ON THE DISTRIBUTION OF THE MAXIMUM LATENT ROOT OF A POSITIVE DEFINITE SYMMETRIC RANDOM MATRIX

TAKESI HAYAKAWA

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1. Summary

In this paper, we consider the distribution of the maximum latent root of a certain positive definite symmetric random matrix. For this purpose, we give a useful transformation of a symmetric matrix and calculate its Jacobian. We also give some useful expansion formulas for zonal polynomials (A. T. James [3]).

Recently Sugiyama [7] and Sugiyama and Hukutomi [8] gave the density functions of maximum latent roots of a central Wishart matrix when the covariance matrix $\Sigma=I$, and of a multivariate Beta matrix and a multivariate F-matrix in the central case.

Here we derive the density functions of maximum latent roots of a multivariate non-central Beta matrix, a non-central Wishart matrix and a multivariate central quadratic form with the covariance matrix Σ, and we also derive the density function of maximum canonical correlation coefficient. The notations in this paper are due to A. T. James [4] and A. G. Constantine [1].

2. Some useful transformation

In this section, we treat some transformation which is useful in the sequel.

Let S be a positive definite symmetric random matrix. As is well known, S can be decomposed into the product of an orthogonal matrix H and a symmetric matrix $\lambda_1 \oplus V$ such that

$$S=H\begin{bmatrix}\lambda_1 & 0\\ 0 & V\end{bmatrix}H', \tag{1}$$

where λ_1 is the maximum latent root of S and V is a positive definite symmetric random matrix which ranges $\lambda_1 I_{p-1} > V > 0$. The first column $\boldsymbol{h}_1$ of H is the corresponding characteristic vector of λ_1. It should be noted that the independent variable of H is only $\boldsymbol{h}_1$ and the remaining part $H_2(p\times\overline{p-1})$ is only a function of $\boldsymbol{h}_1$ such that

$$H_2=f(\boldsymbol{h}_1)=\begin{bmatrix} f_{12}(\boldsymbol{h}_1)\cdots f_{1p}(\boldsymbol{h}_1) \\ \vdots \qquad \vdots \\ f_{p2}(\boldsymbol{h}_1)\cdots f_{pp}(\boldsymbol{h}_1)\end{bmatrix}. \tag{2}$$

LEMMA 1. *Let S be a $p\times p$ positive definite symmetric random matrix. Then the Jacobian of the transformation* (1) *is given by*

$$J(S\rightarrow\lambda_1, \boldsymbol{h}_1, V)=|\lambda_1 I_{p-1}-V|. \tag{3}$$

Y. Tumura [5] considered a Jacobian of (1) in terms of the rotation angles of the orthogonal matrix. His lemma 2.1.2., however, is not convenient for treating directly the distribution problem of maximum latent root of a certain positive definite symmetric random matrix. The proof of the lemma depends on the Hsu's method introduced by W. L. Deemer and I. Olkin [9].

PROOF. Let us differentiate both sides of (1):

$$dS=dH\begin{bmatrix}\lambda_1 & 0\\ 0 & V\end{bmatrix}H'+H\begin{bmatrix}d\lambda_1 & 0\\ 0 & dV\end{bmatrix}H'+H\begin{bmatrix}\lambda_1 & 0\\ 0 & V\end{bmatrix}dH'. \tag{4}$$

Multiply H' from the left and H from the right to obtain

$$H'dSH=H'dH\begin{bmatrix}\lambda_1 & 0\\ 0 & V\end{bmatrix}+\begin{bmatrix}d\lambda_1 & 0\\ 0 & dV\end{bmatrix}+\begin{bmatrix}\lambda_1 & 0\\ 0 & V\end{bmatrix}(H'dH)'. \tag{5}$$

Let $dT=H'dSH$ and $dP=H'dH$. Then

$$dT=dP\begin{bmatrix}\lambda_1 & 0\\ 0 & V\end{bmatrix}+\begin{bmatrix}d\lambda_1 & 0\\ 0 & dV\end{bmatrix}-\begin{bmatrix}\lambda_1 & 0\\ 0 & V\end{bmatrix}dP, \tag{6}$$

since $dP'=-dP$ by its antisymmetry. Here we must note that in $dP=(dP_1, dP_2)=(H'd\boldsymbol{h}_1, H'dH_2)$, dP_2 can be represented by dP_1. In fact

$$dH_2=df(\boldsymbol{h}_1)=\begin{bmatrix} df_{12}(\boldsymbol{h}_1)\cdots df_{1p}(\boldsymbol{h}_1) \\ \vdots \qquad \vdots \\ df_{p2}(\boldsymbol{h}_1)\cdots df_{pp}(\boldsymbol{h}_1)\end{bmatrix}$$

and

$$df_{ij}(\boldsymbol{h}_1)=\frac{\partial f_{ij}}{\partial h_{11}}dh_{11}+\frac{\partial f_{ij}}{\partial h_{21}}dh_{21}+\cdots+\frac{\partial f_{ij}}{\partial h_{p1}}dh_{p1}$$
$$=\left(\frac{\partial f_{ij}}{\partial h_{11}},\frac{\partial f_{ij}}{\partial h_{21}},\cdots,\frac{\partial f_{ij}}{\partial h_{p1}}\right)d\boldsymbol{h}_1$$
$$(i=1,2,\cdots,p;\ j=2,3,\cdots,p).$$

Let $F_{ij}=\left(\frac{\partial f_{ij}}{\partial h_{11}},\frac{\partial f_{ij}}{\partial h_{21}},\cdots,\frac{\partial f_{ij}}{\partial h_{p1}}\right)$. Then, by setting $F=(F_{ij})$, we have

$$dH_2=df(\boldsymbol{h}_1)=F\otimes d\boldsymbol{h}_1$$

where $F\otimes d\boldsymbol{h}_1$ is a direct product of F and $d\boldsymbol{h}_1$. Thus, from $d\boldsymbol{h}_1=Hd\boldsymbol{P}_1$ and $dP_2=H'dH_2$, we have

$$dP_2=H'F\otimes Hd\boldsymbol{P}_1, \tag{7}$$

which establishes the assertion.

