 0$, and so (ii) holds.

The other case is that $\gamma \neq 0$. As was observed after (21), this implies that $\psi = c|z_1|^{2m}$ for some $c > 0$. In this case, if we set $u = 0$, then (17a) becomes

$$Re\,(i\gamma z_1(-i\psi)\psi_1 - R_1\psi_1) = Re\,(\gamma m |z_1|^{4m} - R_1(z) z_1^{m-1} \bar{z}_1^m) = 0. \tag{28}$$

But (28) implies that $\gamma = R_1 = 0$. Again Q has the form (24), so (ii) and (iv) hold in this case, too.

We see that in both of the cases $\mu = 1/2$ and $\mu = 1$ we have (ii), which implies that both $\mathcal{A}^{(1/2)} \neq 0$ and $\mathcal{A}^{(1)} \neq 0$. Thus (ii), (iii), and (iv) all hold.

Now we prove (i). By Lemma 3, it is sufficient to show that $\mathcal{A}^{(3/2)} = 0$ and $\mathcal{A}^{(2)} = 0$. Let us first suppose that $Q \in \mathcal{A}^{(3/2)}$. Then $[\partial/\partial w, Q] \in \mathcal{A}^{(1/2)}$, so by (iii) and (20) we have $Q_0 = \sum (m/2) c_\alpha z_\alpha w^2$. And by Lemma 3 we have

$$Q_1 = \sum c_\alpha z_\alpha z_1 w + R_1(z)$$

and

$$Q_\beta = -\frac{im}{4} c_\beta w^2 + m \sum c_\alpha z_\alpha z_\beta w + R_\beta(z).$$

Now $[L_\gamma, Q] \in \mathcal{A}^{(1)}$, so by (iv) the coefficient of $\partial/\partial w$ in this vector field is given by

$$mc_\gamma\left(-\frac{1}{2}+\frac{i}{4}\right)w^2 - R_\gamma(z) = \lambda m w^2.$$

Since c_γ and λ are real, they must vanish. This yields $R_\gamma = 0$, so $Q = 0$.

Finally, suppose $Q \in \mathcal{A}^{(2)}$. Then $[\partial/\partial w, Q] \in \mathcal{A}^{(1)}$, so by (iv) and (20) we have $Q_0 = \frac{1}{3}\lambda m w^3$. By Lemma 3 we have $Q_1 = (\lambda/2)z_1 w^2 + R_1(z)$ and $Q_\alpha = (\lambda/2) m z_\alpha w^2 + R_\alpha(z)$ for $2 \le \alpha \le n$. Checking the coefficient of u^2 in (17a), we find that $\lambda = 0$. Thus $Q = 0$ by Lemma 3. This completes the proof of (i) and of Lemma 5. $\square$

3. Geometry of the domain

The domain D has the one-parameter family of automorphisms $T_\tau(w, z) = (w + \tau, z)$, $\tau \in R$. By Lemma 2, this induces a one-dimensional subgroup $S_\tau := g \circ T_\tau \circ g^{-1}$ of $\mathrm{Aut}(\Omega)$. Since Ω is bounded, $\mathrm{Aut}(\Omega)$ is a Lie group, so this 1-dimensional subgroup is generated by $Re\, H$ for some holomorphic vector field $H = H_0(\partial/\partial w) + H_1(\partial/\partial z_1) + \cdots + H_n(\partial/\partial z_n)$, where $H_1, \ldots, H_n$ are holomorphic on Ω. By the fact that Ω is pseudoconvex and finite type, we know that the functions $H_1, \ldots, H_n$ extend to be $\mathcal{C}^\infty$ on $\overline{\Omega}$. Repeating the proof of Proposition 2.4 of [BP], we see that $g : D \to \Omega$ extends to a homeomorphism between $\overline{D} \cup \{\infty\}$ and $\overline{\Omega}$. Thus the point $\bar{p} = g(\infty)$ is a fixed point of S_τ and thus a zero of H. Further, it is a *parabolic* fixed point since $\lim_{\tau \to \pm\infty} S_\tau(p) = \bar{p}$ holds for all $p \in \overline{\Omega}$.

The vector field H is generated by the mapping g in the sense that for any function φ on Ω,

$$(Re\, H)(\varphi)|_{p=g(q_0)} = \left.\frac{d\varphi(g(t))}{dt}\right|_{t=0}, \tag{29}$$

where we write $g(t)$ for $g(q_0 + (t, 0, \ldots, 0))$. It was shown in [BP, Proposition 2.4] that

$$|g(t) - \bar{p}| \le C|t|^{-\epsilon} \tag{30}$$

holds for $q_0 \in D$ and $|t| \ge t_0$. Thus we see that at the point $p = g(q) \in \Omega$, we have $|Re\, H(p)| = |dg(q + (t, 0, \ldots, 0))/dt|_{t=0}$.

