Quantum-Mechanical Description of Coherent Spontaneous Emission (*).

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(ricevuto il 28 Aprile 1969)

In this paper we consider the general problem of coherent spontaneous emission of a system of N two-level atoms coupled to a single-mode e.m. field ([1]). The system shall be considered enclosed in a travelling-wave cavity in order to avoid the complications due to the spatial degree of freedom of the e.m. field.

Following DICKE ([1]), the initial state is assumed to have definite population difference M, and co-operation number J; no photon is initially present. The first assumption on the initial state implies that the atoms radiate coherently and the second that the system starts evolving via purely spontaneous emission. The first quantum-mechanical statement of this problem has been lucidly made by DICKE ([1]) several years ago, and the solution was found in the framework of first-order time-dependent perturbation theory. In the following our aim is to remove such limitation, giving a full quantum solution, at least in the case where the co-operation number J is much bigger than 1.

The interaction Hamiltonian is

$$H_{\text{int}} = \hbar K(aJ^+ + a^\dagger J^-)\,, \tag{1}$$

where $J^\pm = \sum_{k=1}^{N} \frac{1}{2}\sigma_k^\pm$ ($\sigma_k^\pm$ are the individual atom flip operators), a is the annihilation operator of the single-mode e.m. field with frequency $\nu = \Delta E/h$ (ΔE being the energy difference between the two levels of the atomic system); and K is the coupling constant. By describing the angular-momentum operators J_i in terms of two harmonic oscillators ([2]) the Hamiltonian (1) can be written as

$$H_{\text{int}} = \hbar K(aa_1 a_2^\dagger + a^\dagger a_1^\dagger a_2)\,, \tag{2}$$

(*) Work supported in part by the Air Force Office of Scientific Research.
(**) Supported in part by the American Optical Corp.
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([1]) R. H. DICKE: *Phys. Rev.*, **93**, 99 (1954).

([2]) J. SCHWINGER: in *Quantum Theory of Angular Momentum*, ed. L. C. BIEDENHARN and H. VAN DAM (New York, 1965), p. 229.

which, as is well known, represents the main features of the coherent coupling of three single-mode boson fields ([3]).

In the following we will concentrate on Hamiltonian (1) and discuss the solutions of the system with N two-level atoms. However with very little effort all our results can be carried over to the processes described by (2).

We write the Schrödinger equation in the interaction picture

$$\frac{1}{i}\frac{\partial}{\partial t}|\psi(t)\rangle = K(a^\dagger J^- + aJ^+)|\psi(t)\rangle . \tag{3}$$

Due to the conservation of $\boldsymbol{J}^2 = J(J+1)$ and $M = J_3 + n = (J_3)_{\text{in}}$ ($n = a^\dagger a$ is the number-of-photons operator) we can simply label the states by the number of photons, and define the amplitude for having n photons at the time t as

$$A(n, t) = \langle n|\psi(t)\rangle$$

with the condition, apart from a phase factor,

$$A(n, 0) = \delta_{n0} . \tag{4}$$

By the definition

$$A(n, t) = i^n a(n, t)$$

the Schrödinger equation becomes

$$\frac{\partial}{\partial t} a(n, t) = [n(J - M + n)(J + M - n + 1)]^{\frac{1}{2}} a(n-1, t) - \\ - [(n + 1)(J - M + n + 1)(J + M - n)]^{\frac{1}{2}} a(n + 1, t) . \tag{5}$$

We can give now a solution for short times of this equation; the range of validity is constrained by the condition that the mean number of photons $\bar{n}(t)$ be much smaller than J. It is easy to check that the solution of (4) and (5) is

$$a(n, t) = \binom{J - M + n}{n}^{\frac{1}{2}} (\text{tgh}\,(J + M)^{\frac{1}{2}} Kt)^n (\text{sech}\,(J + M)^{\frac{1}{2}} Kt)^{J-M+1} \tag{6}$$

and for the probability distribution

$$p(n, t) = \binom{J - M + n}{n} (\text{tgh}^2\,(J + M)^{\frac{1}{2}} Kt)^n (\text{sech}^2\,(J + M)^{\frac{1}{2}} Kt)^{J-M+1}. \tag{6a}$$

In the following we shall be interested in the two limiting cases $M = J$ and $M = 0$.

([3]) For a complete discussion of the relevance of Hamiltonian (2) to «parametric» processes see N. BLOEMBERGER: *Nonlinear Optics* (New York, 1965).