From the above consideration, we need only $d\boldsymbol{P}_1$, dV, $d\lambda_1$ to calculate the Jacobian. Now,

$$\begin{aligned} J(S\to\lambda_1,\boldsymbol{h}_1,V)&=J(dS\to d\lambda_1,d\boldsymbol{h}_1,dV)\\ &=J(dS\to dT)J(dT\to d\lambda_1,d\boldsymbol{h}_1,dV)\\ &=J(dS\to dT)J(dT\to d\lambda_1,d\boldsymbol{P}_1,dV)J(d\boldsymbol{P}_1\to d\boldsymbol{h}_1). \end{aligned} \tag{8}$$

It is easily checked that

$$J(dS\to dT)=1, \qquad J(d\boldsymbol{P}_1\to d\boldsymbol{h}_1)=1$$

and dp_{11}, the first component of $d\boldsymbol{P}_1$, is 0, because $dp_{11}=\boldsymbol{h}_1'd\boldsymbol{h}_1=0$. Thus the transformation (6) is written as

$$\begin{aligned} dt_{11}&=\lambda_1dp_{11}+d\lambda_1-\lambda_1dp_{11}=d\lambda_1\\ dT_{21}&=\lambda_1d\boldsymbol{P}_{21}-Vd\boldsymbol{P}_{21}=(\lambda_1I_{p-1}-V)d\boldsymbol{P}_{21}\\ dT_{22}&=dV+dP_{22}V-VdP_{22}, \end{aligned} \tag{9}$$

where dP_{22} is the submatrix of $dP_2=(dP_{12}'\,dP_{22}')'$ and $dP_1'=(0,d\boldsymbol{P}_{21}')$. From the above relations, we can construct the configuration,

$$
(10) \qquad
\begin{array}{c|c|cccc|cccc}
 & d\lambda_1 & dp_{21} & dp_{31} & \cdots & dp_{p1} & dv_{22} & dv_{23} & \cdots & dv_{pp} \\
\hline
dt_{11} & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
dt_{21} & 0 & \lambda_1 - v_{22} & -v_{23} & \cdots & -v_{2p} & 0 & 0 & \cdots & 0 \\
dt_{31} & 0 & -v_{32} & \lambda_1 - v_{33} & \cdots & -v_{3p} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
dt_{p1} & 0 & -v_{p2} & -v_{p3} & \cdots & \lambda_1 - v_{pp} & 0 & 0 & \cdots & 0 \\
\hline
dt_{22} & 0 & * & * & \cdots & * & 1 & 0 & \cdots & 0 \\
dt_{23} & 0 & * & * & \cdots & * & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
dt_{pp} & 0 & * & * & \cdots & * & 0 & 0 & \cdots & 1
\end{array}
$$

Hence the Jacobian is

$$J(dT \to d\lambda_1, d\boldsymbol{P}_1, dV) = |\lambda_1 I_{p-1} - V|,$$

which completes the proof of the lemma.

From lemma 1, we can give the main principle of deriving the density function of the maximum latent root of a certain symmetric random matrix.

Let S be a positive definite symmetric random matrix with density function $f(S)$. Then the density function of the maximum latent root of S is given by

$$(11) \qquad \int_{\boldsymbol{h}_1'\boldsymbol{h}_1=1} d\boldsymbol{h}_1 \int_{\lambda_1 I_{p-1} > V > 0} |\lambda_1 I - V| f(\lambda_1, V, \boldsymbol{h}_1)\, dV,$$

where λ_1, V and $\boldsymbol{h}_1$ are the same as those in lemma 1.

The following definite integrals are useful for our arguments.

LEMMA 2.

$$(12) \qquad \int_{\lambda_1 I_{p-1} > V > 0} |V|^{(n-p-1)/2} |\lambda_1 I_{p-1} - V| C_\kappa(V) dV$$

$$= \frac{\Gamma_{p-1}\left(\frac{n-1}{2}\right) \Gamma_{p-1}\left(\frac{p+2}{2}\right)}{\Gamma_{p-1}\left(\frac{n+p+1}{2}\right)} \frac{\left(\frac{n-1}{2}\right)_\kappa}{\left(\frac{n+p+1}{2}\right)_\kappa} \lambda_1^{(n+1)(p-1)/2} C_\kappa(\lambda_1 I_{p-1}),$$

(13)
$$\int_{\lambda_1>\lambda_2>\cdots>\lambda_p} \prod_{i=2}^{p} \lambda_i^{(n-p-1)/2} \prod_{1\leq i<j\leq p} (\lambda_i-\lambda_j)C_\kappa(\Lambda_2)d\Lambda_2$$
$$=\frac{\Gamma_{p-1}\left(\frac{p-1}{2}\right)}{\pi^{(p-1)^2/2}} \frac{\Gamma_{p-1}\left(\frac{n-1}{2}\right)\Gamma_{p-1}\left(\frac{p+2}{2}\right)}{\Gamma_{p-1}\left(\frac{n+p+1}{2}\right)} \frac{\left(\frac{n-1}{2}\right)_\kappa}{\left(\frac{n+p+1}{2}\right)_\kappa}$$
$$\cdot\, \lambda_1^{(n+1)(p-1)/2} C_\kappa(\lambda_1 I_{p-1}),$$

where $\Lambda_2=\operatorname{diag}\{\lambda_2,\cdots,\lambda_p\}$ *is a diagonal matrix with latent roots of* V.

