Interpolation mechanism of fuzzy control*

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Abstract The fuzzy control algorithms used commonly at present are all regarded as some interpolation functions, which is in essence equivalent to discrete response functions to be fitted. This means that fuzzy control method is similar to finite element method in mathematical physics, which is a kind of direct manner or numerical method in control systems.

Keywords: fuzzy control, fuzzy inference, base functions, interpolation functions.

A conclusion about fuzzy control has been given[1,2]: a fuzzy controller is in essence an interpolator. However this conclusion is gained in some special way. So people may ask whether all fuzzy control algorithms can be regarded as interpolation functions. In this paper we show that the fuzzy control algorithms used commonly at present can all be regarded as some interpolation functions.

1 Some necessary concepts and signs

Let us review briefly Mamdanian fuzzy control algorithm, taking two-input and one-output controllers as an example, so that some necessary concepts and signs are introduced.

Let X and Y be the universes of input variables and let Z be the universe of output. Denoting $\mathscr{A}=\{\underset{\sim}{A}_i\}_{(1\leqslant i\leqslant n)}$, $\mathscr{B}=\{\underset{\sim}{B}_i\}_{(1\leqslant i\leqslant n)}$, $\mathscr{C}=\{\underset{\sim}{C}_i\}_{(1\leqslant i\leqslant n)}$, where $\underset{\sim}{A}\in\mathscr{F}(X)$, $\underset{\sim}{B}_i\in\mathscr{F}(Y)$ and $\underset{\sim}{C}_i\in\mathscr{F}(Z)$, we regard $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ as linguistic variables. Then n fuzzy inference rules are formed as follows:

$$\text{if } x \text{ is } \underset{\sim}{A}_i \text{ and } y \text{ is } \underset{\sim}{B}_i \text{ then } z \text{ is } \underset{\sim}{C}_i, \tag{1}$$

where $i=1,2,\cdots,n$ and $x\in X, y\in Y$ and $z\in Z$, called base variables. According to Mamdanian algorithm, the inference relation of the ith inference rule is a fuzzy relation from $X\times Y$ to Z, $\underset{\sim}{R}_i\triangleq(\underset{\sim}{A}_i\times\underset{\sim}{B})\times\underset{\sim}{C}_i$, where $\underset{\sim}{R}_i(x,y,z)\triangleq(\underset{\sim}{A}_i(x)\wedge\underset{\sim}{B}_i(y))\wedge\underset{\sim}{C}_i(z)$. As n inference rules should be jointed by "or" (corresponding to set theoretical operator "$\cup$"), the whole inference relation is $\underset{\sim}{R}=\bigcup_{n=1}^{n}\underset{\sim}{R}_i$, i.e.

$$\underset{\sim}{R}_i(x,y,z)=\bigvee_{i=1}^{n}\underset{\sim}{R}_i(x,y,z)=\bigvee_{i=1}^{n}[(\underset{\sim}{A}_i(x)\wedge\underset{\sim}{B}_i(y))\wedge\underset{\sim}{C}_i(z)]. \tag{2}$$

Given $\underset{\sim}{A}'\in\mathscr{F}(X)$, and $\underset{\sim}{B}'\in\mathscr{F}(Y)$, the conclusion of inference $\underset{\sim}{C}'\in\mathscr{F}(Z)$ can be determined as $\underset{\sim}{C}'\triangleq(\underset{\sim}{A}'\times\underset{\sim}{B}')\circ\underset{\sim}{R}$, by use of CRI method, where

$$\underset{\sim}{C}'(z)=\bigvee_{(x,y)\in X\times Y}[(\underset{\sim}{A}'(x)\wedge\underset{\sim}{B}'(y))\wedge\underset{\sim}{R}(x,y,z)]. \tag{3}$$

For a fuzzy controller, because its inputs are crisp quantity, they must be changed into fuzzy sets in order to use (3) as follows:

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$$\underset{\sim}{A}'(x) = \begin{cases} 1, & x = x', \\ 0, & x \neq x', \end{cases} \qquad \underset{\sim}{B}'(y) = \begin{cases} 1, & y = y', \\ 0, & y \neq y', \end{cases}$$

which is called fuzzification. By the above expressions and noticing (2), we can get the inference result $\underset{\sim}{C}'$:

$$\underset{\sim}{C}'(z) = \underset{\sim}{R}(x', y', z) = \bigvee_{i=1}^{n} [(\underset{\sim}{A}_i(x') \wedge \underset{\sim}{B}_i(y')) \wedge \underset{\sim}{C}_i(z)]. \tag{4}$$

We know that $\underset{\sim}{C}'$ should be turned into a crisp number by using some defuzzification methods, in order to be taken as a practical operating quantity, for $\underset{\sim}{C}'$ is a fuzzy set, and the commonly used method is the so-called method of centroid:

$$z' = \int_{z \in \mathbb{Z}} z \underset{\sim}{C}'(z) \mathrm{d}z \Big/ \int_{z \in \mathbb{Z}} \underset{\sim}{C}'(z) \mathrm{d}z. \tag{5}$$