Lemma 6. *The vector field H vanishes to finite order at $\bar{p}$.*

Proof. Let us assume $\bar{p} = 0$ and choose $\delta > 0$ such that $1 - \delta > 2^{-\epsilon}$. Then there is a sequence $t_j \to \infty$ such that

$$|t_j^\epsilon g(t_j)| \ge (1 - \delta) t^\epsilon |g(t)| \tag{31}$$

for $t \geq t_j$. Thus

$$|g(t_j) - g(2t_j)| \geq c|g(t_j)|$$

for $c = 1 - (1-\delta)^{-1}2^{-\epsilon} > 0$. By the mean value theorem, there exists $t_j \leq \xi \leq 2t_j$ such that

$$|t_j g'(\xi)| \geq c|g(t_j)| \geq c2^{-\epsilon}|g(\xi)|, \tag{32}$$

where the last inequality follows from (31). From (30) we have $t_j \leq \xi \leq |Cg(\xi)|^{-1/\epsilon}$. Thus it follows from (32) that $|g'(\xi)|$ is bounded below by a constant times $|g(\xi)|^{1+1/\epsilon}$, and so we conclude that H vanishes to finite order at $\bar{p} = 0$. □

Thus the Taylor expansion of H vanishes to finite order at p_∞, and we may write $H = Q + \ldots$, where $Q \neq 0$ is homogeneous and the dots are terms of higher weight. By Lemma 5, Q is of the form (21), (22), or (24). Let us suppose for the moment that Q has the form (21) and set $\rho = w\bar{w} + z_1^m \bar{z}_1^m + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n$. If $\mu < 0$, then $Re\, H\rho = m\mu\rho + \cdots \leq 0$ holds near the origin. Thus the orbits of $Re\, H$ approach the origin in positive time but not in negative time. So we conclude that for a parabolic vector field with Q given as in (21) we must have $\mu = 0$. In the following lemma, we use the additional fact that

$$H = g_*\left(2\frac{\partial}{\partial w}\right), \tag{33}$$

which is a restatement of (29), and we show that Q cannot have the form (21). Our proof will proceed by a scaling argument.

First we recall the following result, which is an elementary consequence of the definition of the Kobayashi metric.

Proposition 1. *Suppose that $G \subset \mathbf{C}^n$ is an open set, that $\varphi : \Delta \to G$ is a holomorphic mapping defined on the unit disk $\Delta \subset \mathbf{C}$, that $\omega = \{p \in \mathbf{C}^n : \mathrm{dist}(p, \varphi(0)) < \eta\}$ is contained in G, and that $F(p, \xi; G) \geq \epsilon|\xi|$ for some $\epsilon > 0$ and all $p \in \omega$. Then $\varphi(z) \in \omega$ for $|z| < \min\{(1/2), (3/4)\eta\epsilon\}$.*

This is useful because it gives us a criterion for normality of a family of mappings into variable domains.

Proposition 2. *Let $\epsilon, \eta > 0$ and $p_0 \in \mathbf{C}^n$ be given. Let G_ν, $\nu = 1, 2, 3, \ldots$ be domains containing $\omega = \{p \in \mathbf{C}^n : \mathrm{dist}(p_0, p) < \eta\}$, and suppose that $F(p, \xi; G_\nu) \geq \epsilon|\xi|$ for all $p \in \omega$. If $\varphi_\nu : \Delta \to G_\nu$ is holomorphic for $\nu = 1, 2, 3, \ldots$ and if $\lim_{\nu\to\infty} \varphi_\nu(0) = p_0$, then $\{\varphi_\nu\}$ is a normal family on $\{|z| < \min\{(1/2), (3/4)\eta\epsilon\}$.*

Proposition 2 follows easily from Proposition 1.

Lemma 7. *Q has the form (22) or (24), and $\psi = const\, z_1^m \bar{z}_1^m$.*

Proof. By Lemma 5 and the remarks above, it is sufficient to show that if Q has the form (21) with $\mu = 0$, then $Q = 0$. So let us suppose that $Q \neq 0$ and derive a contradiction. Consider the biholomorphic map $R(w, z) = (w + cz_1^{2m}, z_1, \dots, z_n)$ and the scaling $\chi_\nu(w, z) = (\nu^{2m} w, \nu z_1, \nu^m z_2, \dots, \nu^m z_n)$, and note that $R\chi_\nu = \chi_\nu R$. Let us set $\tilde{\Omega} = R^{-1}\Omega$ and $\tilde{\Omega}^{(\nu)} = \chi_\nu \tilde{\Omega}$. If r is as in (16), then by Diederich and Fornaess [DF2] there exists $\delta > 0$ and a smooth function φ on $\overline{\Omega}$ such that $\rho = -e^{\varphi}(-r)^\delta$ is a psh exhaustion of Ω. It follows that $\tilde{\rho}^{(\nu)} = \nu^{-2m\delta} \rho \circ \chi_\nu^{-1} \circ R$ is a psh exhaustion of $\tilde{\Omega}^{(\nu)}$, and near the origin

$$\tilde{\rho}^{(\nu)} = -e^{\varphi^{(\nu)}} \left(- \left(v + Im\, cz_1^{2m} + \psi(z_1, \bar{z}_1) + \sum z_\alpha \bar{z}_\alpha + \tilde{E}^{(\nu)} \right) \right)^\delta$$

where $\varphi^{(\nu)} \to \varphi(0)$ and $\tilde{E}^{(\nu)} \to 0$ as $\nu \to \infty$. Further, as $\nu \to \infty$ the domain $\tilde{\Omega}^{(\nu)}$ tends smoothly on compact subsets to the domain

$$\tilde{\Omega}^* = \left\{ \tilde{\rho}^* = - \left(- \left(v + Im\, cz_1^{2m} + \psi(z_1, \bar{z}_1) + \sum z_\alpha \bar{z}_\alpha \right) \right)^\delta < 0 \right\}.$$

