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# OSCILLATION AND NONOSCILLATION CRITERIA FOR SECOND ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

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## 1. Introduction

We are concerned with the oscillatory (and nonoscillatory) behavior of quasilinear differential equations of the form

$$\text{(A)} \qquad \left(p(t)|y'|^{\alpha-1}y'\right)' + q(t)|y|^{\alpha-1}y = 0, \qquad t \geqq a,$$

where $\alpha$ and $a$ are positive constants and $p(t)$ and $q(t)$ are continuous functions on $[a,\infty)$. We assume throughout the paper that $p(t) > 0$ and $q(t) \geqq 0$ on $[a,\infty)$, and

$$\text{(1.1)} \qquad \int_a^\infty \frac{dt}{\left(p(t)\right)^{1/\alpha}} < \infty.$$

By a solution of (A) we mean a function $y \in C^1[T_y,\infty)$, $T_y \geqq a$, which has the property $p|y'|^{\alpha-1}y' \in C^1[T_y,\infty)$ and satisfies the equation at all points $t \geqq T_y$. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution of (A) is said to be oscillatory if it has an infinite sequence of zeros clustering at $t = \infty$; otherwise a solution is said to be nonoscillatory.

A striking similarity existing between (A) and the linear equation

$$\text{(B)} \qquad \left(p(t)y'\right)' + q(t)y = 0, \qquad t \geqq a,$$

was observed by Mirzov [6,7] and Elbert [1,2], who showed in particular that Sturmian theory (e.g. separation and comparison theorems) for (B) could be extended in a natural way to (A). Thus it is shown that all solutions of (A) are either oscillatory or else nonoscillatory, so that the possibility of coexistence of oscillatory and nonoscillatory solutions is precluded for (A). We say that (A) is oscillatory [resp. nonoscillatory] if all of its solutions are oscillatory [resp. nonoscillatory].

The main objective of this paper is to establish criteria for oscillation and nonoscillation of (A) emphasizing a further similarity between (A) and

(B). The criteria presented in Section 3 will then be used to characterize the phenomena of strong oscillation and nonoscillaton of the differential equation

$$(\mathrm{A}_\lambda) \qquad \left(p(t)|y'|^{\alpha-1}y'\right)' + \lambda q(t)|y|^{\alpha-1}y = 0, \qquad t \geqq a,$$

where $\lambda > 0$ is a parameter. By definition, $(\mathrm{A}_\lambda)$ is *strongly oscillatory* [resp. *strongly nonoscillatory*] if $(\mathrm{A}_\lambda)$ is oscillatory [resp. nonoscillatory] for every $\lambda > 0$; $(\mathrm{A}_\lambda)$ is defined to be *conditionally oscillatory* if there exists a constant $\lambda_0 > 0$ such that $(\mathrm{A}_\lambda)$ is oscillatory for every $\lambda > \lambda_0$ and is nonoscillatory for every $\lambda < \lambda_0$. Such a constant $\lambda_0$ is referred to as the *oscillation constant* of $(\mathrm{A}_\lambda)$. This definition follows that of Nehari [8] for the linear equation

$$(\mathrm{B}_\lambda) \qquad \left(p(t)y'\right)' + \lambda q(t)y = 0, \qquad t \geqq a,$$

and our result for $(\mathrm{A}_\lambda)$ stated in Section 4 is designed for a natural generalization of Nehari's oscillation theorem [8]. The results for (A) and $(\mathrm{A}_\lambda)$ find application to quasilinear degenerate elliptic partial differential equations of the type

$$\sum_{i=1}^{N} D_i\left(|Du|^{m-2}D_iu\right) + c\left(|x|\right)|u|^{m-2}u = 0, \qquad x \in E_a,$$

where $m > 1$, $N \geqq 2$, $D_i = \partial/\partial x_i$, $i = 1, \dots, N$, $D = (D_1, \dots, D_N)$, $E_a = \{x \in R^N : |x| \geqq a\}$, $a > 0$, and $c(t)$ is a nonnegative continuous function on $[a, \infty)$.

In the main body of the paper extensive use is made of the function

$$(1.2) \qquad \pi(t) = \int_t^\infty \frac{ds}{\left(p(s)\right)^{1/\alpha}}, \quad t \geqq a,$$

which is well-defined because of (1.1). We note that the study in the same spirit of (A) in which $p(t)$ satisfies the condition

$$\int_a^\infty \frac{dt}{\left(p(t)\right)^{1/\alpha}} = \infty$$

has already been made by Kusano, Naito and Ogata [4] and Kusano and Yoshida [5].

## 2. Generalized Riccati equation and comparison theorem

In this section of preparatory nature we first demonstrate a close connection between (A) and the first order differential equation

$$u' + q(t) + \alpha\big(p(t)\big)^{-\frac{1}{\alpha}}|u|^{\frac{\alpha+1}{\alpha}} = 0 \tag{2.1}$$

which may well be called a generalized Riccati equation, and then on the basis of this connection we establish a comparison theorem of Hille–Wintner type for a pair of equations (A) and

$$\big(p(t)|z'|^{\alpha-1}z'\big)' + Q(t)|z|^{\alpha-1}z = 0, \qquad t \geqq a, \tag{2.2}$$

where $Q : [a,\infty) \to [0,\infty)$ is a continuous function.

THEOREM 2.1. *The equation* (A) *is nonoscillatory if and only if the generalized Riccati equation* (2.1) *has a solution defined in some neighborhood of infinity* $t = \infty$.

PROOF. If $y(t)$ is a solution of (A) such that $y(t) \neq 0$ for $t \geqq t_0$, then the function

$$u(t) = p(t)\frac{\big|y'(t)\big|^{\alpha-1}y'(t)}{\big|y(t)\big|^{\alpha-1}y(t)} \tag{2.3}$$

satisfies (2.1) for $t \geqq t_0$. Conversely, if $u(t)$ is a solution of (2.1) defined on $[t_0,\infty)$, then it is easy to verify that the function

$$y(t) = \exp\left(\int_{t_0}^{t}\left|\frac{u(s)}{p(s)}\right|^{\frac{1}{\alpha}-1}\frac{u(s)}{p(s)}\,ds\right), \quad t \in [t_0,\infty)$$

satisfies (A) for $t \geqq t_0$.