Note. In (4), $\underset{\sim}{A}'_i(x') \wedge \underset{\sim}{B}_i(y')$ is a number that is independent of z, denoted by λ'_i. So (4) can be rewritten in the following simple form:

$$\underset{\sim}{C}'(z) = \bigvee_{i=1}^{n} (\lambda'_i \wedge \underset{\sim}{C}_i(z)). \tag{6}$$

From the definition for a fuzzy set to be multiplied by a number[3], we know that $(\lambda'_i \wedge \underset{\sim}{C}_i(z)) = (\lambda'_i \cdot \underset{\sim}{C}_i(z))(z)$. So if we put $\underset{\sim}{C}'_i \triangleq \lambda'_i \underset{\sim}{C}_i$, it is easy to know that

$$\underset{\sim}{C}' = \bigcup_{i=1}^{n} \underset{\sim}{C}'_i = \bigcup_{i=1}^{n} \lambda'_i \underset{\sim}{C}_i, \tag{7}$$

where λ'_i is regarded as the agreement degree of the ith rule with respect to (x', y'). These $\underset{\sim}{C}_i$ can also be taken as expansion coefficients so that (7) is just an expansion on $\underset{\sim}{C}_i (i = 1, 2, \cdots, n)$.

For convenience sake, we should introduce several concepts. Given a universe X, let $\mathscr{A} = \{\underset{\sim}{A}_i\}_{(1 \leqslant i \leqslant n)}$ be a family of normal fuzzy sets on X (i.e. $(\forall i)(\exists x_i \in X)(\underset{\sim}{A}_i(x_i) = 1)$) and let x_i be the peak point of $\underset{\sim}{A}_i$. $\mathscr{A}$ is called a fuzzy partition of X, if it satisfies the condition $(\forall i, j)(i \neq j \Rightarrow x_i \neq x_j)$ and

$$(\forall x \in X)\left[\sum_{i=1}^{n} \underset{\sim}{A}_i(x) = 1\right], \tag{8}$$

where $\underset{\sim}{A}_i$ is called a base element of $\mathscr{A}$. Then we also call $\mathscr{A}$ a group of base elements of X. In this paper, when $X = [a, b] \subset \mathbb{R}$ ($\mathbb{R}$ is real number field), we always assume that $a < x_1 < x_2 \cdots < x_n < b$.

2 The interpolation mechanism of Mamdanian algorithm with one input and one output

Let X and Y be the universe of input and output variable respectively, and let $\mathscr{A} = \{\underset{\sim}{A}_i\}_{(1 \leqslant i \leqslant n)}$ and $\mathscr{B} = \{\underset{\sim}{B}_i\}_{(1 \leqslant i \leqslant n)}$ be a fuzzy partition of X and Y respectively. Usually we can suppose that X and Y are respectively a real number interval, i.e. $X = [a, b]$ and $Y = [c, d]$, holding $a < x_1 < x_2 < \cdots < x_n < b$, $c < y_1 < y_2 < \cdots < y_n < d$, where x_i and y_i are the peak point of $\underset{\sim}{A}_i$ and $\underset{\sim}{B}_i$ respectively. When X and Y are general measurable sets, the following result is also true. Moreover we always regard $\underset{\sim}{A}_i$ and $\underset{\sim}{B}_i$ as integrable functions.

Theorem 1. *Under the above conditions, there exists a group of base functions* $\mathscr{A} = \{\underset{\sim}{A}'_i\}_{(1 \leqslant i \leqslant n)}$ *such that Mamdanian algorithm with one input and one output is approximately a unary piecewise interpolation function that takes* $\underset{\sim}{A}'_i$ *as its base functions*:

$$F(x) = \sum_{i=1}^{n} \underset{\sim}{A}'_i(x) y_i, \tag{9}$$

and $\mathscr{A}'$ is just a fuzzy partition of X. Especially when $\{y_i\}_{(1\leqslant i\leqslant n)}$ is an equidistant partition, $\mathscr{A}'$ degenerates into $\mathscr{A}$, i.e.

$$F(x) = \sum_{i=1}^{n} \underset{\sim}{A}_i(x)y_i. \tag{10}$$

Proof. According to Mamdanian algorithm, $\mathscr{A}$ and $\mathscr{B}$ form the inference rules:

$$\text{if } x \text{ is } \underset{\sim}{A}_i \text{ then } y \text{ is } \underset{\sim}{B}_i, \tag{11}$$

and the inference relation of the ith rule is $\underset{\sim}{R}_i = \underset{\sim}{A}_i \times \underset{\sim}{B}_i$, and then the whole inference relation is $\underset{\sim}{R} = \bigcup_{i=1}^{n} \underset{\sim}{R}_i$, i.e.

$$\underset{\sim}{R}_i(x,y) = \bigvee_{i=1}^{n} \underset{\sim}{R}_i(x,y) = \bigvee_{i=1}^{n}(\underset{\sim}{A}_i(x) \wedge \underset{\sim}{B}_i(y)). \tag{12}$$