We may choose c such that $(Im(cz_1^{2m}) + \psi)_{z_1} \neq 0$ on the set $\{z_1 \neq 0 : \psi_{z_1 \bar{z}_1} = 0\}$. Thus $\tilde{\rho}^*$ is strictly psh at all points of $\tilde{\Omega}^* \cap \{z_1 \neq 0\}$. It follows from (13) that

$$\begin{aligned} &\text{for every compact } K \subset \tilde{\Omega}^* \cap \{z_1 \neq 0\} \text{ there exists } c_0 > 0 \text{ such that} \\ &F(\tilde{p}, \xi; \tilde{\Omega}^{(\nu)}) \geq c_0 |\xi| \text{ if } \tilde{p} \in K \end{aligned} \tag{34}$$

for ν large enough that $K \subset \tilde{\Omega}^{(\nu)}$.

Let us fix $p^0 \in \tilde{\Omega}^* \cap \{z_1 \neq 0\}$. Setting $q^{(\nu)} = (w^{(\nu)}, z_1^{(\nu)}, \dots, z_n^{(\nu)}) := g^{-1}\chi_\nu^{-1}(p^0)$, we make the coordinate change

$$\begin{aligned} \hat{z}_j &= z_j - z_j^{(\nu)}, \qquad 1 \leq j \leq n \\ \hat{w} &= w - Re\, w^{(\nu)} - a_0^{(\nu)} i + Re \sum a_j^{(\nu)} (z_j - z_j^{(\nu)}) \end{aligned} \tag{35}$$

where $a_0^{(\nu)} \in \mathbf{R}$ and $a_j^{(\nu)} \in \mathbf{C}$, $1 \leq j \leq n$ are chosen such that (i) in the $\hat{}$-coordinates $q^{(\nu)}$ is given by $(-i\lambda^{(\nu)}; 0, \dots, 0)$, (ii) the point $(0, \dots, 0) \in \partial D$, and (iii) $T_0 \partial D = \{Im\, \hat{w} = 0\}$.

Now in the $\hat{}$-coordinates, the domain D has the form

$$G^{(\nu)} = \{v + P^{(\nu)}(z_1, \bar{z}_1) + z_2\bar{z}_2 + \cdots + z_n \bar{z}_n < 0\},$$

where we have dropped the hats ($\hat{}$) from the notation. We observe that $P^{(\nu)}$ is a polynomial of degree $\leq 2m$, and we may make a change of coordinates

$$\tilde{w} = w + \sum c_j z_1^j \tag{36}$$

so that the new expression for $P^{(\nu)}$ in (36) in the $\tilde{}$-coordinates has no harmonic terms.

Finally we introduce the scaling of coordinates

$$\tilde{w} = \lambda^{(\nu)}\tilde{\tilde{w}}, \quad \tilde{z}_1 = \mu^{(\nu)}\tilde{\tilde{z}}_1, \quad \tilde{z}_\alpha = \sqrt{\lambda^{(\nu)}}\tilde{\tilde{z}}_\alpha, \quad 2 \leq \alpha \leq n. \tag{37}$$

$G^{(\nu)}$ will again be defined in the form (36) in the new coordinate system, and we choose $\mu^{(\nu)}$ such that the largest coefficient in $P^{(\nu)}$ has modulus 1. Now we drop the $\tilde{\tilde{}}$ from our notation. It follows that if we pass to a subsequence $P^{(\nu)}$ converges to a polynomial P^*, and $G^{(\nu)}$ converges smoothly on compact subsets to

$$G^* = \{v + P^*(z_1, \bar{z}_1) + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n < 0\}.$$

Since each $G^{(\nu)}$ is pseudoconvex, it follows that G^* is, too. Thus $P^*(z_1\bar{z}_1)$ is subharmonic.

Now let us consider the mapping $\tilde{h}^{(\nu)} : G^{(\nu)} \to \tilde{\Omega}^{(\nu)}$ that is given by the coordinate changes (35), (36), (37), followed by $\chi_\nu \circ g$. By construction, $\tilde{h}^{(\nu)}(-i, 0, \ldots, 0) = p^0$. Further, we have the estimate (34) in a neighborhood $\{|\tilde{p} - p^0| < c_1\}$.

By Proposition 2, there exists $\kappa > 0$ such that $\{\tilde{h}^{(\nu)}\}$ is a normal family on $V = \{|(w, z) - (-i, 0, \ldots, 0)| < \kappa\}$. Let $\tilde{h} : V \to \tilde{\Omega}^*$ denote the limit of a subsequence of $\{\tilde{h}^{(\nu)}\}$ on V. Now let a_1 be a point near 0 such that $P^*(a_1, \bar{a}_1)_{z_1\bar{z}_1} > 0$. Arguing with psh exhaustion functions, as we did to obtain (34), we see that

$$F(q, \xi; G^{(\nu)}) \geq c|\xi| \tag{38}$$

holds for ν sufficiently large and q sufficiently close to $(-i, a_1, 0, \ldots, 0)$. Now since $\tilde{h}^{(\nu)}$ is biholomorphic and $\operatorname{dist}(h^{(\nu)}(-i, a_1, 0, \ldots, 0), \partial\tilde{\Omega}^{(\nu)})$ is bounded below, the estimate (38) gives the lower bound

$$|\tilde{h}_*^{(\nu)}(-i, a_1, 0, \ldots, 0)\xi| \geq c_2|\xi| \tag{39}$$

for some $c_2 > 0$. Thus $\tilde{h}$ is locally biholomorphic at $(-i, a_1, 0, \ldots, 0)$.