A further analysis of (2.1) yields another characterization for the nonoscillation situation of (A).

THEOREM 2.2. *The equation* (A) *is nonoscillatory if and only if*

$$\int_a^\infty \big(\pi(t)\big)^{\alpha+1}q(t)\,dt < \infty \tag{2.4}$$

*and there exists a continuous function* $u : [t_0,\infty) \to R$, $t_0 \geqq a$, *such that*

(2.5) $\big(\pi(t)\big)^{\alpha}u(t)$ *is bounded on* $[t_0,\infty)$, $\quad \big(\pi(t)\big)^{\alpha}u(t) \geqq -1$ *for* $t \geqq t_0$, *and*

$$(2.6) \qquad \big(\pi(t)\big)^{\alpha+1}u(t)$$
$$\geqq \int_t^\infty \big(\pi(s)\big)^{\alpha+1}q(s)\,ds + (\alpha+1)\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha}u(s)\,ds$$
$$+\alpha\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha+1}\big|u(s)\big|^{\frac{\alpha+1}{\alpha}}\,ds, \quad t \geqq t_0.$$

LEMMA 2.3. *Let $y(t)$ be a solution of* (A) *such that $y(t) \neq 0$ for $t \geqq t_0$. Then, $y(t)$ is bounded on $[t_0, \infty)$ together with*

$$(2.7) \qquad \big(\pi(t)\big)^{\alpha}p(t)\frac{\big|y'(t)\big|^{\alpha-1}y'(t)}{\big|y(t)\big|^{\alpha-1}y(t)}.$$

*Furthermore*

$$(2.8) \qquad \big(\pi(t)\big)^{\alpha}p(t)\frac{\big|y'(t)\big|^{\alpha-1}y'(t)}{\big|y(t)\big|^{\alpha-1}y(t)} \geqq -1, \quad t \geqq t_0$$

*and*

$$(2.9) \qquad \limsup_{t\to\infty}\big(\pi(t)\big)^{\alpha}p(t)\frac{\big|y'(t)\big|^{\alpha-1}y'(t)}{\big|y(t)\big|^{\alpha-1}y(t)} \leqq 0.$$

PROOF OF LEMMA 2.3. We may assume that $y(t) > 0$ for $t \geqq t_0$. Since $p(t)\big|y'(t)\big|^{\alpha-1}y'(t)$ is nonincreasing by (A), we see that $y'(t)$ is eventually of constant sign, that is, $y'(t) > 0$ for $t \geqq t_0$ or there is $t_1 > t_0$ such that $y'(t) < 0$ for $t \geqq t_1$, and that

$$\big(p(s)\big)^{\frac{1}{\alpha}}y'(s) \leqq \big(p(t)\big)^{\frac{1}{\alpha}}y'(t) \quad \text{for} \quad s \geqq t \geqq t_0.$$

Dividing the above by $\big(p(s)\big)^{\frac{1}{\alpha}}$ and integrating it over $[t, \tau]$ gives

$$(2.10) \qquad y(\tau) \leqq y(t) + \big(p(t)\big)^{\frac{1}{\alpha}}y'(t)\int_t^\tau \frac{ds}{\big(p(s)\big)^{\frac{1}{\alpha}}}, \quad \tau \geqq t \geqq t_0.$$

If $y'(t) > 0$ for $t \geqq t_0$, then we have from (2.10)

$$y(\tau) \leqq y(t) + \big(p(t)\big)^{\frac{1}{\alpha}}y'(t)\pi(t), \quad \tau \geqq t \geqq t_0,$$

which shows that $y(t)$ is bounded on $[t_0,\infty)$. If $y'(t)<0$ for $t\geqq t_1$, then $y(t)$ is clearly bounded and, letting $\tau\to\infty$ in (2.10), we have

$$0\leqq y(t)+\big(p(t)\big)^{\frac{1}{\alpha}}y'(t)\pi(t),\quad t\geqq t_0.$$

In either case we obtain

$$\pi(t)\big(p(t)\big)^{\frac{1}{\alpha}}\frac{y'(t)}{y(t)}\geqq -1,\quad t\geqq t_0,$$

of which (2.8) is an immediate consequence.

The relation (2.9) trivially holds if $y'(t)<0$ for $t\geqq t_1$, since in this case the function (2.7) itself is negative for $t\geqq t_1$. If $y'(t)>0$ for $t\geqq t_0$, then there exist positive constants $c_1$ and $c_2$ such that

$$y(t)\geqq c_1\quad\text{and}\quad p(t)\big|y'(t)\big|^{\alpha-1}y'(t)\leqq c_2\quad\text{for}\quad t\geqq t_0,$$

which implies

$$p(t)\frac{\big|y'(t)\big|^{\alpha-1}y'(t)}{\big|y(t)\big|^{\alpha-1}y(t)}\leqq\frac{c_2}{c_1^{\alpha}},\quad t\geqq t_0.$$

Since $\pi(t)\to 0$ as $t\to\infty$, we then conclude that

$$\lim_{t\to\infty}\big(\pi(t)\big)^{\alpha}p(t)\frac{\big|y'(t)\big|^{\alpha-1}y'(t)}{\big|y(t)\big|^{\alpha-1}y(t)}=0.$$

This proves (2.9), and the proof of the lemma is complete.

