CONCENTRATION-CURVE METHODS AND STRUCTURES OF SKEW POPULATIONS

—A METHODOLOGY FOR THE ANALYSIS OF ECONOMIC DATA—

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1. Introduction and summary

In many fields of empirical sciences, skew distributions are so often found that they seemed to be the dominant forms in those fields [13]. The methods based on concentration curves which are regarded as principal tools for the analysis of skew distributions are not yet researched enough as compared with the methods based on the distribution functions.

In part I we try to examine the properties of concentration curves. The definitions are fundamentally based on my recent paper [16], but some notions and descriptive measures are newly induced from the considerations for the effects of inner and outer forces working to the path (concentration curve). In fact, it can be seen that the notions due to curvatures of the path, their variation with respect to the length of the path and so forth are useful to characterize and to decide a certain kind of distributions and their parameters, together with the notions of symmetries and truncations of concentration curves. Namely,

(i) Variations of curvatures of the path are useful to characterize skew distributions, instead of $\dfrac{d\log f(x)}{d\log x}$[1], as is seen in Part II.

(ii) The angle of the tangent to the X-axis at the point (grade) having the *saturation value* (At this point, the curvature is maximized.) gives a *measure of skewness of concentration curve*, especially for Paretoan type curve (2), as is seen in Part II.

(iii) Similarly, the maximum curvature itself gives a *measure of kurtosis of concentration curve*, and this measure can be normalized by dividing with the curvature of circular arc (having generally the next form: $(X-\xi)^2+(Y-1+\xi)^2=\xi^2+(1-\xi)^2$, if both curves have the same area.

(iv) We can suggest the *relative Gini coefficient* (say R.G.) in comparison

[1] See [3] and [12].

with the Gini coefficient of the above circular arc, if both paths have the same length. In this case, obviously $0 \leqq \text{R.G.} \leqq 1$.

Other notions: symmetries and truncations also will show that the Paretoan type (2) is one of the most representative families in concentration curves. Furthermore, they can give us the decomposition of Gini coefficients or mean differences.

Pareto distribution (log-exponential distribution)

$$(1) \qquad \mathrm{P}(X \leqq x) = 1 - \left(\frac{\theta}{x}\right)^{\alpha}; \qquad 0 < \theta \leqq x,\ \alpha > 0$$

is already known as the most typical skew distribution [13], [14]. Particularly, this distribution has remarkable features at the following points:
(i) The mean value μ does not exist for $\alpha \leqq 1$. The variance σ^2 does not exist for $\alpha \leqq 2$. Generally the nth moment does not exist for $\alpha \leqq n$. But the mean deviation δ and the mean difference Δ exist for $\alpha > 1$.
(ii) The central limit theorem sometimes does not hold for the sum of independent random variables, all having the same distribution function (1). Namely the distribution (1) belongs to the domain of attraction of a stable law with the characteristic exponent α in the case of $0 < \alpha < 2$, because its verification is given by the theorem ([7], p. 175) as follows:

$$1) \qquad \frac{F(-x)}{1-F(x)} = 0 \qquad \text{as} \quad x \to \infty\,,$$

$$2) \qquad \frac{1-F(x)+F(-x)}{1-F(kx)+F(-kx)} = \frac{\left(\dfrac{\theta}{x}\right)^{\alpha}}{\left(\dfrac{\theta}{kx}\right)^{\alpha}} = k^{\alpha} \qquad \text{as} \quad x \to \infty\,,$$

for every constant $k > 0$.
(iii) As is well known, the geometric mean of this distribution is equal to $\theta e^{1/\alpha}$ [12].

Now, the results in Part I show that the family of the Paretoan type distributions:

$$(2) \qquad f(x; k, a, b) = \frac{A}{x^k}; \qquad 0 < a \leqq x \leqq b < \infty,\ -\infty < k < \infty,\ A > 0\,,$$

gives the most representative concentration curves.

This distribution[1] expresses directly the double or single upper truncation of the Pareto distribution (1) in the case of $k > 1$. And a limiting distribution of this gives us sometimes the Pareto distribution itself and

[1] The frequency function (2) is even more actual than (1), because each member of the population concerned is restricted very often within finite limits by the condition of existence and activity, and because (1) has nothing of μ, σ, Δ and δ especially for $0 \leqq \alpha \leqq 1$.

sometimes the single upper truncation of the Pareto distribution. This definition is convenient to us for the argument with regard to the concentration curves of the distribution (1), since the distribution (1) has not often the concentration curve [16].

The main results of Part II can be enumerated as follows:

(i) The Pareto or Paretoan type distribution can be distinguished into seven types according to

$$k<0,\ k=0,\quad 0<k<1,\ k=1,\quad 1<k<2,\ k=2 \quad \text{and } k>2\ ,$$

by its concentration curve [13].

(ii) The measure of skewness of concentration curve can be given by $\cos 2\theta$, as is seen in Part I, using the angle θ of the tangent to the X-axis at the grade X_s of the saturation value x_s (maximizing the curvature $\rho=1/\mu f(x)\left(1+\frac{x^2}{\mu^2}\right)^{3/2}$ of the concentration curve of the distribution (2)). This is determined by only the Pareto coefficient α in (1) or k in (2) as $1-\frac{2}{3}k$. And $\cos 2\theta=\frac{\mu^2-x_s^2}{\mu^2+x_s^2}$ exists for $0\leqq k\leqq 3$ and varies in $[0, 1]$. On the other hand, the Pearson's skewness $S_p=\frac{|\text{mode}-\mu|}{\sigma}$ $\left(=\sqrt{\frac{\alpha-2}{\alpha}}\right)$ exists only for $\alpha>2$ and also the skewness $S=\frac{\mathrm{E}(x-\mu)^3}{\sigma^{3/2}}=\frac{2(\alpha+1)}{\alpha-3}\sqrt{\frac{\alpha-2}{\alpha}}$ exists only for $\alpha>2$ in (1).

(iii) The Paretoan type distribution, existing in the interval $[a, b]$ and having $k=1.5$ is self-symmetric and two Paretoan type distributions (2), existing in the same interval $[a, b]$ and having respectively $k=1.5+p$ and $k=1.5-p$ for any real number p, are mutually symmetric, regarding to the diagonal line $X+Y=1$ of the concentration curve. The measure in (ii) is based on these results.

In Part III we applied the results in Parts I and II to some economical data with regard to the national wealth in Japan, which were reported by the Economic Planning Agency in the Government in Japan. (The author ever joined in this survey as a member of sampling designers [11].) At the same time, we suggested the notion *limit concentration coefficient*. The interpretations and structural model-buildings are as yet tentative and abstract ones. But Pareto coefficients calculated from the data will point out some of the most usual types in practice within concentration curves of the Paretoan type (2).

Now, if we always ought to say the conclusion in a word, we are obliged to confess that skew distributions and concentration curves seemed to be typically represented by the Paretoan distribution (2), as if the Gaussian distribution was the most representative one in distribution functions and that the concentration curve method is suitable to explain the collective phenomena due to the interactions between groups, classes and individuals.

PART I. THEORETICAL APPROACHES TO CONCENTRATION CURVES

1. General notions and definitions [16]

[I] Concentration curve

DEFINITION 1. The concentration curve $\Lambda(X)$ is a certain kind of path ordered and normalized. Let

$$x_1, x_2, \cdots, x_N \tag{3}$$

be a sequence of positive real numbers and

$$x_{(1)}, x_{(2)}, \cdots, x_{(N)} \tag{4}$$

be its ordered one. Furthermore, denoting

$$s_{(n)} = \sum_{i=1}^{n} x_{(i)} ,$$

we can formulate the empirical concentration curve $\Lambda_N(X)$ of (3) in [0, 1] as follows:

$$Y = \Lambda_N(X) = \begin{cases} 0 & \text{for } 0 \leqq X < \dfrac{1}{N} , \\ \dfrac{s_{(n)}}{s_{(N)}} & \text{for } \dfrac{n}{N} \leqq X < \dfrac{n+1}{N} , \qquad n = 1, 2, \cdots, N-1 \\ 1 & \text{for } X = 1 . \end{cases} \tag{5}$$

And also in such cases that the sequence (3) contains k ($<N$) different numbers $x_1', \cdots, x_k'$ and each x_i' has the frequency ν_i and that (3) can be extended to a sequence containing infinite numbers, concentration curves $\Lambda_N(X)$ and $\Lambda(X) = \lim_{n\to\infty} \Lambda_N(X)$ can be defined as well as (5) [16].

In this paper, these concentration curves of the given sequence of positive real numbers are usually treated in the form of concentration polygon or their limit function [16]. Furthermore, the concentration curves of continuous frequency function of positive variate x can be obviously expressed by the formulae:

$$
(6) \qquad
\begin{aligned}
X &= \int_0^x f(x)\,dx \\
Y &= \Lambda(X) = \frac{1}{\mu}\int_0^x xf(x)\,dx\ .
\end{aligned}
$$

In this paper we treat only the frequency functions of positive variate x as far as we do not note. And in this case, we easily obtain

$$
(7) \qquad \Lambda'(X) = \frac{x}{\mu} \qquad \text{and} \qquad \Lambda''(X) = \frac{1}{\mu f(x)}\ .
$$

Conversely, if two frequency functions: $f_1(x)$ and $f_2(x)$ give the same concentration curve, then $f_1(x)$ is equal to $cf_2(cx)$, because we have $\frac{x_1}{\mu_1} = \frac{x_2}{\mu_2}$ and $\mu_1 f_1(x_1) = \mu_2 f_2(x_2)$ from (7).

[II] Location

DEFINITION 2. A location parameter of concentration curve is given by the grade (or fraction) of mean scale X_μ which can be defined as X satisfying

$$
(8) \qquad \Lambda'(X) = 1, \qquad \text{if } \Lambda'(X) \text{ exists.}
$$

(See Fig. 1.)

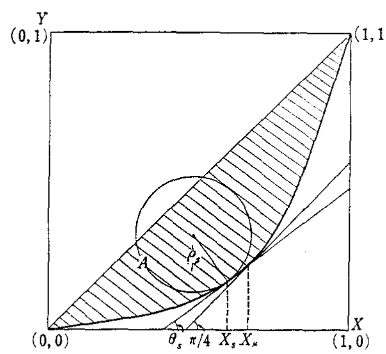

Fig. 1. Illustration of notions with respect to concentration curve (Lorenz diagram)

[III] Dispersion

DEFINITION 3. A dispersion parameter of concentration curve is given by the Gini coefficient (concentration coefficient) which can be defined by

$$
(9) \qquad G = \frac{\Delta}{2\mu}\ ,
$$

as the relative mean difference, where

$$
(10) \qquad \Delta = \frac{1}{N(N-1)} \sum_{\substack{i \neq j \\ i,j=1,2,\cdots,N}} \sum |x_i - x_j| \qquad \text{for the sequence (1)}
$$

and

$$
(11) \qquad \Delta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| f(x) f(y)\,dx\,dy
$$

for continuous frequency functions.