Now we consider the mappings $h^{(\nu)} := R\tilde{h}^{(\nu)} : V \to \Omega^*$, where

$$\Omega^* = R\tilde{\Omega}^* = \left\{v + \psi(z_1, \bar{z}_1) + \sum z_\alpha\bar{z}_\alpha < 0\right\}.$$

We observe that the coordinate changes (35), (36), (37) preserve the orbits of $\partial/\partial u$ and in fact map $\partial/\partial w$ to a real constant multiple of itself. The map $h^{(\nu)} := R\tilde{h}^{(\nu)} : G^{(\nu)} \to \Omega^{(\nu)} := R\tilde{\Omega}^{(\nu)}$

has the property that

$$h_*^{(\nu)}\left(\frac{\partial}{\partial w}\right) = c_\nu (\chi_\nu)_* H. \tag{40}$$

It is easily seen that the limit of a vector field under homogeneous scaling will yield only the terms of weight 0, i.e., $\lim_{\nu\to\infty}(\chi_\nu)_* H = Q$. Since $h^{(\nu)}$ converges to $h = R\tilde{h}$, we see that $|c_\nu|$ is bounded above, and by (39), $h^{(\nu)}$ converges to a locally biholomorphic mapping, so $|c_\nu|$ is bounded below away from zero. Thus we have

$$h_*\left(\frac{\partial}{\partial w}\right) = cQ \tag{41}$$

for some real $c \neq 0$.

Next we will use (40) and (41) to show that h may be continued to a holomorphic mapping of $V + (\mathbf{R} \times \{0\})$ to Ω^*. We may assume that $h(V)$ is relatively compact in Ω^*. In the form (21), if $\gamma \neq 0$, then the functions w, $\psi = |z_1|^{2m}$ and $|z_2|^2 + \cdots + |z_n|^2$ are constant on the orbits of $Re\, Q$; otherwise, if $\gamma = 0$, then z_1 in addition is constant. In either case, the orbit of $h(V)$ under $Re\, Q$ is relatively compact in Ω^*. Thus by (34) there exists $\epsilon > 0$ and $\eta > 0$ so that

$$F(p, \xi; \Omega^*) \geq \epsilon|\xi| \tag{42}$$

in an η-neighborhood of the orbit of $h(V)$. Since $Re(\partial/\partial w)$ is an isometry, we have

$$F(q, \xi; \tilde{G}^*) \geq \epsilon|\xi| \tag{43}$$

for $q \in V + (\mathbf{R} \times \{0\})$ and some other $\epsilon > 0$. Now we may use (40), (41), (42), and Proposition 2 to conclude that $\{h^{(\nu)}\}$ is a normal family on $V + (\mathbf{R} \times \{0\})$, and we again denote the limit by h. By the estimate (43), it follows that h is a biholomorphic mapping from $V + (\mathbf{R} \times \{0\})$ to its image.

Finally, let σ denote the orbit of $Re\, Q$ passing through $h(-i, a_1, 0, \ldots, 0)$. For any point $A \in \sigma$, there is a (small) neighbornood U containing A on which h^{-1} is defined. By the estimates (42) and (43), the Jacobian matrix $h_*^{(\nu)}$ is bounded above and below on a fixed neighborhood of $h^{-1}(A)$. Thus $(h^{(\nu)})^{-1}(A)$ converges to $h^{-1}(A)$ as $\nu \to \infty$. Thus, by Proposition 2, $\{(h^{(\nu)})^{-1}\}$ is a normal family on a neighborhood of $\overline{\sigma}$. But this is a contradiction, since $h^{-1}(\sigma) = \{(-i + t, a_1, 0, \ldots, 0) : t \in \mathbf{R}\}$ is not bounded. This completes the proof of Lemma 7. □

Remark. We note that if $P(z_1, \bar{z}_1)$ is subharmonic and homogeneous, then the domain $\{v + P(z_1, \bar{z}_1) + z_2\bar{z}_2 + \cdots + z_n\bar{z}_n < 0\}$ is hyperbolic. (We may use the change of variables $\tilde{w} = w + cz_1^{2m}$ to obtain a bounded function that is strictly psh at any given point of the domain.) □

4. Parabolic orbits

We let $H = g_*(2(\partial/\partial w))$ denote our parabolic vector field, and we let $f_t = \exp(t\, Re\, H)$ denote the family of automorphisms of Ω obtained by exponentiating. For a point $p_0 \in \Omega$, we let $p_t = (w(t), z(t)) = f_t(p_0)$ denote the orbit under $Re\, H$. In this section, we study the asymptotic behavior of the orbit $t \mapsto p_t$ as $t \to \pm\infty$. By Lemmas 5 and 7, we have $H = Q + \cdots$ where either $Q = Q^{(1/2)}$ is as in (22) or $Q = Q^{(1)}$ is as in (24). In all cases the orbits approach 0 tangentially to the z_1-axis, but there is an essential difference: if $\mu = 1/2$, then $w(t)$ is essentially imaginary (i.e., the w-coordinate approaches the boundary essentially normally), whereas in the more singular case $\mu = 1$, $v(t) = O(u(t)^2)$.