For a given input $x' \in X$, being similar to (4), we have $\underset{\sim}{B}'(y) = \bigvee_{i=1}^{n}(\underset{\sim}{A}_i(x') \wedge \underset{\sim}{B}_i(y))$. And by (5) we can get a crisp response value y':

$$y' = \int_c^d y\underset{\sim}{B}'(y)\mathrm{d}y \Big/ \int_c^d \underset{\sim}{B}'(y)\mathrm{d}y. \tag{13}$$

Let $h_1 = y_1 - c$, $h_i = y_i - y_{i-1}(i = 2, 3, \cdots, n)$ and $h = \max\{h_i \mid 1 \leqslant i \leqslant n\}$. Because $\mathscr{A}$ and $\mathscr{B}$ are fuzzy partitions, they have Kronecker's property: $\underset{\sim}{A}_i(x_i) = \delta_{ij} = \underset{\sim}{B}_i(y_i)$. Based on the definition of definite integral, we have

$$y' = \frac{\int_c^d y\underset{\sim}{B}'(y)\mathrm{d}y}{\int_c^d \underset{\sim}{B}'(y)\mathrm{d}y} \approx \frac{\sum_{i=1}^{n} y_i\underset{\sim}{B}'(y_i)h_i}{\sum_{i=1}^{n} \underset{\sim}{B}'(y_i)h_i}$$

$$= \frac{\sum_{i=1}^{n} h_i\left[\bigvee_{k=1}^{n}(\underset{\sim}{A}_k(x') \wedge \underset{\sim}{B}_k(y_i))\right]y_i}{\sum_{i=1}^{n} h_i\left[\bigvee_{k=1}^{n}(\underset{\sim}{A}_k(x') \wedge \underset{\sim}{B}_k(y_i)\right]}$$

$$= \frac{\sum_{i=1}^{n} h_i\underset{\sim}{A}_i(x')y_i}{\sum_{i=1}^{n} h_i\underset{\sim}{A}_i(x')}.$$

Denote $\alpha_i(x') \triangleq h_i \Big/ \sum_{j=1}^{n} h_j\underset{\sim}{A}_j(x')$, and $\underset{\sim}{A}'_i(x') \triangleq \alpha_i(x')\underset{\sim}{A}_i(x')$. Then

$$y' \approx \sum_{i=1}^{n} \alpha_i(x')\underset{\sim}{A}_i(x')y_i = \sum_{i=1}^{n} \underset{\sim}{A}'_i(x')y_i, \tag{14}$$

which means that y' is approximately represented as a unary piecewise interpolation function taking $\underset{\sim}{A}'_i$ as its base functions. Now we write $\mathscr{A}' = \{\underset{\sim}{A}'_i\}_{(1\leqslant i\leqslant n)}$. If $F(x) = \sum_{i=1}^{n} \underset{\sim}{A}'_i(x)y_i$, then we get (9).

Then it is easy to prove that $\mathscr{A}'$ satisfies the conditions of fuzzy partition. At last, when $\{y_i\}_{(1\leqslant i\leqslant n)}$ is an equidistant partition, $(\forall i)(h_i = h)$, by (8) we have

$$\alpha_i(x) = h_i \Big/ \sum_{j=1}^{n} h_j\underset{\sim}{A}_j(x) = 1 \Big/ \sum_{j=1}^{n} \underset{\sim}{A}_j(x) = 1.$$

So $\underset{\sim}{A}'_i = \underset{\sim}{A}_i$, i.e. $\mathscr{A}' = \mathscr{A}$. Thus $F(x) = \sum_{i=1}^{n} \underset{\sim}{A}_i(x) y_i$ which is just (10). Q.E.D.

Note. From $\underset{\sim}{A}'_i(x) = \alpha_i(x)\underset{\sim}{A}_i(x)$, we know that $\underset{\sim}{A}'_i(x)$ is a kind of weighted form of $\underset{\sim}{A}_i(x)$. $\alpha_i(x)$ is only related to h_i and $\underset{\sim}{A}_i(x)$, and $\underset{\sim}{A}'_i(x)$ is in essence the same as $\underset{\sim}{A}_i(x)$. Besides, $\underset{\sim}{A}_i$, the fuzzy sets as antecedents of inference, are just the base functions of interpolation, and $\underset{\sim}{B}_i$, the fuzzy sets as consequents of inference, exhibit only their peak points y_i but have nothing to do with their shapes as membership functions under given conditions, in (9), which is very important for designing a fuzzy controller.

3 The interpolation mechanism of Mamdanian algorithm with two inputs and one output

Reviewing the signs in sec. 2, let X, Y and Z be real number intervals: $X = [a, b]$, $Y = [c, d]$ and $Z = [e, f]$, where z_i is the peak point of $\underset{\sim}{C}_i$ holding that $e < z_1 < z_2 < \cdots < z_n < f$. And suppose that $\underset{\sim}{A}_i$, $\underset{\sim}{B}_i$ and $\underset{\sim}{C}_i$ are integrable functions.

