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CONSISTENCY THEORY FOR SM-METHODS

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Dedicated to Professor Károly Tandori on the occasion of his 70th birthday

B. Przybylski [13] proved that the so-called iteration product of two regular matrices is equivalent in a certain sense to summability methods defined by a sequence of matrices which are called SM-methods. Thereby he used for double sequences the following definition of convergence: A double sequence (written as an infinite matrix) is called convergent to a scalar α, if each column is convergent and the sequence of their limits converges to α. Among other results he proved that the domain of SM-methods is a separable FK-space and that in case of regular SM-methods a bounded consistency theorem holds if the methods fulfill certain additional assumptions. The question arises whether one may use gliding hump techniques for the development of consistency theorems on the base of theorems of Mazur–Orlicz type proved in several papers of the first and second author using factor sequence methods originally due to G. M. Petersen (cf. the proof of the bounded consistency theorem in [11]) and the third author in his very early paper [25]. Partially the answer is positive under a restriction: Up to now, gliding hump methods in consistency theory are based on the consideration of the sections of sequences and of weak sectional convergence and — as we demonstrate by a very simple example — the domain of an SM-method may be large whereas the set of all members of the domain being weakly sectionally convergent is small. Furthermore, there are situations in consistency investigations where the domain is not fat enough in the sense that for a given sequence there does not exist sufficiently many factor sequences of it in the domain. A way out of these difficulties is to consider for double sequences a more general notion of convergence which leads to SM-methods which are well-known and called 'Einschachtelungsverfahren' in German (see [27]).

In Section 2 we introduce some partly known double sequence spaces and equip some of them with FH-topologies where $H := \Omega$ is the space of all double sequences. In particular, we consider the space $\mathcal{C}_c$, $\mathcal{C}_{uc}$, $\widehat{\mathcal{C}}$ and $\mathcal{C}_p$ of all double sequences being convergent in the sense mentioned at first, being uniformly convergent to a scalar, being convergent in the mentioned generalized sense and being convergent in the sense of A. Pringsheim, respectively. In Section 3 we recall the definition of the iteration product of two matrix methods and we introduce SM-methods in a more general setting as it was done by B. Przybylski. For example, this definition contains as special cases the SM-methods considered by B. Przybylski and those examined by M. Stieglitz, in particular, almost convergence due to G. G. Lorentz and invariant means considered by P. A. Raimi [16] and several other authors. Furthermore, as a stronger method then the SM-method considered by B. Przybylski we get the mentioned 'Einschachtelungsverfahren' which play an important rôle in the last section of the paper. In the important cases of SM-methods the domain $\mathcal{C}_{\mathcal{A}}$ proves to be an FK-space, however, contrary to the result of B. Przybylski, in case of domains $\mathcal{C}_{\mathcal{A}}$ it is not necessarily separable. At the end of the section we consider the set $W_{\mathcal{A}}$ of all members of the domain $\widehat{\mathcal{C}}_{\mathcal{A}}$ being weakly sectionally convergent where $\mathcal{A}$ is an SM-method. Motivated by an example which shows that $W_{\mathcal{A}}$ may be very small (for example equal to c_0) whereas the domain is big (for example equal to m), we introduce a bigger subspace $V_{\mathcal{A}}$ which proves to be suitable for consistency considerations in Section 4.

The main results of the paper are contained in Section 4: In the first part of Section 4 we prove that $V_{\mathcal{A}}$ has an oscillating property, the ABSOLUTE SP_OSCP, which allows the application of a theorem of Mazur–Orlicz type [4] on $V_{\mathcal{A}} \cap Y$ where Y is a sequence space having a gliding hump property, namely SIGNED P_GHP. But, since $\widehat{\mathcal{C}}_{\mathcal{A}}$ is not separable in general, we cannot use the mentioned theorem of Mazur–Orlicz type to get consistency theorems for SM-methods (but, for example, only for an SM-method and a stronger matrix method). To get general consistency theorems we follow a similar way as in [4]: We prove a non-summability theorem and deduce the desired theorem of Mazur–Orlicz type and consistency theorems.

Closing the introduction we should remark that, in case of matrix methods, this non-summability theorem corresponds to [25, Satz 3.2 (non-summability statement)] and the mentioned oscillating property of $W_{\mathcal{A}}$ and $V_{\mathcal{A}}$ is a summability statement which corresponds to [25, Satz 3.2 (summability statement)].

1. Notation and preliminaries

Let ω denote the linear space of all scalar (real or complex) sequences. By a sequence space E we shall mean any linear subspace of ω. A sequence space E with a locally convex topology τ is called a K-space if the inclusion map $i : (E, \tau) \to \omega$ is continuous where ω has the topology τ_ω of coordinatewise convergence. A K-space with a Fréchet topology is called an FK-space. If, in addition, the topology is normable then it is called a BK-space. More general, if H is a topological vector (Hausdorff) space then an FH-space is a locally convex Fréchet space E such that E is a linear subspace of H and the inclusion map $i : E \longrightarrow H$ is continuous. We will assume throughout familiarity with the standard FH- and FK-spaces and their natural topologies as well as the properties enjoyed by these spaces (see e.g., [23], [7]). We refer the reader to [5] for a discussion of L_φ-spaces, a class of K-spaces which includes, as a proper subset, the class of separable FK-spaces and domains of operator valued matrices.

For a sequence space the β-dual of E is given by

$$E^\beta = \left\{ x \in \omega \,\Big|\, \sum_k x_k y_k \quad \text{converges for each } y \in E \right\}.$$

Let e denote the sequence of ones and let $e^k = (\delta_{jk})_{j=1}^\infty$ be the k^{th} coordinate vector. For $x \in \omega$, $n \in \mathbf{N}$ the n^{th} section of x is

$$P_n(x) = \sum_{k=1}^n x_k e^k.$$

If (E, F) is a dual pair then $\sigma(E, F)$, $\tau(E, F)$ denotes the weak topology and the Mackey topology respectively. For a sequence space E and a linear subspace F of E^β, (E, F) is a dual pair under the natural bilinear form

$$\langle x, y \rangle = \sum_k x_k y_k.$$

If E is a K-space containing φ, the space of finitely non-zero sequences, we let

$$W_E = \left\{ x \in E \,\big|\, P_n(x) \longrightarrow x\big(\sigma(E, E')\big) \right\},$$

$$B_E = \left\{ x \in E \,\big|\, \left\{ P_n(x) \,|\, n \in \mathbf{N} \right\} \text{ is bounded in } E \right\}$$

where E' denotes the topological dual of E.

If $A = (a_{nk})$ is an infinite matrix with scalar entries the application domain

$$\omega_A = \left\{ x \in \omega \,\Big|\, \sum_k a_{nk} x_k \text{ converges for each } n \in \mathbf{N} \right\}$$

and the domain

$$E_A = \left\{ x \in \omega_A \,\Big|\, Ax = \left(\sum_k a_{nk}\, x_k \right)_{n=1}^\infty \in E \right\}$$

with respect to an FK-space E admit a natural FK-topology (see [23]). For $x \in c_A$ we write $\lim_A x = \lim Ax$.

DEFINITION AND REMARK 1.1. Let $A := (E, L)$ be a (linear) summability method, that is, E is a subspace of ω and L is a linear functional on E. If $\varphi \subset E$ then we put $a_k := L(e^k)$ $(k \in \mathbf{N})$ and use the notations

$$I_A := \left\{ x = (x_k) \in E \,\middle|\, x \in \left\{ (a_k) \right\}^{\beta} \right\},$$

$$\Lambda_A : I_A \longrightarrow \mathbf{K},\ x \longrightarrow L(x) - \sum_k a_k x_k \quad \text{where} \quad \mathbf{K} \in \{\mathbf{R}, \mathbf{C}\},$$

$$\Lambda_A^{\perp} := \left\{ x \in I_A \mid \Lambda_A(x) = 0 \right\}.$$

Note, in case of matrix methods A the set I_A is called *inset* and was introduced by Wilansky [22]; he proved $W_{c_A} = B_{c_A} \cap \Lambda_A^{\perp}$ which is leading to the considerations in Section 4.

For the sake of completeness we take over the following definitions from [4].

A sequence $\left(y^{(j)}\right)$ in ω is called a *block sequence* if there exists an index sequence (γ_j) with $\gamma_1 = 1$ and $y_k^{(j)} = 0$ for $k < \gamma_j$ and $\gamma_{j+1} \leqq k$ $(j \in \mathbf{N})$. Therefore, each $y^{(j)}$ has the representation $y^{(j)} = \sum_{k=\gamma_j}^{\gamma_{j+1}-1} y_k^{(j)} e^k$ and the *coordinatewise sum* $y = \sum_j y^{(j)}$ is the limit of this series in (ω, τ_ω) and is given by $y_k = y_k^{(j)}$ for $\gamma_j \leqq k < \gamma_{j+1}$ and $j \in \mathbf{N}$.

A sequence $\left(y^{(j)}\right)$ in ω is called a *step 1-block sequence with respect to an index sequence* (k_i) *with* $k_1 = 1$ if there exists an increasing sequence (i_j) in $\mathbf{N}$ such that:

(α) $i_{3j-1} < i_{3j}$ $(j \in \mathbf{N})$,

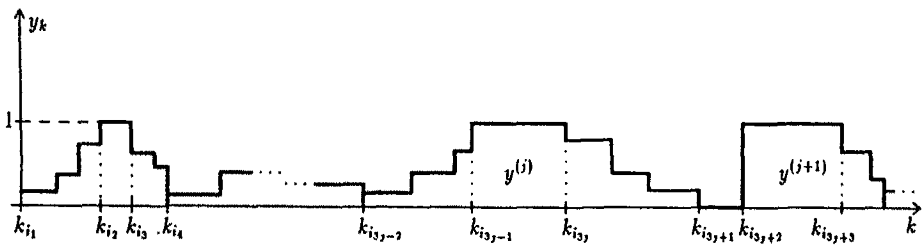
(β) $y_k^{(j)} = \begin{cases} 0 & \text{if } k < k_{i_{3j-2}} \text{ or } k \geqq k_{i_{3j+1}} \\ 1 & \text{if } k_{i_{3j-1}} \leqq k < k_{i_{3j}} \end{cases}$ $(j \in \mathbf{N})$ and $y_1^{(1)} \geqq 0$,

(γ) $\left(y_k^{(j)}\right)$ is monotonic for $k_{i_{3j-2}} - 1 \leqq k \leqq k_{i_{3j-1}}$ and for $k_{i_{3j}} - 1 \leqq k \leqq k_{i_{3j+1}}$ $(j \in \mathbf{N})$,

(δ) $y = \sum_j y^{(j)}$ (coordinatewise sum) is constant for $k_i \leqq k < k_{i+1}$ $(i \in \mathbf{N})$.

