LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

A

Sequent Correlations in Stochastic Point Processes - II.

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(ricevuto il 21 Maggio 1966)

In a previous contribution to this journal [1] (referred to as paper I) the idea of sequent correlations in evolutionary stochastic point processes has been introduced with special reference to the cascade theory of cosmic-ray showers. If x is the continuous parameter over which the « particles » are distributed and t the parameter with respect to which the « process » under consideration evolves, then the central quantity of interest is the random variable $\mathrm{d}N(x, t)$ denoting the number of particles with parametric values between x and $x+\mathrm{d}x$ at t. As the process evolves, the particles undergo changes in their associated parametric values and hence the idea of the primitive parametric value and the consequent random variable $\mathrm{d}M(x, t)$ have been useful in the description of electromagnetic cascades as has been amply demonstrated in references [1] and [2]. Such correlation functions have been found to be very useful in the formulation of kinetic theory in terms of generalized Boltzmann equations [3,4] and also in the theory of random noise [5]. In this latter we wish to point out the existence of another type of sequent correlation and the utility of the characteristic functional in the description of physical phenomena.

Though the notion of the characteristic functional is well known, its utility in point processes has been recognized only recently (see for example HARRIS [6]). As point processes, we can define a family of characteristic functionals which will generate different types of product densities and sequent correlations. Let us con-

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(1) S. K. SRINIVASAN and K. S. S. IYER: *Nuovo Cimento*, **33**, 273 (1964).
(2) S. K. SRINIVASAN and K. S. S. IYER: *Zeit. Phys.*, **182**, 243 (1965).
(3) S. K. SRINIVASAN: *Zeit. Phys.* (1966) (in press).
(4) S. K. SRINIVASAV: *A novel approach to the kinetic theory offluids, II. Onset of turbulent motion.* I.I.T. Mathematics Report No. 15, April 1966.
(5) P. I. KUZNESTSON, R. L. STRATONOVICH and V. I. TIKHONOV: *Nonlinear Transformations of Stochastic Processes*, Chap. I (London, 1965).
(6) T. E. HARRIS: *Theory of Branching Processes* (Berlin, 1963).

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sider Θ defined by

$$\Theta[q, t] = \mathscr{E}\left\{\exp\left[i\int_x \mathrm{d}N(x, t) q(x)\right]\right\}, \tag{1}$$

where $\mathscr{E}$ denotes the expectation value of the quantity inside the curly brackets. Using the properties of $\mathrm{d}N(x, t)$ as explained in ref. (1), we can rewrite (1) as

$$\Theta[q, t] = 1 + \sum_{k=1}^{\infty} \frac{(i)^k}{k!} \sum_{h=1}^{k} \sum_{i} C^k_{h(i)} \int\int \dots \int f_h(x_1, x_2, \dots, x_h, t) \cdot \\ \cdot [q(x_1)]^{l_1} [q(x_2)]^{l_2} \dots [q(x_h)]^{l_h} \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_h , \tag{2}$$

where f_h is the product density of degree h. $C^k_{h(i)}$ is the partition coefficient (see ref. (7)) denoting the number of ways in which the « complexion » characterized by the set of numbers $l_1, l_2, \dots, l_h$ $(l_1 + l_2 + \dots + l_h = k)$ can arise. The symbol i denotes a typical complexion, the summation over i being understood to mean the sum over all possible complexions. From (2) it follows

$$(-i)^h \left. \frac{\delta^h \Theta}{\delta q(x_1)\, \delta q(x_2) \dots \delta q(x_h)} \right|_{q=0} = f_h(x_1, x_2, \dots, x_h, t) . \tag{3}$$

a result which projects the importance of the characteristic functional in point processes.

Let us deal with a simple example of the distribution of random points on a line of extension $(0, t)$. If we associate with each random point occurring at x a deterministic function $g(x)$, then the distribution of the random variable defined by

$$y(t) = \sum_i g(x_i) = \int_x g(x)\, \mathrm{d}N(x, t) \tag{4}$$

can be obtained if we are in possession of Θ. In fact if $h(\lambda)$ is the moment-generating function of $y(t)$ in the sense that

$$\mathscr{E}[[y(t)]^n] = (i)^n \left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^n h(\lambda)\bigg|_{\lambda=0} , \tag{5}$$

then

$$h(\lambda) = \Theta[\lambda g, t] . \tag{6}$$

A formula like (6) is useful in kinetic theory if we adopt the Gibbsian grand-ensemble approach and identify the random points with molecules and x with the phase space (see for example references (3) and (4)).

Likewise the sequent product density introduced by RAMAKRISHNAN and RADHA (8)

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(7) A. RAMARISHNAN: *Proc. Camb. Soc.*, **49**, 473 (1953).
(8) A. RAMARKRISHNAN and T. K. RADHA: *Proc. Camb. Phil. Soc.*, **57**, 843 (1961).

can be generated by the characteristic functional Ψ defined by

$$\Psi[\theta, t_1, t_2] = \mathscr{E}\left\{\exp\left[i\int_{x_1}\int_{x_2} \mathrm{d}N(x_1, t_1)\,\mathrm{d}N(x_2, t_2)\,\theta(x_1, x_2)\right]\right\}, \tag{7}$$

the functional derivatives of Ψ with respect to θ at $\theta=0$ yielding the desired sequent densities. These densities are useful in cascade theory and the relevant correlations in t can be obtained by the method given above.