It was well known that $\frac{1}{2}\frac{N-1}{N}G$ is equal to the area A surrounded by the concentration polygon and the egaritarian line: $Y=X$ for (3); and, $\frac{1}{2}G$ is equal to the area A surrouned by the concentration curve (6) and the egaritarian line for continuous frequency functions, namely

$$G=1-2\int_0^1 \Lambda(X)\,dX \qquad \text{(see Fig. 1)}. \tag{12}$$

Hence,

$$\begin{aligned} &0\leqq G\leqq \frac{N}{N-1} && \text{for (1)} \\ &0\leqq G\leqq 1 && \text{for continuous frequency functions.} \end{aligned} \tag{13}$$

Therefore, if we define the improved mean difference Δ' as follows:

$$\Delta'=\frac{1}{N^2}\sum_{i,j=1,\cdots,N}\sum |x_i-x_j|\,, \tag{10$'$}$$

then we have always

$$0\leqq G\leqq 1\,. \tag{13$'$}$$

By the way, for mean deviation δ, we have $\frac{\delta}{\mu}=2(Y_\mu-X_\mu)$ (see Part I, [III]). Especially, let s be the length of the path (concentration curve). Then

$$G\leqq\frac{2s^2}{\pi}-1\,, \qquad \text{where} \quad s=\frac{1}{\mu}\int_0^\infty\sqrt{\mu^2+x^2}\,f(x)\,dx\,, \tag{14}$$

considering the circular arc: $(X-\xi)^2+(Y-1+\xi)^2=\xi^2+(1-\xi)^2$, $0\leqq X\leqq 1$.

Using the above A, G and s, it can be said that the concentration curve is the path maximizing A under the given sequence or frequency function.

Now, let θ be the angle of tangent of the concentration curve to the X axis and let ρ be the radius of curvature at X. Then we can add some definitions:

[IV] Skewness

DEFINITION 4. The measure of skewness S of concentration curve can be defined by $S=\cos 2\theta_s=\frac{\mu^2-x_s^2}{\mu^2+x_s^2}$, if θ_s is the angle of tangent at the grade X_s of x_s (saturation value), giving the maximum curvature $\frac{1}{\rho_s}$ of concentration curve. (See Fig. 1 and Definition 5). Obviously we can see $-1\leqq S\leqq 1$.

[V] Sharpness

DEFINITION 5. The measure of kurtosis of concentration curve can be defined by ρ_s itself and normalized as the form $1-\frac{\rho_s}{R}$ where R expresses the radius of circular arc having the same concentration coefficient with the given concentration curve. Then, we have $0\leqq 1-\frac{\rho_s}{R}\leqq 1$.

2. On symmetries of concentration curves

[I] Self-symmetrical concentration curves

DEFINITION 6. If the given concentration curve is symmetrical with respect to the diagonal, drawn at the right angles to the diagonal of equal distribution, this concentration curve can be defined as self-symmetrical (see Fig. 2).

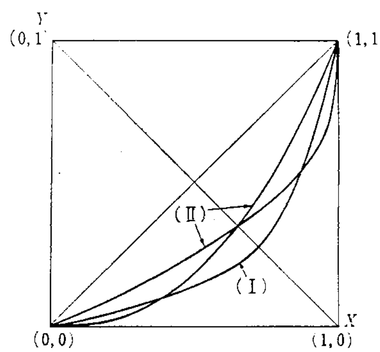

Fig. 2. Symmetrical concentration curves

The above mentioned implies the same meaning as that the ordinary concentration curve, cumulated from the lowest grade, and the extraordinary concentration curve, cumulated from the highest grade, are symmetrical with respect to the egaritarian diagonal. Now, we can express the self-symmetrical concentration curve $\Lambda(X)$ as follows: If $Y=\Lambda(X)$, the relation

$$1-X=\Lambda(1-Y) \tag{15}$$

holds as well. Furthermore, if $\Lambda(X)$ is differentiable,

$$\Lambda'(1-\Lambda(X))\Lambda'(X)=1 \tag{16}$$

and

$$-\Lambda''(1-\Lambda(X))\Lambda'^2(X)+\Lambda''(X)\Lambda'(1-\Lambda(X))=0\ . \tag{17}$$

Last of all

$$\frac{\Lambda''(X)}{\Lambda''(1-\Lambda(X))}=\Lambda'^3(X)\ . \tag{18}$$

Considering the fundamental relation (7) with respect to concentration curves having continuous[1] frequency function, (18) can be transformed

[1] If $f(x)$ is continuous, $\Lambda(X)$ has also the second derivative (see [8] and [17]).

into the following:

$$\frac{f\left(\frac{\mu^2}{x}\right)}{f(x)}=\left(\frac{x}{\mu}\right)^3 \qquad \text{for every fixed value } x\,. \tag{19}$$

Because, if we put $\Lambda'(X)=\frac{x}{\mu}$ and $\Lambda'(1-\Lambda(X))=\frac{x'}{\mu}$, we obtain $\frac{xx'}{\mu^2}=1$ from (16). Accordingly, we have

THEOREM 1. (See Champernowne [6][1].) *If $\Lambda(X)$ is the concentration curve of the given continuous frequency function $f(x)$, the necessary and sufficient condition under which $\Lambda(X)$ is self-symmetrical is that $f(x)$ satisfies the relation* (19).

Indeed, for example, the log-normal distribution [1] *satisfies* (19) *and the Paretoan type distribution* (2) *having $k=1.5$ satisfies it, too.*

[II] Mutual symmetrical concentration curves

DEFINITION 7. When the given two concentration curves $\Lambda_1(X)$ and $\Lambda_2(X)$ are mutual symmetrical with respect to the diagonal $X+Y=1$, we call the relation between them to be mutual symmetrical (see Fig. 2).

This relation can be expressed as follows:

$$\text{If} \quad Y=\Lambda_1(X)\,, \qquad \text{then} \quad 1-X=\Lambda_2(1-Y)\,. \tag{20}$$

And (18) can also be replaced as

$$\frac{\mu_2 f_2(x')}{\mu_1 f_1(x)}=\left(\frac{x}{\mu_1}\right)^3, \qquad xx'=\mu_1\mu_2 \qquad \text{for every fixed } x\,. \tag{21}$$

Thus we have

THEOREM 2. *If $\Lambda_1(X)$ and $\Lambda_2(X)$ are the concentration curves of the given continuous frequency functions $f_1(x)$ and $f_2(x)$, each having non-zero mean μ_1 or μ_2, the necessary and sufficient condition under which they are mutually symmetrical is that $f_1(x)$ and $f_2(x)$ satisfy* (21).

The Paretoan type distributions, having respectively $k=1.5-p$ and $k=1.5+p$ and having the same interval, satisfy this condition.

[III] If the point (X_μ, Y_μ) corresponding to the mean value μ is on that diagonal line,

$$X_\mu+\Lambda(X_\mu)=1\,. \tag{22}$$

On the other hand

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[1] M. G. Kendall afforded the allied formula in [6].

$$\begin{aligned}\Lambda(X_\mu)&=\frac{1}{\mu}\int_{-\infty}^{\mu} xf(x)\,dx\\&=\frac{1}{\mu}\int_{-\infty}^{\mu}(x-\mu)f(x)\,dx+\int_{-\infty}^{\mu}f(x)\,dx\\&=\frac{-\int_{-\infty}^{\infty}|x-\mu|f(x)\,dx}{2\mu}+X_\mu\ .\end{aligned}$$

Therefore, using the mean deviation δ,

$$2F(\mu)-\frac{\delta}{2\mu}=1\ ,$$

or

$$F(\mu)=\frac{1}{2}+\frac{\delta}{4\mu}\ ,\qquad \text{where}\quad F'(x)=f(x)\ . \tag{23}$$

THEOREM 3. *On the concentration curve of the given continuous frequency function, the point* (X_μ, Y_μ) *corresponding to the mean value* μ *is on the diagonal line*: $X+Y=1$ *if and only if* $f(x)$ *satisfies the formula* (23).

Naturally, Gibrat distribution and the Paretoan type distribution having $k=1.5$ satisfy this condition.

3. Truncations of concentration curve

[I] Single truncated concentration curves

DEFINITION 8. We can define the single truncated concentration curve as the concentration curve of the single truncated distribution function of the given frequency function (see Fig. 3).

Let $\Lambda(X)$ be the concentration curve of a continuous frequency function. Moreover, let $\Lambda_1(X;\xi)$ and $\Lambda_2(X;\xi)$ be respectively the upper and lower truncated concentration curve, where $\xi=F(x_0)$, $-\infty<x_0<\infty$ (hence $0<\xi<1$) and $F'(x)=f(x)$. By Definition 8, $\Lambda_1(X;\xi)$ and $\Lambda_2(X;\xi)$ are respectively the concentration curves of $\mathrm{P}(X\leqq x\mid X\leqq x_0)=\dfrac{F(x)}{F(x_0)}$ and $\mathrm{P}(X\geqq x\mid X\geqq x_0)=\dfrac{1-F(x)}{1-F(x_0)}$.