Furthermore, it is called a *step 1-block sequence* if there exists an index sequence (k_i) such that $\left(y^{(j)}\right)$ is a step 1-block sequence with respect to (k_i).

In particular, any step 1-block sequence $\left(y^{(j)}\right)$ fulfills $0 \leqq y_k^{(j)} \leqq 1$ $(j \in \mathbf{N})$ and $\sup_j \left\| y^{(j)} \right\|_{bv} = 2$. The definition of a step 1-block sequence is illustrated by the following picture.



In the following definitions the convergence of the sums is coordinatewise.

DEFINITION 1.2. A sequence space E containing φ has the *signed pointwise gliding hump property* (SIGNED P_GHP) if for each $x \in E$ and every block sequence $(y^{(j)})$ with $\sup_j \|y^{(j)}\|_{bv} < \infty$ there exist a subsequence $(y^{(j_k)})$ of $(y^{(j)})$ and a sequence (h_k) in $\{1, -1\}$ such that $yx \in E$ where $y := \sum_k h_k y^{(j_k)}$.

For convenience we use in the following the notion of a *strong subsequence* $(y^{(j_k)})$ of a sequence $(y^{(j)})$ in the sense that $\mathbf{N} \setminus \{ j_k \mid k \in \mathbf{N}\}$ is an infinite subset of $\mathbf{N}$.

DEFINITION 1.3. Let E be a sequence space containing φ. E is defined to have the *signed pointwise oscillating property* (SIGNED P_OSCP) if for each $x \in E$ and any index sequence (k_i) with $k_1 = 1$ there exists a strong subsequence $(y^{(\nu_j)})$ of a step 1-block sequence $(y^{(\nu)})$ with respect to (k_i) such that there exists a sequence (h_j) in $\{1, -1\}$ with $yx \in E$ where $y := \sum_j h_j y^{(\nu_j)}$.

Furthermore, E has the *absolute strong pointwise oscillating property* (ABSOLUTE SP_OSCP) if in the definition of the SIGNED P_OSCP we may choose $(y^{(\nu)})$ such that for each subsequence $(y^{(\nu_j)})$ of it and each sequence (h_j) in $\{-1, 1\}$ we get $yx \in E$ where $y := \sum_j h_j y^{(\nu_j)}$.

2. Some double sequence spaces

We start this section by introducing several partly known double sequence spaces.

$$\Omega := \{ x = (x_{n\nu}) \mid \forall\, n, \nu \in \mathbf{N} : x_{n\nu} \in \mathbf{K} \} \quad \textit{(set of all double sequences)},$$

$$\mathcal{M} := \{ x \in \Omega \mid \forall\, \nu \in \mathbf{N} : (x_{n\nu})_n \in m \},$$

that $\mathcal{C}_r$ and $\mathcal{M}_u \cap \mathcal{C}_p$ endowed with $\| \ \|_u$ are Banach-spaces is already stated in [10, Theorem 1].

THEOREM 2.3. (a) *$\widehat{\mathcal{M}}$ is a non-separable FH-space ($H := \Omega$) and the FH-topology $\widehat{\tau}$ of $\widehat{\mathcal{M}}$ is generated by the seminorms q_ν ($\nu \in \mathbf{N}$) and q where q_ν is defined in 2.2 and*

$$q(x) := \Big\| \big(\limsup_n |x_{n\nu}| \big)_\nu \Big\|_\infty \qquad \big(x = (x_{n\nu}) \in \widehat{\mathcal{M}}\big).$$

(b) *$\widehat{\mathcal{C}}$ is a closed non-separable subspace of $\widehat{\mathcal{M}}$ and both $\mathcal{C}_c$ and $\mathcal{C}_{c_0}$ are closed subspaces of the FH-space $\widehat{\mathcal{C}}$.*

(c) *The FH-space ($H := \Omega$) $\mathcal{C}_{c_0}$ has AK (in the sense of the sections defined in (1) that is, each $x = (x_{n\nu}) \in \mathcal{C}_{c_0}$ has the representation*

$$(2) \qquad x = \lim_r \widehat{x}^{[r]} = \sum_{\nu=1}^{\infty} \lim_n x_{n\nu} e^{\nu} + \lim_r \sum_{n,\nu=1}^{r} \big(x_{n\nu} - \lim_\mu x_{\mu\nu} \big) e^{n\nu}.$$

Consequently, both $\mathcal{C}_{c_0}$ and $\mathcal{C}_c$ are separable FH-spaces ($H := \Omega$).

PROOF. The statement that $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{C}}$ are not separable may be proved with the same argument as in Remark 2.2.

(a) For a proof of the statement that $(\widehat{\mathcal{M}}, \widehat{\tau})$ is an FH-space we may argue quite similar to [26, 2.1 and 2.1a]. To that end we consider the FH-space $\mathcal{M}$ and the seminorms

$$\chi_\nu : \mathcal{M} \longrightarrow \mathbf{R},\ (x_{n\nu}) \longrightarrow \limsup_n |x_{n\nu}| \qquad (\nu \in \mathbf{N}).$$

Obviously, χ_ν ($\nu \in \mathbf{N}$) is continuous, thus

$$q : \mathcal{M} \longrightarrow \mathbf{R},\ x = (x_{n\nu}) \longrightarrow \sup_\nu \chi_\nu(x) = \sup_\nu \limsup_n |x_{n\nu}|$$

is semicontinuous and thus

$$\widehat{\mathcal{M}} = \big\{ x \in \mathcal{M} \bigm| q(x) < \infty \big\}$$

is an FH-space.

(b) Now we are going to prove the closedness of $\widehat{\mathcal{C}}$ in $(\widehat{\mathcal{M}}, \widehat{\tau})$. To that end let $\big(x^{(r)}\big)$ be a sequence in $\widehat{\mathcal{C}}$ converging to an $x \in \widehat{\mathcal{M}}$. It is easy to check that the functional

$$\widehat{\mathcal{C}}\text{-lim} : \widehat{\mathcal{C}} \longrightarrow \mathbf{K} : x \longrightarrow a_x$$

is continuous on $(\widehat{\mathcal{C}},\widehat{\tau})$. Thus there exists an $a_x \in \mathbf{K}$ with $a_x = \lim_{r\to\infty} \widehat{\mathcal{C}}\text{-}\lim x^{(r)} = \lim_{r\to\infty} a_{x^{(r)}}$. However, using

$$q\left(x - x^{(r)}\right) \overset{r\to\infty}{\longrightarrow} 0, \quad \limsup_n \left|x_{n\nu}^{(r)} - a_{x^{(r)}}\right| \overset{\nu\to\infty}{\longrightarrow} 0 \quad (r \in \mathbf{N})$$

$$\text{and} \quad a_x = \lim_{r\to\infty} a_{x^{(r)}}$$

we get $x \in \widehat{\mathcal{C}}$ and $a_x = \widehat{\mathcal{C}}\text{-}\lim x$ by the estimation

$$\limsup_n |x_{n\nu} - a_x| \leqq q\left(x - x^{(r)}\right) + \limsup_n \left|x_{n\nu}^{(r)} - a_{x^{(r)}}\right| + \left|a_{x^{(r)}} - a_x\right|$$

where we choose first of all r sufficiently large such that on the right hand side of the inequality the first and third term is smaller than a given ε and then ν_0 such that the second term is smaller than ε for any $\nu \geqq \nu_0$.

In the next step we prove the closedness of $\mathcal{C}_c$ in $\widehat{\mathcal{C}}$, therefore in $\widehat{\mathcal{M}}$. If $\left(x^{(r)}\right)$ is a sequence in $\mathcal{C}_c$ converging to an $x \in \widehat{\mathcal{C}}$, then for each $\nu \in \mathbf{N}$ the sequence $\left(\left(x_{n\nu}^{(r)}\right)_n\right)_r$ is a Cauchy sequence in $\left(c, \|\ \|_\infty\right)$, thus convergent to a $y_\nu \in c$; therefore $y_\nu = (x_{n\nu})_n$ and consequently $x \in \mathcal{C}$, thus $x \in \mathcal{C}_c = \widehat{\mathcal{C}} \cap \mathcal{C}$. Finally, the closedness of $\mathcal{C}_{c_0}$ in $\mathcal{C}_c$ and therefore in $\widehat{\mathcal{C}}$ comes from the continuity of $\mathcal{C}_c\text{-}\lim : \mathcal{C}_c \longrightarrow \mathbf{K}$ and $\mathcal{C}_{c_0} = \operatorname{Kern}(\mathcal{C}_c\text{-}\lim)$.

(c) The proof of the (modified) AK-property of $\mathcal{C}_{c_0}$ is straightforward, we omit it. The separability of $\mathcal{C}_{c_0}$ is an easy consequence of the AK-property. Since $\mathcal{C}_{c_0}$ has codimension 1 and is closed in $\mathcal{C}_c$, the separability of $\mathcal{C}_{c_0}$ implies the separability of $\mathcal{C}_c$. □

An essential point in the proof of the fact that $\widehat{\mathcal{C}}$ is an FH-space and $\mathcal{M}_u \cap \mathcal{C}_p$ is a Banach space ($H := \Omega$) is the restriction to $\mathcal{M}$ and $\mathcal{M}_u$, respectively, in the definition of those spaces: The (natural) topological structure of $\mathcal{C}_e$ and $\mathcal{C}_p$ is more complicated as the following theorem shows.

THEOREM 2.4. *$\mathcal{C}_e$ and $\mathcal{C}_p$ are LFH-spaces and not one of both is an FH-space ($H := \Omega$).*

(For a given locally convex space H a locally convex space (E,τ) is called LFH-space if E is the union of an increasing sequence of FH-spaces E_μ and τ is the (locally convex) inductive limit topology on E with respect to the inclusion maps from E_μ into E ($\mu \in \mathbf{N}$); LFH-spaces were introduced as IFH-spaces in [3].)