PROOF OF THEOREM 2.2. (The "only if" part.) Let $y(t)$ be a solution of (A) such that $y(t)\neq 0$ for $t\geqq t_0$. Define $u(t)$ by (2.3). By Theorem 2.1 $u(t)$ is a solution of (2.1) on $[t_0,\infty)$. We now multiply (2.1) by $\big(\pi(t)\big)^{\alpha+1}$ and integrate over $[t,\tau]$, $\tau\geqq t\geqq t_0$, to obtain

$$(2.11)\qquad \big(\pi(\tau)\big)^{\alpha+1}u(\tau)-\big(\pi(t)\big)^{\alpha+1}u(t)$$
$$+(\alpha+1)\int_t^{\tau}\big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha}u(s)\,ds+\int_t^{\tau}\big(\pi(s)\big)^{\alpha+1}q(s)\,ds$$
$$+\alpha\int_t^{\tau}\big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha+1}\big|u(s)\big|^{\frac{\alpha+1}{\alpha}}\,ds=0,\quad \tau\geqq t\geqq t_0.$$

In view of the boundedness of $\big(\pi(t)\big)^{\alpha}u(t)$ (cf. Lemma 2.3) we see that

$$\big(\pi(\tau)\big)^{\alpha+1}u(\tau)=\pi(\tau)\cdot\big(\pi(\tau)\big)^{\alpha}u(\tau)\to 0\quad\text{as}\quad\tau\to\infty,$$

and

$$\left|\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha}u(s)\,ds\right| \leqq \int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\left|\big(\pi(s)\big)^{\alpha}u(s)\right| ds < \infty,$$

$$\left|\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha+1}\left|u(s)\right|^{\frac{\alpha+1}{\alpha}} ds\right|$$

$$\leqq \int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\left|\big(\pi(s)\big)^{\alpha}u(s)\right|^{\frac{\alpha+1}{\alpha}} ds < \infty$$

for any $t \geqq t_0$. Therefore, letting $\tau \to \infty$ in (2.11), we find that $\int_t^\infty \pi^{\alpha+1}(s)q(s)\,ds$ is convergent, i.e., (2.4) holds, and

$$\big(\pi(t)\big)^{\alpha+1}u(t) = \int_t^\infty \big(\pi(s)\big)^{\alpha+1}q(s)\,ds$$

$$+(\alpha+1)\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha}u(s)\,ds$$

$$+\alpha\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha+1}\left|u(s)\right|^{\frac{\alpha+1}{\alpha}} ds, \quad t \geqq t_0,$$

establishing (2.6) with equality sign. That $\big(\pi(t)\big)^{\alpha}u(t) \geqq -1$ for $t \geqq t_0$ follows from Lemma 2.3.

(The "if" part.) Suppose that (2.4) holds and let $u(t)$ be a continuous function having the properties (2.5) and (2.6). Let $C[t_0,\infty)$ be the Fréchet space of all continuous functions on $[t_0,\infty)$ with the topology of uniform convergence on every compact subinterval of $[t_0,\infty)$. Consider the set

$$V = \left\{ v \in C[t_0,\infty) : -1 \leqq v(t) \leqq \big(\pi(t)\big)^{\alpha}u(t), \quad t \geqq t_0 \right\}, \tag{2.12}$$

which is a closed convex subset of $C[t_0,\infty)$. Define the mapping $F : V \to C[t_0,\infty)$ by

$$\pi(t)(Fv)(t) = \int_t^\infty \big(\pi(s)\big)^{\alpha+1}q(s)\,ds + (\alpha+1)\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}v(s)\,ds \tag{2.13}$$

$$+\alpha\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\left|v(s)\right|^{\frac{\alpha+1}{\alpha}} ds, \quad t \geqq t_0.$$

If $v \in V$, then from (2.13), (2.12) and (2.6) it follows that

$$(Fv)(t) \leqq \big(F(\pi^{\alpha}u)\big)(t) \leqq \big(\pi(t)\big)^{\alpha}u(t), \quad t \geqq t_0,$$

and

$$\pi(t)\big[(Fv)(t)+1\big]$$
$$\geqq \int_t^\infty (p(s))^{-\frac{1}{\alpha}}\Big[\alpha\big|v(s)\big|^{\frac{\alpha+1}{\alpha}}+(\alpha+1)v(s)+1\Big]\,ds \geqq 0, \quad t \geqq t_0,$$

where we used here also the property that the function $\alpha|\xi|^{\frac{\alpha+1}{\alpha}}+(\alpha+1)\xi$ is strictly increasing for $\xi \geqq -1$, i.e.,

$$\alpha|\xi|^{\frac{\alpha+1}{\alpha}}+(\alpha+1)\xi+1 \geqq 0 \quad \text{holding for} \quad \xi \geqq -1. \tag{2.14}$$

This shows that $F$ maps $V$ into itself. It can be shown in a routine manner that $F$ is continuous and $F(V)$ is relatively compact in the topology of $C[t_0,\infty)$. Therefore, by the Schauder–Tychonoff fixed point theorem, there exists an element $v \in V$ such that $v(t)=(Fv)(t)$, $t \geqq t_0$. Define $w(t) = v(t)/\big(\pi(t)\big)^{\alpha}$. Then, in view of (2.13), $w(t)$ satisfies the integral equation

$$\big(\pi(t)\big)^{\alpha+1}w(t)=\int_t^\infty \big(\pi(s)\big)^{\alpha+1}q(s)\,ds$$
$$+(\alpha+1)\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha}w(s)\,ds$$
$$+\alpha\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}}\big(\pi(s)\big)^{\alpha+1}\big|w(s)\big|^{\frac{\alpha+1}{\alpha}}\,ds, \quad t \geqq t_0.$$

Differentiating the above and dividing by $\big(\pi(t)\big)^{\alpha+1}$ shows that $w(t)$ solves the generalized Riccati equation (2.1) for $t \geqq t_0$, and the desired conclusion that (A) is nonoscillatory follows from Theorem 2.1. This completes the proof.

We note that the "only if" part of Theorem 2.2 provides the following simple oscillation criterion for (A).