PROOF. (12) is a Beta integral which is given by A. G. Constantine [1]. (13) is obtained from (12) by transforming $V=H_2\Lambda_2H_2'$ where H_2 is a $\overline{p-1}\times\overline{p-1}$ orthogonal matrix, and by integrating over $0(p-1)$, i.e.,

$$\int_{0(p-1)} dH_2^* = \frac{\pi^{(p-1)^2/2}}{\Gamma_{p-1}\left(\frac{p-1}{2}\right)}.$$

(13) is the same result as given by T. Sugiyama [7].

3. Some expansions of zonal polynomial

(14)
$$C_\kappa(A\oplus B)=\sum_\tau\sum_\nu a_{\tau\nu}^\kappa C_\tau(A)C_\nu(B)$$

where A and B are symmetric matrices of any order, respectively, and $A\oplus B$ stands for the direct sum of A and B. The summation is over all partitions τ of k_1 and ν of k_2 such that $k_1+k_2=k$ and κ is a partition of k.

(15)
$$C_\nu(A)C_\sigma(A)=\sum_\theta b_{\nu\sigma}^\theta C_\theta(A)$$

where A is a symmetric matrix, $C_\nu(A)$ and $C_\sigma(A)$ are zonal polynomials which correspond to a partition ν of t and σ of m, respectively, and the summation is over all partitions θ of s satisfying

(16)
$$t+m=s.$$

We do not know the explicit formulas of $a_{\tau\nu}^\kappa$ and $b_{\nu\sigma}^\theta$, but we give the tables of $a_{\tau\nu}$ and $b_{\nu\sigma}^\theta$ in lower orders at the end of this paper. We must note there

$$C_\kappa(A)=\frac{c(\kappa)}{1\cdot3\cdots(2k-1)}Z_\kappa(A),$$

where $c(\kappa)$ is the degree of the representation $[2\kappa]$ of the symmetric group of $2k$ symbols. We read tables (I) and (II) as follows:

$$Z_{(2)}(A\oplus B)=Z_{(2)}(A)+Z_{(2)}(B)+2Z_{(1)}(A)Z_{(1)}(B),$$

$$Z_{(1)}(A)Z_{(2)}(A)=\frac{1}{5}Z_{(3)}(A)+\frac{4}{5}Z_{(21)}(A).$$

These tables are calculated from the ones due to A. T. James [4].

4. The density function of the maximum latent root of a quadratic form $Z=XAX'$

Let the $p\times N$ ($p<N$) matrix variate X be a sample matrix from a p-variate normal population with mean $\mathbf{0}$ and covariance matrix Σ, and A be a positive definite symmetric matrix. T. Hayakawa [6] obtained the density function of $Z=XAX'$ and of latent roots of Z as

$$(17)\qquad \frac{1}{\Gamma_p\left(\frac{N}{2}\right)|2\Sigma|^{N/2}|A|^{p/2}}|Z|^{(N-p-1)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-\frac{1}{2}\Sigma^{-1}Z\right)C_\kappa(A^{-1})}{k!\,C_\kappa(I_N)},$$

and

$$(18)\qquad \frac{\pi^{p^2/2}}{\Gamma_p\left(\frac{p}{2}\right)\Gamma_p\left(\frac{N}{2}\right)|2\Sigma|^{N/2}|A|^{p/2}}\left(\prod_{i=1}^{p}\lambda_i\right)^{(N-p-1)/2}\prod_{i<j}(\lambda_i-\lambda_j)$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-\frac{1}{2}\Sigma^{-1}\right)C_\kappa(A^{-1})C_\kappa(\Lambda)}{k!\,C_\kappa(I_p)C_\kappa(I_N)},$$

respectively, where $\Lambda=\mathrm{diag}\{\lambda_1,\cdots,\lambda_p\}$ is a diagonal matrix with latent roots. Using an expansion formula (14), we can decompose $C_\kappa(\Lambda)$ into the form

$$C_\kappa(\Lambda)=\sum_{\tau,\nu}a^{\kappa}_{\tau\nu}C_\tau(\lambda_1)C_\nu(\Lambda_2),$$

where $\Lambda_2=\mathrm{diag}\{\lambda_2,\cdots,\lambda_p\}$. Thus (18) is rewritten as

$$(19)\qquad \frac{\pi^{p^2/2}}{\Gamma_p\left(\frac{p}{2}\right)\Gamma_p\left(\frac{N}{2}\right)|2\Sigma|^{N/2}|A|^{p/2}}\lambda_1^{(N-p-1)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-\frac{1}{2}\Sigma^{-1}\right)C_\kappa(A^{-1})}{k!\,C_\kappa(I_p)C_\kappa(I_N)}$$

$$\cdot\sum_{\tau,\nu}a^{\kappa}_{\tau\nu}C_\tau(\lambda_1)\left(\prod_{i=2}^{p}\lambda_i\right)^{(N-p-1)/2}\prod_{1\le i<j\le p}(\lambda_i-\lambda_j)C_\nu(\Lambda_2).$$

Hence, the integration of (18) with respect to $\lambda_2, \cdots, \lambda_p$ by the use of (13) gives the density function of λ_1 as

$$(20)\quad \frac{\pi^{p/2}\Gamma_{p-1}\left(\frac{N-1}{2}\right)\Gamma_{p-1}\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma_p\left(\frac{N}{2}\right)\Gamma_{p-1}\left(\frac{N+p+1}{2}\right)|2\Sigma|^{N/2}|A|^{p/2}}\lambda_1^{Np/2-1}$$
$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa\left(-\frac{1}{2}\Sigma^{-1}\right)C_\kappa(A^{-1})}{k!\,C_\kappa(I_p)C_\kappa(I_N)}\sum_{\tau,\nu}a_{\tau\nu}^{\kappa}\frac{\left(\frac{N-1}{2}\right)_\nu}{\left(\frac{N+p+1}{2}\right)_\nu}C_\tau(\lambda_1)C_\nu(\lambda_1 I_{p-1}).$$

THEOREM 1. *Let $Z=XAX'$ be distributed with density function* (17). *Then the density function of the maximum latent root of Z is given by* (20).