Now we may define Ω locally as

$$r(w, z) = v + |z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2 + u\varphi(u, z) + E(z) < 0, \tag{44}$$

where φ satisfies (5). Since Ω is pseudoconvex,

$$E = O\left(|z_1|^{2m+1} + \sum(|z_1^{2m} z_\alpha| + |z_1 z_\alpha^2| + |z_\alpha^3|)\right). \tag{45}$$

In the first case, $\mu = 1/2$, we may make a change of coordinates by an $(n-1)\times(n-1)$ (real) orthogonal matrix acting on the variables $z_2, \dots, z_n$, followed by a dilation, so that $Q = Q^{(1/2)} = 2P_2$ (as in (23)). Thus we may assume

$$\begin{aligned} \frac{1}{2}H &= (wz_2 + G_0)\frac{\partial}{\partial w} + \left(\frac{1}{m}z_1 z_2 + G_1\right)\frac{\partial}{\partial z_1} \\ &\quad + \left(-\frac{i}{2}w + z_2^2 + G_2\right)\frac{\partial}{\partial z_2} + \sum_{\alpha=3}^{n}(z_2 z_\alpha + G_\alpha)\frac{\partial}{\partial z_\alpha}, \end{aligned}$$

where G_j denotes terms of higher weight. Thus the orbit $t \mapsto p_t = (w(t), z(t))$ satisfies the equations

$$\begin{aligned} wz_2 + G_0 &= \dot{w} \\ \frac{1}{m}z_2 z_1 + G_1 &= \dot{z}_1 \\ -\frac{i}{2}w + z_2^2 + G_2 &= \dot{z}_2 \\ z_2 z_\alpha + G_\alpha &= \dot{z}_\alpha, \qquad 3 \le \alpha \le n. \end{aligned} \tag{46}$$

From (44), we have

$$|z_1|^{2m} + \sum |z_\alpha|^2 = O(|w|) \tag{47}$$

for points $(w,z) \in \Omega$. Thus if G_j vanishes with weight at least μ_j, we have $G_j(w,z) = O(|w|^{\mu_j})$. From the first equation of (46) we have

$$z_2 = w^{-1}\dot{w} - w^{-1}G_0 \tag{48}$$

and so

$$\dot{z}_2 = w^{-2}(w\ddot{w} - (\dot{w})^2) - \frac{d}{dt}(w^{-1}G_0). \tag{49}$$

Now we see from (46) and (47) that $\dot{w} = O(|w|^{3/2})$, $\dot{z}_1 = O(|w|^{(1/2)+(1/2m)})$, and $\dot{z}_\alpha = O(|w|)$. Thus it follows that

$$\frac{d}{dt}(w^{-1}G_0) = \sum_{j=0}^{n} \frac{\partial}{\partial z_j}(w^{-1}G_0)\dot{z}_j = O\left(|w|^{1+\frac{1}{2m}}\right),$$

where we use the notation $z_0 = w$. We substitue (48) and (49) into the third equation of (46) and divide by w to obtain

$$-\frac{i}{2} - \frac{2\dot{w}G_0}{w^3} + \frac{G_0^2}{w^3} + \frac{G_2}{w} = \frac{d}{dt}\left(\frac{\dot{w}}{w^2}\right) - \frac{1}{w}\frac{d}{dt}(w^{-1}G_0),$$

which we may rewrite as

$$-\frac{i}{2} + E_2 = \frac{d}{dt}\left(\frac{\dot{w}}{w^2}\right), \tag{50}$$

where $E_2 = O(|w|^{1/2m})$. We integrate this with respect to t, and it follows that $w^{-2}\dot{w} \sim -(i/2)t$ in the sense that $\lim_{t\to\infty} t^{-1}w^{-2}\dot{w} = -i/2$. Integrating again, we obtain $w^{-1} \sim it^2/4$, or $w \sim -4it^{-2}$. Substituting this back into (50), we have

$$-\frac{i}{2} + O\left(|t|^{-\frac{1}{m}}\right) = \frac{d}{dt}\left(\frac{\dot{w}}{w^2}\right).$$

Integrating this gives

$$\frac{\dot{w}}{w^2} = -\frac{it}{2} + O\left(|t|^{1-\frac{1}{m}}\right)$$

and

$$w = -4it^{-2} + O\left(|t|^{-2-\frac{1}{m}}\right), \tag{51}$$

which, by (48), gives

$$z_2 = -2t^{-1} + O\left(|t|^{-\frac{1}{m}}\right). \tag{52}$$

Further, by (47), we have

$$z_1 = O\left(|t|^{-\frac{1}{m}}\right), \text{ and } z_\alpha = O\left(|t|^{-1}\right) \text{ for } 2 \le \alpha \le n. \tag{53}$$

Lemma 8. *As $t \to \pm\infty$, we have*

(i) $w(t) = -4it^{-2} + O(|t|^{-2-(1/m)})$.