THEOREM 2.4. *The equation* (A) *is oscillatory if*

$$\int_a^\infty \big(\pi(t)\big)^{\alpha+1}q(t)\,dt=\infty. \tag{2.15}$$

It is now natural to ask what can be said about the oscillatory behavior of (A) with $q(t)$ satisfying

$$\int_a^\infty \big(\pi(t)\big)^{\alpha+1}q(t)\,dt<\infty. \tag{2.16}$$

An answer to this somewhat delicate question will be given in the next section with the help of a Hille–Wintner type comparison theorem below.

THEOREM 2.5. *Consider the equations* (A) *and* (2.2) *subject to the conditions* (1.1) *and*

$$\int_a^\infty \big(\pi(t)\big)^{\alpha+1} q(t)\,dt < \infty, \quad \int_a^\infty \big(\pi(t)\big)^{\alpha+1} Q(t)\,dt < \infty. \tag{2.17}$$

*Suppose that*

$$\int_t^\infty \big(\pi(s)\big)^{\alpha+1} q(s)\,ds \leqq \int_t^\infty \big(\pi(s)\big)^{\alpha+1} Q(s)\,dt, \quad t \geqq t_0. \tag{2.18}$$

*Then, the nonoscillation of* (2.2) *implies that of* (A), *or equivalently, the oscillation of* (A) *implies that of* (2.2).

PROOF. Assume that (2.2) is nonoscillatory. Then, by the "only if" part of Theorem 2.2 applied to (2.2), there exists a continuous function $u : [t_0, \infty) \to R$, $t_0 \geqq a$, satisfying (2.5) and

$$\begin{aligned}\big(\pi(t)\big)^{\alpha+1} u(t) \geqq& \int_t^\infty \big(\pi(s)\big)^{\alpha+1} Q(s)\,ds \\ &+ (\alpha+1)\int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}} \big(\pi(s)\big)^{\alpha} u(s)\,ds \\ &+ \alpha \int_t^\infty \big(p(s)\big)^{-\frac{1}{\alpha}} \big(\pi(s)\big)^{\alpha+1} |u(s)|^{\frac{\alpha+1}{\alpha}}\,ds, \quad t \geqq t_0.\end{aligned} \tag{2.19}$$

Using (2.18) in (2.19), we see that $u(t)$ satisfies the integral inequality (2.6) for $t \geqq t_0$, and hence that (A) is nonoscillatory by the "if" part of Theorem 2.2. This finishes the proof.

## 3. Generalized Euler equation and oscillation criteria

The goal of this section is to prove the following theorem giving oscillation and nonoscillation criteria for (A) subject to (1.1) and (2.16).

THEOREM 3.1. *Suppose that* (1.1) *and* (2.16) *are satisfied.*
(i) (A) *is oscillatory if*

$$\liminf_{t\to\infty} \big(\pi(t)\big)^{-1} \int_t^\infty \big(\pi(s)\big)^{\alpha+1} q(s)\,ds > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{3.1}$$

(ii) (A) *is nonoscillatory if*

$$(\pi(t))^{-1}\int_t^\infty (\pi(s))^{\alpha+1}q(s)\,ds \leqq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{3.2}$$

*for all sufficiently large t.*

Our proof of this theorem makes use of the observation that the analysis of (A) is reduced to that of the simpler equation

$$(t^\delta|y'|^{\alpha-1}y')' + q(t)|y|^{\alpha-1}y = 0, \qquad t \geqq a, \tag{3.3}$$

where $\delta$ is a constant such that $\delta > \alpha$. In fact, for any fixed $\delta > \alpha$, the change of variables $(t,y) \to (\tau, Y)$ given by

$$\tau = (\pi(t))^{-\frac{\alpha}{\delta-\alpha}}, \quad Y(\tau) = y(t) \tag{3.4}$$

transforms (A) into

$$(\tau^\delta|\dot{Y}|^{\alpha-1}\dot{Y})^{\dot{}} + Q(\tau)|Y|^{\alpha-1}Y = 0, \qquad \tau \geqq \tau_0, \tag{3.5}$$

where $\tau_0 = (\pi(a))^{-\alpha/(\delta-\alpha)}$,

$$Q(\tau) = \left(\frac{\delta-\alpha}{\alpha}\right)^{\alpha+1}(p(t))^{\frac{1}{\alpha}}(\pi(t))^{\frac{\delta}{\delta-\alpha}}q(t), \tag{3.6}$$

and the dot denotes differentiation with respect to $\tau$.

Note that (3.3) is a special case of (A) in which $p(t) = t^\delta$ satisfies (1.1) and defines, according to (1.2), the function $\pi(t)$ to be

$$\pi(t) = \frac{\alpha}{\delta-\alpha}t^{-\frac{\delta-\alpha}{\alpha}}, \quad t \geqq a. \tag{3.7}$$

THEOREM 3.2. *The equation* (3.3) *is oscillatory if*

$$\int_a^\infty q(t)\,dt = \infty \tag{3.8}$$

*and*

$$\int_a^\infty \frac{1}{t}\left[t^{\alpha+1-\delta}q(t) - \left(\frac{\delta-\alpha}{\alpha+1}\right)^{\alpha+1}\right]dt = \infty. \tag{3.9}$$

PROOF. Suppose that (3.3) has a nonoscillatory solution $y(t)$ which may be assumed to be positive for $t \geqq t_0$. Since $t^\delta |y'(t)|^{\alpha-1} y'(t)$ is nonincreasing for $t \geqq t_0$ by (3.3), either $y'(t) > 0$ for $t \geqq t_0$ or there is $t_1 > t_0$ such that $y'(t) < 0$ for $t \geqq t_1$. We claim that the first case does not happen. Suppose the contrary: $y'(t) > 0$ for $t \geqq t_0$. Then, integrating (3.3) on $[t_0, t]$ and using the fact that $y(t) \geqq c$, $t \geqq t_0$, for some constant $c > 0$, we have

$$t^\delta \big(y'(t)\big)^\alpha = t_0^\delta \big(y'(t_0)\big)^\alpha - \int_{t_0}^t q(s)\big(y(s)\big)^\alpha \, ds$$

$$\leqq t_0^\delta \big(y'(t_0)\big)^\alpha - c^\alpha \int_{t_0}^t q(s)\, ds, \quad t \geqq t_0,$$

from which, because of (3.8), it follows that $t^\delta \big(y'(t)\big)^\alpha \to -\infty$ as $t \to \infty$. This contradiction proves that $y'(t) < 0$ for $t \geqq t_1$.