Theorem 2. *Under the above conditions, there exists a group of base functions* $\Phi = \{\phi_i\}_{(1 \leqslant i \leqslant n)}$ *such that Mamdanian algorithm with two inputs and one output is approximately a binary piecewise interpolation function taking* ϕ_i *as its base functions, as follows:*

$$F(x, y) = \sum_{i=1}^{n} \phi_i(x, y) z_i. \tag{15}$$

Proof. Let $h_1 = z_1 - e$, $h_i = z_i - z_{i-1} (i = 2, 3, \cdots, n)$ and $h = \max\{h_i |_{1 \leqslant i \leqslant n}\}$. From (4) and (5), we have

$$\begin{aligned} z' &= \frac{\int_e^f z \underset{\sim}{C}'(z) \mathrm{d}z}{\int_e^f \underset{\sim}{C}'(z) \mathrm{d}z} \approx \frac{\sum_{i=1}^{n} z_i \underset{\sim}{C}'(z_i) h_i}{\sum_{i=1}^{n} \underset{\sim}{C}'(z_i) h_i} \\ &= \frac{\sum_{i=1}^{n} h_i (\underset{\sim}{A}_i(x') \wedge \underset{\sim}{B}_i(y')) z_i}{\sum_{i=1}^{n} h_i (\underset{\sim}{A}_i(x') \wedge \underset{\sim}{B}_i(y'))} \\ &= \sum_{i=1}^{n} \beta_i(x', y') (\underset{\sim}{A}_i(x') \wedge \underset{\sim}{B}_i(y')) z_i = \sum_{i=1}^{n} \phi_i(x', y') z_i, \end{aligned} \tag{16}$$

where we have put $\beta_i(x', y') \triangleq h_i \Big/ \sum_{i=1}^{n} h_i (\underset{\sim}{A}_i(x') \wedge \underset{\sim}{B}_i(y'))$ and $\phi_i(x', y') \triangleq \beta_i(x', y') \cdot (\underset{\sim}{A}_i(x') \wedge \underset{\sim}{B}_i(y'))$. If writing $F(x, y) = \sum_{i=1}^{n} \phi_i(x, y) z_i$, then we get (15). Q.E.D.

4 A note on completeness of inference rules

The writing of the group of rules as (1) is a convention in engineering, but has its shortcomings that it is not the case of mathematical treatments and may cause some misunderstandings because no "cross terms" (for example, "if x is $\underset{\sim}{A}_i$ and y is $\underset{\sim}{B}_j$ then $\cdots$") emerge.

Reference [2] has pointed out that inference and mapping are the same thing to some extent. From the point of view of mapping, the inference rules of fuzzy control can be represented as the following mapping (taking the two-input and one-output case as an example):

$$f^{*}: \mathscr{A}^{*} \times \mathscr{B}^{*} \rightarrow \mathscr{C}^{*},$$
$$(\underset{\sim}{A}_i^{*}, \underset{\sim}{B}_j^{*}) \mapsto f^{*}(\underset{\sim}{A}_i^{*}, \underset{\sim}{B}_j^{*}) = \underset{\sim}{C}_{ij}^{*}, \tag{17}$$

where $\mathscr{A}^{*} = \{\underset{\sim}{A}_i^{*}\}_{(1 \leqslant i \leqslant p)}$, $\mathscr{B}^{*} = \{\underset{\sim}{B}_j^{*}\}_{(1 \leqslant j \leqslant q)}$ and $\mathscr{C}^{*} = \{\underset{\sim}{C}_{ij}^{*}\}_{(1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q)}$, which is a linguistic variable on X, Y and Z respectively. Taking account of the base variables of these linguistic variables, (17) can also be regarded as the following inference rules:

$$\text{if } x \text{ is } \underset{\sim}{A}_i^{*} \text{ and } y \text{ is } \underset{\sim}{B}_j^{*} \text{ then } z \text{ is } \underset{\sim}{C}_{ij}^{*}, \tag{18}$$

which is clearly a complete group of inference rules. According to Mamdanian algorithm, the inference relation of the (i, j)th reference rule is $\underset{\sim}{R}_{ij}^{*} = \underset{\sim}{A}_i^{*} \times \underset{\sim}{B}_j^{*} \times \underset{\sim}{C}_{ij}^{*}$. So the whole inference relation of these $p \times q$ inference rules is $\underset{\sim}{R}^{*} = \bigcup_{i=1}^{p} \bigcup_{j=1}^{q} \underset{\sim}{R}_{ij}^{*}$, i.e.

$$\underset{\sim}{R}^{*}(x, y, z) = \bigvee_{i=1}^{p} \bigvee_{j=1}^{q} (\underset{\sim}{A}_i^{*}(x) \wedge \underset{\sim}{B}_j^{*}(y) \wedge \underset{\sim}{C}_{ij}^{*}(z)). \tag{19}$$