COROLLARY 1. *Let $Z=XAX'$ be distributed with density function* (17). *Then the density function of the maximum latent root of a determinantal equation $|Z-\lambda\Sigma|=0$ is given by setting $\Sigma=I_p$ in* (20).

COROLLARY 2. *Let Z be a Wishart matrix on N degrees of freedom with covariance matrix Σ. Then the density function of the maximum latent root of Z is given by setting $A=I_N$ in* (20).

COROLLARY 3. *Let Z be a Wishart matrix on N degrees of freedom with covariance matrix Σ. Then the density function of the maximum latent root of $\Sigma^{-1/2}Z\Sigma^{-1/2}$ is given by setting $A=I_N$ and $\Sigma=I_p$ in* (20).

Corollary 3 is easily obtained in another way. In fact, let Z be a Wishart matrix on N degrees of freedom. Then by the use of transformation (1) the joint density function of λ_1, V and H is

$$(21)\quad \frac{1}{2^{pN/2}\Gamma_p\left(\frac{N}{2}\right)}\exp\left(-\frac{1}{2}\lambda_1\right)\lambda_1^{(N-p-1)/2}$$
$$\cdot\operatorname{etr}\left(-\frac{1}{2}V\right)|V|^{(N-p-1)/2}|\lambda_1 I-V|.$$

Then the integration (21) over H gives

$$(22)\quad \frac{\pi^{p/2}}{2^{pN/2}\Gamma\left(\frac{p}{2}\right)\Gamma_p\left(\frac{N}{2}\right)}\exp\left(-\frac{\lambda_1}{2}\right)\lambda_1^{(N-p-1)/2}$$
$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\left(-\frac{1}{2}\right)^k|V|^{(N-p-1)/2}|\lambda_1 I-V|C_\kappa(V).$$

Hence, using (12) we obtain the density function of λ_1 as

$$(23)\qquad \frac{\pi^{p/2}\Gamma_{p-1}\left(\frac{N-1}{2}\right)\Gamma_{p-1}\left(\frac{p+2}{2}\right)}{2^{pN/2}\Gamma\left(\frac{p}{2}\right)\Gamma_p\left(\frac{N}{2}\right)\Gamma_{p-1}\left(\frac{N+p+1}{2}\right)}\exp\left(-\frac{1}{2}\lambda_1\right)\lambda_1^{Np/2-1}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{k!}\frac{\left(\frac{N-1}{2}\right)_\kappa}{\left(\frac{N+p+1}{2}\right)_\kappa}C_\kappa\left(-\frac{1}{2}\lambda_1 I_{p-1}\right).$$

This is the same result as given by T. Sugiyama [7].

Note. If we compare the terms of the kth degrees in (23) and in corollary 3, then we have a linear relation

$$(24)\qquad \exp\left(-\frac{1}{2}\lambda_1\right)\frac{\left(\frac{N-1}{2}\right)_\kappa}{\left(\frac{N+p+1}{2}\right)_\kappa}C_\kappa(\lambda_1 I_{p-1})$$

$$=\sum_{\tau,\nu}a_{\tau\nu}^{\kappa}\frac{\left(\frac{N-1}{2}\right)_\nu}{\left(\frac{N+p+1}{2}\right)_\nu}C_\tau(\lambda_1)C_\nu(\lambda_1 I_{p-1}).$$

5. The density function of the maximum latent root of a non-central Wishart matrix with covariance matrix $\Sigma=I_p$

Let S be a non-central Wishart matrix on n degrees of freedom and with covariance matrix $\Sigma=I_p$. Then we may start from the form of the density function of S

$$(25)\qquad \operatorname{etr}\left(-\frac{1}{2}\Omega\right)\frac{1}{2^{pn/2}\Gamma_p\left(\frac{n}{2}\right)}|S|^{(n-p-1)/2}\operatorname{etr}\left(-\frac{1}{2}S\right)$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{\left(\frac{n}{2}\right)_\kappa}\frac{C_\kappa\left(\frac{1}{4}\Omega\right)C_\kappa(S)}{k!\,C_\kappa(I_p)},$$

since the maximum root is invariant under any orthogonal transformation (cf. A. T. James [3]). Let us decompose S as (1), i.e.,

$$S=H\begin{bmatrix}\lambda_1 & 0\\ 0 & V\end{bmatrix}H'$$

and expand $C_\kappa(\lambda_1\oplus V)$ as

$$C_\kappa(\lambda_1\oplus V)=\sum_{\tau,\nu} a^\kappa_{\tau\nu}C_\tau(\lambda_1)C_\nu(V),$$

by (14). Then (25) is rewritten as

$$\text{(26)}\qquad \operatorname{etr}\left(-\frac{1}{2}\Omega\right)\frac{1}{2^{pn/2}\Gamma_p\left(\frac{n}{2}\right)}\lambda_1^{(n-p-1)/2}\exp\left(-\frac{1}{2}\lambda_1\right)$$

$$\cdot\sum_{k=0}^{\infty}\frac{1}{\left(\frac{n}{2}\right)_\kappa}\frac{C_\kappa\left(\frac{1}{4}\Omega\right)}{k!\,C_\kappa(I_p)}\sum_{\tau,\nu}a^\kappa_{\tau\nu}C_\tau(\lambda_1)$$

$$\cdot\operatorname{etr}\left(-\frac{1}{2}V\right)|V|^{(n-p-1)/2}|\lambda_1 I-V|\,C_\nu(V),$$

since the Jacobian is $|\lambda_1 I_{p-1}-V|$.