Let us consider the function $u(t)$ defined by (2.3), i.e. $u(t) = t^\delta |y'(t)|^{\alpha-1} \cdot y'(t)/|y(t)|^{\alpha-1} y(t)$. Then, $u(t) < 0$, $t \geqq t_1$, and we see that $t^{-(\delta-\alpha)} u(t)$ is bounded on $[t_1, \infty)$ (cf. (2.5) and (3.7)), and that $u(t)$ satisfies

$$u'(t) + q(t) + \alpha t^{-\frac{\delta}{\alpha}} |u(t)|^{\frac{\alpha+1}{\alpha}} = 0, \quad t \geqq t_1 \tag{3.10}$$

(cf.(2.1)). Multiplying (3.10) by $t^{-(\delta-\alpha)}$ and integrating on $[t_1, t]$ gives

$$\begin{aligned} &t^{-(\delta-\alpha)} u(t) - t_1^{-(\delta-\alpha)} u(t_1) \\ &+(\delta-\alpha) \int_{t_1}^t s^{\alpha-\delta-1} u(s)\, ds + \int_{t_1}^t s^{\alpha-\delta} q(s)\, ds \\ &+\alpha \int_{t_1}^t s^{-\frac{\delta}{\alpha}+\alpha-\delta} |u(s)|^{\frac{\alpha+1}{\alpha}} ds = 0, \quad t \geqq t_1. \end{aligned} \tag{3.11}$$

We now use the Young inequality

$$AB \leqq \frac{A^a}{a} + \frac{B^b}{b},$$

where $A$, $B$, $a$, and $b$ are positive constants with $1/a + 1/b = 1$, to estimate the integrand of the first integral in (3.11) as follows

$$\begin{aligned} (\delta-\alpha) s^{\alpha-\delta-1} |u(s)| &= (\alpha+1)^{\frac{\alpha}{\alpha+1}} s^{\alpha-\delta-\frac{\alpha}{\alpha+1}} |u(s)| \cdot (\delta-\alpha)(\alpha+1)^{-\frac{\alpha}{\alpha+1}} s^{-\frac{1}{\alpha+1}} \\ &\leqq \frac{\alpha}{\alpha+1} \Big[(\alpha+1)^{\frac{\alpha}{\alpha+1}} s^{\alpha-\delta-\frac{\alpha}{\alpha+1}} |u(s)|\Big]^{\frac{\alpha+1}{\alpha}} \end{aligned}$$

$$+\frac{1}{\alpha+1}\left[(\delta-\alpha)(\alpha+1)^{-\frac{\alpha}{\alpha+1}}s^{-\frac{1}{\alpha+1}}\right]^{\alpha+1}$$

$$=\alpha s^{-\frac{\delta}{\alpha}+\alpha-\delta}\left|u(s)\right|^{\frac{\alpha+1}{\alpha}}+\left(\frac{\delta-\alpha}{\alpha+1}\right)^{\alpha+1}s^{-1}.$$

We then see from (3.11) that

$$t^{-(\delta-\alpha)}u(t)-t_1^{-(\delta-\alpha)}u(t_1)+\int_{t_1}^{t}s^{\alpha-\delta}q(s)\,ds+\alpha\int_{t_1}^{t}s^{-\frac{\delta}{\alpha}+\alpha-\delta}\left|u(s)\right|^{\frac{\alpha+1}{\alpha}}ds$$

$$=(\delta-\alpha)\int_{t_1}^{t}s^{\alpha-\delta-1}\left|u(s)\right|ds$$

$$\leqq\alpha\int_{t_1}^{t}s^{-\frac{\delta}{\alpha}+\alpha-\delta}\left|u(s)\right|^{\frac{\alpha+1}{\alpha}}ds+\left(\frac{\delta-\alpha}{\alpha+1}\right)^{\alpha+1}\int_{t_1}^{t}\frac{ds}{s}$$

which implies

$$t^{-(\delta-\alpha)}u(t)-t_1^{-(\delta-\alpha)}u(t_1)$$

$$+\int_{t_1}^{t}\frac{1}{s}\left[s^{\alpha+1-\delta}q(s)-\left(\frac{\delta-\alpha}{\alpha+1}\right)^{\alpha+1}\right]ds\leqq 0,\quad t\geqq t_1.$$

Letting $t\to\infty$ in the above and using (3.9), we conclude that $t^{-(\delta-\alpha)}u(t)\to-\infty$ as $t\to\infty$, which contradicts the boundedness of $t^{-(\delta-\alpha)}u(t)$ on $[t_1,\infty)$. This completes the proof.

We now consider the following particular case of (3.3):

$$\left(t^{\delta}|y'|^{\alpha-1}y'\right)'+\lambda t^{\delta-\alpha-1}|y|^{\alpha-1}y=0,\qquad t\geqq a,\tag{3.12}$$

$\lambda$ being a positive parameter, which is called a generalized Euler equation. It can be shown that (3.11) is conditionally oscillatory and its oscillation constant is $\left[(\delta-\alpha)/(\alpha+1)\right]^{\alpha+1}$.