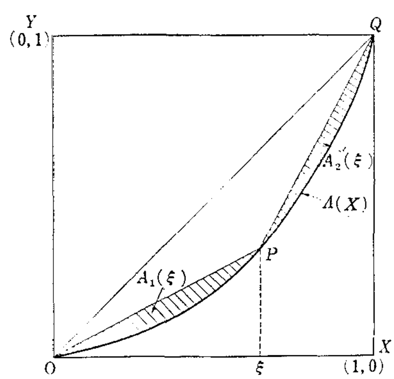


Fig. 3. Illustration of single truncated concentration curves

Considering the properties of the arc OP, the cord OP and the area $A_1(\xi)$ surrounded with them in Fig. 3, the following relations hold between $\Lambda(X)$ and $\Lambda_1(X;\xi)$. Namely

$$\Lambda(\xi)\,\Lambda_1(X;\xi)=\Lambda(\xi X) \qquad \text{for} \quad 0\leqq X\leqq 1\,, \tag{24}$$

$$\frac{\mu_1(\xi)}{\mu}=\frac{\Lambda(\xi)}{\xi}=\Lambda'(\xi X_{\mu_1}) \tag{25}$$

and

$$G_1(\xi)=1-2\frac{\int_0^{\xi}\Lambda(X)\,dX}{\xi\Lambda(\xi)} \tag{26}$$

where $\mu_1(\xi)$, X_{μ_1} and $G_1(\xi)$ express respectively the mean, the grade of mean scale and the concentration ratio of $\Lambda_1(X;\xi)$. Because, $\Lambda(X)$ $(0\leqq X\leqq\xi)$ is in accordance with the curve of $\Lambda_1(X;\xi)$, shortened respectively with proportions ξ and $\Lambda(\xi)$ in the direction of X and Y. From (24) we can easily get the following:

THEOREM 4. *The concentration curve $\Lambda(X)$ of any continuous frequency function is expressible as*

$$\Lambda(X)=\frac{(1-G)\exp\left(-2\int_X^1\frac{d\xi}{\xi\{1-G_1(\xi)\}}\right)}{X\{1-G_1(X)\}} \qquad \textit{for any } X, \quad 0<X\leqq 1\,, \tag{27}$$

where G expresses the concentration coefficient of $\Lambda(X)$.

PROOF. Differentiating (26) with respect to ξ, we obtain

$$2\Lambda(\xi)=\Lambda(\xi)\{1-G_1(\xi)-\xi G_1'(\xi)\}+\xi\Lambda'(\xi)\{1-G_1(\xi)\}\,,$$

accordingly

$$\frac{\Lambda'(\xi)}{\Lambda(\xi)}=\frac{1+G_1(\xi)+\xi G_1'(\xi)}{\xi\{1-G_1(\xi)\}}\,.$$

Integrating with respect to ξ from X to 1, we have (27).

COROLLARY 1. *A continuous frequency function $f(x)$ is given by the form:*

$$f(x)=\frac{A}{x^k} \qquad k<1,\ \infty>x\geqq a\begin{cases}\geqq 0 & \textit{for} \quad k\leqq 0\\ >0 & \textit{for} \quad 0<k<1\,,\end{cases} \tag{28}$$

if and only if the concentration coefficient or concentration curve of its *arbitrary upper truncated distribution remains the same.*

PROOF. If we put

$$G_1(X)=G\ ,\qquad 0\leqq G\leqq 1$$

for arbitrary X $(0<X<1)$ in (27), we have

$$\Lambda(X)=X^{(1+G)/(1-G)}\ .$$

Hence, we can get easily (28) from this and (7). In this case, considering the formula (24), we have

COROLLARY 2. *If and only if the given frequency function satisfies* (28), *it holds that*

$$\Lambda(X)\,\Lambda(Y)=\Lambda(XY) \tag{29}$$

for arbitrary X, Y, where $0\leqq X,\ Y\leqq 1$.

Similarly, considering the lower truncated concentration curve $\Lambda_2(X;\xi)$, we obtain the following formulae:

$$\{1-\Lambda(\xi)\}\Lambda_2(X;\xi)+\Lambda(\xi)=\Lambda\{(1-\xi)X+\xi\}\qquad \text{for}\quad 0\leqq X\leqq 1\ , \tag{30}$$

$$\frac{\mu_2(\xi)}{\mu}=\frac{1-\Lambda(\xi)}{1-\xi}=\Lambda'\{\xi+(1-\xi)X_{\mu_2}\}\ , \tag{31}$$

and

$$G_2(\xi)=1-2\frac{\int_\xi^1\{\Lambda(X)-\Lambda(\xi)\}\,dX}{(1-\xi)\{1-\Lambda(\xi)\}}\ , \tag{32}$$

where $\mu_2(\xi)$, X_{μ_2} and $G_2(\xi)$ express respectively the mean, the grade of mean scale and the concentration coefficient of $\Lambda_2(X;\xi)$. From these relations we have

THEOREM 5. *The concentration curve $\Lambda(X)$ of any continuous frequency function is expressible as*

$$1-\Lambda(X)=\frac{(1+G)\exp\left(-2\int_0^X\frac{d\xi}{(1-\xi)\{1+G_2(\xi)\}}\right)}{(1-X)\{1+G_2(X)\}}\qquad \textit{for}\quad 0\leqq X<1\ . \tag{33}$$

COROLLARY 3. (See N. Bhattacharya [2].) "*Suppose we have an income distribution over the range x_0 to ∞ $(x_0>0)$ and suppose we consider truncated forms of this distribution over (x_t,∞) where $x_t\geqq x_0$. It is proved that the concentration curve (and the concentration coefficient) for this truncated distribution will be independent of x_t if and only if the* *income distribution has the Pareto form," so far as $\alpha>1$ in* (1) (*and if $\alpha<1$, naturally μ and $\Lambda(X)$ do not exist*).

PROOF. If we put $G_2(X)=G$ $(0\leqq G\leqq 1)$ in (30), we can get easily

$$1-\Lambda(X)=(1-X)^{(1-G)/(1+G)} \tag{34}$$

This belongs to (1) (so far as $\alpha\geqq 1$) from (7).

COROLLARY 4. *If and only if the given concentration curve belongs to Pareto form* (1) *with $\alpha>1$, it holds that*

$$\{1-\Lambda(Y)\}\Lambda(X)+\Lambda(Y)=\Lambda\{(1-Y)X+Y\} \tag{35}$$

for arbitrary X and Y, where $0\leqq X, Y\leqq 1$.

[II] The relationships between single truncated curves

From (25) and (31), it can be easily seen that

$$\xi\Lambda'(\xi X_{\mu_1})+(1-\xi)\,\Lambda'\{\xi+(1-\xi)X_{\mu_2}\}=1\ . \tag{36}$$

On the other hand, considering again the area of Fig. 3 and knowing that the area of the triangular OPQ is equal to $\xi-\Lambda(\xi)$, we obtain

$$G=\xi\Lambda(\xi)G_1(\xi)+(1-\xi)\{1-\Lambda(\xi)\}G_2(\xi)+\xi-\Lambda(\xi)\ . \tag{37}$$

These formulae give relationships between single truncated curves.

[III] Doubly truncated concentration curves and relationships between them

DEFINITION 9. For an interval $I=[a, b]$, $(0<a<b<1)$, we can define the doubly truncated concentration curve $\Lambda(X; I)$ of the concentration curve $\Lambda(X)$ of the given frequency function $f(x)$ as the concentration curve of the doubly truncated frequency function of $f(x)$ (see Fig. 4).

In this case, as well as in [I], we obtain the following relations:

$$\Lambda(X;\ I)=\frac{\Lambda\{(b-a)X+a\}-\Lambda(a)}{\Lambda(b)-\Lambda(a)} \tag{38}$$

$$0\leqq X\leqq 1\ ,$$

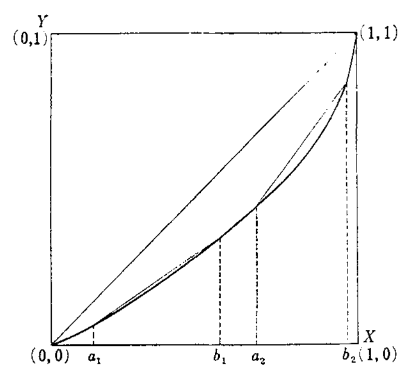


Fig. 4 A relationship of two doubly truncated concentration curves

$$\frac{\Lambda(b)-\Lambda(a)}{b-a}=\Lambda'(X_{\mu(I)}\,;I)=\frac{\mu(I)}{\mu}\,, \tag{39}$$

and

$$G(I)=1-2\frac{\int_a^b\{\Lambda(X)-\Lambda(a)\}\,dX}{(b-a)\{\Lambda(b)-\Lambda(a)\}}\,. \tag{40}$$

(38), (39) and (40) are also respectively the generalizations of (24) and (30); (25) and (31); (26) and (32).

Now, let a be a fixed value and let b be a variable ξ over $(a, 1]$. Then we obtain the generalization of (33).

THEOREM 6. *Any concentration curve $\Lambda(\xi)$ of the given continuous frequency function can be expressed by $G(a, \xi)$ as follows:*

$$\Lambda(X)=\Lambda(a)+\frac{C_1\exp\left(2\int_a^X\frac{d\xi}{\{1-G(a,\xi)\}(\xi-a)}\right)}{\{1-G(a,X)\}(X-a)}\qquad a<\xi\leqq 1\,, \tag{41}$$

where C_1 is determined by the condition $\Lambda(1)=1$.

In the same way, if a is a variable ξ and b is a constant a, then we have

COROLLARY 5. *The formula (41) holds too for $0\leqq\xi<a$, if we decide C_1 by $\Lambda(0)=0$ and we define $G(a, b)=-G(b, a)$.*

Now, let $\Lambda(X; I_1)$ and $\Lambda(X; I_2)$ be two doubly truncated concentration curves, respectively defined over the intervals $I_1=[a_1, b_1]$ and $I_2=[a_2, b_2]$, where $I_1\cap I_2=\phi$.

DEFINITION 10. The joint concentration curve $\Lambda(X; I_1+I_2)$ of two doubly truncated curves, each defined by definition 9, is the concentration curve of the frequency function:

$$g(x)=\begin{cases}\dfrac{f(x)}{b_1-a_1+b_2-a_2} & \text{for}\quad \alpha_1\leqq x\leqq\beta_1\quad\text{and}\quad \alpha_2\leqq x\leqq\beta_2\\[2ex] 0 & \text{otherwise,}\end{cases}$$

where $F(\alpha_1)=a_1$ and $F(\beta_1)=b_1$; $F(\alpha_2)=a_2$ and $F(\beta_2)=b_2$; $F'(x)=f(x)$ (see Fig. 4).