(ii) $z_1(t) = O(|t|^{-2/m})$.

(iii) $z_2(t) = -2t^{-1} + O(|t|^{-1-(1/m)})$.

(iv) $z_\alpha(t) = O(|t|^{-1-(2/m)})$ *for* $3 \le \alpha \le n$.

Proof. We have already shown (i) and (iii). To show (ii), we note that by (53), we have $z_1 = O(|t|^{-\delta})$ for $\delta = -1/m$, and by (52), the second equation of (46) has the form $-2z_1/mt + O(|t|^{-1-(1/m)-\delta}) = \dot{z}_1$, or

$$t^{-\frac{2}{m}} \frac{d}{dt}\left(z_1 t^{\frac{2}{m}}\right) = O\left(|t|^{-1-\frac{1}{m}-\delta}\right) \tag{54}$$

Multiplying this by $t^{2/m}$, integrating, and then dividing by $t^{2/m}$, we have $z_1 = O(t^{-2/m}\log|t|)$. If we substitute this improved estimate for z_1 into the second equation of (46), then we have (54) with $\delta > 1/m$. Repeating our procedure, we obtain (ii).

The proof of (iv) is similar: by (53), we have $z_\alpha = O(|t|^{-1-\delta})$ with $\delta = 0$. By (i), (ii), and (53), we have $G_\alpha = O(|t|^{-2-(2/m)})$, the "worst" monomial that can appear in G_α being $z_2^2 z_1$. Thus the fourth equation of (46) is

$$t^{-2}\frac{d}{dt}(t^2 z_\alpha) = O\left(t^{-2-\delta-\frac{1}{m}}\right) + O\left(|t|^{-2-\frac{2}{m}}\right).$$

Proceeding as in the previous paragraph, we obtain (iv). $\square$

In the case $\mu = 1$, we may write the vector field as

$$\frac{1}{2}H = (w^2 + G_0)\frac{\partial}{\partial w} + \left(\frac{1}{m}z_1 w + G_1\right)\frac{\partial}{\partial z_1} + \sum(z_\alpha w + G_\alpha)\frac{\partial}{\partial z_\alpha},$$

where G_j denotes terms of higher weight. Thus the orbit $t \mapsto p_t = (w(t), z(t))$ satisfies the equations

$$\begin{aligned} w^2 + G_0 &= \dot{w} \\ \frac{1}{m}wz_1 + G_1 &= \dot{z}_1 \\ wz_\alpha + G_\alpha &= \dot{z}_\alpha, \qquad 2 \le \alpha \le n. \end{aligned} \tag{55}$$

Since $\mathrm{wt}(G_0) > 2$, we have $G_0 = O(|w|^{2+(1/2m)})$ by (47). Under the change of variable $w^* = -1/w$, the first equation of (55) becomes

$$1 + (w^*)^2 G_0((w^*)^{-1}, z) = \dot{w}^* \tag{56}$$

or

$$1 + A(w^*) + iB(w^*) = \dot{w}^*, \tag{57}$$

where $|A(w^*)|, |B(w^*)| = O(|w^*|^{-1/2m})$. By elementary estimates on (57), we obtain $w^*(t) = t + O(t^{1-(1/2m)})$, from which it follows that

$$\begin{aligned} u(t) &= -t^{-1} + O\left(t^{-1-\frac{1}{2m}}\right) \\ v(t) &= O\left(t^{-1-\frac{1}{2m}}\right). \end{aligned} \tag{58}$$

By (56) and (58) we have

$$z_1(t) = O\left(t^{-\frac{1}{2m}}\right) \qquad \text{and} \qquad z_\alpha = O\left(t^{-\frac{1}{2}}\right). \tag{59}$$

Lemma 9. $c^{-1}t^{-2} \le |r(w(t), z(t))| \le ct^{-2}$.

Proof. Since $Re\, H$ is tangential to $\partial\Omega$, $Re\, H(r)$ is a smooth function on $\overline{\Omega}$ that vanishes at the boundary. Thus near 0 there is a smooth function h such that $Re\, H(r) = hr$. By (44) and (55) we have

$$\begin{aligned} Re\, H(r) &= 2Re\left((w^2 + G_0)\left(\frac{-i}{2} + \frac{1}{2}(u\varphi + E_u\right)\right. \\ &\quad + \left(\frac{1}{m}z_1 w + G_1\right)(m z_1^{m-1}\bar{z}_1^m + (u\varphi + E)_{z_1}) \\ &\quad \left. + \sum (z_\alpha w + G_\alpha)(\bar{z}_\alpha + (u\varphi + E)_{z_\alpha})\right) \\ &= (-2u(v + |z_1|^{2m} + |z_2|^2 + \cdots + |z_n|^2) + \cdots), \end{aligned}$$

where the last set of dots indicates terms of higher weight. It follows that $\mathrm{wt}(h - 2u) > 1$. Since $r < 0$ on Ω, we may divide by r and obtain

$$\frac{d}{dt}\log r = \frac{\dot{r}}{r} = (2u + \cdots) = -2t^{-1} + O\left(t^{-1-\frac{1}{2m}}\right).$$

Here the dots indicate terms of weight > 1, and we have substituted the powers of t obtained in (59). Integrating this equation and then exponentiating, we obtain $r = e^c t^{-2} e^{O(t^{-1/2m})}$, which proves the lemma. $\square$

We will use the notation $A = O(t^{\mu+0})$ to mean that $A = O(t^{\mu+\epsilon})$ for each $\epsilon > 0$.