The main problem is to integrate out V from (26). Unfortunately the results obtained so far are not available to do so directly and hence we need to reformulate (26) so that the known formulas can be applied. The part of reformulation is

$$\text{(27)}\qquad \operatorname{etr}\left(-\frac{1}{2}V\right)|V|^{(n-p-1)/2}|\lambda_1 I-V|\,C_\nu(V).$$

First, by the expansion

$$\text{(28)}\qquad \operatorname{etr}\left(-\frac{1}{2}V\right)=\sum_{l=0}^{\infty}\sum_{\sigma}\frac{1}{l!}\left(-\frac{1}{2}\right)^l C_\sigma(V),$$

(27) can be rewritten as

$$\text{(29)}\qquad |V|^{(n-p-1)/2}|\lambda_1 I-V|\sum_{l=0}^{\infty}\sum_{\sigma}\frac{1}{l!}\left(-\frac{1}{2}\right)^l C_\sigma(V)C_\nu(V).$$

Second, by using (12), (29) is expressed in the form

$$\text{(30)}\qquad |V|^{(n-p-1)/2}|\lambda_1 I-V|\sum_{l=0}^{\infty}\sum_{\sigma}\frac{1}{l!}\left(-\frac{1}{2}\right)^l\sum_{\theta}b^\theta_{\nu\sigma}C_\theta(V).$$

Hence the integration with respect to V over $\lambda_1 I>V>0$ gives

$$\lambda_1^{(n+1)(p-1)/2}\frac{\Gamma_{p-1}\left(\frac{n-1}{2}\right)\Gamma_{p-1}\left(\frac{p+2}{2}\right)}{\Gamma_{p-1}\left(\frac{n+p+1}{2}\right)} \tag{31}$$

$$\cdot\sum_{l=0}^{\infty}\sum_{\sigma}\frac{1}{l!}\left(-\frac{1}{2}\right)^l\sum_{\theta}b_{\nu\sigma}^{\theta}\frac{\left(\frac{n-1}{2}\right)_\theta}{\left(\frac{n+p+1}{2}\right)_\theta}C_\theta(\lambda_1 I_{p-1}).$$

Thus the joint density of λ_1 and H is as follows:

$$\operatorname{etr}\left(-\frac{1}{2}\Omega\right)\frac{\Gamma_{p-1}\left(\frac{n-1}{2}\right)\Gamma_{p-1}\left(\frac{p+2}{2}\right)}{2^{pn/2}\Gamma_p\left(\frac{n}{2}\right)\Gamma_{p-1}\left(\frac{n+p+1}{2}\right)} \tag{32}$$

$$\cdot\lambda_1^{np/2-1}\exp\left(-\frac{1}{2}\lambda_1\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{\left(\frac{n}{2}\right)_\kappa}\left(\frac{-1}{2}\right)^k\frac{C_\kappa\left(\frac{1}{4}\Omega\right)}{k!\,C_\kappa(I_p)}$$

$$\cdot\sum_{\tau,\nu}a_{\tau\nu}^{\kappa}C_\tau(\lambda_1)\sum_{l=0}^{\infty}\sum_{\sigma}\frac{1}{l!}\left(-\frac{1}{2}\right)^l\sum_{\theta}b_{\nu\sigma}^{\theta}\frac{\left(\frac{n-1}{2}\right)_\theta}{\left(\frac{n+p+1}{2}\right)_\theta}C_\theta(\lambda_1 I_{p-1}).$$

To obtain the density function of λ_1, we only integrate (32) with respect to H by using the spherical integral, that is,

$$\frac{1}{2}\int_{O(1)}dH=\frac{\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}.$$

Thus the density function of λ_1 is

$$\operatorname{etr}\left(-\frac{1}{2}\Omega\right)\frac{\Gamma_{p-1}\left(\frac{n-1}{2}\right)\Gamma_{p-1}\left(\frac{p+2}{2}\right)}{2^{pn/2}\Gamma\left(\frac{p}{2}\right)\Gamma_p\left(\frac{n}{2}\right)\Gamma_{p-1}\left(\frac{n+p+1}{2}\right)} \tag{33}$$

$$\cdot\lambda_1^{np/2-1}\exp\left(-\frac{1}{2}\lambda_1\right)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{\left(\frac{n}{2}\right)_\kappa}\left(-\frac{1}{2}\right)^k\frac{C_\kappa\left(\frac{1}{4}\Omega\right)}{k!\,C_\kappa(I_p)}$$

$$\cdot\sum_{\tau,\nu}a_{\tau\nu}^{\kappa}C_\tau(\lambda_1)\sum_{l=0}^{\infty}\sum_{\sigma}\frac{1}{l!}\left(-\frac{1}{2}\right)^l\sum_{\theta}b_{\nu\sigma}^{\theta}\frac{\left(\frac{n-1}{2}\right)_\theta}{\left(\frac{n+p+1}{2}\right)_\theta}C_\theta(\lambda_1 I_{p-1}).$$

THEOREM 2. *Let S be a non-central Wishart matrix on n degrees of freedom with covariance matrix $\Sigma=I_p$ and the non-central parameters Ω. Then the density function of maximum latent root λ_1 is given by* (33).

6. The density function of the maximum latent root of a non-central Beta-matrix

Let S_1 be a non-central Wishart matrix on n_1 degrees of freedom and S_2 be a central Wishart matrix on n_2 degrees of freedom with the covariance matrix Σ, respectively. Let S_1 and S_2 be independent. A non-central Beta-matrix R is defined as

$$R=(S_1+S_2)^{-1/2}S_1(S_1+S_2)^{-1/2}, \tag{34}$$

and Wilks' statistics of likelihood ratio criterion for testing equality of the mean vectors is, then,

$$|I-R|=|S_2(S_1+S_2)^{-1}|.$$

A. G. Constantine [1], [2] considered the distribution of

i) latent roots $\lambda_1>\cdots>\lambda_p$ of R

and

ii) $\operatorname{tr} S_1S_2^{-1}=\sum_{i=1}^{p}\dfrac{\lambda_i}{1-\lambda_i}$.