THEOREM 3.3. *Suppose that $\delta>\alpha$ and $\lambda>0$.*
(i) (3.12) *is oscillatory if*

$$\lambda>\left(\frac{\delta-\alpha}{\alpha+1}\right)^{\alpha+1}.\tag{3.13}$$

(ii) (3.12) *is nonoscillatory if*

$$\lambda \leqq \left(\frac{\delta-\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{3.14}$$

PROOF. The equation (3.12) is a special case of (3.3) with $q(t) = \lambda t^{\delta-\alpha-1}$. Since the conditions (3.8) and (3.9) hold for this function $q(t)$ provided $\lambda$ satisfies (3.13), the first statement follows from Theorem 3.2.

The equation (3.12) with $\lambda = \left[(\delta-\alpha)/(\alpha+1)\right]^{\alpha+1}$ is nonoscillatory, since it has a nonoscillatory solution $y(t) = t^{-(\delta-\alpha)/(\alpha+1)}$. This fact combined with the Sturmian comparison theorem proved by Elbert [1] shows that the second statement is true.

Oscillation and nonoscillation criteria for (3.3) now follow.

THEOREM 3.4. *Suppose that $\delta > \alpha$.*
(i) (3.3) *is oscillatory if*

$$\liminf_{t\to\infty} t^{\frac{\delta-\alpha}{\alpha}} \int_t^\infty s^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)} q(s)\, ds > \frac{\alpha(\delta-\alpha)^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{3.15}$$

(ii) (3.3) *is nonoscillatory if*

$$t^{\frac{\delta-\alpha}{\alpha}} \int_t^\infty s^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)} q(s)\, ds \leqq \frac{\alpha(\delta-\alpha)^\alpha}{(\alpha+1)^{\alpha+1}} \tag{3.16}$$

*for all sufficiently large t.*

PROOF. (i) Suppose that (3.15) holds. There exist positive constants $\lambda^*$ and $T$ such that $\lambda^* > \left[(\delta-\alpha)/(\alpha+1)\right]^{\alpha+1}$, $T \geqq a$, and

$$t^{\frac{\delta-\alpha}{\alpha}} \int_t^\infty s^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)} q(s)\, ds > \frac{\alpha}{\delta-\alpha}\lambda^* \quad \text{for} \quad t \geqq T.$$

Since

$$\frac{\alpha}{\delta-\alpha}\lambda^* = t^{\frac{\delta-\alpha}{\alpha}} \int_t^\infty s^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)} (\lambda^* s^{\delta-\alpha-1})\, ds,$$

we have

$$\int_t^\infty s^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)} q(s)\, ds > \int_t^\infty s^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)} (\lambda^* s^{\delta-\alpha-1})\, ds, \quad t \geqq T.$$

Noting that the generalized Euler equation (3.12) with $\lambda = \lambda^*$ is oscillatory by Theorem 3.3 and applying Theorem 2.5, we conclude that (3.3) is oscillatory.

(ii) Let $\lambda_0 = \left[(\delta-\alpha)/(\alpha+1)\right]^{\alpha+1}$. The equation (3.12) with $\lambda = \lambda_0$ is clearly nonoscillatory. Since (3.16) can be written as

$$\int_t^\infty s^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)} q(s)\,ds \leqq \frac{\alpha}{\delta-\alpha}\lambda_0 t^{-\frac{\delta-\alpha}{\alpha}} = \int_t^\infty s^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)}\left(\lambda_0 s^{\delta-\alpha-1}\right) ds,$$

it follows from Theorem 2.5 that (3.3) is nonoscillatory provided (3.16) is satisfied.

PROOF OF THEOREM 3.1. We are now ready to prove Theorem 3.1 stated at the beginning of this section. Let a constant $\delta > \alpha$ be fixed, and introduce the new variables $(\tau, Y)$ defined by (3.4). Then (3.3) transforms into (3.5) with the coefficient $Q(\tau)$ given by (3.6). Theorem 3.4 applied to (3.5) shows that (3.5) is oscillatory if

$$\liminf_{\tau\to\infty} \tau^{\frac{\delta-\alpha}{\alpha}} \int_\tau^\infty \sigma^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)} Q(\sigma)\,d\sigma > \frac{\alpha(\delta-\alpha)^\alpha}{(\alpha+1)^{\alpha+1}} \tag{3.17}$$

and that (3.5) is nonoscillatory if

$$\tau^{\frac{\delta-\alpha}{\alpha}} \int_\tau^\infty \sigma^{-\frac{\alpha+1}{\alpha}(\delta-\alpha)} Q(\sigma)\,d\sigma \leqq \frac{\alpha(\delta-\alpha)^\alpha}{(\alpha+1)^{\alpha+1}} \tag{3.18}$$

for all sufficiently large $\tau$. It is a matter of easy computation to verify that the inequalities (3.17) and (3.18) transform back to (3.1) and (3.2), respectively, which are directly applicable to the original equation (A). This completes the proof.

We conclude this section with an oscillation theorem of slightly different nature.

THEOREM 3.5. *Suppose that* (1.1) *and* (2.16) *are satisfied. The equation* (A) *is oscillatory if*

$$\limsup_{t\to\infty} \left(\pi(t)\right)^{-1} \int_t^\infty \left(\pi(s)\right)^{\alpha+1} q(s)\,ds > 1. \tag{3.19}$$

PROOF. Suppose that (A) is nonoscillatory. Let $y(t)$ be a solution of (A) such that $y(t) > 0$ for $t \geqq t_0$. Consider the function $u(t)$ defined by (2.3). According to Theorem 2.2 and Lemma 2.3, $u(t)$ satisfies (2.5), (2.6) and (2.9). From (2.6) and (2.14) we see that

$$\left(\pi(t)\right)^{\alpha+1} u(t) + \pi(t)$$
$$\geqq \int_t^\infty \left(p(s)\right)^{-\frac{1}{\alpha}} \left[\alpha\left|\left(\pi(s)\right)^\alpha u(s)\right|^{\frac{\alpha+1}{\alpha}} + (\alpha+1)\left(\pi(s)\right)^\alpha u(s) + 1\right] ds$$

$$+\int_t^\infty \big(\pi(s)\big)^{\alpha+1} q(s)\,ds \geqq \int_t^\infty \big(\pi(s)\big)^{\alpha+1} q(s)\,ds, \quad t \geqq t_0,$$

which implies

$$\big(\pi(t)\big)^{-1} \int_t^\infty \big(\pi(s)\big)^{\alpha+1} q(s)\,ds \leqq \big(\pi(t)\big)^{\alpha} u(t) + 1, \quad t \geqq t_0.$$

Taking the upper limit of the above as $t \to \infty$ and taking (2.9) into account, we find

$$\limsup_{t\to\infty} \big(\pi(t)\big)^{-1} \int_t^\infty \big(\pi(s)\big)^{\alpha+1} q(s)\,ds \leqq 1,$$

which contradicts (3.19). The proof is thus complete.