PROOF. Obviously, $\mathcal{C}_e = \bigcup_\mu E_\mu$ where

$$E_\mu := \left\{x = (x_{n\nu}) \in \Omega \,\middle|\, \forall \nu \geqq \mu : (x_{n\nu})_n \in m \text{ and}\right.$$

(i) $\Phi(x) \geqq 0$ for each $x \in m$ with $x_n \geqq 0$ $(n \in \mathbf{N})$,
(ii) $\Phi(e) = 1$,
(iii) $\Phi\big((x_{\sigma(n)})_n\big) = \Phi(x)$ for any $x \in m$.

Then V_σ denotes the space of all $x \in m$ such that all invariant means are equal. Note, if $\sigma(n) := n + 1$ $(n \in \mathbf{N})$ then $f = V_\sigma$. From [16] (see also [17]) we have

$$V_\sigma = \Big\{ x = (x_k) \in m \,\Big|\, \exists\, \alpha_x \in \mathbf{K} : \quad \frac{1}{n}\sum_{j=1}^{n} x_{\sigma^{j-1}(\nu)} \overset{n\to\infty}{\longrightarrow} \alpha_x \text{ uniformly in } \nu \in \mathbf{N} \Big\} \tag{3}$$

and we may easily check that the restriction $x \in m$ is needless, that is

$$V_\sigma = \Big\{ x = (x_k) \in \omega \,\Big|\, \exists\, \alpha_x \in \mathbf{K} : \quad \frac{1}{n}\sum_{j=1}^{n} x_{\sigma^{j-1}(\nu)} \overset{n\to\infty}{\longrightarrow} \alpha_x \text{ uniformly in } \nu \in \mathbf{N} \Big\}.$$

Let $A^{(\nu)} = \big(a_{nk}^{(\nu)}\big)$ be defined by

$$a_{nk}^{(\nu)} := \begin{cases} \frac{1}{n} & \text{if } n \in \mathbf{N} \text{ and } k = \sigma^{j-1}(\nu) \text{ and } j = 1, \dots, n \\ 0 & \text{otherwise} \end{cases} \qquad (\nu \in \mathbf{N}).$$

Then for the SM-method $\mathcal{A} = \big(A^{(\nu)}\big)$ we get the statement

$$V_\sigma = \mathcal{F}_{\mathcal{A}} = (\mathcal{C}_{uc})_{\mathcal{A}} \subsetneqq m \quad \text{and} \quad \mathcal{F}\text{-}\lim_{\mathcal{A}} x = \alpha_x \quad \text{for } x \in V_\sigma \ (\alpha_x \text{ as in (3)}).$$

PROPOSITION 3.3 (B. Przybylski [13]). *Every IPM-method is equivalent to a C_c-SM-method, and every C_c-SM-method is equivalent to an IPM-method. In particular:*

(a) *Let $\Theta = (\vartheta_{n\nu})$ and $\Lambda = (\lambda_{\nu k})$ be infinite matrices and let $\mathcal{A} = \big(A^{(\nu)}\big)$ with $A^{(\nu)} = \big(a_{nk}^{(\nu)}\big)$ be the sequence of matrices defined by*

$$a_{nk}^{(\nu)} := \begin{cases} \lambda_{\nu k} & \textit{for all } \nu \in \mathbf{N} \textit{ and } n = 1 \\ \sum_{\mu=1}^{n-1} \vartheta_{\nu\mu}\lambda_{\mu k} & \textit{for all } \nu \in \mathbf{N} \textit{ and } n \geqq 2 \end{cases} \qquad (k \in \mathbf{N}),$$

Then the $\mathcal{C}_c$-SM-method $\mathcal{A}$ and the IPM-method $\Theta[\Lambda]$ are equivalent, that is $C_{c\mathcal{A}} = c_{\Theta[\Lambda]}$ with consistency.

(b) *Let $\mathcal{A} = \left(A^{(\nu)}\right)$ with $A^{(\nu)} = \left(a_{nk}^{(\nu)}\right)$ be a sequence of infinite matrices, let ρ be a one-to-one map from $\mathbf{N}^2$ to $\mathbf{N}$ such that $\rho(n,k) < \rho(n,k+1)$ for all $n,k \in \mathbf{N}$ and let $\Theta = (\vartheta_{n\nu})$ and $\Lambda = (\lambda_{\nu k})$ be the matrices defined by*

$$(4) \qquad \lambda_{\mu k} := \begin{cases} a_{1k}^{(\nu)} & \text{if } \mu = \rho(\nu,1) \text{ and } \nu \in \mathbf{N} \\ a_{nk}^{(\nu)} - a_{n-1,k}^{(\nu)} & \text{if } \mu = \rho(\nu,n) \text{ and } \nu \in \mathbf{N},\ n \geqq 2 \end{cases} \qquad (k \in \mathbf{N})$$

and

$$(5) \qquad \vartheta_{\nu\mu} := \begin{cases} 0 & \text{if } \mu \neq \rho(\nu,n) \text{ for each } n \in \mathbf{N} \\ 1 & \text{if } \mu = \rho(\nu,n) \text{ for some } n \in \mathbf{N} \end{cases} \qquad (k \in \mathbf{N}).$$

Then the $\mathcal{C}_c$-SM-method $\mathcal{A}$ and the IPM-method $\Theta[\Lambda]$ are equivalent.

REMARK 3.4. There are regular IPM-methods ($\mathcal{C}_c$-SM-methods) such that there exists no b-stronger and consistent regular matrix method (see [13, Theorem 2.2.2] and [14, Remark on page 10]). This fact immediately implies that regular IPM-methods ($\mathcal{C}_c$-SM-methods) are not matrix methods in general. Furthermore, from the introduction of Section 2.2 in [13] we know that regular $\mathcal{C}_c$-SM-methods $\mathcal{A} = \left(A^{(\nu)}\right)$ such that there exists no b-stronger and consistent regular matrix method must necessarily fulfill $\chi\left(A^{(\nu)}\right) = 0$ for each $\nu \in \mathbf{N}$. Consequently, in general a regular IPM-method is not equivalent to a $\mathcal{C}_c$-SM-method $\mathcal{A} = \left(A^{(\nu)}\right)$ where $A^{(\nu)}$ is regular for some $\nu \in \mathbf{N}$.

REMARK 3.5. Let $\mathcal{A} = \left(A^{(\nu)}\right)$ be a $\mathcal{C}_c$-SM-method such that each matrix $A^{(\nu)}$ is regular and $m \cap c_{A^{(i)}} \supset m \cap c_{A^{(i+1)}}$ $(i \in \mathbf{N})$. Then (see [12, Corollary of Theorem 4.3.2]) there exists a regular matrix B such that

$$m \cap c_B = \bigcap_{\nu} \left(m \cap c_{A^{(\nu)}}\right).$$

In particular, from the bounded consistency theorem we get

$$\lim_B x = \lim_{A^{(\nu)}} x \qquad (x \in m \cap c_B \text{ and } \nu \in \mathbf{N}),$$

hence the matrices $A^{(\nu)}$ $(\nu \in \mathbf{N})$ are pairwise consistent on $m \cap c_B$. As a consequence we get

$$m \cap C_{c\mathcal{A}} = \bigcap_{\nu} \left(m \cap c_{A^{(\nu)}}\right) = m \cap c_B.$$

PROOF. (a) Let $f \in (C_{c\mathcal{A}})'$ be given. Then, on account of the seminorms generating the FK-topology (see 3.7(b)), there exists an $r \in \mathbf{N}$, $f_\nu \in c'_{A^{(\nu)}}$ $(\nu = 1, \ldots, r)$ and $g \in c'$ with

$$f(x) = \sum_{\nu=1}^{r} f_\nu(x) + g\big(\big(\lim_{A^{(\nu)}} x\big)_\nu\big) \qquad (x \in C_{c\mathcal{A}}).$$

Using the representation of the continuous linear functionals on the FK-spaces $c_{A^{(\nu)}}$ and c (see e.g., [23]) we get the representation (7). The converse statement is true since it is true in case of matrix domains and since

$$T : C_{c\mathcal{A}} \longrightarrow c,\ x \longrightarrow \big(\lim_{A^{(\nu)}} x\big)_\nu$$

is a continuous linear map where both the domain and the range space carries its FK-topology.

(b) The statement $W_{\widehat{C}_\mathcal{A}} = W_{C_{c\mathcal{A}}}$ is an immediate consequence of $\overline{\varphi} \supset W_{C_{c\mathcal{A}}}$ and the closedness of $C_{c\mathcal{A}}$ in the FK-space $\widehat{C}_\mathcal{A}$. Furthermore, since $\lim_\mathcal{A}$ and $\lim_{A^{(\nu)}}$ are continuous on the FK-space $C_{c\mathcal{A}}$ and since weak sectional convergence implies section boundedness we obviously get $W_{C_{c\mathcal{A}}} \subset B_{C_{c\mathcal{A}}} \cap \Lambda_\mathcal{A}^\perp \cap \bigcap_\nu \Lambda_{A^{(\nu)}}^\perp$. Conversely, let $x \in B_{C_{c\mathcal{A}}} \cap \Lambda_\mathcal{A}^\perp \cap \bigcap_\nu \Lambda_{A^{(\nu)}}^\perp$ be given. Then $x \in B_{c_{A^{(\nu)}}} \cap \Lambda_{A^{(\nu)}}^\perp = W_{c_{A^{(\nu)}}}$ $(\nu \in \mathbf{N})$. Using (7) we get $x \in W_{C_{c\mathcal{A}}}$.