For a given input (x, y), being similar to the previous discussion, we have

$$\underset{\sim}{C}'(z) = \bigvee_{i=1}^{p} \bigvee_{j=1}^{q} [(\underset{\sim}{A}_i^{*}(x') \wedge \underset{\sim}{B}_j^{*}(y')) \wedge \underset{\sim}{C}_{ij}^{*}(z)]. \tag{20}$$

If $k \triangleq (i-1)q + j$, $\underset{\sim}{A}_k \triangleq \underset{\sim}{A}_i^{*}$, $\underset{\sim}{B}_k \triangleq \underset{\sim}{B}_j^{*}$, $\underset{\sim}{C}_k \triangleq \underset{\sim}{C}_{ij}^{*}$, $z_k \triangleq z_{ij}$ (where z_{ij} is the peak point of C_{ij}^{*}) and $n \triangleq pq$, then (20) changes back into (4). By using the definition, if $h_{ij} \triangleq h_k$, $\beta_{ij}(x, y) \triangleq h_{ij} / \sum_{i=1}^{p} \sum_{j=1}^{q} h_{ij}(\underset{\sim}{A}_i^{*}(x) \wedge \underset{\sim}{B}_j^{*}(y))$ and $\phi_{ij}(x, y) \triangleq \beta_{ij}(x, y)(\underset{\sim}{A}_i^{*}(x) \wedge \underset{\sim}{B}_j^{*}(y))$, then $\beta_{ij}(x, y) = \beta_k(x, y)$ and $\phi_{ij}(x, y) = \phi_k(x, y)$, where $k = (i-1)q + j$. Then (15) can be written as

$$F(x, y) = \sum_{i=1}^{p} \sum_{j=1}^{q} \beta_{ij}(x, y)(\underset{\sim}{A}_i^{*}(x) \wedge \underset{\sim}{B}_j^{*}(y)) z_{ij} = \sum_{i=1}^{p} \sum_{j=1}^{q} \phi_{ij}(x, y) z_{ij}, \tag{21}$$

which is a typical binary piecewise interpolation function.

5 The interpolation mechanism of (+ , ·)-centroid algorithm

The (+ , ·)-centroid algorithm is proposed in ref. [5] which is thought by some people to be more convenient than (∨ , ∧)-centroid algorithm, i.e. Mamdanian algorithm.

Theorem 3. *Under the conditions in Theorem* 1, (+ , ·)*-centroid algorithm with one input and one output has the same conclusion as in Theorem* 1.

Proof. In (12), after (∨ , ∧) is replaced by (+ , ·), we have

$$\underset{\sim}{R}(x, y) = \sum_{i=1}^{n} \underset{\sim}{R}_i(x, y) = \sum_{i=1}^{n} \underset{\sim}{A}_i(x) \underset{\sim}{B}_i(y). \tag{22}$$

For a given input x', $\underset{\sim}{B}'(y) = \sum_{i=1}^{n} \underset{\sim}{A}_i(x') \underset{\sim}{B}_i(y)$. From (13), we know

$$y' \approx \frac{\sum_{i=1}^{n} y_i \underset{\sim}{B}'(y_i) h_i}{\sum_{i=1}^{n} \underset{\sim}{B}'(y_i) h_i} = \frac{\sum_{i=1}^{n} h_i \left(\sum_{k=1}^{n} \underset{\sim}{A}_k(x') \underset{\sim}{B}_k(y) \right) y_i}{\sum_{i=1}^{n} h_i \left(\sum_{k=1}^{n} \underset{\sim}{A}_k(x') \underset{\sim}{B}_k(y) \right)}$$
$$= \frac{\sum_{i=1}^{n} h_i \underset{\sim}{A}_i(x) y_i}{\sum_{i=1}^{n} h_i \underset{\sim}{A}_i(x')} = \sum_{i=1}^{n} \alpha_i(x') \underset{\sim}{A}_i(x') y_i = \sum_{i=1}^{n} \underset{\sim}{A}'_i(x') y_i,$$

which is just (14). The other results in the theorem are clear. Q.E.D.

The theorem means that $(+,\cdot)$-centroid algorithm is in essence the same as Mamdanian algorithm in the case with one input and one output.

Theorem 4. *Under the conditions in Theorem 2, there exists a group of base functions* $\Psi = \{\psi_i\}_{(1\leqslant i\leqslant n)}$ *such that* $(+,\cdot)$*-centroid algorithm with two inputs and one output is approximately a binary piecewise interpolation function taking* ψ_i *as its base functions, as follows:*

$$F(x,y) = \sum_{i=1}^{n} \psi_i(x,y) z_i. \tag{23}$$

Furthermore, when $\mathscr{A}^*$ *and* $\mathscr{B}^*$ *are fuzzy partitions and* $\{z_i\}_{(1\leqslant i\leqslant n)}$ *is an equidistant partition, we have*

$$F(x,y) = \sum_{i=1}^{n} \underset{\sim}{A}_i(x)\underset{\sim}{B}_i(y) z_i = \sum_{i=1}^{p}\sum_{j=1}^{q} \underset{\sim}{A}_i^*(x)\underset{\sim}{B}_j^*(y) z_{ij}. \tag{24}$$