Here we consider the density function of λ_1, the maximum latent root of R in the non-central case. We note that the distribution of roots of R is invariant under the simultaneous transformation such that

$$S_1\to\frac{1}{2}\Sigma^{-1/2}S_1\Sigma^{-1/2}, \qquad S_2\to\frac{1}{2}\Sigma^{-1/2}S_2\Sigma^{-1/2}.$$

We may, therefore, assume that the joint density function of S_1 and S_2 is

$$\operatorname{etr}(-\Omega)\frac{1}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}|S_1|^{(n_1-p-1)/2}|S_2|^{(n_2-p-1)/2} \cdot \operatorname{etr}(-(S_1+S_2))\,{}_0F_1\left(\frac{n_1}{2};\Omega,S_1\right), \tag{35}$$

where Ω is a symmetric matrix of non-centrality parameters. By the transformation of S_1 and S_2 such that

$$G=S_1+S_2, \qquad R=G^{-1/2}S_1G^{-1/2},$$

the density function of R and G is

$$\text{(36)} \qquad \operatorname{etr}(-\Omega)\frac{1}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\operatorname{etr}(-G)|G|^{(n_1+n_2-p-1)/2}$$
$$\cdot {}_0F_1\left(\frac{n_1}{2}; \Omega, GR\right)|R|^{(n_1-p-1)/2}|I-R|^{(n_2-p-1)/2}.$$

To turn out G from (36), we use the formula

$$\text{(37)} \qquad \int_{G>0}\operatorname{etr}(-G)|G|^{(n_1+n_2-p-1)/2}C_\kappa(RG)dG=\Gamma_p\left(\frac{n_1+n_2}{2}; \kappa\right)C_\kappa(R),$$

and thus the density function of R is

$$\text{(38)} \qquad \operatorname{etr}(-\Omega)\frac{\Gamma_p\left(\frac{n_1+n_2}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}|R|^{(n_1-p-1)/2}|I-R|^{(n_2-p-1)/2}$$
$$\cdot {}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \Omega, R\right).$$

Now, we decompose R into λ_1, H and V as lemma 1,

$$R=H\begin{bmatrix}\lambda_1 & 0\\ 0 & V\end{bmatrix}H'.$$

Since the Jacobian is $|\lambda_1 I-V|$, we can rewrite (38) in terms of λ_1, V and H as

$$\text{(39)} \qquad \operatorname{etr}(-\Omega)\frac{\Gamma_p\left(\frac{n_1+n_2}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\lambda_1^{(n_1-p-1)/2}(1-\lambda_1)^{(n_2-p-1)/2}$$
$$\cdot|V|^{(n_1-p-1)/2}|\lambda_1 I-V||I-V|^{(n_2-p-1)/2}$$
$$\cdot {}_1F_1\left(\frac{n_1+n_2}{2}; \frac{n_1}{2}; \Omega, \lambda_1\oplus V\right).$$

Now, we reformulate the part including V, that is,

$$\text{(40)} \qquad |V|^{(n_1-p-1)/2}|\lambda_1 I-V||I-V|^{(n_2-p-1)/2}C_\kappa(\lambda_1\oplus V).$$

By using (14), (40) can be written as

$$|V|^{(n_1-p-1)/2}|\lambda_1 I-V||I-V|^{(n_2-p-1)/2}\sum_{\tau,\nu} a^{\kappa}_{\tau\nu}C_\tau(\lambda_1)C_\nu(V) \tag{41}$$

and then by the expansion of generalized binomial series, that is,

$$|I-V|^{(n_2-p-1)/2}=\sum_{l=0}^{\infty}\sum_{\sigma}\left(\frac{-n_2+p+1}{2}\right)_\sigma \frac{C_\sigma(V)}{l!}, \tag{42}$$

(41) can be rewritten as

$$|V|^{(n_1-p-1)/2}|\lambda_1 I-V|\sum_{l=0}^{\infty}\sum_{\sigma}\left(\frac{-n_2+p+1}{2}\right)_\sigma \frac{1}{l!} \cdot\sum_{\tau,\nu} a^{\kappa}_{\tau\nu}C_\tau(\lambda_1)C_\nu(V)C_\sigma(V). \tag{43}$$

Then if we use (15), we have

$$|V|^{(n_1-p-1)/2}|\lambda_1 I-V|\sum_{\tau,\nu} a^{\kappa}_{\tau\nu}C_\tau(\lambda_1) \cdot\sum_{l=0}^{\infty}\sum_{\sigma}\left(\frac{-n_2+p+1}{2}\right)_\sigma \frac{1}{l!}\sum_{\theta} b^{\theta}_{\nu\sigma}C_\theta(V). \tag{44}$$

Hence, the integration of (44) with respect to V over $\lambda_1 I>V>0$ gives, by the use of (12),

$$\lambda_1^{(n_1+1)(p-1)/2}\frac{\Gamma_{p-1}\left(\frac{n_1-1}{2}\right)\Gamma_{p-1}\left(\frac{p+2}{2}\right)}{\Gamma_{p-1}\left(\frac{n_1+p+1}{2}\right)}\sum_{\tau,\nu} a^{\kappa}_{\tau\nu}C_\tau(\lambda_1) \cdot\sum_{l=0}^{\infty}\sum_{\sigma}\left(\frac{-n_2+p+1}{2}\right)_\sigma \frac{1}{l!}\sum_{\theta} b^{\theta}_{\nu\sigma}\frac{\left(\frac{n_1-1}{2}\right)_\theta}{\left(\frac{n_1+p+1}{2}\right)_\theta}C_\theta(\lambda_1 I_{p-1}). \tag{45}$$