The statement $\Lambda_\mathcal{A}^\perp \cap \bigcap_\nu \Lambda_{A^{(\nu)}}^\perp \subset \Lambda_{A^\bullet}^\perp$ immediately follows by

$$\lim_{A^\bullet} x = \lim_\nu \sum_k a_k^{(\nu)} x_k = \lim_\nu \lim_{A^{(\nu)}} x = \lim_\mathcal{A} x = \sum_k a_k x_k$$

$$\Big(x \in \Lambda_\mathcal{A}^\perp \cap \bigcap_\nu \Lambda_{A^{(\nu)}}^\perp\Big). \qquad \square$$

In case of $\widehat{C}$-methods $\mathcal{A}$ with $\varphi \subset C_{c\mathcal{A}}$ the set $W_\mathcal{A}$ is a subset of $C_{c\mathcal{A}}$ (see 3.8(b)) and — as the following example shows — it may be very small in comparison to the domain $\widehat{C}_\mathcal{A}$. This fact implies that $W_\mathcal{A}$ does not play an important rôle for $\widehat{C}$-SM-methods in connection with theorems of Mazur–Orlicz type and consistency considerations. As the following section proves, a way out is given by the set

$$V_\mathcal{A} := B_\mathcal{A} \cap \Lambda_\mathcal{A}^\perp \cap \Lambda_{A^\bullet}^\perp \quad \text{where} \quad B_\mathcal{A} := B_{\widehat{C}_\mathcal{A}}$$

(note the comment to 3.8(b)). Obviously, we have $W_\mathcal{A} \subset V_\mathcal{A}$ in general and $V_\mathcal{A} = B_\mathcal{A} \cap \Lambda_\mathcal{A}^\perp$ if $a_k^{(\nu)} = 0$ $(k, \nu \in \mathbf{N})$. In case of matrix methods, that is $A^{(\nu)} = A^{(1)}$ $(\nu \in \mathbf{N})$, we get $V_\mathcal{A} = B_\mathcal{A} \cap \Lambda_\mathcal{A}^\perp = W_\mathcal{A}$.

REMARK 3.9. *If $\mathcal{A} = (A^{(\nu)})$ is a $\widehat{C}$-SM-method then $V_{\mathcal{A}}$ is an FK-AB-space.* For a proof we note that $I_{\mathcal{A}} = \widehat{C}_{\mathcal{A}} \cap \{(a_k)\}^{\beta}$ and $I_{\mathcal{A}^\bullet} = c_{\mathcal{A}^\bullet} \cap \{(a_k)\}^{\beta}$ are intersections of FK-spaces thus FK-spaces. Since $\lim_{\mathcal{A}}$ and $\lim_{\mathcal{A}^\bullet}$ are continuous on $I_{\mathcal{A}}$ and $I_{\mathcal{A}^\bullet}$, respectively, $\Lambda_{\mathcal{A}}^{\perp}$ and $\Lambda_{\mathcal{A}^\bullet}^{\perp}$ are FK-spaces too. Altogether, $V_{\mathcal{A}} = B_{\mathcal{A}} \cap \Lambda_{\mathcal{A}}^{\perp} \cap \Lambda_{\mathcal{A}^\bullet}^{\perp}$ is an FK-space. On account of the continuity on φ of the (standard) seminorms generating the FK-topologies of $\Lambda_{\mathcal{A}}^{\perp}$ and $\Lambda_{\mathcal{A}^\bullet}^{\perp}$ with respect to the FK-topology of $B_{\mathcal{A}}$ we get the AB-property of $V_{\mathcal{A}}$ by the AB-property of $B_{\mathcal{A}}$.

EXAMPLE 3.10. Let $\mathcal{A} = (A^{(\nu)})$ with $A^{(\nu)} = (a_{nk}^{(\nu)})$ be defined by

$$a_{nk}^{(\nu)} := \begin{cases} \frac{1}{k} & \text{if } k = n \leqq \nu \\ \frac{1}{\nu} & \text{if } k = n > \nu \\ 0 & \text{otherwise} \end{cases} \quad (k, n, \nu \in \mathbf{N}).$$

Then $C_{c\mathcal{A}} = c = C_{c_0\mathcal{A}}$ and $\widehat{C}_{\mathcal{A}} = m = \widehat{C}_{0\mathcal{A}}$. Furthermore, $\widehat{C}_{\mathcal{A}}$ is a non-separable FK-space and $W_{\mathcal{A}} = c_0 \subsetneqq V_{\mathcal{A}} = B_{\mathcal{A}} = \widehat{C}_{\mathcal{A}} = m$.

For the sake of completeness we make the following remark.

REMARK 3.11. The domain $C_{e\mathcal{A}}$ of a C_e-method $\mathcal{A}$ is an LFK-space (that is an LFH-space with $H := \omega$) and not an FK-space in general. This is an immediate consequence of 2.4 and the representation $C_e = \bigcup_{\mu} E_{\mu}$ in the proof of 2.4.

4. Consistency of SM-methods via gliding hump methods

In the first part of this section we apply a theorem of Mazur–Orlicz type being a main result in [4] and telling us that the implication

$$Y \subset F \Longrightarrow Y \subset W_F$$

holds for each sequence space Y containing φ and having the SIGNED P_OSCP and for each separable FK-space (more general, each L_{φ}-space) F. The following theorem yields this application.

THEOREM 4.1. *Let $\mathcal{A}$ be a $\widehat{C}$-SM-method with $\varphi \subset C_{c\mathcal{A}}$. Both $W_{\mathcal{A}}$ and $V_{\mathcal{A}}$ have the* ABSOLUTE SP_OSCP. *(The statement on $W_{\mathcal{A}}$ remains true under the weaker assumption $\varphi \subset \widehat{C}_{\mathcal{A}}$.)*

PROOF. From [4, Theorem 3.1] we know that W_E has the ABSOLUTE SP_OSCP whenever E is an FK-space containing φ. Consequently, $W_{\mathcal{A}}$ has

We proceed inductively: If $\nu_1,\dots,\nu_{t-1}$ and $n(\nu_1),\dots,n(\nu_{t-1})$ and $k_1,\dots,k_{t-1}$ are determined then for $\varepsilon := 2^{-t}$ and $\widetilde{k} := k_{t-1}$ we choose in virtue of (9), (10) and (12) a $\nu_t \in \mathbf{N}$ with $\nu_{t-1} < \nu_t$; in particular

$$\text{(14)}\quad \sum_{k=1}^{k_{t-1}} \left|a_k^{(\nu)} - a_k\right| |x_k| < 2^{-t} \quad \text{and} \quad \left|\sum_k a_k^{(\nu)} x_k - \alpha\right| < 2^{-t} \quad (\nu \geqq \nu_t).$$

On account of the choice of ν_t and in virtue of (10) and (11) there exists an $n(\nu_t)$ with $n(\nu_t) > n(\nu_{t-1})$ and

$$\text{(15)}\qquad \left|\sum_k a_{nk}^{(\nu)} x_k - \alpha\right| < 2^{-t+1} \qquad (\nu_{t-1} \leqq \nu \leqq \nu_t,\ n(\nu_t) \leqq n),$$

$$\text{(16)}\qquad \sum_{k=1}^{k_{t-1}} \left|a_{nk}^{(\nu)} - a_k^{(\nu)}\right| |x_k| < 2^{-t+1} \qquad (\nu \leqq \nu_t,\ n(\nu_t) \leqq n).$$

Now, we may pick $k_t \in \mathbf{N}$ with $k_t > k_{t-1}$ and

$$\text{(17)}\quad \left|\sum_{k=r}^{s} a_{nk}^{(\nu)} x_k\right| < 2^{-t},\ \left|\sum_{k=r}^{s} a_k x_k\right| < 2^{-t} \quad \text{and} \quad \left|\sum_{k=r}^{s} a_k^{(\nu)} x_k\right| < 2^{-t}$$

$$(\nu \leqq \nu_t,\ n \leqq n(\nu_t),\ k_t \leqq r \leqq s).$$

Let (l_i) be any given index sequence. Then, without loss of generality, we may assume that the index sequence (k_t) chosen above is a subsequence of (l_i).

Now we are going to define a step 1-block sequence $\left(y^{(\mu)}\right)$ with respect to (k_i), thus, with respect to (l_i).

Let $(f_j)_{j=1}^{\infty}$ be the sequence of partial sums of the series

$$1 - 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \dots + \frac{1}{n} + \dots + \frac{1}{n} - \frac{1}{n} - \dots - \frac{1}{n} + \dots$$

where $\frac{1}{n}$ appears in the series $2n$ times, the first n-times with a plus sign, the second n times with a minus sign. (The sequence (f_j) was used, for example, by G. M. Petersen in [11] proving the bounded consistency theorem due to Mazur, Orlicz and Brudno.) It is known that (f_j) has the following properties:

$$0 \leqq f_j \leqq 1 \qquad (j \in \mathbf{N}),$$

$$f_j = \begin{cases} 1 & \text{if } j \in \mathbf{N} \text{ and } j = \mu^2,\ \mu \in \mathbf{N} \\ 0 & \text{if } j \in \mathbf{N} \text{ and } j = \mu(\mu+1),\ \mu \in \mathbf{N}, \end{cases}$$

$$f_j \geqq f_{j+1} \quad \text{if} \quad \mu^2 \leqq j < \mu(\mu+1), \mu \in \mathbf{N},$$

$$f_j \leqq f_{j+1} \quad \text{if} \quad \mu(\mu+1) \leqq j < (\mu+1)^2, \mu \in \mathbf{N}.$$

We define $y = (y_k)$ and $y^{(\mu)}$, $\mu \in \mathbf{N}$, by $y = \sum_\mu y^{(\mu)}$ (coordinatewise sum) and

$$y_k^{(0)} := \begin{cases} 1 & \text{if } k < k_1 \\ 0 & \text{otherwise} \end{cases} \qquad (k \in \mathbf{N})$$

and, if $\mu \geqq 1$,

$$y_k^{(\mu)} := \begin{cases} f_j & \text{if } k_{j-1} \leqq k < k_j \text{ and } \mu(\mu+1) \leqq j < (\mu+1)(\mu+2) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$y_k = \begin{cases} 1 & \text{if } k < k_1 \\ f_j & \text{if } k_{j-1} \leqq k < k_j, j \in \mathbf{N} \end{cases} \qquad (k \in \mathbf{N}).$$

Obviously $\left(y^{(\mu)}\right)$ is a step 1-block sequence with respect to (k_i). First of all we prove $yx \in V_{\mathcal{A}}$. Then, checking this proof, we may state that $zx \in V_{\mathcal{A}}$ for $z = \sum_i h_i\, y^{(\mu_i)}$ where (h_i) is any sequence in $\{-1,1\}$ and $\left(y^{(\mu_i)}\right)$ is any subsequence of $\left(y^{(\mu)}\right)$. That will prove $V_{\mathcal{A}}$ to have the ABSOLUTE SP_OSCP.