Proof. In (2), after $(\vee,\wedge)$ is replaced by $(+,\cdot)$, $\underset{\sim}{R}(x,y,z) = \sum_{i=1}^{n}\underset{\sim}{R}_i(x,y,z)$. For a given input (x',y'), $\underset{\sim}{C}'(z) = \sum_{i=1}^{n}\underset{\sim}{A}_i(x')\underset{\sim}{B}_i(y')\underset{\sim}{C}_i(z)$. From (5), we know

$$z' \approx \frac{\sum_{i=1}^{n} z_i \underset{\sim}{C}'(z_i) h_i}{\sum_{i=1}^{n}\underset{\sim}{C}'(z_i) h_i} = \frac{\sum_{i=1}^{n} h_i\Big(\sum_{k=1}^{n} h_i \underset{\sim}{A}_i(x')\underset{\sim}{B}_i(y)\Big) z_i}{\sum_{i=1}^{n} h_i \underset{\sim}{A}_i(x')\underset{\sim}{B}_i(y)}$$

$$= \sum_{i=1}^{n}\gamma_i(x',y')\underset{\sim}{A}_i(x')\underset{\sim}{B}_i(y') z_i = \sum_{i=1}^{n}\psi_i(x',y') z_i,$$

where $\gamma_i(x',y') \triangleq h_i \Big/ \sum_{i=1}^{n} h_i \underset{\sim}{A}_i(x')\underset{\sim}{B}_i(y')$ and $\psi_i(x',y') \triangleq \gamma_i(x',y')\underset{\sim}{A}_i(x')\underset{\sim}{B}_i(y')$. Let $F(x,y) = \sum_{i=1}^{n}\psi_i(x,y) z_i$. Then we get (23). If $\gamma_{ij}(x,y) = \gamma_k(x,y)$ and $\psi_{ij}(x,y) = \psi_k(x,y)$, where $k = (i-1)q + j$, then (23) can be written as

$$F(x,y) = \sum_{i=1}^{p}\sum_{j=1}^{q}\psi_{ij}(x,y) z_{ij} = \sum_{i=1}^{p}\sum_{j=1}^{q}\gamma_{ij}(x,y)\underset{\sim}{A}_i^*(x)\underset{\sim}{B}_j^*(y). \tag{25}$$

Especially, when $\mathscr{A}^*$ and $\mathscr{B}^*$ are fuzzy partitions and $\{z_i\}_{(1\leqslant i\leqslant n)}$ is an equidistant partition, it is easy to prove the $\gamma_{ij}(x,y) \equiv 1$. So (25) is true. Q.E.D.

6 The interpolation mechanism of simple inference algorithm

Simple inference algorithm is proposed in refs. [6, 7] which is thought by some people to be a kind of quick and simple algorithm with respect to fuzzy inference.

In the algorithm, the fuzzy sets representing inference consequents are replaced by numbers. For example, inference rules with two inputs and one output are as follows:

$$\text{If } x \text{ is } \underset{\sim}{A}_i \text{ and } y \text{ is } \underset{\sim}{B}_i \text{ then } z \text{ is } z_i. \tag{26}$$

Then for a given input (x',y') the response value z' is calculated in the following steps:

Step 1.

$$\lambda_i = \underset{\sim}{A}_i(x')\underset{\sim}{B}_i(y') \ (\text{or } \lambda_i = \underset{\sim}{A}_i(x') \wedge \underset{\sim}{B}_i(y')). \tag{27}$$

Step 2.

$$z' = \sum_{i=1}^{n} \lambda_i z_i \Big/ \sum_{i=1}^{n} \lambda_i . \tag{28}$$

Theorem 5. *Let $\mathscr{A} = \{\underset{\sim}{A}_i\}_{(1 \leqslant i \leqslant n)}$ be a fuzzy partition of X. Then simple inference algorithm with one input and one output is just a unary piecewise interpolation function taking $\underset{\sim}{A}_i$ as its base functions, as (10).*

Proof. Here the inference rules are as: if x is $\underset{\sim}{A}_i$ then y is y_i. For a given x', clearly $\lambda_i = \underset{\sim}{A}_i(x')$. From (28) we have

$$y' = \frac{\sum_{i=1}^{n} \lambda_i y_i}{\sum_{i=1}^{n} \lambda_i} = \frac{\sum_{i=1}^{n} \underset{\sim}{A}_i(x') y_i}{\sum_{i=1}^{n} \underset{\sim}{A}_i(x')} = \sum_{i=1}^{n} \underset{\sim}{A}_i(x') y_i . \tag{29}$$

Taking $F(x) = \sum_{i=1}^{n} \underset{\sim}{A}_i(x) y_i$, we get (10). Q.E.D.