Hence, the joint density function of λ_1 and H is

$$\operatorname{etr}(-\Omega)\frac{\Gamma_p\left(\frac{n_1+n_2}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)}\frac{\Gamma_{p-1}\left(\frac{n_1-1}{2}\right)\Gamma_{p-1}\left(\frac{p+2}{2}\right)}{\Gamma_{p-1}\left(\frac{n_1+p+1}{2}\right)} \cdot\lambda_1^{n_1p/2-1}(1-\lambda_1)^{(n_2-p-1)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n_1+n_2}{2}\right)_\kappa}{\left(\frac{n_1}{2}\right)_\kappa}\frac{C_\kappa(\Omega)}{k!C_\kappa(I_p)} \cdot\sum_{\tau,\nu} a^{\kappa}_{\tau\nu}C_\tau(\lambda_1)\sum_{l=0}^{\infty}\sum_{\sigma}\left(\frac{-n_2+p+1}{2}\right)_\sigma \frac{1}{l!}\sum_{\theta} b^{\theta}_{\sigma\nu}\frac{\left(\frac{n_1-1}{2}\right)_\theta}{\left(\frac{n_1+p+1}{2}\right)_\theta}C_\theta(\lambda_1 I_{p-1}). \tag{46}$$

To obtain the density function of λ_1, we only integrate (46) with respect to H by using the spherical integral. Thus

$$(47)\qquad \operatorname{etr}(-\Omega)\frac{B_{p-1}\left(\frac{n_1-1}{2},\frac{p+2}{2}\right)}{B_p\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\frac{\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}\lambda_1^{n_1p/2-1}(1-\lambda_1)^{(n_2-p-1)/2}$$

$$\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n_1+n_2}{2}\right)_\kappa}{\left(\frac{n_1}{2}\right)_\kappa}\frac{C_\kappa(\Omega)}{k!\,C_\kappa(I_p)}\sum_{\tau,\nu}a_{\tau,\nu}^{\kappa}C_\tau(\lambda_1)\sum_{l=0}^{\infty}\sum_{\sigma}\left(\frac{-n_2+p+1}{2}\right)_\sigma\frac{1}{l!}$$

$$\cdot\sum_{\theta}b_{\sigma\nu}^{\theta}\frac{\left(\frac{n_1-1}{2}\right)_\theta}{\left(\frac{n_1+p+1}{2}\right)_\theta}C_\theta(\lambda_1 I_{p-1})\,.$$

THEOREM 3. *Let R be a non-central Beta-matrix defined by* (34). *Then the density function of the maximum latent root of R is given by* (47).

Note. The density function of λ_1 in the central case can be easily obtained from (3), (12) and (38). It is

$$(48)\qquad \frac{\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}\frac{B_{p-1}\left(\frac{n_1-1}{2},\frac{p+2}{2}\right)}{B_p\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\lambda_1^{n_1p/2-1}(1-\lambda_1)^{(n_2-p-1)/2}$$

$$\cdot{}_2F_1\left(\frac{n_1-1}{2},\frac{-n_2+p+1}{2};\frac{n_1+p+1}{2};\lambda_1 I_{p-1}\right),$$

which is the result given by T. Sugiyama and K. Hukutomi [8].

7. The density function of the maximum canonical correlation coefficient in the non-null case

Suppose the variates $x_1,\cdots,x_p$, $y_1,\cdots,y_q$ $(p\leqq q)$ are normally distributed with zero means and covariance matrix Σ. If $\rho_1,\cdots,\rho_p$ are the canonical correlation coefficients between $(x_1,\cdots,x_p)$ and $(y_1,\cdots,y_q)$ and $\lambda_1,\cdots,\lambda_p$ are the maximum likelihood estimates from a sample of size n, $n\geqq p+q$, then the density function of $\lambda_1,\cdots,\lambda_p$ is given by A. G. Constantine [1] as

(49) $$\frac{\pi^{p^2/2}\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p\left(\frac{q}{2}\right)\Gamma_p\left(\frac{n-p}{2}\right)\Gamma_p\left(\frac{p}{2}\right)}|I-P|^{n/2}|\Lambda|^{(q-p-1)/2}|I-\Lambda|^{(n-q-p-1)/2}$$

$$\cdot\prod_{1\leqq i<j\leqq p}(\lambda_i-\lambda_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n}{2}\right)_\kappa\left(\frac{n}{2}\right)_\kappa}{\left(\frac{q}{2}\right)_\kappa}\frac{C_\kappa(P)C_\kappa(\Lambda)}{k!\,C_\kappa(I_p)},$$

where $P=\mathrm{diag}(\rho_1,\cdots,\rho_p)$ and $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_p)$. Using (14) and (15), we can rewrite (49) as

(50) $$\frac{\pi^{p^2/2}\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p\left(\frac{q}{2}\right)\Gamma_p\left(\frac{n-p}{2}\right)\Gamma_p\left(\frac{p}{2}\right)}|I-P|^{n/2}\lambda_1^{(q-p-1)/2}(1-\lambda_1)^{(n-q-p-1)/2}$$

$$\cdot\left(\prod_{i=2}^{p}\lambda_i\right)^{(q-p-1)/2}\prod_{1\leqq i<j\leqq p}(\lambda_i-\lambda_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n}{2}\right)_\kappa\left(\frac{n}{2}\right)_\kappa}{\left(\frac{q}{2}\right)_\kappa}\frac{C_\kappa(P)}{k!\,C_\kappa(I_p)}$$

$$\cdot\sum_{\tau,\nu}a_{\tau\nu}^{\kappa}C_\tau(\lambda_1)\sum_{l=0}^{\infty}\sum_{\sigma}\left(\frac{-n+q+p+1}{2}\right)_\sigma\frac{1}{l!}\sum_{\theta}b_{\nu\sigma}^{\theta}C_\theta(\Lambda_2),$$

where $\Lambda_2=\mathrm{diag}(\lambda_2,\cdots,\lambda_p)$. Thus, using (13) we get the density function of λ_1 as