Using (17) and the fact that y is constant for $k_{j-1} \leqq k < k_j$ ($j \in \mathbf{N}$) it is easy to prove the convergence of the series

$$\sum_k a_k y_k x_k, \quad \sum_k a_k^{(\nu)} y_k x_k \quad \text{and} \quad \sum_k a_{nk}^{(\nu)} y_k x_k \qquad (n, \nu \in \mathbf{N});$$

in particular $yx \in \Omega_{\mathcal{A}}$ and $yx \in \bigcap_\nu \left\{ \left(a_k^{(\nu)}\right)_k \right\}^\beta$.

Altogether, $yx \in V_{\mathcal{A}}$ is proved if we have verified

(i) $\sup_r p\left(yx^{[r]}\right) = \sup_{r,\nu} \limsup_n \left| \sum_{k=1}^r a_{nk}^{(\nu)} x_k y_k \right| = \sup_{r,\nu} \left| \sum_{k=1}^r a_k^{(\nu)} x_k y_k \right| < \infty,$

(ii) $\sup_r p_\nu\left(yx^{[r]}\right) = \sup_{n,r} \left| \sum_{k=1}^r a_{nk}^{(\nu)} x_k y_k \right| < \infty \quad (\nu \in \mathbf{N}),$

(iii) $yx \in \Lambda_{\mathcal{A}^\bullet}^\perp \subset c_{\mathcal{A}^\bullet}$,

(iv) $yx \in \Lambda_{\mathcal{A}}^{\perp} \subset \mathcal{C}_{e\mathcal{A}}$ where $\mathcal{A}$ is considered as $\mathcal{C}_e$-SM-method.
Note, (i) includes $\mathcal{A}(yx) \in \mathcal{M}$. Therefore (i) and (iv) imply $yx \in \widehat{\mathcal{C}}_{\mathcal{A}}$.
(i): We know $x \in B_{\mathcal{A}}$, that is

$$\sup_{r,\nu} \limsup_{n} \left| \sum_{k=1}^{r} a_{nk}^{(\nu)} x_k \right| = \sup_{r,\nu} \left| \sum_{k=1}^{r} a_k^{(\nu)} x_k \right| =: M < \infty$$

and

$$\sup_{n,r} \left| \sum_{k=1}^{r} a_{nk}^{(\nu)} x_k \right| =: N_\nu < \infty \qquad (\nu \in \mathbf{N}).$$

We subdivide the estimation into some cases.
$\nu < \nu_1$ and $r < k_2$:

$$\left| \sum_{k=1}^{r} a_k^{(\nu)} x_k y_k \right| \leqq \|y\|_\infty \sup_{\rho < \nu_1} \sum_{k=1}^{k_1 - 1} \left| a_k^{(\rho)} x_k \right| + 2^{-1} =: N_0 < \infty.$$

$\nu < \nu_1$ and $k_j \leqq r < k_{j+1}$ (for a suitable $j > 1$):

$$\left| \sum_{k=1}^{r} a_k^{(\nu)} x_k y_k \right| \leqq N_0 + \|y\|_\infty \sum_{\mu=1}^{j-1} \sup_{\rho < \nu_1} \left| \sum_{k=k_\mu}^{k_{\mu+1}-1} a_k^{(\rho)} x_k \right| + \|y\|_\infty \sup_{\rho < \nu_1} \left| \sum_{k=k_j}^{r} a_k^{(\rho)} x_k \right|$$

$$\leqq N_0 + \sum_{\mu=1}^{j-1} 2^{-\mu} + 2^{-j} \leqq N_0 + 2.$$

$\nu_{j-1} \leqq \nu < \nu_j$ and $r < k_{j-2}$ (for a suitable $j > 2$; see (14)):

$$\left| \sum_{k=1}^{r} a_k^{(\nu)} x_k y_k \right| \leqq \|y\|_\infty \sum_{k=1}^{r} \left| a_k^{(\nu)} - a_k \right| |x_k| + \left| \sum_{k=1}^{r} a_k x_k y_k \right|$$

$$\leqq 2^{-j+1} + \underbrace{\sup_{\mu} \left| \sum_{k=1}^{\mu} a_k x_k y_k \right|}_{=:\overline{N}} \leqq 1 + \overline{N} < \infty.$$

$\nu_{j-1} \leqq \nu < \nu_j$ and $k_{j-2} \leqq r < k_{j-1}$ (for a suitable $j > 1$):

$$\left| \sum_{k=1}^{r} a_k^{(\nu)} x_k y_k \right| \leqq 1 + \overline{N} + \left| \sum_{k=k_{j-2}}^{r} a_k^{(\nu)} x_k y_k \right| \leqq 1 + \overline{N} + 2M < \infty.$$

$\nu_{j-1} \leqq \nu < \nu_j$ *and* $k_{j-1} \leqq r < k_j$ *(for a suitable* $j > 1$*):*

$$\left|\sum_{k=1}^{r} a_k^{(\nu)} x_k y_k\right| \leqq 1 + \overline{N} + 4M < \infty.$$

$\nu_{j-1} \leqq \nu < \nu_j$ *and* $k_\rho \leqq r < k_{\rho+1}$ *(for suitable* $j > 1$ *and* $\rho \geqq j$*):*

$$\left|\sum_{k=1}^{r} a_k^{(\nu)} x_k y_k\right| \leqq 1 + \overline{N} + 4M + \|y\|_\infty \sum_{\mu=j+1}^{\rho} \left|\sum_{k=k_\mu}^{k_{\mu+1}-1} a_k^{(\nu)} x_k\right|$$

$$\leqq 1 + \overline{N} + 4M + 1 \leqq \overline{N} + 4M + 2 < \infty.$$

Altogether we have proved

$$\sup_{r,\nu}\left|\sum_{k=1}^{r} a_k^{(\nu)} x_k y_k\right| \leqq \overline{N} + 4M + N_0 + 3 < \infty,$$

that is (i).

(ii): Let $\nu \in \mathbf{N}$ be given. Then we choose $\xi \geqq 1$ with $\nu \leqq \nu_\xi$. Quite similarly to the cases in the proof of (i) we get following estimations.

$n \leqq n(\nu_\xi)$ *and* $r \in \mathbf{N}$:

$$\left|\sum_{k=1}^{r} a_{nk}^{(\nu)} x_k y_k\right| \leqq \sup_{n \leqq n(\nu_\xi)} \sum_{k=1}^{k_\xi} \left|a_{nk}^{(\nu)}\right| |x_k| + 1 < \infty.$$

$t \geqq \xi$, $n(\nu_t) \leqq n \leqq n(\nu_{t+1})$ *and* $r \in \mathbf{N}$:

$$\left|\sum_{k=1}^{r} a_{nk}^{(\nu)} x_k y_k\right| \leqq \sum_{k=1}^{k_{t-1}} \left|a_{nk}^{(\nu)} - a_k^{(\nu)}\right| |x_k| + \sup_\mu p\big(yx^{[\mu]}\big) + 4N_\nu + 1$$

$$\leqq \sup_\mu p\big(yx^{[\mu]}\big) + 4N_\nu + 2 < \infty.$$

Altogether we have proved (ii).

(iii): We are going to prove

$$yx \in c_{A^\bullet} \quad \text{and} \quad \lim_{A^\bullet} yx = \sum_k a_k y_k x_k,$$

that is

$$\forall\,\varepsilon > 0 \quad \exists\,\widetilde{\nu} \in \mathbf{N} \quad \forall\,\nu \geqq \widetilde{\nu}: \left|\sum_k a_k^{(\nu)} y_k x_k - \sum_k a_k y_k x_k\right| < \varepsilon.$$

Let $\xi \geqq 3$ and $\nu \in \mathbf{N}$ with $\nu_{\xi-1} \leqq \nu \leqq \nu_\xi$ be given. Then we get the estimations

$$\left|\sum_k a_k^{(\nu)} y_k x_k - \sum_k a_k y_k x_k\right|$$

$$\leqq \sum_{k=1}^{k_{\xi-2}-1} \left|a_k^{(\nu)} - a_k\right| |x_k| + \left|\sum_{k=k_{\xi-2}}^{k_\xi-1} a_k^{(\nu)} y_k x_k\right| + \left|\sum_{k=k_\xi}^{\infty} a_k^{(\nu)} y_k x_k\right| + \left|\sum_{k=k_{\xi-2}}^{\infty} a_k y_k x_k\right|$$

$$\leqq 2^{-\xi+1} + \left|\sum_{k=k_{\xi-2}}^{k_\xi-1} a_k^{(\nu)} y_k x_k\right| + \|y\|_\infty \sum_{\mu=\xi}^{\infty} \left|\sum_{k=k_\mu}^{k_{\mu+1}-1} a_k^{(\nu)} x_k\right| + 2^{-\xi+3}$$

$$\leqq 2^{-\xi+1} + \left|\sum_{k=k_{\xi-2}}^{k_\xi-1} a_k^{(\nu)} y_k x_k\right| + 2^{-\xi+1} + 2^{-\xi+3}$$

and furthermore

$$\left|\sum_{k=k_{\xi-2}}^{k_\xi-1} a_k^{(\nu)} y_k x_k\right| \leqq |f_{\xi-1}| \left|\sum_{k=k_{\xi-2}}^{k_\xi-1} a_k^{(\nu)} x_k\right| + |f_\xi - f_{\xi-1}| \left|\sum_{k=k_{\xi-1}}^{k_\xi-1} a_k^{(\nu)} x_k\right|$$

$$\leqq |f_{\xi-1}| \left|\sum_{k=k_{\xi-2}}^{k_\xi-1} a_k^{(\nu)} x_k\right| + \underbrace{|f_\xi - f_{\xi-1}|}_{\leqq \frac{1}{\xi}} 2M$$

and

$$\left|\sum_{k=k_{\xi-2}}^{k_\xi-1} a_k^{(\nu)} x_k\right|$$

$$\leqq \left|\sum_{k=1}^{\infty} a_k^{(\nu)} x_k - \alpha\right| + \sum_{k=1}^{k_{\xi-2}-1} \left|a_k^{(\nu)} - a_k\right| |x_k| + \left|\sum_{k=k_{\xi-2}}^{\infty} a_k x_k\right| + \left|\sum_{k=k_\xi}^{\infty} a_k^{(\nu)} x_k\right|$$

$$\leqq 2^{-\xi+1} + 2^{-\xi+1} + 2^{-\xi+3} + 2^{-\xi+1} \leqq 2^{-\xi+4};$$

altogether we have

$$\left|\sum_k a_k^{(\nu)} y_k x_k - \sum_k a_k y_k x_k\right| \leqq 7 \cdot 2^{-\xi+2} + \frac{2}{\xi} M$$

which proves $yx \in \Lambda_{A^\bullet}^{\perp}$, thus (iii).