REMARK. Theorems 3.1 and 3.5 can be considered as a natural generalization of the classical oscillation criteria of Hille [3] for the linear equation (B) with $p(t) \equiv 1$.

## 4. Strong oscillation and nonoscillation

We are now in a position to discuss the problem of strong oscillation and nonoscillation for the equation

$$(\mathrm{A}_\lambda) \qquad \big(p(t)|y'|^{\alpha-1}y'\big)' + \lambda q(t)|y|^{\alpha-1}y = 0, \qquad t \geqq a,$$

where $p(t)$, $q(t)$, $\alpha$ and $a$ are as in the preceding sections. Our main result here shows that the situations for strong oscillation and strong nonoscillation of $(\mathrm{A}_\lambda)$ can be completely characterized. We need only to consider the case where the coefficient $q(t)$ satisfies (2.16), since otherwise (A) is oscillatory by Theorem 2.4, so that $(\mathrm{A}_\lambda)$ is strongly oscillatory.

THEOREM 4.1. *Suppose that* (1.1) *and* (2.16) *are satisfied.*
(i) $(\mathrm{A}_\lambda)$ *is strongly oscillatory if and only if*

$$\limsup_{t\to\infty} \big(\pi(t)\big)^{-1} \int_t^\infty \big(\pi(s)\big)^{\alpha+1} q(s)\,ds = \infty. \tag{4.1}$$

(ii) $(\mathrm{A}_\lambda)$ *is strongly nonoscillatory if and only if*

$$\lim_{t\to\infty} \big(\pi(t)\big)^{-1} \int_t^\infty \big(\pi(s)\big)^{\alpha+1} q(s)\,ds = 0. \tag{4.2}$$

PROOF. (i) Let $(\mathrm{A}_\lambda)$ be strongly oscillatory. From (ii) of Theorem 3.1 it follows that

$$\limsup_{t\to\infty}\big(\pi(t)\big)^{-1}\int_t^\infty\big(\pi(s)\big)^{\alpha+1}\big(\lambda q(s)\big)\,ds \geqq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \tag{4.3}$$

for every $\lambda>0$. But this implies (4.1), since otherwise (4.3) would be violated for sufficiently small values of $\lambda$.

If (4.1) holds, then

$$\limsup_{t\to\infty}\big(\pi(t)\big)^{-1}\int_t^\infty\big(\pi(s)\big)^{\alpha+1}\big(\lambda q(s)\big)\,ds=\infty>1 \quad \text{for every} \quad \lambda>0,$$

and so $(\mathrm{A}_\lambda)$ is oscillatory for $\lambda>0$ by Theorem 3.5, implying the strong oscillation of $(\mathrm{A}_\lambda)$.

(ii) Let $(\mathrm{A}_\lambda)$ be strongly nonoscillatory. From the proof of Theorem 3.5 we see that

$$\limsup_{t\to\infty}\big(\pi(t)\big)^{-1}\int_t^\infty\big(\pi(s)\big)^{\alpha+1}\big(\lambda q(s)\big)\,ds \leqq 1 \quad \text{for every} \quad \lambda>0.$$

The arbitrariness of $\lambda$ then implies that

$$\limsup_{t\to\infty}\big(\pi(t)\big)^{-1}\int_t^\infty\big(\pi(s)\big)^{\alpha+1}q(s)\,ds=0,$$

which is equivalent to (4.2).

If (4.2) holds, then

$$\lim_{t\to\infty}\big(\pi(t)\big)^{-1}\int_t^\infty\big(\pi(s)\big)^{\alpha+1}\big(\lambda q(s)\big)\,ds=0 \text{ for every } \lambda>0,$$

and from (ii) of Theorem 3.1 it follows that $(\mathrm{A}_\lambda)$ is nonoscillatory for every $\lambda>0$. Thus $(\mathrm{A}_\lambda)$ is strongly nonoscillatory.

From Theorem 4.1 we conclude that the equation $(\mathrm{A}_\lambda)$ is conditionally oscillatory if and only if either

$$0<\lim_{t\to\infty}\big(\pi(t)\big)^{-1}\int_t^\infty\big(\pi(s)\big)^{\alpha+1}q(s)\,ds<\infty \tag{4.4}$$

or

$$0\leqq\liminf_{t\to\infty}\big(\pi(t)\big)^{-1}\int_t^\infty\big(\pi(s)\big)^{\alpha+1}q(s)\,ds \tag{4.5}$$
$$<\limsup_{t\to\infty}\big(\pi(t)\big)^{-1}\int_t^\infty\big(\pi(s)\big)^{\alpha+1}q(s)\,ds<\infty.$$

Information about the oscillation constant of a conditionally oscillatory equation $(A_\lambda)$ is provided by the following theorem, where use is made of the notation

$$(4.6)\qquad \begin{cases} q_* = \liminf\limits_{t\to\infty} \left(\pi(t)\right)^{-1} \displaystyle\int_t^\infty \left(\pi(s)\right)^{\alpha+1} q(s)\, ds, \\ q^* = \limsup\limits_{t\to\infty} \left(\pi(t)\right)^{-1} \displaystyle\int_t^\infty \left(\pi(s)\right)^{\alpha+1} q(s)\, ds. \end{cases}$$