From this definition and by the same reason for (24), (25), (26) and so on, we get

$$\Lambda(X;\, I_1+I_2)=\begin{cases}q_1\Lambda\left(\dfrac{X}{p_1};\, I_1\right) & \text{for}\quad 0\leqq X\leqq p_1\\[2ex] q_2\Lambda\left(\dfrac{X}{p_2};\, I_2\right)+q_1 & \text{for}\quad p_1<X\leqq 1\,,\end{cases} \tag{42}$$

where

$$p_1=\frac{b_1-a_1}{(b_1-a_1)+(b_2-a_2)} \quad \text{and} \quad p_2=\frac{b_2-a_2}{(b_1-a_1)+(b_2-a_2)} ;$$

$$q_1=\frac{\Lambda(b_1)-\Lambda(a_2)}{\{\Lambda(b_1)-\Lambda(a_1)\}+\{\Lambda(b_2)-\Lambda(a_2)\}} \quad \text{and}$$

$$q_2=\frac{\Lambda(b_2)-\Lambda(a_2)}{\{\Lambda(b_1)-\Lambda(a_1)\}+\{\Lambda(b_2)-\Lambda(a_2)\}} ,$$

$$\mu(I_1+I_2)=p_1\mu(I_1)+p_2\mu(I_2) , \tag{43}$$

and

$$\mu(I_1+I_2)G(I_1+I_2)=p_1^2\mu(I_1)G(I_1)+p_2^2\mu(I_2)G(I_2)+p_1p_2|\mu(I_1)-\mu(I_2)| \tag{44}$$

or

$$\Delta'(I_1+I_2)=p_1^2\Delta'(I_1)+p_2^2\Delta'(I_2)+2p_1p_2|\mu(I_1)-\mu(I_2)| ,$$

where Δ' means the improved mean difference (see Part I).

Inversely, we have

DEFINITION 11. The subtracted concentration curve $\Lambda(X; I_1-I_2)$ of two doubly truncated concentration curves, respectively defined over the intervals $I_1=[a_1, b_1]$ and $I_2=[a_2, b_2]$, where $a_1<a_2$ and $b_1=b_2$, is the concentration curve of the frequency function $g(x)$:

$$g(x)=\begin{cases} \dfrac{f(x)}{a_2-a_1} & \text{for } \alpha_1\leqq x\leqq\alpha_2 \\ 0 & \text{otherwise} , \end{cases}$$

where $F(\alpha_1)=a_1$ and $F(\alpha_2)=a_2$

(see Fig. 5).

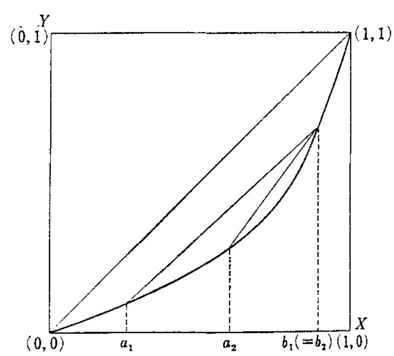


Fig. 5. A relationship of two doubly truncated concentration curves

Therefore, we have also

$$\{\Lambda(a_2)-\Lambda(a_1)\}\Lambda(X; I_1-I_2)=\Lambda(a_1+(a_2-a_1)X)-\Lambda(a_1) , \tag{45}$$

$$\mu(I_1-I_2)=p_1\mu(I_1)-p_2\mu(I_2) , \tag{46}$$

and

$$\mu(I_2-I_1)G(I_2-I_1)=p_1^2\mu(I_1)G(I_1)-p_2^2\mu(I_1)G(I_1)-p_1p_2|\mu(I_1)-\mu(I_2)| \tag{47}$$

or

$$\Delta'(I_2-I_1)=p_1^2\Delta'(I_1)-p_2^2\Delta'(I_2)-2p_1p_2|\mu(I_1)-\mu(I_2)| , \tag{48}$$

where

$$p_1=\frac{b_1-a_1}{a_2-a_1} \quad \text{and} \quad p_2=\frac{b_2-a_2}{a_1-a_2} .$$

If $a_1=a_2$ and $b_1>b_2$, the above results hold for $p_1=\dfrac{b_1-a_1}{b_1-b_2}$ and $p_2=\dfrac{b_2-a_2}{b_1-b_2}$.

The formulae (42)~(44) can be extended to the case of n doubly truncated curves, respectively defined over the intervals $I_1, I_2, \cdots, I_n$ where $I_h \cap I_k=\phi$ for $h\neq k$ and $h, k=1, 2, \cdots, n$. Namely,

$$(49)\qquad A\left(X;\sum_{i=1}^{n} I_i\right)=\begin{cases} q_1A\left(\dfrac{X}{p_1};I_1\right) & \text{for}\quad 0\leqq X\leqq p_1 \\ q_2A\left(\dfrac{X}{p_2};I_2\right)+q_1 & \text{for}\quad p_1<X\leqq p_1+p_2 \\ \cdots\cdots\cdots\cdots & \\ q_nA\left(\dfrac{X}{p_n};I_n\right)+q_1+\cdots+q_{n-1} & \\ & \text{for}\quad p_1+p_2+\cdots+p_{n-1}<X\leqq 1 , \end{cases}$$

where

$$p_j=\frac{b_j-a_j}{\sum_{i=1}^{n}(b_i-a_i)} \quad \text{and} \quad q_j=\frac{A(b_j)-A(a_j)}{\sum_{i=1}^{n}\{A(b_i)-A(a_i)\}} ; \qquad j=1, 2, \cdots, n ,$$

$$(50)\qquad \mu\left(\sum_{i=1}^{n} I_i\right)=\sum_{i=1}^{n} p_i\mu(I_i) ,$$

and

$$(51)\qquad \Delta'\left(\sum_{i=1}^{n} I_i\right)=\sum_{i=1}^{n} p_i^2\Delta'(I_i)+\sum_{\substack{h\neq k\\ h,k=1,2,\cdots,n}}\sum p_hp_k|\mu(I_h)-\mu(I_k)| .$$

Therefore, we have

LEMMA 1. *$\Delta'\left(\sum_{i=1}^{n} I_i\right)$ is decomposable into the within mean difference $\Delta'_{\text{with.}}$ and between mean difference $\Delta'_{\text{bet.}}$:*

$$(52)\qquad \Delta'\left(\sum_{i=1}^{n} I_i\right)=\Delta'_{\text{with.}}+\Delta'_{\text{bet.}} ,$$

where

$$\Delta'_{\text{with.}}=\sum_{i=1}^{n} p_i^2\Delta'(I_i)$$

and

$$\Delta'_{\text{bet.}} = \sum_{\substack{h\neq k\\ h,k=1,2,\cdots,n}}\sum p_hp_k|\mu(I_h)-\mu(I_k)| .$$

The above mentioned holds for not differentiable or discrete (at the countable points) concentration curves. We can apply the above results, for example, to the concentration curves of a finite sequence of positive real numbers:

$$x_{11}, \cdots, x_{1N_1}; \quad x_{21}, \cdots, x_{2N_2}; \quad x_{k1}, \cdots, x_{kN_k}; \quad x_{n1}, \cdots, x_{nN_n}, \tag{53}$$

where

$$x_{ki} \leqq x_{kj} \qquad \text{for} \quad i<j \quad \text{and} \quad k=1, \cdots, n$$

and

$$x_{ki} \leqq x_{hj} \qquad \text{for} \quad k<h \quad \text{and} \quad h, k=1, \cdots, n,$$

where $i=1, \cdots, N_k$, $j=1, \cdots, N_h$.

Denoting

$$N=\sum_{h=1}^{n} N_h, \qquad p_h=\frac{N_h}{N}, \qquad \mu=\frac{1}{N}\sum_{\substack{h=1,\cdots,n \\ i=1,\cdots,N_h}}\sum x_{hi}, \qquad \mu_h=\frac{\sum_{i=1}^{N_h} X_{hi}}{N_h},$$

$$\Delta_h'=\frac{1}{N_h^2}\sum_{i,j=1,\cdots,N_h}\sum |x_{hi}-x_{hj}|, \qquad \Delta_h=\frac{N_h}{N_h-1}\Delta_h', \tag{54}$$

$$\Delta'=\frac{1}{N^2}\sum_{\substack{h,k=1,\cdots,n \\ i=1,\cdots,N_h \\ j=1,\cdots,N_k}}\sum\sum\sum |x_{hi}-x_{kj}| \qquad \text{and} \qquad \Delta=\frac{N}{N-1}\Delta',$$

we can easily get

$$\mu=\sum_{h=1}^{n} p_h\mu_h \tag{55}$$

and

$$\Delta'=\sum_{h=1}^{n} p_h^2\Delta_h' + \sum_{\substack{h\neq k \\ h,k=1,\cdots,n}}\sum p_h p_k|\mu_h-\mu_k| \tag{56}$$

or

$$\frac{N-1}{N}\Delta=\sum_{h=1}^{n} p_h^2\frac{N_h-1}{N_h}\Delta_h + \sum_{\substack{h\neq k \\ h,k=1,\cdots,n}}\sum p_h p_k|\mu_h-\mu_k|. \tag{57}$$

Now, for any continuous curve, we have

DEFINITION 12. We can define the concentration curve of any subset S of $I=[0, 1]$ as the join of intervals I_j, $j=1, \cdots, m$, where $\bigcup_{j=1}^{m} I_j=S$ and $I_j \cap I_k=\phi$ for $j, k=1, \cdots, m$, $j\neq k$. Similarly the join and meet of any n subsets S_i, $i=1, \cdots, n$ of I are definable as the join of intervals

$I_{j'}$, $j'=1, \cdots, m'$, where $\bigcup_{j'=1}^{m'} I_{j'} = \bigcup_{i=1}^{n} S_i$ or $\bigcap_{i=1}^{n} S_i$, $I_{j'} \cap I_{k'} = \phi$ for $j', k' = 1, \cdots, m'$, $j' \neq k'$.

Let $L(S)$ be the length of S. Then, especially we have

THEOREM 7. *If* $S_i \cap S_j = \phi$ $(i \neq j,\ i, j=1, \cdots, n)$ *for* n *subsets* S_i *of* I,

$$\{\sum L(S_i)\}\mu\left(\bigcup_{i=1}^{n} S_i\right) = \sum_{i=1}^{n} L(S_i)\mu(S_i) \tag{58}$$

and

$$\{\sum L(S_i)\}^2 \Delta'\left(\bigcup_{i=1}^{n} S_i\right) = \sum_{i=1}^{n} L(S_i)^2 \Delta'(S_i) + \sum_{\substack{h \neq k \\ h, k=1, \cdots, n}}\sum L(S_h)L(S_k)|\mu(S_h) - \mu(S_k)| \,. \tag{59}$$

PROOF. The above formulae result directly from suitable bracketing in (51).