(iv): We have to verify

$$yx \in \mathcal{C}_{e\mathcal{A}} \quad \text{and} \quad \mathcal{C}_e\text{-}\lim_{\mathcal{A}} yx = \sum_k a_k y_k x_k,$$

that is

$$\forall\, \varepsilon > 0 \quad \exists\, \widetilde{\nu} \in \mathbf{N} \quad \forall\, \nu \geqq \widetilde{\nu} \quad \exists\, n(\nu) \in \mathbf{N} \quad \forall\, n \geqq n(\nu) :$$

$$\left|\sum_k a_{nk}^{(\nu)} y_k x_k - \sum_k a_k y_k x_k\right| < \varepsilon.$$

To this end let $\nu, \xi, t \in \mathbf{N}$ with $t \geqq \xi + 2$ and $\nu_{\xi-1} \leqq \nu \leqq \nu_\xi$ be given. Then, using the estimations in the proof of (iii) for $n(\nu_t) \leqq n \leqq n(\nu_{t+1})$ we get the estimations

$$\left|\sum_k a_{nk}^{(\nu)} y_k x_k - \sum_k a_k y_k x_k\right|$$

$$\leqq \sum_{k=1}^{k_{t-1}-1} \left|a_{nk}^{(\nu)} - a_k^{(\nu)}\right| |x_k| + \sum_{k=1}^{k_{\xi-2}-1} \left|a_k^{(\nu)} - a_k\right| |x_k| + \left|\sum_{k=k_{\xi-2}}^{k_{t-1}-1} a_k^{(\nu)} y_k x_k\right|$$

$$+ \left|\sum_{k=k_{\xi-2}}^{\infty} a_k y_k x_k\right| + \left|\sum_{k=k_{t-1}}^{k_{t+1}-1} a_{nk}^{(\nu)} y_k x_k\right| + \left|\sum_{k=k_{t+1}}^{\infty} a_{nk}^{(\nu)} y_k x_k\right|$$

$$\leqq 2^{-t+1} + 2^{-\xi+1} + 4 \cdot 2^{-\xi+2} + \frac{2}{\xi} M + \|y\|_\infty \sum_{\mu=\xi}^{t-2} \left|\sum_{k=k_\mu}^{k_{\mu+1}-1} a_k^{(\nu)} x_k\right|$$

$$+ \|y\|_\infty \sum_{\mu=\xi-1}^{\infty} \left|\sum_{k=k_{\mu-1}}^{k_\mu-1} a_k x_k\right| + \left|\sum_{k=k_{t-1}}^{k_{t+1}-1} a_{nk}^{(\nu)} y_k x_k\right| + \|y\|_\infty \sum_{\mu=t+1}^{\infty} \left|\sum_{k=k_\mu}^{k_{\mu+1}-1} a_{nk}^{(\nu)} x_k\right|$$

$$\leqq 2^{-t+1} + 2^{-\xi+1} + 4 \cdot 2^{-\xi+2} + \frac{2}{\xi} M + 2^{-\xi+1} + 2^{-\xi+3} + \left|\sum_{k=k_{t-1}}^{k_{t+1}-1} a_{nk}^{(\nu)} y_k x_k\right| + 2^{-t}$$

and furthermore

$$\left|\sum_{k=k_{t-1}}^{k_{t+1}-1} a_{nk}^{(\nu)} y_k x_k\right| \leqq |f_t| \left|\sum_{k=k_{t-1}}^{k_{t+1}-1} a_{nk}^{(\nu)} x_k\right| + |f_{t+1} - f_t| \left|\sum_{k=k_t}^{k_{t+1}-1} a_{nk}^{(\nu)} x_k\right|$$

$$\leqq |f_t| \left|\sum_{k=k_{t-1}}^{k_{t+1}-1} a_{nk}^{(\nu)} x_k\right| + \underbrace{|f_{t+1} - f_t|}_{\leqq \frac{1}{t}} 2N_\nu$$

and

$$\left|\sum_{k=k_{t-1}}^{k_{t+1}-1} a_{nk}^{(\nu)} x_k\right|$$

$$\leqq \left|\sum_{k=1}^{\infty} a_{nk}^{(\nu)} x_k - \alpha\right| + \sum_{k=1}^{k_{t-1}-1} \left|a_{nk}^{(\nu)} - a_k^{(\nu)}\right| |x_k| + \sum_{k=1}^{k_{\xi-2}-1} \left|a_k^{(\nu)} - a_k\right| |x_k|$$

$$+ \left|\sum_{k=k_{\xi-2}}^{k_\xi - 1} a_k^{(\nu)} x_k\right| + \sum_{\mu=\xi}^{t-2} \left|\sum_{k=k_\mu}^{k_{\mu+1}-1} a_k^{(\nu)} x_k\right| + \left|\sum_{k=k_{\xi-2}}^{\infty} a_k x_k\right| + \sum_{\mu=t+1}^{\infty} \left|\sum_{k=k_\mu}^{k_{\mu+1}-1} a_{nk}^{(\nu)} x_k\right|$$

$$\leqq 2^{-\xi+1} + 2^{-t+1} + 2^{-\xi+1} + 4 \cdot 2^{-\xi+2} + 2^{-\xi+1} + 2^{-\xi+2} + 2^{-t};$$

altogether we have

$$\left|\sum_k a_{nk}^{(\nu)} y_k x_k - \sum_k a_k y_k x_k\right| \leqq 14 \cdot 2^{-\xi+2} + \frac{2}{t} N_\nu + \frac{2}{\xi} M$$

which proves $yx \in \Lambda_{\mathcal{A}}^{\perp}$, thus (iv). □

As an immediate corollary we get the following theorem of Mazur–Orlicz type in case of $V_{\mathcal{A}}$. Note, in case of $W_{\mathcal{A}}$ we have a corresponding result which is contained in [4, Corollary 3.6] since $\widehat{C}_{\mathcal{A}}$ is an FK-space.

THEOREM 4.2 (of Mazur–Orlicz type). *Let $\mathcal{A}$ be a $\widehat{C}$-SM-method with $\varphi \subset C_{c\mathcal{A}}$ and Y be a sequence space including φ and having the* SIGNED P_GHP. *Then*

$$Y \cap V_{\mathcal{A}} \subset F \Longrightarrow Y \cap V_{\mathcal{A}} \subset W_F$$

holds for any L_φ-space F, in particular, for any separable FK-space F.

PROOF. By [4, Remark 2.3(c)] $Y \cap V_{\mathcal{A}}$ has the SIGNED P_OSCP since Y has the SIGNED P_GHP and $V_{\mathcal{A}}$ has the ABSOLUTE SP_OSCP by Theorem 4.1. Thus we get the implication in question by [4, Theorem 3.4(a)]. □

REMARK 4.3. If in Theorem 4.2 we consider $C_{c\mathcal{A}}$ instead of $\widehat{C}_{\mathcal{A}}$ and if we restrict $V_{\mathcal{A}}$ to $C_{c\mathcal{A}}$, that is we consider $V_{\mathcal{A}} \cap C_{c\mathcal{A}}$, then the statement in Theorem 4.2 fails in general since in case of Example 3.10 we have $V_{\mathcal{A}} \cap C_{c\mathcal{A}} = c$.

Now, in a proved way, see for example [4, Corollary 3.7], on the basis of Theorem 4.2 we can state a general consistency theorem:

COROLLARY 4.4 (consistency). *Let $\mathcal{A}$ be a $\widehat{C}$-SM-method with $\varphi \subset C_{c\mathcal{A}}$, let B be any summability method such that the domain c_B is a separable FK-space (or more generally, an L_φ-space) and the limit functional* $\lim_B : c_B \longrightarrow \mathbf{K}$ *is continuous and linear. Furthermore, let F and Y be sequence spaces containing φ such that Y has the* SIGNED P_GHP *and $(Y \cap V_{\mathcal{A}}) + F \subset \widehat{C}_{\mathcal{A}} \cap c_B$. Then the consistency of $\mathcal{A}$ and B on F implies the consistency on $(Y \cap V_{\mathcal{A}}) + F$.*

PROOF. It runs analogously to the proof of [4, Corollary 3.7]. □

This consistency theorem has a mar in itself: We cannot use it when $\mathcal{B}$ is a $\widehat{C}$-SM-method since in general the domain of $\mathcal{B}$ is not an L_φ-space as Example 3.10 shows. We are going to prove, that Theorem 4.1 (and thus Theorem 4.2 and Corollary 4.4) also holds if we replace F and W_F by a domain $\widehat{C}_{\mathcal{B}}$ of an $\widehat{C}$-SM-method $\mathcal{B}$ and $\Lambda_{\mathcal{B}}{}^{\perp}$, respectively. For that, analogously to Section 3 of [4], we first of all prove a non-summability theorem in case of C_e-SM-methods generalizing [4, Theorem 3.2].

THEOREM 4.5 (non-summability). *Let $\mathcal{A} = \left(A^{(\nu)}\right)$ be a sequence of matrices with $\varphi \subset C_{c\mathcal{A}}$. If $x \in C_{e\mathcal{A}} \setminus \Lambda_{\mathcal{A}}^{\perp}$ is given, then there exists an index sequence (k_i) such that for each strong subsequence $\left(y^{(\mu_j)}\right)$ of any step 1-block sequence $\left(y^{(\mu)}\right)$ with respect to (k_i) we have $z := yx \notin C_{e\mathcal{A}}$ where $y := \sum_j h_j y^{(\mu_j)}$ (coordinatewise sum) and (h_j) is any sequence in $\mathcal{S} := \left\{x \in \mathbf{K} \mid |x| = 1\right\}$.*

PROOF. First of all we make some considerations in advance. Without loss of generality we assume $x \in \widehat{C}_{\mathcal{A}}$. (Otherwise we choose $\widehat{\nu} \in \mathbf{N}$ such that $\left(A^{(\nu)}x\right)_{\nu \geqq \widehat{\nu}} \in \widehat{C}$ and prove the statement of the theorem for $\widehat{\mathcal{A}} := \left(A^{(\nu)}\right)_{\nu \geqq \widehat{\nu}}$.) Furthermore, we assume $x \notin \Lambda_{\mathcal{A}}^{\perp}$ which is equivalent to the statement that one of the following properties (i) and (ii) is fulfilled:

(i) *There exists an index sequence (η_ν) such that* $\lim_\nu \sum_{k=1}^{\eta_\nu - 1} a_k x_k \neq C_e\text{-}\lim_{\mathcal{A}} x$.