For $M = J$:

$$p(n, t) = \left(\frac{\bar{n}(t)}{1 + \bar{n}(t)}\right)^n \frac{1}{1 + \bar{n}(t)}, \tag{7}$$

where

$$\bar{n}(t) = \sinh^2 (2J)^{\frac{1}{2}} Kt, \tag{8}$$

which is obviously a chaotic distribution with mean photon number given by $\bar{n}(t)$ ([4]); its dispersion is

$$(\Delta n)^2 = \bar{n}(t)[1 + \bar{n}(t)] \tag{9}$$

and its range of validity is for $\bar{n}(t) \ll 2J$, and therefore up to

$$Kt \simeq \frac{1}{(2J)^{\frac{1}{2}}};$$

for $M = 0$:

$$p(n, t) \simeq \exp[-\bar{n}(t)] \frac{(\bar{n}(t))^n}{n!} \tag{10}$$

is a Poisson distribution where

$$\bar{n}(t) = J \operatorname{tgh}^2 (J)^{\frac{1}{2}} Kt \tag{11}$$

and

$$(\Delta n)^2 = \bar{n}(t)$$

and is valid for

$$Kt \ll \frac{1}{J^{\frac{1}{2}}}.$$

We now study (4) and (5) when the previous constraints are not met $\left(i.e.\ t > 1/(2J)^{\frac{1}{2}}\right)$. By defining a new amplitude

$$A(n, t) = [G(n)]^{\frac{1}{2}} a(n, t), \tag{12}$$

where

$$G(n) = \frac{F(n) F(J - M + n) F(J + M - n)}{F(n-1) F(J - M + n - 1) F(J + M - n - 1)}$$

and

$$F(n) = 2^{n/2} \Gamma(n/2 + 1),$$

we derive from (5)

$$\frac{\partial A}{\partial t}(n, t) = KG(n)[A(n-1, t) - A(n+1, t)]. \tag{13}$$

We notice that a good approximation for $G(n)$ is for all n

$$G(n) \simeq [(1 + n)(J - M + n + 1)(J + M - n + 1)]^{\frac{1}{2}}, \tag{14}$$

([4]) This solution has been found in a discussion of the parametric amplifier by B. R. MOLLOW and R. J. GLAUBER: *Phys. Rev.*, **160**, 1076, 1097 (1967).

where use has been made of

$$\frac{\Gamma(n/2+1)}{\Gamma(n/2+\frac{1}{2})} \simeq \left(\frac{n+1}{2}\right)^{\frac{1}{2}} .$$

Instructed by the nature of the classical solution of the problem we set

$$n = (J+M)\sin^2\theta_n \tag{15}$$

and we notice that a variation of n by one unit gives us a very small variation of θ,

$$\Delta\theta_n = \theta_{n+1} - \theta_{n-1} \simeq \left[\frac{1}{(n+1)(J+M-n+1)}\right]^{\frac{1}{2}} , \tag{16}$$

thus hinting to the possibility of a transition to the continuous variable θ. We now expand the r.h.s. of (13) in a Taylor series in $\Delta\theta_n$; we have

$$A(n-1,t) - A(n+1,t) \simeq -K\left[\left(\frac{\partial A}{\partial\theta}\right)_{\theta_n}(\Delta\theta_n) + \frac{1}{2}\left(\frac{\partial^2 A}{\partial\theta^2}\right)_{\theta_n}(\Delta\theta_n)^2 + \dots\right] \tag{17}$$

and we are willing to neglect derivatives of order higher than 1. This is possible as long as $\partial A/\partial\theta$ does not vary considerably in the small interval $\Delta\theta_n$; this condition is certainly met if we wait a time t^* such that $\bar{n}(t^*) \gg 1$, still being $\bar{n}(t^*) \ll J$.

Using eqs. (15)-(17) we get the partial differential equation

$$\frac{\partial A}{\partial t}(\theta,t) + K[(J-M+1) + (J+M)\sin^2\theta]^{\frac{1}{2}}\frac{\partial A}{\partial\theta}(\theta,t) = 0 \tag{18}$$

and going over to the new variable

$$u = \int_0^\theta \mathrm{d}\varphi \frac{1}{[J-M+1+(J+M)\sin^2\varphi]^{\frac{1}{2}}} \tag{19}$$

we write (18) as

$$\frac{\partial A}{\partial t} + K\frac{\partial A}{\partial u} = 0 . \tag{20}$$

As is well known this equation has as its solution

$$A(u,t) = F(u - Kt) ,$$

where $F(u)$ has to be determined by the initial conditions (amplitude distribution for $t = t^*$). From (12) for the probability distribution in n we get

$$p(n,t) = |a(n,t)|^2 = \frac{1}{G(n)} A^2(n,t) \tag{21}$$

and noticing that $1/G(n) = (\Delta u)_n$, we write (21) as

$$p(n, t) = F^2(u_n - Kt)\,\Delta u_n\,. \tag{22}$$

The normalization conditions is (passing to the continuum in u)

$$\sum_{n=0}^{J+M} p(n, t) = \int_{-\infty}^{+\infty} du\, F^2(u - Kt) = 1\,, \tag{23}$$

which is evidently satisfied at all times if

$$\int_{-\infty}^{+\infty} du\, F^2(u) = 1\,.$$

The solution $F^2(u-Kt)$ represents a probability distribution in u which does not deform as the time varies. This, however, does not mean that the probability distribution in n does remain constant, as is clear from the nonlinear relationship between n and u_n (eqs. (15) and (18))

$$n = \frac{(J+M)(J-M+1)}{2J+1}\, sd^2\left([2J+1]^{\frac{1}{2}} u_n \,\middle|\, \frac{J+M}{2J+1}\right), \tag{24}$$

where sd is one of the Jacobi elliptic functions ([5]).