COROLLARY 6. *Any improved mean difference* Δ' *of the sequence* (3) *of finite observations or continuous frequency function having non-zero mean is decomposable into the within mean difference* $\Delta'_{\text{with.}}$ *and the between mean difference* $\Delta'_{\text{bet.}}$:

$$\Delta' = \Delta'_{\text{with.}} + \Delta'_{\text{bet.}} \tag{60}$$

where

$$\Delta'_{\text{with.}} = \sum_{i=1}^{n} p(C_i)^2 \Delta'(C_i) \,,$$

$$\Delta'_{\text{bet.}} = \sum_{\substack{h \neq k \\ h, k=1, \cdots, n}}\sum p(C_h)p(C_k)|\mu(C_h) - \mu(C_k)|$$

and

$$p(C_i) = L(C_i)/\sum L(C_i) \,,$$

each C_i *means a class in the given sequence of observations or a subset of the sample space of the given continuous frequency function* $(\mu \neq 0)$, *satisfying* $\bigcup_{i=1}^{n} C_i = I$ *(total) and* $C_i \cap C_j = \phi$ *for* $i, j=1, \cdots, n$, $i \neq j$; *and* $L(C_i)$ *means the number* N_i *of observations in* C_i *for the sequence of observations and otherwise means the probability of* C_i *for the frequency function, as if any variance is decomposable in the same way.*

PROOF. The above is deduced from the fact that any sequence of observations or continuous frequency functions having non-zero mean have continuous concentration curves, for example, as concentration polygons of sequences.

PART II. CONCRETE APPROACHES TO TYPICAL CONCENTRATION CURVES [15]

1. Concentration curves of various frequency functions

From Part I we can suppose that skew-type distributions, putting the Paretoan distribution (1) at the head, have typical concentration curves. Now, we can really induce various kinds of concentration curve through the transformation (7) from frequency functions. Tables 1 and 2 show some results of the above.

Generally speaking, we can enumerate Pareto, Zipf and Yule, etc.[1] as skew-type frequency functions (see Table 1) and on the other hand the Pearson system distribution and so on give us the opposed forms as in Table 2. We can concretely see that only the former satisfies

$$yy''=ky'^2 \quad ^{2)} \tag{61}$$

or allied equation if we put $y=\Lambda'(X)$. In fact, skew distributions have such definition that they coincide with (1) for sufficiently large x [13]. But we understand its meaning as below.

Let s and ρ be respectively the length and the radius of curvature of the concentration curve and let θ be the angle of the tangent of the path defined in the section 1 of Part I. Considering

$$\rho=\frac{ds}{d\theta}=\sec\theta\frac{dX}{d\theta}$$

and

$$y=\Lambda'(X)=\tan\theta \qquad \text{for} \quad 0\leqq\theta\leqq\frac{\pi}{2},$$

we have

$$y'=\sec^2\theta\frac{d\theta}{dX}=\frac{1}{\rho}\sec^3\theta$$

and

$$y''=\frac{3}{\rho^2}\sec^4\theta\tan\theta-\frac{1}{\rho^2}\sec^4\theta\frac{d\rho}{ds}.$$

Therefore, we obtain

---

1) For example, c.f. M. G. Kendall [10] and H. A. Simon [13].

2) $\frac{yy''}{y'^2}=-\frac{Ef(x)}{Ex}=-\frac{d\log f(x)}{d\log x}$ ([3]).

(62) $$\frac{yy''}{y'^2}=3\sin^3\theta-\cos\theta\sin\theta\frac{d\rho}{ds},$$

namely

(63) $$-\left|\frac{d\rho}{ds}\right|<\frac{yy''}{y'^2}<3+\left|\frac{d\rho}{ds}\right|.$$

Consequently, from the formula (61) we can define skew distributions as the following: the path $\Lambda(X)$ of the continuous frequency function having non-zero mean are skew if and only if $\frac{d\rho}{ds}\to\rho_1$ (constant) as $X\to 1$. And furthermore, if ρ is everywhere finite, the given continuous frequency function is skew.

2. A classification of the concentration curve of Paretoan type distribution (2)

Considering the formulae (7) and the condition $\Lambda(0)=0$ and $\Lambda(1)=1$, we can obtain the solutions of the differential equation (61), which have different seven types, as is shown in Table 3 and Figs. 6~13, according to the value of k. Comparing the solutions with the formulae (24) and (30) in Part I and using the range r $(=b-a)$ of (2), we can interpret the roles of parameters β and γ, as is shown in Table 4. Especially, on the Pareto distribution (1) (the limit curve of the type (G) in Table 3), we can see that the mean value is one of the most important parameter in its concentration curve, in contrast with the geometrical mean in its frequency function [12] and that the parameter β is concerned with only k.

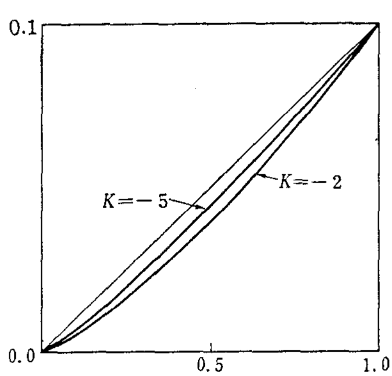


Fig. 6. $k<0$

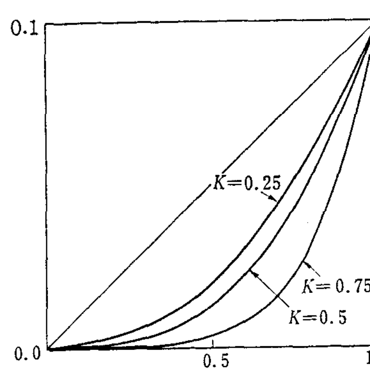


Fig. 7. $0<k<1$

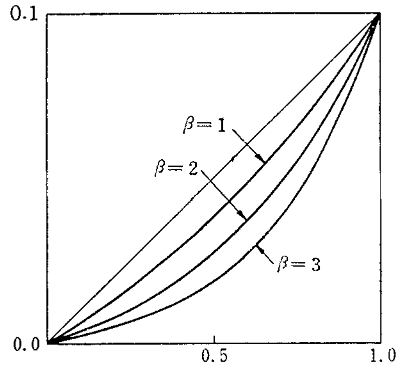


Fig. 8. $k=1$

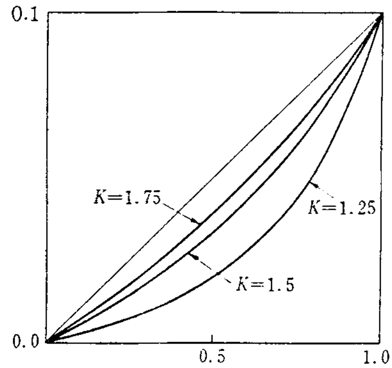


Fig. 9. $1<k<2$, $\gamma=0.5$

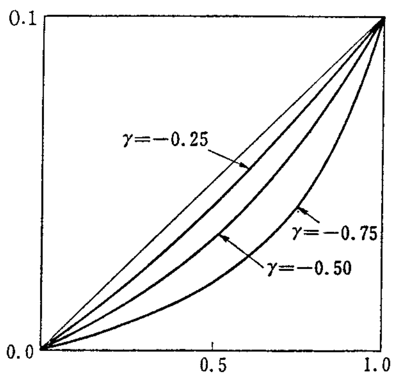


Fig. 10. $k=1.5$

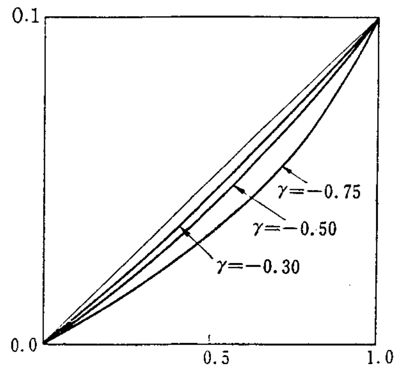


Fig. 11. $k=2$

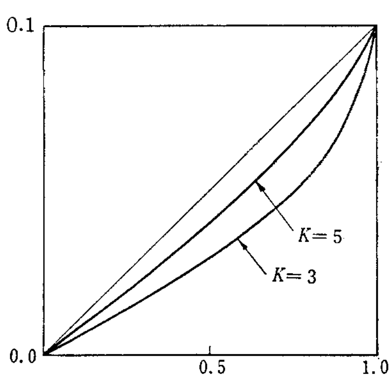


Fig. 12. $k>2$

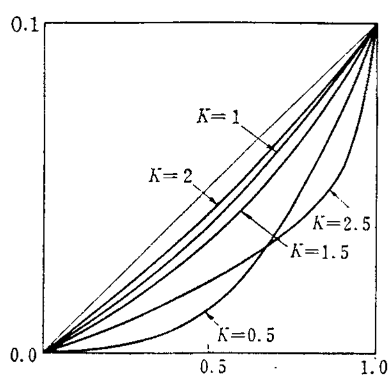


Fig. 13. $-\infty<k<+\infty$

3. Some analysis of the concentration curve of the Paretoan type distribution (2)

In this section we shall analyze the concentration curves of (2) from the following points of view.

[I] Location

The grade X_μ of mean value, defined in the section 1 of Part I, satisfies $\Lambda'(X_\mu)=1$ for continuous frequency function. Therefore, we obtain easily the results in Table 5.

[II] Dispersions

The area A defined in the section 1 of Part I is equal to one-half of the Gini coefficient. The results calculated from (12) are represented in Table 6 and Figs. 14 and 15. By the way, the mean deviation is equal to $2\mu(Y_\mu - X_\mu)$ and can be easily calculated from the results of Tables 3 and 5.

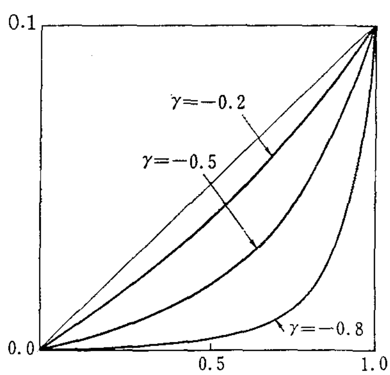


Fig. 14. k=1.25

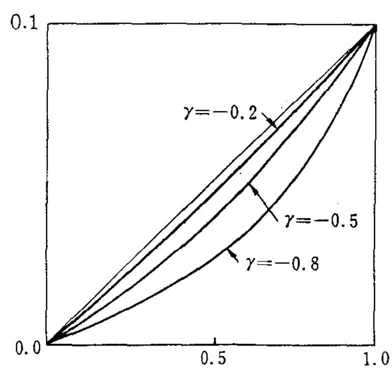


Fig. 15. k=1.75

[III] Skewness

If we put $\dfrac{d\rho}{ds}=0$ in (62), we have directly

$$\sin\theta=\sqrt{\frac{k}{3}} \qquad 0\leqq k\leqq 3\ .$$

Hence $S=\cos 2\theta=1-\dfrac{2}{3}k$ (obviously $[-1\leqq S\leqq 1]$). Thus also the saturation value x_S in the preliminary is represented by

$$x_S=\mu y_S=\frac{\mu}{\sqrt{\dfrac{3}{k}-1}} \qquad \text{for} \quad 0<k<3\ .$$

Table 7 expresses the above results.