(ii) $\sup_\nu \left| \sum_{k=1}^{\nu} a_k x_k \right| = \infty$.

In both cases (i) and (ii) we will choose an index sequence (k_i) depending on $x \in \widehat{\mathcal{C}}_{\mathcal{A}}$ and information resulting from this fact and $\varphi \subset \mathcal{C}_{c\mathcal{A}}$.

For all index sequences (k_i) and (n_i) and each sequence $z \in \omega$ we use the notations

$$\sum_k a_{n_i k}^{(\nu)} z_k = A_i^{(\nu)} + \widetilde{A}_i^{(\nu)} + A_i^* + B_i^{(\nu)} + C_i^{(\nu)} \quad (i, \nu \in \mathbf{N})$$

where the convergence of $\sum_k a_{n_i k}^{(\nu)} z_k$ is assumed and

$$A_i^{(\nu)} := \sum_{k=1}^{k_i - 1} \left(a_{n_i k}^{(\nu)} - a_k^{(\nu)} \right) z_k, \quad \widetilde{A}_i^{(\nu)} := \sum_{k=1}^{k_i - 1} \left(a_k^{(\nu)} - a_k \right) z_k,$$

$$A_i^* := \sum_{k=1}^{k_i - 1} a_k z_k, \quad B_i^{(\nu)} := \sum_{k=k_i}^{k_{i+1}-1} a_{n_i k}^{(\nu)} z_k, \quad C_i^{(\nu)} := \sum_{k=k_{i+1}}^{\infty} a_{n_i k}^{(\nu)} z_k.$$

In both cases (i) and (ii) we shall construct the index sequences (k_i) and (n_i) [and (ν_i)] such that

$$\left(A_i^{(\nu)} \right) \in \mathcal{C}_{c0}, \quad \left(\widetilde{A}_i^{(\nu)} \right) \in \mathcal{C}_{c0} \quad \text{and} \quad \left(C_i^{(\nu)} \right) \in \mathcal{C}_{c0}$$

where z is chosen as described in the theorem. Then it is immediate that each of the following conditions implies $z \notin \mathcal{C}_{e\mathcal{A}}$:

(α) $(A_i^*) \in c$ and $\left(B_i^{(\nu)} \right) \notin \mathcal{C}_e$.

(β) $(A_{i_j}^*) \notin c$ and $\left(B_{i_j}^{(\nu)} \right) \in \mathcal{C}_{c0}$ where (i_j) is a suitable index sequence.

Let $x \in \widehat{\mathcal{C}}_{\mathcal{A}}$ and $d := \widehat{\mathcal{C}}\text{-}\lim_{\mathcal{A}} x$ and let (η_ν) with $\eta_1 = 1$ be any given index sequence. (Later on we will fix (η_ν) based on properties (i) and (ii).) Since $\varphi \subset \mathcal{C}_{c\mathcal{A}}$ and $x \in \widehat{\mathcal{C}}_{\mathcal{A}}$ we may choose — quite similarly to the proof of Theorem 4.1 — index sequences (μ_t), (k_t), (ν_t) and $\left(n(\nu_t) \right)$. Let $\varepsilon := 2^{-1}$, $\mu_1 := 1$ and $k_1 := \eta_{\mu_1}$. We put $\nu_0 := 1$ and pick $\nu_1 > 1$ in virtue of (10) and (12); in particular we have

$$\sum_{k=1}^{k_1 - 1} \left| a_k^{(\nu)} - a_k \right| |x_k| < 2^{-1} \quad (\nu \geqq \nu_1).$$

Then we choose $n_1 := n(\nu_1)$ in virtue of (10) and (11) such that

$$\left| \sum_k a_{nk}^{(\nu_1)} x_k - d \right| < 2^{-0} = 1 \qquad \left(n \geqq n(\nu_1) \right),$$

$$\sum_{k=1}^{k_1-1} \left| a_{nk}^{(\nu)} - a_k^{(\nu)} \right| |x_k| < 2^{-0} = 1 \qquad \left(\nu \leqq \nu_1 \text{ and } n \geqq n(\nu_1) \right).$$

For $n(\nu_1)$ we may pick a μ_2 with $\mu_2 > \mu_1$ and, if $k_2 := \eta_{\mu_2}$,

$$\left| \sum_{k=r}^{s} a_{nk}^{(\nu)} x_k \right| < 2^{-1} \quad \text{and} \quad \left| \sum_{k=r}^{s} a_k^{(\nu)} x_k \right| < 2^{-1}$$
$$\left(\nu \leqq \nu_1,\ n \leqq n(\nu_1),\ k_2 \leqq r \leqq s \right).$$

Continuing inductively we may choose (μ_t), (k_t) with $k_t := \eta_{\mu_t}$, (ν_t) and (n_t) with $n_t := n(\nu_t)$ such that

$$\sum_{k=1}^{k_t-1} \left| a_k^{(\nu)} - a_k \right| |x_k| < 2^{-t} \qquad (\nu \geqq \nu_t),$$
$$\left| \sum_k a_{nk}^{(\nu)} x_k - d \right| < 2^{-t+1} \qquad \left(\nu_{t-1} \leqq \nu \leqq \nu_t,\ n(\nu_t) \leqq n \right),$$
$$\sum_{k=1}^{k_t-1} \left| a_{nk}^{(\nu)} - a_k^{(\nu)} \right| |x_k| < 2^{-t+1} \qquad \left(\nu \leqq \nu_t,\ n(\nu_t) \leqq n \right),$$
$$\left| \sum_{k=r}^{s} a_{nk}^{(\nu)} x_k \right| < 2^{-t} \quad \text{and} \quad \left| \sum_{k=r}^{s} a_k^{(\nu)} x_k \right| < 2^{-t}$$
$$\left(\nu \leqq \nu_t,\ n \leqq n(\nu_t),\ k_{t+1} \leqq r \leqq s \right).$$

Now, let $\left(y^{(\mu)}\right)$ be any step 1-block sequence with respect to (k_t). Then we consider in both cases strong subsequences $\left(y^{(\mu_j)}\right)$ of $\left(y^{(\mu)}\right)$. Using $\|y\|_\infty \leqq 1$ for $y := \sum\limits_j h_j y^{(\mu_j)}$ (coordinatewise sum) where $h_j \in \mathcal{S}$ $(j \in \mathrm{N})$ and noting (δ) in the definition of a step 1-block sequence for $z := yx$ we get the estimations

$$\left| A_i^{(\nu)} \right| \leqq \sum_{k=1}^{k_i-1} \left| a_{n_i k}^{(\nu)} - a_k^{(\nu)} \right| |x_k| < 2^{-i+1} \qquad \left(\xi \in \mathrm{N},\ i \geqq \xi + 2,\ \nu_{\xi-1} \leqq \nu \leqq \nu_\xi \right)$$

thus $\left(A_i^{(\nu)}\right) \in \mathcal{C}_{c_0}$. Furthermore (we assume the convergence of the series)

$$\left| C_i^{(\nu)} \right| = \left| \sum_{k=k_{i+1}}^{\infty} a_{n_i k}^{(\nu)} z_k \right| \leqq \sum_{r=i+1}^{\infty} \left| \sum_{k=k_r}^{k_{r+1}-1} a_{n_i k}^{(\nu)} x_k \right| \leqq \sum_{r=i+1}^{\infty} 2^{-r} = 2^{-i}$$
$$\left(\xi \in \mathrm{N},\ i \geqq \xi + 2,\ \nu_{\xi-1} \leqq \nu \leqq \nu_\xi \right),$$

thus $\left(C_i^{(\nu)}\right) \in \mathcal{C}_{c_0}$ too.

Now, we are going to fix (η_ν) depending on (i) and (ii). In case of (i) we may pick (η_ν) such that $\eta_1 = 1$ and

$$\alpha := \lim_\nu \sum_{k=1}^{\eta_\nu - 1} a_k x_k \neq d; \tag{18}$$

furthermore we may assume

$$\left| \sum_{k=\eta_\nu}^{\eta_{\nu+\mu}-1} a_k x_k \right| < 2^{-\nu} \qquad (\nu, \mu \in \mathbf{N}). \tag{19}$$

For this (η_ν) let (μ_i), (k_i), (ν_i) and (n_i) be chosen as described above. Furthermore, let $\left(y^{(\mu)}\right)$ be any step 1-block sequence with respect to (k_i) and let $\left(y^{(\gamma_j)}\right)$ be any strong subsequence of $\left(y^{(\mu)}\right)$. Since it is strong there exists an index sequence (j_r) with $\gamma_{j_r} + 1 \neq \gamma_{j_r+1}$ for each $r \in \mathbf{N}$. Roughly speaking, if we consider $y := \sum_j y^{(\gamma_j)}$ (coordinatewise sum) then there exists a 0-block between $y^{(\gamma_{j_r})}$ and $y^{(\gamma_{j_r+1})}$; in particular, if (i_j) is chosen as in the definition of a step 1-block sequence, then we have

$$y_k = \begin{cases} 0 & \text{of } k < k_{i_{3\gamma_1-2}} \text{ or } k_{i_{3\gamma_{j_r}+1}} \leqq k < k_{i_{3\gamma_{j_r+1}-2}} \quad (r \in \mathbf{N}) \\ 1 & \text{if } k_{i_{3\gamma_j-1}} \leqq k < k_{i_{3\gamma_j}} \quad (j \in \mathbf{N}) \end{cases} \tag{20}$$

as we may verify with (β) in the definiton of a step 1-block sequence by the aid of the picture following that definiton. We will prove $z := yx \notin C_{e\mathcal{A}}$ where $y := \sum_j h_j y^{(\gamma_j)}$ (coordinatewise sum) and $h_j \in \mathcal{S}$ $(j \in \mathbf{N})$.