If however we are interested in the sequent product densities of particles that are created at a certain t_1 and particles that are found at t_2, it is clear that we have to deal with the products of the type $\mathrm{d}\mathscr{M}(x_1, t_1)\,\mathrm{d}N(x_2, t_2)$, where $\mathrm{d}\mathscr{M}(x_1, t_1)$ defined over the product space of x_1 and t_1 denotes the number of particles produced between t_1 and $t_1+\mathrm{d}t_1$, $\mathrm{d}N(x_2, t_2)$ being of course defined a susual over the x_2 space. In this case, it is convenient to define the double characteristic functional by

$$\Phi[\theta, \chi; t_1, t_2] = \mathscr{E}\left\{\exp\left[i\int_{x_1}\int_{x_2} \mathrm{d}\mathscr{M}(x_1, t_1)\,\mathrm{d}N(x_2, t_2)\,\theta(x_1)\,\chi(x_2)\right]\right\}, \tag{8}$$

where the integration under the exponential is to be performed over the variables x_1 and x_2. An interesting analogue involving the sequent product densities of particles produced at certain values of t and of particles that exist at certain other values of t can be derived.

In paper I, we have dealt with the random variable $\mathrm{d}M(x_1, t_1, x_2, t_2)$ representing the number of particles that are produced between t_1 and $t_1+\mathrm{d}t_1$ with primitive parametric values between x_1 and $x_1+\mathrm{d}x_1$ and have a parametric value between x_2 and $x_2+\mathrm{d}x_2$ at t_2 $(t_2>t_1)$. Therein we have also discussed a method of arriving at such a sequent correlation by the use of invariant embedding technique of BELLMAN *et al.* [9]. In this note, we shall show how we can obtain differential equations for the characteristic functional by the invariant embedding technique by dealing with the correlations in electromagnetic cascades using the same approximation as in Sect. 4 of paper I. Let us take $\Phi^i[\theta, \chi; t_1, t_2|E_0]$ as the characteristic functional corresponding to a primary of i-th type ($i=1$ and 2 denote electron and photon respectively) of energy E_0. To be precise, we embed the process corresponding to (t_1, t_2) into a class of processes corresponding to $(t_1-\Delta, t_2-\Delta)$ and let Δ tend to zero. Using simple probability arguments as in paper I, we obtain

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}\right)\Phi^i(\theta, \chi; t_1, t_2|E_0) = -\int R^i(E'|E_0)\,\Phi^i[\theta, \chi; t_1, t_2|E_0]\,\mathrm{d}E' + \int R^i(E'|E_0)\,\Phi^{3-i}[\theta, \chi; t_1, t_2|E_0-E']\,\Phi^1[\theta, \chi; t_1, t_2|E']\,\mathrm{d}E', \tag{9}$$

where $R^i(E'|E_0)$ is defined exactly in the same way as in paper I. It is interesting to note that eqs. (9) are valid for both the cases $t_1>t_2$ and $t_1<t_2$ under properly chosen initial conditions. When $t_1<t_2$ the initial conditions imposed on (9) are

(9) R. E. BELLMAN, R. KALABA and G. M. WING: *Journ. Math. Phys.*, **1**, 280 (1960).

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given by

$$(10)\qquad \begin{cases} \Phi^1[\theta, \chi; 0, t_2|E_0] = 0\,, \\ \Phi^2[\theta, \chi; 0, t_2|E_0] = \int R^2(E'|E_0)\exp[\Theta(E')]\Theta^1[\chi; t_2|E']\,\Theta^1[\chi; t_2|E_0 - E']\,\mathrm{d}E'\,, \end{cases}$$

while for $t_2 < t_1$ the conditions are given by

$$(11)\qquad \begin{cases} \Phi^1[\theta, \chi; t_1, 0|E_0] = \exp[\chi(E_0)]\Psi^1[\theta; t_1|E_0]\,, \\ \Phi^2[\theta, \chi; t_1, 0|E_0] = 0\,, \end{cases}$$

where Θ^1 is the characteristic functional of electrons that are found at t_2 while Ψ^1 is the characteristic functional of electrons that are produced between t_1 and $t_1 + \mathrm{d}t_1$. Differential equations for $F^i(E_1, t_1; E_2, t_2|E_0)$ defined by

$$(12)\qquad F^i(E_1, t_1; E_2, t_2|E_0)\,\mathrm{d}E_1\mathrm{d}t_1\mathrm{d}E_2 = \mathscr{E}\{\mathrm{d}\mathscr{M}(E_1, t_1)\,\mathrm{d}N(E_2, t_2)\}$$

can be obtained by functionally differentiating Φ^i with respect to $\theta(E_1)$ and $\chi(E_2)$ at $\theta = \chi = 0$ and solved by Mellin transform technique. Explicit Mellin transform solutions for F^i have been obtained and we do not propose to present them in this letter as they can be readily derived by following the procedure outlined in Sect. **4** of paper I.

Finally we wish to remark that sequent product densities of the type defined by (7) are useful in dealing with two-time correlations in a plasma. However in this case, the embedding technique is not applicable since the process is not regenerative in character. Nevertheless it is possible to deal with Ψ by analysing the events that occur in an infinitesimal interval following t_1.

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Two of us (S.K.S. and R.V.) have great pleasure in acknowledging many useful discussions with Prof. A. Ramakrishnan particularly on the properties of the characteristic functional.