[IV] Kurtosis

If $k<0$, $\Lambda(X)$ belongs to the type (A) and

$$\rho_0=0$$

and

$$\rho_1=\frac{\{(k-1)^2+(k-2)^2\}^{3/2}}{(k-1)(k-2)}=-\frac{(1+\beta^2)^{3/2}}{\beta(\beta-1)} \qquad \text{as} \quad a\to 0\ .$$

ρ increases when X increases and when k decreases. By the definition in the section 1 of Part I, $\rho_0=0$ gives a kurtosis. But if $k\geqq 3$, $\Lambda(X)$ belongs to type (G) and

$$\rho_0=\frac{\{(k-1)^2+(k-2)^2\}^{3/2}}{(k-1)^2(k-2)^2}=\frac{(1+\beta^2)^{3/2}}{\beta(\beta-1)}\ ,$$

and

$$\rho_1=0 \qquad \text{as} \quad b\to\infty\ .$$

In this case, ρ decreases whenever X increases and ρ and k increase or decrease together. Therefore, we have $\rho_S=0$. These results can distinguish the types of (2) in their graphs.

Table 1. Properties of skew distributions [15]

Distributions	Pareto	Zipf	Yule	Levy Pareto	Inverse of truncated random normal variate
$f(x)$	$\frac{\alpha\theta^{\alpha}}{x^{1+\alpha}}$, $0<\theta\leqq x\leqq\infty$	$\frac{A}{x^{\rho}}$	$\frac{k\Gamma(x)\Gamma(\rho+1)}{\Gamma(x+\rho+1)}$	$\frac{1}{\sqrt{2\pi}}e^{-1/2x}x^{-3/2}$, $x>0$	$\sqrt{\frac{2}{\pi}}e^{-1/2x^2}x^{-2}$, $x>0$
Type of concentration curve	$yy''=(1+\alpha)y'^2$, when $\alpha>0$	$yy''=\rho y'^2$, when μ exists	Provided x to be continuous in the frequency function: $f(x)=\frac{k\rho!}{(x+\rho)(x+\rho-1)\cdots x}$ $yy''=y'^2\left[\frac{yk}{y+\rho/\mu}+\frac{y}{y+(\rho-1)/\mu}+\cdots+1\right]$, where μ expresses mean value and ρ expresses integer	$yy''=\left(\frac{3}{2}-\frac{\mu}{2y}\right)y'^2$	$yy''=\left(2-\frac{\mu^2}{y^2}\right)y'^2$
Remarks			$yy''=(\rho+1)y'^2$ as $y\to\infty$	$yy''=\frac{3}{2}y'^2$ as $y\to\infty$	$yy''=2y'^2$ as $y\to\infty$

Table 2. Properties of non-skew distributions [15]

[I] Discrete type	Poisson	Negative binomial
Frequency function $f(x)$	$e^{-\lambda}\dfrac{\lambda^x}{x!}$, $x=1,2,\cdots$	$q^{-n}\dfrac{\Gamma(n+x)}{x!\Gamma(n)}\left(\dfrac{p}{q}\right)^x$, $x=0,1,2,\cdots$, $q=1+p$, $n>0$, $p>0$
Type of concentration curve	Provided x to be continuous in the frequency function approximated by $x!\sim\sqrt{2\pi}\,x^{x+1/2}e^{-x}$ for sufficient large x, $y''=\dfrac{\lambda}{2}y'^3\log y$	Provided x to be continuous in the frequency function: $f(x)=q^{-n}\dfrac{(n+x-1)\cdots(x+1)}{\Gamma(n)}\left(\dfrac{p}{q}\right)^x$, $y''=-\mu y'^2\left(\log\dfrac{p}{q}+\dfrac{1}{n+x-1}+\dfrac{1}{n+x-2}+\cdots+\dfrac{1}{x+1}\right)$, where $x=\mu y$

[II] Continuous-type	Exponential	Weibull	Gibrat	Truncated Chauchy
Frequency function $f(x)$	$\Lambda e^{-\lambda x}$, $x>0$	$b\dfrac{x^{b-1}}{\theta}e^{-x^b/\theta}$, $x\geqq 0$, $b,\theta>0$	$\Phi'\left(\dfrac{\log x-\lambda}{\theta}\right)$	$\dfrac{2}{\pi}\dfrac{\alpha}{1+(x-\mu)^2}$, $\mu+k\geqq x>\mu$, $k=\tan\dfrac{\pi}{2\alpha}$
Type of concentration curve	$y''=y'^2$	$yy''=\left\{1-b+b\Gamma\left(\dfrac{1}{b}+1\right)y^b\right\}y'^2$	$yy''=\left(\dfrac{\log y}{\lambda^2}+\dfrac{3}{2}\right)y'^2$	$y''=\pi(my-\mu)y'$ where $m=\dfrac{\mu}{\alpha}-\dfrac{2}{\pi}\log\cos\dfrac{\pi}{2\alpha}=\dfrac{\mu}{\alpha}+\dfrac{1}{\pi}\log(1+k^2)+\infty$

	Gauss	Γ	B
Frequency function $f(x)$	$\dfrac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\dfrac{1}{2}\dfrac{(x-\mu)^2}{\sigma}\right\}$, $\mu\neq 0$	$\dfrac{1}{a\Gamma(p)}\left(\dfrac{x}{a}\right)^{p-1}e^{-x/a}$, $x>0$	$\dfrac{1}{B(p,q)}x^{p-1}(1-x)^{q-1}$, $0<x<1$
Type of concentration curve	$y''=\dfrac{\mu^2}{\sigma}(y+1)y'^2$	$yy''=\{(1-p)+py\}y'^2$	$yy''=\left\{q-p-\dfrac{(1-q)/(p+q)}{p+q-py}\right\}y'^2$

Table 3. Classification of the concentration curves of (2)

Types	k	Concentration curve	Limit curve
(A)	$k<0$	$Y=\frac{(1+\gamma X)^{\beta}-1}{(1+\gamma)^{\beta}-1}$, where $1<\beta<2$ and $\gamma>0$ *	$Y=X^{\beta}$ as $a\to 0$ *
(B)	$k=0$	(i) $Y=X^2$ as $A\lessgtr 1$ (ii) $Y=X$ as $A=1$	
(C)	$0<k<1$	$Y=\frac{(1+\gamma X)^{\beta}-1}{(1+\gamma)^{\beta}-1}$, where $2<\beta\leqq\infty$ and $\gamma>0$ *	$Y=X^{\beta}$ as $a\to 0$ *
(D)	$k=1$	$Y=\frac{e^{\beta X}-1}{e^{\beta}-1}$, $\beta>0$ (Zipf)	
(E)	$1<k<2$	$Y=\frac{(1+\gamma X)^{\beta}-1}{(1+\gamma)^{\beta}-1}$, where $-\infty\leqq\beta<0$ and $-1<\gamma<0$ (Mandelbrot)	
(F)	$k=2$	$Y=\frac{\log(1+\gamma X)}{\log(1+\gamma)}$, where $-1<\gamma<0$	
(G)	$k>2$	$Y=\frac{(1+\gamma X)^{\beta}-1}{(1+\gamma)^{\beta}-1}$, where $0<\beta<1$ and $-1<\gamma<0$ *	$1-Y=(1-X)^{\beta}$ as $b\to\infty$ *

* Generally β is equal to $(k-2)/(k-1)$ in (A), (C), (E) and (G)

Table 4. Properties of parameters of the concentration curves in Table 3

Types	Parameters		Limit curve
	β	γ	
(A)	$\beta>\frac{b}{\mu}$ *	$\frac{F(a)}{1-F(a)}$	$\beta=\frac{b}{\mu}$ as $a\to 0$
(B)	$\beta=\frac{r}{\mu}$ for $A\neq 1$		
(C)	$\frac{b}{\mu F(a)}>\beta>\frac{b}{\mu}$ *	$\frac{F(a)}{1-F(a)}$	$\beta=\frac{b}{\mu}$ as $a\to 0$
(D)	$\beta=\frac{r}{\mu}$		
(E)	$-\frac{r}{\mu F}<\beta$ *		
(F)			
(G)	$\beta<\frac{a}{\mu F(b)}$ *	$-F(b)$	$\beta=\frac{a}{\mu}$ as $b\to\infty$

* Generally β is equal to $(r+b\gamma)/\mu\gamma$ in (A), (C), (E) and (G).

Table 5. Variations of X_μ with respect to k

Types	k	X_μ	Remarks	Limit curve	Remarks
(A)	$k<0$	$X_\mu<\beta^{1/(1-\beta)}$ *	X_μ decreases when γ increases	$X_\mu=\left(\frac{1-k}{2-k}\right)^{1-k}=\beta^{1/(1-\beta)}$ as $a\to 0$	$X_\mu=\frac{1}{e}$ as $k\to-\infty$
(B)	$k=0$	(i) $X_\mu=\frac{1}{2}$ for $A\gtrless 1$ (ii) indeterminable for $A=1$			
(C)	$0<k<1$	$X_\mu<\beta^{1/(1-\beta)}$ *	X_μ decreases when γ increases	$X_\mu=\beta^{1/(1-\beta)}>\frac{1}{\beta}$ as $a\to 0$	
(D)	$k=1$	$X_\mu=\frac{1}{\beta}\log\frac{e^\beta-1}{\beta}>\frac{1}{2}$	X_μ decreases when β increases		
(E)	$1<k<2$	*	X_μ decreases when γ increases X_μ increases when k increases		
(F)	$k=2$	$X_\mu=-\frac{1}{\gamma}+\frac{1}{\log(1+\gamma)}$	X_μ increases when γ decreases		
(G)	$k>2$	$X_\mu<1-\beta^{1/(1-\beta)}$ *	X_μ increases when γ decreases	$X_\mu=1-\beta^{1/(1-\beta)}<1-\frac{1}{\beta}$ as $b\to\infty$	$X_\mu=1-\frac{1}{e}$ as $k\to\infty$

* For (A), (C), (E) and (G), generally $X_\mu=\dfrac{\left[\dfrac{(1+\gamma)^\beta-1}{\gamma\beta}\right]^{1/(\beta-1)}-1}{\gamma}$ and X_μ increases when k increases for the same interval $[a, b]$.