Let $\rho, \lambda \in \mathbf{N}$ with $1 \leqq \lambda < \rho$ be given. By that and (19) we get

$$\left| \sum_{k=1}^{k_\rho - 1} a_k z_k - \sum_{k=1}^{k_\lambda - 1} a_k z_k \right| = \left| \sum_{k=k_\lambda}^{k_\rho - 1} a_k z_k \right|$$

$$\leqq \sum_{\mu=\nu_\lambda}^{\nu_\rho - 1} \left| \sum_{k=\eta_\mu}^{\eta_{\mu+1}-1} a_k x_k \right| \leqq \sum_{\mu=\nu_\lambda}^{\nu_\rho - 1} 2^{-\mu} \leqq 2^{-\nu_\lambda + 1}$$

implying the existence of

$$\lim_i \sum_{k=1}^{k_i - 1} a_k z_k$$

implying the existence of

$$\lim_i \sum_{k=1}^{k_i-1} a_k z_k$$

and thus $(A_i^*) \in c$. Now, we prove $\left(B_i^{(\nu)}\right) \notin \mathcal{C}_e$. By construction we know

$$z_k = \begin{cases} h_j x_k & \text{if } k_{i_{3\gamma_j - 1}} \leqq k < k_{i3\gamma_j} \quad (j \in \mathbf{N}) \\ 0 & \text{if } k_{i_{3\gamma_{j_r}+1}} \leqq k < k_{i3\gamma_{j_r}+2} \quad (r \in \mathbf{N}). \end{cases}$$

Therefore $B^{(\nu)}_{i_{3\gamma_{j_r}+1}} = 0$, thus $\left(B^{(\nu)}_{i_{3\gamma_{j_r}+1}}\right) \in \mathcal{C}_{c_0}$. Then $\left(B_i^{(\nu)}\right) \notin \mathcal{C}_e$ is proved if $\left(B^{(\nu)}_{i_{3\gamma_{j_r}-1}}\right) \notin \mathcal{C}_{c_0}$.

However, putting $i := i_{3\gamma_{j_r}-1}$ and $j := i_{3\gamma_{j_\xi}+1}$, this follows by the identities (ξ is fixed, $r \geqq \xi + 2$)

$$B_i^{(\nu_j)} = \sum_{k=k_i}^{k_{i+1}-1} a_{n_i k}^{(\nu_j)} z_k = h_{j_r} \sum_{k=k_i}^{k_{i+1}-1} a_{n_i k}^{(\nu_j)} x_k$$

$$= h_{j_r}\left(\sum_k a_{n_i k}^{(\nu_j)} x_k - \sum_{k=1}^{k_j-1} a_k x_k\right) - h_{j_r} \underbrace{\sum_{k=1}^{k_i-1}\left(a_{n_i k}^{(\nu_j)} - a_k^{(\nu_j)}\right) x_k}_{=:B_i^{(\nu_j)}(x)}$$

$$-h_{j_r} \underbrace{\sum_{k=1}^{k_i-1}\left(a_k^{(\nu_j)} - a_k\right) x_k}_{=:\widetilde{A}_i^{(\nu_j)}(x)} - h_{j_r} \sum_{k=k_{j+1}}^{k_i-1} a_k x_k - h_{j_r} \underbrace{\sum_{k=k_{i+1}}^{\infty} a_{n_i k}^{(\nu_j)} x_k}_{=:C_i^{(\nu_j)}(x)}$$

since $\left(\sum_k a_{n_i k}^{(\nu_j)} x_k - \sum_{k=1}^{k_j-1} a_k x_k\right)_{i,j} \in \widehat{\mathcal{C}}$ with $\widehat{\mathcal{C}}$-limit $d - \alpha \neq 0$ [see (18)],

$$\left|\sum_{k=k_{j+1}}^{k_i-1} a_k x_k\right| \leqq \sum_{\mu=j+1}^{\infty} \left|\sum_{k=k_\mu}^{k_{\mu+1}-1} a_k x_k\right| \leqq 2^{-j} \quad \text{[see (19)]}$$

and because of $\left(B_i^{(\nu_j)}(x)\right)_{i,j} \in \mathcal{C}_{c_0}$, $\left(\widetilde{A}_i^{(\nu_j)}(x)\right)_{i,j} \in \mathcal{C}_{c_0}$ and $\left(C_i^{(\nu_j)}(x)\right)_{i,j} \in \mathcal{C}_{c_0}$. Altogether we have proved $(A_i^*) \in c$ and $\left(B_i^{(\nu)}\right)_{i,\nu} \notin \mathcal{C}_e$, thus $z \notin \mathcal{C}_{e\mathcal{A}}$ by (α).

In case (ii), without loss of generality, we may assume

$$\sup_{\nu} \Re\left(\sum_{k=1}^{\nu} a_k x_k\right) = \infty.$$

Therefore, we may choose an index sequence (η_ν) such that $\eta_1 = 1$ and

$$\Re\left(\sum_{k=\eta_\nu}^{\eta_{\nu+1}-1} a_k x_k\right) \geqq \nu + \sum_{k=1}^{\eta_\nu - 1} |a_k x_k| \geqq 0 \qquad (\nu \in \mathbf{N}). \tag{21}$$

Again, for this (η_ν) let (μ_i), (ν_i) and (n_i) be chosen as is described above. Furthermore, let $\left(y^{(\mu)}\right)$ be any step 1-block sequence with respect to (k_i), let $\left(y^{(\gamma_j)}\right)$ be any strong subsequence of $\left(y^{(\mu)}\right)$ and let $\left(y^{(\gamma_{j_r})}\right)$ be chosen as in case (i). We are going to prove $z := yx \notin \mathcal{C}_{e\mathcal{A}}$ where $y := \sum_j h_j y^{(\gamma_j)}$ and $h_j \in \mathcal{S}$ $(j \in \mathbf{N})$.

Considering $i := i_{3\gamma_{j_r}+1}$ we get by (20) and the definition of z the statement $B_i^{(\nu)} = 0$, thus $\left(B_i^{(\nu)}\right) \in \mathcal{C}_{c_0}$, and also $(A_i^*) \notin c$ since by (21)

$$|A_i^*| = \left|\sum_{k=1}^{k_i - 1} a_k z_k\right| \geqq |h_{j_r}| \left|\sum_{k=k_{i_{3\gamma_{j_r}-1}}}^{k_{i_{3\gamma_{j_r}+1}}-1} a_k x_k y_k^{(\gamma_{j_r})}\right| - \left|\sum_{k=1}^{k_{i_{3\gamma_{j_r}-1}}-1} a_k z_k\right|$$

$$\geqq \Re\left(\sum_{k=k_{i_{3\gamma_{j_r}-1}}}^{k_{i_{3\gamma_{j_r}}}-1} a_k x_k\right) - \sum_{k=1}^{k_{i_{3\gamma_{j_r}-1}}-1} |a_k z_k| \geqq 3\gamma_{j_r} \xrightarrow{r\to\infty} \infty.$$

Altogether we have shown $z \notin \mathcal{C}_{e\mathcal{A}}$ by (β) and the statement of the theorem is proved. □

THEOREM 4.6 (of Mazur–Orlicz type). *Let E be a sequence space with $\varphi \subset E$. If E has the* SIGNED P_OSCP *then*

$$E \subset \widehat{\mathcal{C}}_{\mathcal{B}} \Longrightarrow E \subset \Lambda_{\mathcal{B}}^{\perp}$$

holds for any $\widehat{\mathcal{C}}$-SM-method $\mathcal{B}$ with $\varphi \subset \mathcal{C}_{c\mathcal{B}}$.

PROOF. Application of Theorem 4.5 in case of $x \in \widehat{\mathcal{C}}_{\mathcal{B}}$. □

Note, Example 3.10 shows that Theorem 4.6 fails in general if we replace $\Lambda_{\mathcal{B}}^{\perp}$ by $W_{\mathcal{B}}$.

COROLLARY 4.7 (consistency). *Let E, F be sequence spaces containing φ and let $\mathcal{A},\mathcal{B}$ be $\widehat{\mathcal{C}}$-SM-methods with $\varphi \subset C_{c\mathcal{A}} \cap C_{c\mathcal{B}}$ and $E+F \subset \widehat{C}_{\mathcal{A}} \cap \widehat{C}_{\mathcal{B}}$. If $\mathcal{A}$ and $\mathcal{B}$ are consistent on F (that is $\lim_{\mathcal{A}} x = \lim_{\mathcal{B}} x$ for each $x \in F$) and E has the* SIGNED P_OSCP *then $\mathcal{A}$ and $\mathcal{B}$ are consistent on $E+F$.*

PROOF. Take over the proof of [4, Corollary 3.7] word for word. □

REMARK 4.8. The statement in 4.6 and 4.7 remains true if we replace the $\widehat{\mathcal{C}}$-SM-method $\mathcal{B}$ and $\mathcal{A}$ or $\mathcal{B}$, respectively,
- by any $\mathcal{C}_e$-SM-method or by any
- summability method such that the domain X is an L_φ-space and the limit functional is a linear and continuous functional from X into $\mathbf{K}$.

Closing the paper we ask, whether Theorem 4.6 remains true if we replace $\Lambda_{\mathcal{B}}^{\frac{1}{\mathcal{B}}}$ by $V_{\mathcal{B}}$. The answer is positive, if, in addition E is an FK-AB-space and if we restrict the considerations to $\widehat{\mathcal{C}}$-SM-methods $\mathcal{B}$ fulfilling $E \subset c_{\mathcal{B}\bullet}$. For example, E is an FK-AB-space if $E := Y \cap V_{\mathcal{A}}$ where $\mathcal{A}$ is a $\widehat{\mathcal{C}}$-SM-method and Y is an FK-AB-space having the SIGNED P_GHP (see Remark 3.9). Note, $V_{\mathcal{A}}$ is a BK-AB-space and thus $Y \cap V_{\mathcal{A}}$ is an FK-AB-space as intersection of FK-AB-spaces.

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