Theorem 6. *Simple inference algorithm with two inputs and one output is just a binary piecewise interpolation function. When $\lambda_i = \underset{\sim}{A}_i(x) \wedge \underset{\sim}{B}_i(y)$, $F(x,y) = \sum_{i=1}^{n} \theta_i(x,y) z_i$, where $\theta_i(x,y) = (\underset{\sim}{A}_i(x) \wedge \underset{\sim}{B}_i(y)) \Big/ \sum_{i=1}^{n} (\underset{\sim}{A}_i(x) \wedge \underset{\sim}{B}_i(y))$; when $\lambda_i = \underset{\sim}{A}_i(x) \underset{\sim}{B}_i(y)$ and $\mathscr{A}^*$ and $\mathscr{B}^*$ are fuzzy partitions, $F(x,y) = \sum_{i=1}^{n} \underset{\sim}{A}_i(x) \underset{\sim}{B}_i(y) z_i$.*

Proof. When $\lambda_i = \underset{\sim}{A}_i(x) \wedge \underset{\sim}{B}_i(y)$, from (28), we have

$$F(x,y) = z = \sum_{i=1}^{n} (\underset{\sim}{A}_i(x) \wedge \underset{\sim}{B}_i(y)) z_i \Big/ \sum_{i=1}^{n} (\underset{\sim}{A}_i(x) \wedge \underset{\sim}{B}_i(y)) = \sum_{i=1}^{n} \theta_i(x,y) z_i ;$$

when $\lambda_i = \underset{\sim}{A}_i(x) \underset{\sim}{B}_i(y)$, as $\mathscr{A}^*$ and $\mathscr{B}^*$ are fuzzy partitions,

$$\sum_{k=1}^{n} \underset{\sim}{A}_k(x) \underset{\sim}{B}_k(y) = \sum_{i=1}^{p} \sum_{j=1}^{q} \underset{\sim}{A}_i^*(x) \underset{\sim}{B}_j^*(y) = \Big(\sum_{i=1}^{p} \underset{\sim}{A}_i^*(x)\Big)\Big(\sum_{j=1}^{q} \underset{\sim}{B}_j^*(y)\Big) = 1 .$$

Hence

$$F(x,y) = z = \sum_{i=1}^{n} \underset{\sim}{A}_i(x) \underset{\sim}{B}_i(y) z_i \Big/ \sum_{i=1}^{n} (\underset{\sim}{A}_i(x) \underset{\sim}{B}_i(y)) = \sum_{i=1}^{n} \underset{\sim}{A}_i(x) \underset{\sim}{B}_i(y) z_i .$$

Q.E.D.

7 The interpolation mechanism of function inference algorithm

Function inference algorithm is proposed in ref. [8], which is the generalization of simple inference algorithm. We take the two-input and one-putput case as an example, where the inference rules are as follows:

$$\text{if } x \text{ is } \underset{\sim}{A}_i \text{ and } y \text{ is } \underset{\sim}{B}_i \text{ then } z \text{ is } z_i(x,y). \tag{30}$$

Clearly in (26), if constant points z_i are replaced by variable points $z_i(x,y)$, then the expression becomes (30).

For a given input (x', y'), $\lambda \triangleq g_i(\underset{\sim}{A}_i(x), \underset{\sim}{B}_i(y'))$ where g_i is a general operator, for example, $g_i = \wedge$ (or $\cdot$). Then similar to (28),

$$z' = \sum_{i=1}^{n} \lambda_i z_i(x', y') \Big/ \sum_{i=1}^{n} \lambda_i .$$

Theorem 7. *There exists a group of base functions $\xi_i(x, y)$ $(i = 1, 2, \cdots, n)$ such that function inference algorithm is just a binary piecewise interpolation function with variable nodal points as follows:*

$$F(x, y) = \sum_{i=1}^{n} \xi_i(x, y) z_i(x, y), \tag{31}$$

where variable nodal points mean that the nodal points $z_i(x, y)$ are functions of (x, y).

Proof. If $\xi_i(x, y) = g_i(\underset{\sim}{A}_i(x), \underset{\sim}{B}_i(y')) \Big/ \sum_{i=1}^{n} g_i(\underset{\sim}{A}_i(x), \underset{\sim}{B}_i(y'))$ then

$$F(x, y) = z = \sum_{i=1}^{n} \lambda_i z_i(x, y) \Big/ \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \xi_i(x, y) z_i(x, y). \qquad \text{Q.E.D.}$$

Especially when $\lambda_i = \cdot$, $\xi_i(x, y) = \underset{\sim}{A}_i(x) \underset{\sim}{B}_i(y)$. So (31) becomes the following simple form:

$$F(x, y) = \sum_{i=1}^{n} \underset{\sim}{A}_i(x) \underset{\sim}{B}_i(y) z_i(x, y). \tag{32}$$

Note. Usually $z_i(x, y)$ are taken as linear functions: $z_i(x, y) = a_i x + b_i y + c_i$, where coefficients a_i, b_i anc c_i should be determined by some method.