Lemma 10. *$z_1(t) = O(t^{-1/m})$, and $z_\alpha(t) = O(t^{-1+0})$ for $2 \le \alpha \le n$.*

Proof. G_1 vanishes with weight $> 1+(1/2m)$ at $(w,z) = (0,0)$ and so $G_1 = O(|w|^2 + |wz_\alpha| + |z_\alpha^2 z_1^2| + |z_\alpha|^3 + |z_1|^{2m+2})$. By (59) $G_1 = O(t^{-1-(1/m)})$, and the second equation in (55) becomes

$$\left(\frac{-1}{mt} + O\left(t^{-1-\frac{1}{2m}}\right)\right) z_1 + O\left(t^{-1-\frac{1}{m}}\right) = \dot{z}_1,$$

or

$$t^{-\frac{1}{m}} \frac{d}{dt}\left(t^{\frac{1}{m}} z_1\right) = O\left(t^{-1-\frac{1}{m}}\right).$$

We multiply this by $t^{1/m}$, integrate with respect to t, and then divide by $t^{1/m}$ to obtain $z_1(t) = O(t^{-1/m}\log t)$. When we substitute this new estimate for z_1 into G_1, we have $G_1 = O(t^{-1-(2/m)+0})$. Thus the second equation in (55) is

$$\left(\frac{-1}{mt} + O\left(t^{-1-\frac{1}{2m}}\right)\right) z_1 + O\left(t^{-1-\frac{2}{m}+0}\right) = \frac{-z_1}{mt} + O\left(t^{-1-\frac{3}{2m}+0}\right) + O\left(t^{-1-\frac{2}{m}+0}\right) = \dot{z}_1.$$

Solving this again, we obtain $z_1 = O(t^{-1/m})$.

To take care of $z_2, \ldots, z_n$, we note that G_α vanishes with weight $> 3/2$, and so $G_\alpha = O(|w|^2 + \sum_\beta(|wz_\beta z_1| + |wz_\beta^2| + |z_\beta^3 z_1| + |z_\beta^2 z_1^{m+1}| + |z_\beta z_1^{2m+1}|) + |z_1^{3m+1}|)$. We may assume that $z_\beta = O(t^{-(1/2)-\epsilon})$ for some $\epsilon \ge 0$, and so the third equation in (55) becomes

$$t^{-1}\frac{d}{dt}(tz_\alpha) = O(t^{-2}) + O\left(t^{-\frac{3}{2}-\epsilon-\frac{1}{m}}\right) + O(t^{-2-2\epsilon}).$$

If $\epsilon < 1/2$, then we multiply by t, integrate with respect to t, and divide by t, to obtain $z_\alpha = ct^{-1}\log t + O(t^{-(1/2)-\epsilon-(1/m)})$. Thus we improve the old ϵ by adding $1/m$ until we reach $z_\alpha = O(t^{-1}\log t)$. $\square$

5. Final rescaling

The goal of this section is to make a final rescaling of the domain Ω and complete the proof of the theorem. We let $f_t = \exp(t\,Re\,H)$ denote the family of automorphisms of Ω obtained by exponentiating the vector field H. We will assume first that the vector field has the form $H = Q^{(1)} + \ldots$. At the end of this section we discuss the case $\mu = 1/2$.

We now perform a scaling of coordinates that is very similar to that of Section 1. We let $\hat{z} = z - z(t)$ and $\hat{w} = w - w^*(t)$, where we define $w^*(t) = u(t) + iv^*(t)$ with $v^* < 0$ chosen so that $r(w^*(t), z(t)) = 0$, i.e., $v^* = -|z_1(t)|^{2m} - \sum |z_\alpha(t)|^2 - u(t)\varphi(u(t), z(t)) - E(z(t))$. In the new coordinates, the defining function of the domain is given by

$$\begin{aligned} r(w,z) &= r(\hat{w} + w^*(t), \hat{z} + z(t)) \\ &= \hat{v} + (|\hat{z}_1 + z_1(t)|^{2m} - |z_1(t)|^{2m}) + \sum(|\hat{z}_\alpha|^2 + 2Re\, \bar{z}_\alpha(t) z_\alpha) \\ &\quad + \hat{u}\varphi(\hat{u} + u(t), \hat{z} + z(t)) + u(t)[\varphi(\hat{u} + u(t), \hat{z} + z(t)) \\ &\quad - \varphi(u(t), z(t))] + E(\hat{z} + z(t)) - E(z(t)). \end{aligned}$$