From eq. (22) the K-th moment of the photon distribution at a certain time t can be calculated as

$$\langle n^k(t)\rangle = \sum_{n=0}^{J+M} n^k p(n, t) = \int_{-\infty}^{+\infty} du\, n^k(u) F^2(u - Kt)\,, \tag{25}$$

where $n(u)$ is given by eq. (24).

An exact calculation of the properties of the photon distribution is really quite complicated to do in practice, however a simple approximate result for the mean number of photons is obtained by setting in eq. (24) $u = Kt$ ([6]), *i.e.*

$$\bar{n}(t) = \frac{(J+M)(J-M+1)}{2J+1}\, sd^2\left(k[2J+1]^{\frac{1}{2}} t \,\middle|\, \frac{J+M}{2J+1}\right). \tag{26}$$

From this we gather that $\bar{n}(t)$ is a periodic function of the time, with period

$$T = \frac{1}{K(2J+1)^{\frac{1}{2}}}\left(\pi + \ln\frac{2J+1}{J-M+1}\right). \tag{27}$$

([5]) See M. ABRAMOVITZ and I. STEGUN: *Handbook of Mathematical Functions* (New York).

([6]) This result would be exact if $F^2(u-tk)$ were a very sharp function, in the limit $\delta(u-tk)$. In the actual situation $F^2(u-tk)$ is never a δ-function, but its shape is determined by the short-time solutions (6) and (6a). The effect of this is essentially to multiply (26) by $f(t)$ which is almost constant and approximately equal to 1, having its minimum for $t = T/2$, and being a few percent different from 1 in the worst case $M = J$.

We now discuss formulae (26) and (27) in the two limiting cases $M = J$ and $M = 0$.

For $M = J$, *i.e.* when all atoms are initially excited, the *sd* function has period

$$T = \frac{1}{K(2J+1)^{\frac{1}{2}}}\left[\pi + \ln(2J+1)\right] \tag{28}$$

and its shape around its maximum (25), which occurs at $T/2$, is very well approximated by the function [5]

$$\overline{n}(t) = 2J \operatorname{sech}^2 K(2J+1)^{\frac{1}{2}}(t - T/2)\,. \tag{29}$$

We note that the ratio between the duration of a single spike and the repetition time T is given by

$$R = \frac{2}{\ln(2J+1) + \pi} \ll 1\,.$$

For $M = 0$, the so-called superradiant state, the period is shorter, namely

$$T = \frac{1}{K(2J+1)^{\frac{1}{2}}}\left[\pi + \ln 2\right] \tag{30}$$

and the shape of $\overline{n}(t)$ is much smoother [5] and $R \sim 1$.

We turn to the dispersion of the photon distribution. Again the use of (25) to evaluate $(\Delta n)^2 = \langle n^2(t)\rangle - \langle n(t)\rangle^2$ is in practice quite arduous, but we can get a fairly good approximate estimate from the formula

$$\Delta n = \left(\frac{\Delta n}{\Delta u}\right)_{\overline{n}} (\Delta u)_0 = G(\overline{n})(\Delta u)_0\,. \tag{31}$$

This formula needs at least two comments. First it cannot give accurate results when $n(u)$, as given by (24), varies too dramatically in the neighbourhood of $u = tK$ as u varies by $(\Delta u)_0$. Second, $(\Delta u)_0$ is the constant dispersion in the u-distribution to be calculated from the requirement that the short-time solution be joined to the solution of the partial differential equation (20), *i.e.*

$$(\Delta u)_0 = (\Delta n)_{t=t^*} \frac{1}{G(n^*)}\,. \tag{32}$$

For $M = J$, we have from (31), (32), (7), (8) and (9),

$$(\Delta u)_0 = \frac{1}{(2J+1)^{\frac{1}{2}}} \tag{33}$$

and

$$\Delta n \simeq \overline{n}(t)\left(1 - \frac{\overline{n}(t)}{2J}\right)^{\frac{1}{2}} \tag{34}$$

with the warning that, according to the first comment after eq. (31), the factor in the square root should be replaced by a nonzero one for $\bar{n}(t)$ extremely close to its maximum value $2J$.

The essential feature of (34) is that the dispersion of the photon distribution tends to conserve the chaotic nature of the initial distribution appropriate to purely spontaneous emission.

For $M=0$, eqs. (31), (32), (9) and (10) lead to

$$\Delta n = (\bar{n}(t))^{\frac{1}{2}} \left(1 - \frac{n(t)^2}{J^2}\right)^{\frac{1}{2}}, \tag{35}$$

which is similar but not quite the same as the dispersion of a binomial distribution. For $\bar{n}(t) \ll J$ it is hardly distinguishable from a Poisson distribution, and the state of the radiation field is coherent [7]. This is exactly what one would have expected after observing that in such a range of $\bar{n}$ the source of the radiation field (*i.e.* the transverse component of the total angular momentum) is essentially classical. A more complete description of this work shall be given in a separate paper.

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We want to thank Prof. R. J. GLAUBER for useful discussions and a reading of the manuscript.

([7]) R. J. GLAUBER: *Phys. Rev.*, **130**, 2529 (1963).