Table 6. Variation of Gini coefficient with respect to k

Types	k	G	Remarks	Limit curve
(A)	$k<0$	$<\frac{\beta-1}{\beta+1}\ \left(=\frac{1}{3-2k}\right)^*$	G increases when k increases	$G=\frac{\beta-1}{\beta+1}$ as $a\to 0$
(B)	$k=0$	(i) $\frac{1}{3}$ for $A\gtrless 1$ (ii) 0 for $A=1$		
(C)	$0<k<1$	$<\frac{\beta-1}{\beta+1}$ *	G increases when k increases	$G=\frac{\beta-1}{\beta+1}$ as $a\to 0$
(D)	$k=1$	$1-\frac{2}{\beta}+\frac{2}{e^{\beta}-1}$ $1-\frac{2}{\beta}<G<-1-\frac{2}{\gamma}$	G increases when β increases	
(E)	$1<k<2$	(ii) $1-2\dfrac{\frac{\log(1+\gamma)}{\gamma}-1}{(1+\gamma)^{-1}-1}$, for $k=1.5$ *	G increases when γ decreases	
(F)	$k=2$	$-1-\frac{2}{\gamma}+\frac{2}{\log(1+\gamma)}$	G increases when γ decreases $G=0$ as $\gamma\to 0$ and $G=1$ as $\gamma\to -1$	
(G)	$k>2$	$<\frac{\beta-1}{\beta+1}$ *	G decreases when k increases	$G=\frac{\beta-1}{\beta+1}$ as $b\to\infty$

* Generally $G=1-2\dfrac{\dfrac{(1+\gamma)^{\beta+1}-1}{\gamma(\beta+1)}-1}{(1+\gamma)^{\beta}-1}$ for (A), (C) and (E), except $k=1.5$ and (G).

Table 7. Variation of the skewness with respect to k

Types	k	S	Saturation value	Remarks
(A)	$k<0$	$\frac{1-\beta^2}{1+\beta^2}$ **	$x_S=\mu\beta$ as $a\to 0$	
(B)	$k=0$	1	does not exist	
(C)~(F)	$0<k<2$	$1-\frac{2}{3}k$ *	$\dfrac{\mu}{\sqrt{\frac{3}{k}-1}}$	$-1<S<0$ for $0<k<1.5$ $S=0$ for self symmetrical case: $k=1.5$ $0<S<1$ for $k>1.5$
(G)	(i) $2\leqq k<3$ (ii) $k\geqq 3$	$1-\frac{2}{3}k$ * $\frac{1-\beta^2}{1+\beta^2}$ **	$\dfrac{\mu}{\sqrt{\frac{3}{k}-1}}$, $x_S=\mu\beta$ as $b\to\infty$	$0<S\leqq 1$

* S expresses the inclination of the direction to the center of curvature from the path for that direction at the mean point.

** The formula of S in (C)~(F) and (G)(i) can be extended to $[-\infty,\infty]$. And in this case, $S=1-2k/3$ is equal to $3/G$ at the limit concentration curve.

PART III. PRACTICAL APPROACHES TO CONCENTRATION CURVES

[I] Empirical findings

Table 8 is tabulated from the reports of the national wealth survey in Japan (1955). We can easily see from the table that the type (G) in Table 3 (and $2<k<3$) is dominant form for the group of enterprises in Japan.

If the Pareto distribution (1) holds in practice, we can get

$$\frac{3}{4}<X_\mu<1\,, \qquad \frac{1}{3}<G<1 \qquad \text{(see Table 8)} \quad \text{and} \quad -1<S<-\frac{1}{3}$$

from Tables 5, 6 and 7.

By the way, if $S>0$ in definition 4, the direction toward the center of curvature at X_S from the point (X_S, Y_S) on the path inclines to the right-hand for that direction at X_μ.

[II] Some tentative interpretations of skew distributions and their concentration curves [3], [5] and [13]

It seems to me that the concentration curve method is especially suitable for the analysis of system or organization. In this case we propose the next notion.

DEFINITION 13. Let us put $a=X$ and $b=X+\Delta X$ in (39) and (40). Then, we have

$$\frac{\mu(X, X+\Delta X)}{\mu}=\frac{\Delta\Lambda(X)}{\Delta X}$$

and

$$G(X, X+\Delta X)=1-2\frac{\int_X^{X+\Delta X}\{\Lambda(\xi)-\Lambda(X)\}d\xi}{\Delta X\Delta\Lambda(X)}.$$

We define the limit (or local) concentration coefficient $g(X)$ by

$$\lim_{\Delta X\to 0}\frac{G(X, X+\Delta X)}{\Delta X}\ .$$

Then, obviously we have

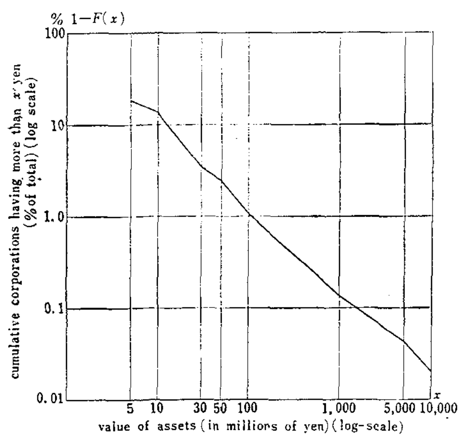
$$g(X)=\frac{1}{6}\frac{\Lambda''(X)}{\Lambda'(X)} \tag{64}$$

from the above definition. And using the above notion, the relation (1) can be expressed by

$$\frac{d}{dX}\left\{\frac{1}{g(X)}\right\}=-\alpha \tag{65}$$

from the relation (61).

In such cases of economical phenomena as shown in Figs. 16, 17 and 18, expressing a constitution of national capital and, at the same time, reflecting organizations and inner oppositions of capitals in a nation, it seemed that distributions and concentration curves can not be explained enough without the concepts of competition, organization, oligopoly, monopoly, etc. "The birth-and-death rules", "the law of proportionate

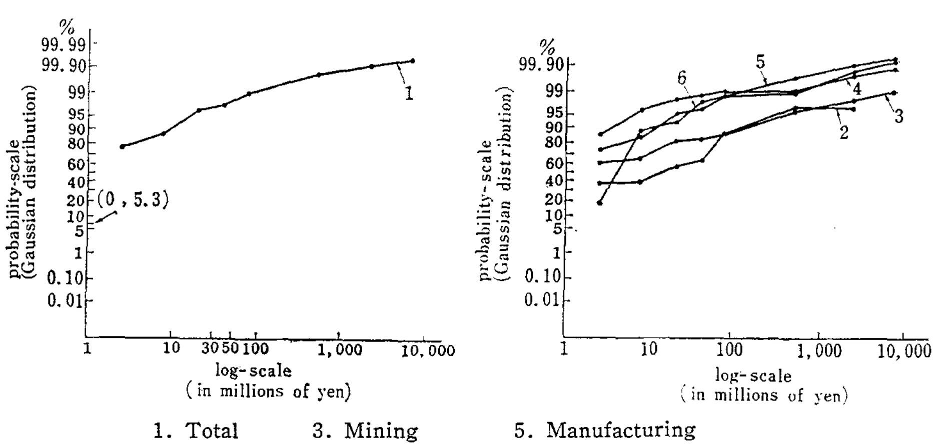


* $1-F(x)=\left(\frac{\theta}{x}\right)^{\alpha}$ and

$\hat{\alpha}=\frac{1}{2G}+\frac{1}{2}=1.06517$

in Table 8.

Fig. 16. Asset size distribution of total industry



1. Total
2. Fisheries
3. Mining
4. Construction
5. Manufacturing
6. Finance and insurance

Fig. 17. Asset size distributions within some industrie

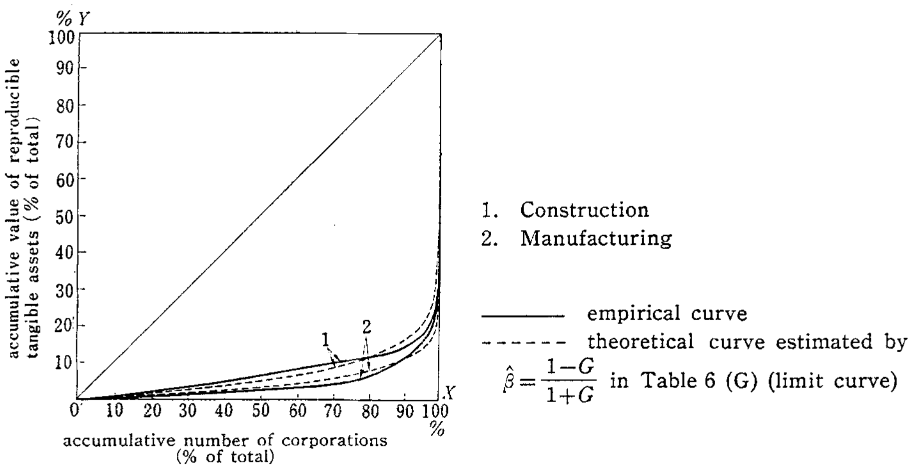


Fig. 18. Empirical curves and theoretical curves

growth", etc. look like too superficial view of the matter nowadays.

For example, the relation (65) may be interpreted as follows:

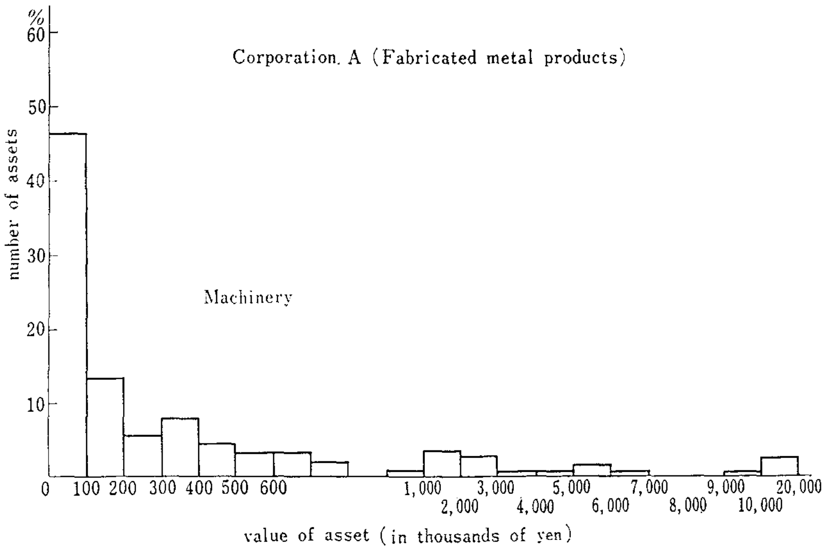
$\frac{d}{dX}\left\{\frac{1}{g(X)}\right\}$ and $\frac{1}{g(X)}$ in (65) respectively concerned to the force of organization or oligopolization of capital at the grade X—so to say, "statistical gravity" working to X in the system (or field) (that is concerned with sometimes markets, sometimes productions and so forth), having the action of centralization; and the force of competitions or struggles [5] between capitals (or enterprises) in the neighbourhood of the grade X around share—so to say, "statistical restriction" against the above gravity in the system concerned, where y is generally a variable consisting of trend, periodicity and uncertainty component and X gives the grade with respect to the trend of y, but at the present time we suppose y to be constructed only by a trend satisfying (65).