(51) $$\frac{\pi^{p/2}\Gamma_p\left(\frac{n}{2}\right)\Gamma_{p-1}\left(\frac{q-1}{2}\right)\Gamma_p\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma_p\left(\frac{q}{2}\right)\Gamma_p\left(\frac{n-q}{2}\right)\Gamma_{p-1}\left(\frac{q+p+1}{2}\right)}|I-P|^{n/2}\lambda_1^{qp/2-1}$$

$$\cdot(1-\lambda_1)^{(n-q-p-1)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\left(\frac{n}{2}\right)_\kappa\left(\frac{n}{2}\right)_\kappa}{\left(\frac{q}{2}\right)_\kappa}\frac{C_\kappa(P)}{k!\,C_\kappa(I_p)}\sum_{\tau,\nu}a_{\tau\nu}^{\kappa}C_\tau(\lambda_1)$$

$$\cdot\sum_{l=0}^{\infty}\sum_{\sigma}\left(\frac{-n+q+p+1}{2}\right)_\sigma\frac{1}{l!}\sum_{\theta}b_{\nu\sigma}^{\theta}\frac{\left(\frac{q-1}{2}\right)_\theta}{\left(\frac{q+p+1}{2}\right)_\theta}C_\theta(\lambda_1 I_{p-1}).$$

THEOREM 4. *Let* $(x_1,\cdots,x_p,\ y_1,\cdots,y_q)$, $p\leqq q$, *be distributed with* $p+q$ *variate normal distribution with mean* **0** *and covariance matrix* Σ. *Then the density function of the maximum canonical correlation coefficient* λ_1 *of maximum likelihood estimate from a sample of size* n, $n\geqq p+q$, *is given by* (51).

Appendix

(I) The table of $Z_\lambda(A\oplus B)=\sum_{\mu,\nu} a^{\lambda}_{\mu\nu} Z_\mu(A)Z_\nu(B)$

	$Z_{(1)}(A)$	$Z_{(1)}(B)$
$Z_{(1)}$	1	1

	$Z_{(2)}(A)$	$Z_{(2)}(B)$	$Z_{(1^2)}(A)$	$Z_{(1^2)}(B)$	$Z_{(1)}(A)Z_{(1)}(B)$
$Z_{(2)}$	1	1			2
$Z_{(1^2)}$			1	1	2

	$Z_{(3)}(A)$	$Z_{(3)}(B)$	$Z_{(21)}(A)$	$Z_{(21)}(B)$	$Z_{(1^3)}(A)$	$Z_{(1^3)}(B)$	$Z_{(2)}(A)Z_{(1)}(B)$	$Z_{(1)}(A)Z_{(2)}(B)$	$Z_{(1^2)}(A)Z_{(1)}(B)$	$Z_{(1)}(A)Z_{(1^2)}(B)$
$Z_{(3)}$	1	1					3	3		
$Z_{(21)}$			1	1			4/3	4/3	5/3	5/3
$Z_{(1^3)}$					1	1			3	3

	$Z_{(4)}(A)$	$Z_{(4)}(B)$	$Z_{(31)}(A)$	$Z_{(31)}(B)$	$Z_{(2^2)}(A)$	$Z_{(2^2)}(B)$	$Z_{(21^2)}(A)$	$Z_{(21^2)}(B)$	$Z_{(1^4)}(A)$	$Z_{(1^4)}(B)$	$Z_{(3)}(A)Z_{(1)}(B)$	$Z_{(1)}(A)Z_{(3)}(B)$
$Z_{(4)}$	1	1									4	4
$Z_{(31)}$			1	1							6/5	6/5
$Z_{(2^2)}$					1	1						
$Z_{(21^2)}$							1	1				
$Z_{(1^4)}$									1	1		

	$Z_{(2)}(A)Z_{(2)}(B)$	$Z_{(21)}(A)Z_{(1)}(B)$	$Z_{(1)}(A)Z_{(21)}(B)$	$Z_{(2)}(A)Z_{(1^2)}(B)$	$Z_{(1^2)}(A)Z_{(2)}(B)$	$Z_{(1^2)}(A)Z_{(1^2)}(B)$	$Z_{(1^3)}(A)Z_{(1)}(B)$	$Z_{(1)}(A)Z_{(1^3)}(B)$
$Z_{(4)}$	6							
$Z_{(31)}$	4/3	14/5	14/5	7/3	7/3			
$Z_{(2^2)}$	8/3	4	4			10/3		
$Z_{(21^2)}$		5/2	5/2	5/3	5/3	8/3	3/2	3/2
$Z_{(1^4)}$						6	4	4

(II) The table of $Z_\mu(A)Z_\nu(A)=\sum_\lambda b^{\lambda}_{\mu\nu} Z_\lambda(A)$

		$Z_{(2)}$	$Z_{(1^2)}$		$Z_{(3)}$	$Z_{(21)}$	$Z_{(1^3)}$		$Z_{(4)}$	$Z_{(31)}$	$Z_{(2^2)}$	$Z_{(21^2)}$	$Z_{(1^4)}$
$Z_{(1)}$	$Z_{(1)}$	1/3	2/3	$Z_{(2)}$	1/5	4/5		$Z_{(3)}$	1/7	6/7			
				$Z_{(1^2)}$		1/2	1/2	$Z_{(21)}$		2/9	2/9	5/9	
								$Z_{(1^3)}$				3/5	2/5
$Z_{(2)}$								$Z_{(2)}$	3/35	8/21	8/15		
								$Z_{(1^2)}$		1/3		2/3	
$Z_{(1^2)}$								$Z_{(2)}$		1/3		2/3	
								$Z_{(1^2)}$			1/6	8/15	3/10

Entries not shown in the table mean zero for table (I) and table (II), respectively.

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