THEOREM 4.2. *Suppose that $0 < q_* \leqq q^* < \infty$. Then the oscillation constant $\lambda_0$ of the equation $(A_\lambda)$ satisfies*

$$(4.7)\qquad \frac{1}{q^*}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \leqq \lambda_0 \leqq \min\left\{\frac{1}{q^*}, \frac{1}{q_*}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\right\}.$$

*If in particular $q_* = q^*$, then*

$$(4.8)\qquad \lambda_0 = \frac{1}{q^*}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} = \frac{1}{q_*}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

PROOF. Let $\lambda \in (0, \lambda_0)$. Then, $(A_\lambda)$ is nonoscillatory, and so by Theorems 3.1 and 3.5 we have

$$\liminf_{t\to\infty} \left(\pi(t)\right)^{-1} \int_t^\infty \left(\pi(s)\right)^{\alpha+1} \left(\lambda q(s)\right) ds = \lambda q_* \leqq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

and

$$\limsup_{t\to\infty} \left(\pi(t)\right)^{-1} \int_t^\infty \left(\pi(s)\right)^{\alpha+1} \left(\lambda q(s)\right) ds = \lambda q^* \leqq 1,$$

from which, letting $\lambda \to \lambda_0-$, we obtain

$$\lambda_0 \leqq \frac{1}{q_*}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \quad \text{and} \quad \lambda_0 \leqq \frac{1}{q^*}.$$

Let $\lambda \in (\lambda_0, \infty)$. Since $(A_\lambda)$ is oscillatory, from Theorem 3.1 we see that

$$\limsup_{t\to\infty} \left(\pi(t)\right)^{-1} \int_t^\infty \left(\pi(s)\right)^{\alpha+1} \left(\lambda q(s)\right) ds = \lambda q^* \geqq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

which, in the limit as $\lambda \to \lambda_0+$, implies

$$\lambda_0 \geqq \frac{1}{q^*}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

The truth of (4.7) is thus verified. It is clear from (4.7) that (4.8) holds in the case $q_* = q^*$. This completes the proof.

It would be of interest to observe that the results developed above can be applied to the study of the oscillatory behavior of degenerate elliptic partial differential equations of the type

$$\sum_{i=1}^{N} D_i\big(|Du|^{m-2}D_i u\big) + c\big(|x|\big)\,|u|^{m-2}u = 0, \qquad x \in E_a, \tag{4.9}$$

where $m > 1$, $N \geqq 2$, $D_i = \partial/\partial x_i$, $i = 1,\dots,N$, $D = (D_1,\dots,D_N)$, $E_a = \{x \in R^N : |x| \geqq a\}$, $a > 0$, and $c : [a,\infty) \to [0,\infty)$ is continuous. Our attention will be focused on *radial* solutions of (4.9), that is, those solutions which depend only on $|x|$. As it is easily verified, a radial function $u = y\big(|x|\big)$ is a solution of (4.9) in $E_a$ if and only if $y(t)$ is a solution of the ordinary differential equation

$$\big(t^{N-1}|y'|^{m-2}y'\big)' + t^{N-1}c(t)|y|^{m-2}y = 0, \qquad t \geqq a, \tag{4.10}$$

which is a special case of (3.3) in which $\delta = N-1$, $\alpha = m-1$ and $q(t) = t^{N-1}c(t)$. Note that the condition (1.1) holds if and only if $N > m$, in which case the function $\pi(t)$ defined by (1.2) reduces to

$$\pi(t) = \frac{m-1}{N-m}t^{-\frac{N-m}{m-1}}, \quad t \geqq a.$$

Theorems 2.4, 3.4 and 3.5 specialized to (4.10) yield the following result.

COROLLARY 4.3. *Suppose that* $N > m$.

(i) *All nontrivial radial solutions of* (4.9) *are oscillatory if*

$$\int_a^\infty t^{N-1-\frac{m}{m-1}(N-m)}c(t)\,dt = \infty. \tag{4.11}$$

(ii) *Suppose that*

$$\int_a^\infty t^{N-1-\frac{m}{m-1}(N-m)}c(t)\,dt < \infty. \tag{4.12}$$

*Then, all nontrivial radial solutions of* (4.9) *are oscillatory if*

$$\liminf_{t\to\infty} t^{\frac{N-m}{m-1}} \int_t^\infty s^{N-1-\frac{m}{m-1}(N-m)}c(s)\,ds > \frac{(m-1)(N-m)^{m-1}}{m^m}$$

*or if*

$$\limsup_{t\to\infty} t^{\frac{N-m}{m-1}} \int_t^\infty s^{N-1-\frac{m}{m-1}(N-m)} c(s)\,ds > \left(\frac{N-m}{m-1}\right)^{m-1}.$$

(iii) *Under the condition* (4.12) *all nontrivial radial solutions of* (4.9) *are nonoscillatory if*

$$\limsup_{t\to\infty} t^{\frac{N-m}{m-1}} \int_t^\infty s^{N-1-\frac{m}{m-1}(N-m)} c(s)\,ds < \frac{(m-1)(N-m)^{m-1}}{m^m}.$$

One can speak of strong oscillation and nonoscillation of the equation

$$(4.9_\lambda) \qquad \sum_{i=1}^{N} D_i\big(|Du|^{m-2} D_i u\big) + \lambda c\big(|x|\big)\,|u|^{m-2}u = 0, \qquad x \in E_a,$$

on the understanding that only radial solutions are the object of consideration, and one can easily derive criteria for strong oscillation and nonoscillation of $(4.9_\lambda)$ in this sense from Theorem 4.1. The details are left to the reader. We notice that the oscillatory behavior of $(4.9_\lambda)$ in the case $m \geqq N$ has already been investigated in the papers [4,5].

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