In this point of view, the statistical differential equation (65) represents an equilibrium (no matter whether probabilistic laws hold or not), as the result of the multiplication or the cancelling of both previous forces opposed to each other and in this case α nearly connected with the skewness of concentration curve suggests a specific value of this system. Furthermore, this equilibrium of the interaction generates the same effect as the force—so to say "skew force"—working in the vertical direction to the path $\Lambda(X)$ at each point of its path with the proportional size to its curvature (the value of y, giving its maximum, gives a measure of kurtosis as shown in Part II, 3); $\Lambda(X)$ itself gives a certain kind of potentiality of the fraction X in this system.

The process of some break-down and its recuperation of this equilibrium brings some variation on α; in fact, in empirical data α and μ

increase almost year by year [5][1]. Therefore it can be said that α expresses even the step of development of capital accumulation.

Consequently, we understand that the relations (61) show a static



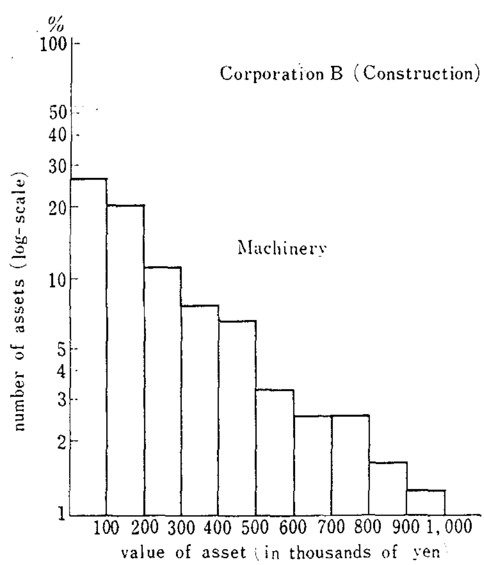


Fig. 19. Asset distribution within some corporation*

* Source of data: the results of the pretest for the national wealth survey in Japan 1955.

[1] cf. Saburo Shiomi, *Japan's Finance and Taxation 1740-1956*, Columbia Univ. Press, (1957), 156-157.

dynamic balancing model under an ideal condition of this system and the work of any other forces (political, economical, social, physical, etc.) will add to this another terms or factors such as in Table 1.

In some other cases, for example, Figs. 19~21, expressing inner constitutions of individual capital (or enterprise) and reflecting organization of production, concentration curves may be interpreted as a result of the interaction between the force of centralization of production on one hand and the force of polarization of production on the other.

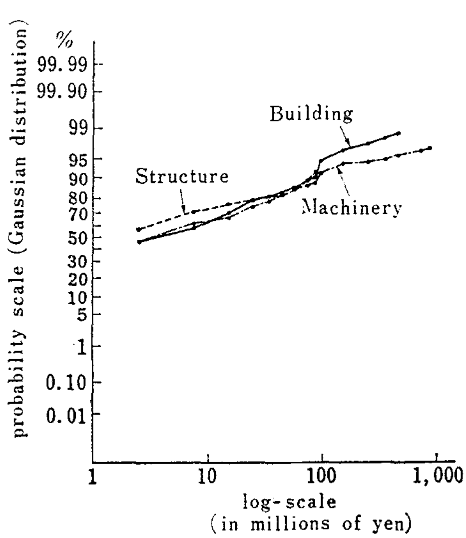


Fig. 20. Asset size distribution within corporation N

But the above model buildings are outlined on merely tentative assumption and therefore, in future, should be more precisely decribed and determined from the more essential principles and mechanisms.

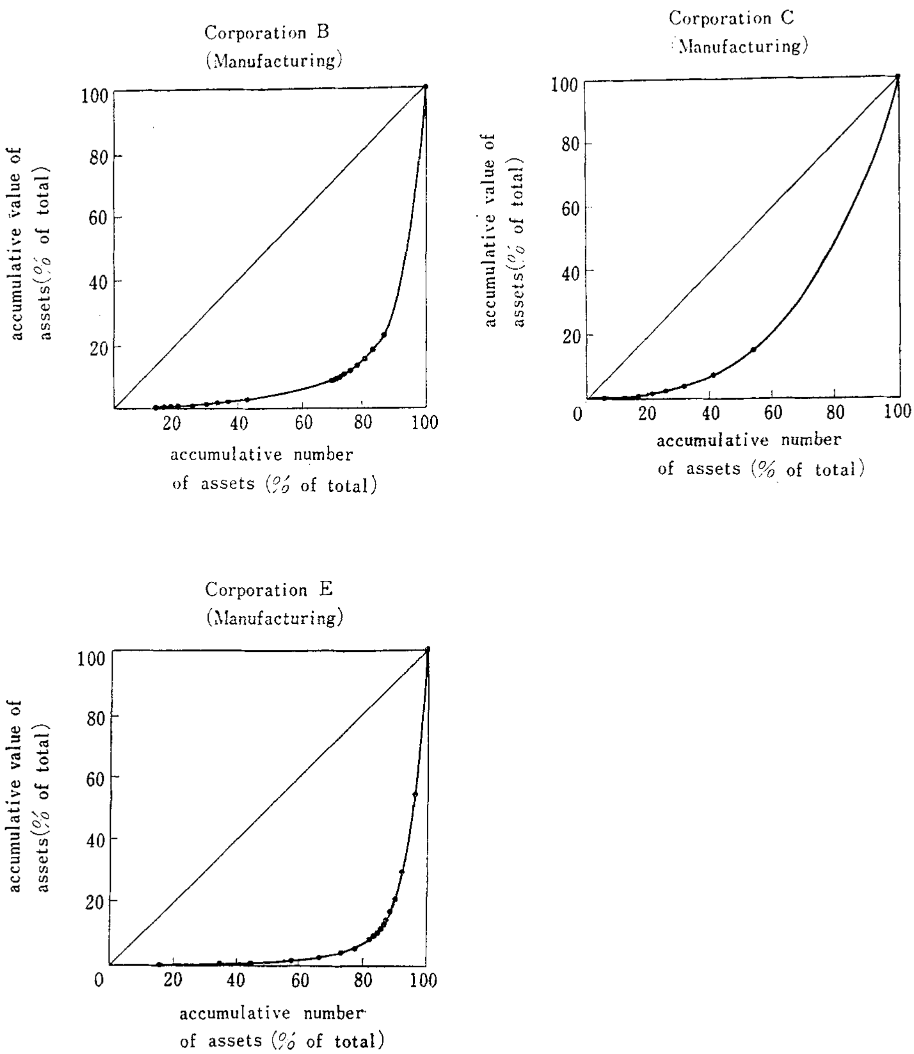


Fig. 21. Concentration curve of value to asset within some corporation

Table 8. Asset share distributions and concentration ratios (Gini's coefficient) with in each industry

Industry *	Estimated corporations **	Estimated reproducible tangible assets (in millions of yen) **	Concentration ratios G ***	Pareto's constants α ****	$\frac{\sigma}{\mu}$
Total	357272	6153141	0.88469	1.06517	24.53954
Agriculture & Forestry	3523	4125	0.95898	1.02139	8.53403
Fisheries	1286	43656	0.81410	1.11418	10.78741
Mining	668	191581	0.93435	1.03513	4.98658
Coal	75	123705	0.69710	1.21726	1.90098
Other minings	593	67876	0.94300	1.03023	7.91028
Construction	16070	144384	0.85304	1.08614	23.19385
Manufacturing	109007	2601331	0.90767	1.05086	18.04110
Food & kindred products	22560	230301	0.81454	1.11384	11.08739
Textiles	17110	467934	0.89680	1.05754	15.28596
Wood & lumber	21079	144448	0.58626	1.35286	2.22943
Paper & allied products	5614	142972	0.96360	1.01889	12.69969
Chemical & allied products	8503	466723	0.92195	1.04233	10.41299
Glass, ceramic stone & clay products	4012	93720	0.88867	1.06264	14.45083
Metals	5306	454512	0.95622	1.02289	17.26955
Fabricated metal products	7585	43118	0.85806	1.08271	10.74839
Machinery, excl. electrical	12982	541073	0.94949	1.02660	14.38368
Other manufacturings	4256	16525	0.53224	1.43943	5.60697
Wholesale & Retail	180737	1400452	0.72550	1.18918	8.25166
Wholesale	83719	1077512	0.73984	1.17582	6.31216
Retail	97018	322940	0.55342	1.40348	13.16724
Department	161	51806	0.75974	1.15812	3.16998
Other retails	96857	271133	0.47058	1.56252	18.92804
Finance & Insurance	13544	170852	0.94813	1.02736	18.92804
Bank & trust	86	105917	0.69770	1.21664	2.07389
Insurance	3564	10354	0.98660	1.00679	16.79095
Other finances & insurances	9894	54580	0.83233	1.10072	12.09409
Real estate	4693	43466	0.93016	1.03754	9.88277
Public utilities	5768	1388739	0.97516	1.01274	13.12603
Transport	5636	404523	0.93462	1.03497	8.71483
Local railway	107	138497	0.63586	1.28634	1.93758
Motor vehicle	2640	48904	0.78641	1.13580	3.01286
Water transport	799	174580	0.94608	1.02850	5.59630
Warehouses	1008	17265	0.68422	1.23076	5.65814
Other transports	1082	25275	0.90312	1.05363	13.94602
Communication	43	14358	0.75584	1.16151	2.57376
Power supplying	89	969857	0.85599	1.08412	2.44890
Electricity	17	905199	0.39853	1.75460	0.80532
Gas	72	64657	0.91266	1.04785	4.62680
Service	21976	164551	0.75077	1.16598	11.25720

* Foundation of classification: Statistical Standards Bureau Administrative Management Agency: Standard industrial classification in Japan, 1954, 330-343.

** Source of data: I. Nakayama (supervisor) [11].

*** G is estimated by the formula: $2\times$ area of concentration polygon.

**** α is estimated by the formula: $1/2G+1/2$.

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