Now we make another change of coordinates $\hat{\hat{z}} = \hat{z}$ and $\hat{\hat{w}} = \hat{w} + \sum 2i\bar{z}_\alpha(t) z_\alpha$. In the $\hat{\hat{}}$-coordinates, but with the $\hat{}$ omitted from all coordinates, the defining function has the form

$$\begin{aligned} r &= v + (|z_1 + z_1(t)|^{2m} - |z_1(t)|^{2m}) + \sum z_\alpha \bar{z}_\alpha \\ &\quad + \left(u - \sum Re\,(2i\bar{z}_\alpha(t) z_\alpha)\right) \varphi\left(u - \sum Re\, 2i\bar{z}_\alpha(t) z_\alpha + u(t), z + z(t)\right) \\ &\quad + u(t)[\varphi\left(u - \sum Re\, 2i\bar{z}_\alpha(t) z_\alpha + u(t), z + z(t)\right) - \varphi(u(t), z(t))] \\ &\quad + E(\hat{z} + z(t)) - E(z(t)). \end{aligned} \tag{60}$$

Now we make a scaling $w = t^{-2}\tilde{w}$, $z_1 = t^{-1/m}\tilde{z}_1$, $z_\alpha = t^{-1}\tilde{z}_\alpha$. We substitute the $\tilde{}$-coordinates into (60) and multiply the defining equation by t^2 to obtain

$$\begin{aligned} r^{(t)} &= v + (|z_1 + t^{\frac{1}{m}} z_1(t)|^{2m} - |t^{\frac{1}{m}} z_1(t)|^{2m}) + \sum z_\alpha \bar{z}_\alpha \\ &\quad + \left[\left(u - \sum Re\,(2i\bar{z}_\alpha(t) t z_\alpha)\right) \varphi\left(t^{-2}u - \sum Re\, 2i\bar{z}_\alpha(t) t^{-1} z_\alpha + u(t), t^{-\mu} z + z(t)\right)\right] \\ &\quad + t^2 u(t)\left[\varphi\left(t^{-2}u - \sum Re\, 2i\bar{z}_\alpha(t) t^{-1} z_\alpha + u(t), t^{-\mu} z + z(t)\right) - \varphi(u(t), z(t))\right] \\ &\quad + t^2[E(t^{-\mu} z + z(t)) - E(z(t))], \end{aligned} \tag{61}$$

where $t^{-\mu} z = (t^{-\frac{1}{m}} z_1, t^{-1} z_2, \ldots, t^{-1} z_n)$.

Let us analyze the behavior of the various expressions. First, by Lemma 8, we may pass to a subsequence and assume that $z_1(t) t^{1/m}$ converges to a limit β. Also by Lemma 8, $Re\, 2i\bar{z}_\alpha(t) = O(t^{-1+0})$, and it follows from (5) that

$$\varphi(t^{-2}u - Re\, 2i\bar{z}_\alpha(t) t^{-1} z_\alpha + u(t), t^{-\mu} z + z(t)) = O(t^{-1-\frac{1}{m}+0}).$$

Thus it follows the first term in brackets [ ] in (61) tends to 0 as $t \to \infty$. For the second bracketed term, we see from (5) and Lemma 8 that

$$\left[\varphi\left(t^{-2}u - \sum Re\, 2i\bar{z}_\alpha(t)t^{-1}z_\alpha + u(t), t^{-\mu}z + z(t)\right) - \varphi(u(t), z(t))\right]$$
$$= O(t^{-1-\frac{1}{m}+0}) + c(t^{-2}u - Re\, 2i\bar{z}_\alpha(t)t^{-1}z_\alpha) = O(t^{-1-\frac{1}{m}+0}).$$

Thus $t^2u(t)$ times this term vanishes as $t \to \infty$. By Lemma 8 and (45), we have $E(t^{-\mu}z + z(t)) = O(t^{-2-(1/m)+0})$, so t^2 times the last term vanishes as $t \to \infty$.

Let h_t denote the holomorphic mapping obtained by the composition of f_t and the coordinate transformations $z \mapsto \hat{z}$ and the $z \mapsto \tilde{z}$. If $G_t = \{r^{(t)} < 0\}$, then $h_t : \Omega \to G_t$ is biholomorphic, and the domains G_t converge to the domain $G = \{v + |z_1 - \beta|^{2m} - |\beta|^{2m} + \sum |z_\alpha|^2 < 0\}$, which is equivalent to $\{v + |z_1|^{2m} + \sum |z_\alpha|^2 < 0\}$ by translation. The point $h_t(p_t) = h_t(f_t(p_0))$ is given by the $\tilde{}$-coordinates of the point p_t and is equal to $(it^{-2}(v^*(t) - v(t)), 0, \ldots, 0)$. Since

$$v^*(t) - v(t) = r(w^*(t), z(t)) - r(w(t), z(t)) = -r(w(t), z(t)),$$

it follows from Lemma 9 that p_t is bounded and bounded away from $\partial\Omega$. Thus $\{h_{t_j}(p_0)\}$ has a subsequence that converges to a point of G. It follows then from Lemma 2 that there is a subsequence of mappings h_{t_j} that converges to a biholomorphic mapping $h : \Omega \to G$, which completes the proof of the theorem in the case $\mu = 1$.

The main difficulties encountered in the final rescaling procedure above were due to the fact that $u(t)$ is relatively large, compared to $v(t)$. If $\mu = 1/2$, however, then by (i) of Lemma 8, $u(t) = o(v(t))$. Thus the arguments above carry through more easily in this case, and the theorem is proved.

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