8 The interpolation machanism of characteristic expansion inference algorithm

Characteristic expansion inference algorithm is proposed in ref. [9], which is thought to be of quicker effect than Mamdanian algorithm, but as pointed out in ref. [9], the algorithm is equivalent to Mamdanian algorithm. Considering the inference rules as (1), given (antecedent) inputs $\underset{\sim}{A}'$ and $\underset{\sim}{B}'$ we should first calculate the characteristic coefficients:

$$\alpha_i = \bigvee_{x \in X} (\underset{\sim}{A}'(x) \wedge \underset{\sim}{A}_i(x)), \; B_i = \bigvee_{y \in Y} (\underset{\sim}{B}'(y) \wedge \underset{\sim}{B}_i(y)). \tag{33}$$

Then by means of (7), writing $\underset{\sim}{C}'_i \triangleq (\alpha_i \wedge \beta_i) \underset{\sim}{C}_i$, we have

$$\underset{\sim}{C}'_i(z) = \bigvee_{i=1}^{n} [(\alpha_i \wedge \beta_i) \wedge \underset{\sim}{C}_i(z)]. \tag{34}$$

For given crisp inputs x' and y', by using fuzzification (see (4)), we get $\underset{\sim}{A}'$ and $\underset{\sim}{B}'$. Substituting $\underset{\sim}{A}'$ and $\underset{\sim}{B}'$ into (33) yields $\alpha_i = \underset{\sim}{A}_i(x')$ and $\beta_i = \underset{\sim}{B}_i(y')$. And α_i and β_i are substituted into (34). Then $\underset{\sim}{C}'(z) = \bigvee_{i=1}^{n} [(\underset{\sim}{A}_i(x') \wedge \underset{\sim}{B}_i(y')) \wedge \underset{\sim}{C}_i(z)]$ which is just the same as (4). Therefore we have the following result.

Theorem 8. *Under the conditions in Theorem 2, characteristic expansion inference algorithm is approximately a binary piecewise interpolation function as* (15).

9 A general fuzzy control algorithm

From the above results we can give a general fuzzy control algorithm (taking the two-input and one-output case as an example) as follows.

Theorem 9. *Let $\mathscr{C}$ be a fuzzy partition and let $\underset{\sim}{A}_i$, $\underset{\sim}{B}_i$ and $\underset{\sim}{C}_i$ be integrable functions. Then a fuzzy controller with two inputs and one output is a binary piecewise interpolation function with variable nodal points as follows:*

$$F(x, y) = \sum_{i=1}^{n} \eta_i(x, y) z_i(x, y), \tag{35}$$

where $\eta_i(x,y) = h_i(\underset{\sim}{A}_i(x)T\underset{\sim}{B}_i(y)) \Big/ \sum_{i=1}^{n} h_i(\underset{\sim}{A}_i(x)T\underset{\sim}{B}_i(y))$ *and "T" means a t-norm.*

Clearly the results from Theorems 1—8 can be regarded as corollaries of Theorem 9. Now we consider such a problem: The operations between $\underset{\sim}{A}_i(x)$ and $\underset{\sim}{B}_i(y)$ are derived from Cartesian product between fuzzy sets and t-norm T is a kind of generalized form of intersection operations of fuzzy sets. Are they compatible? The answer is positive (see reference [3]).

We suggest that $T = \cdot$, $\{h_i\}_{(1\leqslant i\leqslant n)}$ is an equidistant partition and $z_i(x,y)\equiv z_i$. Then (35) turns into (24). In the sense of interpolation performance, (24) is much better. (24) can be easily generalized as the multi-input and multi-output case. We take the three-input and two-output case as an example. In this case, the inference rules are: if x is $\underset{\sim}{A}i$ and y is $\underset{\sim}{B}_i$ and z is $\underset{\sim}{C}_i$ then u is $\underset{\sim}{D}_i$ and v is $\underset{\sim}{E}_i$. It is easy to prove that the fuzzy control algorithm is a many-variable vector value piecewise interpolation function:

$$F(x,y,z) = (u,v) = \Big(\sum_{i=1}^{n}\underset{\sim}{A}_i(x)\underset{\sim}{B}_i(y)\underset{\sim}{C}_i(z)u_i,\ \sum_{i=1}^{n}\underset{\sim}{A}_i(x)\underset{\sim}{B}_i(y)\underset{\sim}{C}_i(z)v_i\Big). \quad (36)$$

10 Conclusion

By analyzing the fuzzy control algorithms commonly used at present, we find out that, after making a group of inference rules as (1), the fuzzy sets representing these rules $\underset{\sim}{A}_i$, $\underset{\sim}{B}_i$ and $\underset{\sim}{C}_i$ can be determined where their peak points are denoted by x_i, y_i and z_i, which is equivalent to acquiring a group of samples of inputs and outputs as follows:

$$\{((x_i,y_i),z_i) \mid i = 1,2,\cdots,n\}. \quad (37)$$

If $\underset{\sim}{A}_i$ and $\underset{\sim}{B}_i$ are regarded as a group of base functions in interpolation, some interpolation function, for example (24), can be got. This means that to acquire a group of inference rules (as (1)) and to acquire a group of data being similar to (37) are the same thing. Therefore we have such an idea: fuzzy control method is similar to finite element method in mathematical physics, and it is a kind of direct method of control theory.

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