Geometric Formulation of Classical Mechanics and Field Theory.

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1. – Introduction.

Since the classical work of Cartan [1], the application of the methods of « modern » differential geometry to physical problems has been continuous and has undergone a steady increase in recent years. Apart from their prominent role in the development of the (symplectic) formulation of classical mechanics, as described in the excellent book of Abraham and Marsden [2], their influence in other branches of physics has been considerable. The modern formulation of the theory of gauge (Yang-Mills) fields is probably one of the most conspicuous examples of this trend, but certainly not the only one [3]. Another area in which this geometrical approach has been particularly fruitful is the study of the variational principles in classical field theory (*). Since the early work of Dedecker [4] on the variational calculus, the subject has attracted the attention of many authors [5-18]. As is known (see, *e.g.*, [7, 10, 18] and references therein), the use of the techniques of differential geometry

(*) « Classical » is used here in the sense of « nonquantized ».

and the fibre bundle formalism in particular provides a scheme within which the formulation of the variational principles and the conservation laws (through the Noether theorem [5, 10, 11, 18]) take a specially elegant form. At the same time, the introduction of the Poincaré-Cartan form, a useful tool specially advocated by García-Pérez and Pérez-Rendón [10, 11; see also 18] allows a neat distinction between the ordinary Hamilton principle—which leads to the (Euler) Lagrange equations—and the so-called modified Hamilton principle which, in an adequate formulation [18], leads to the Hamilton equations. As is well known, both variational principles do not always lead to the same set of solutions; they are equivalent, however, when the regularity condition is fulfilled.

As already mentioned, the geometric differential approach has been extensively applied to the formulation of classical mechanics (see, *e.g.*, the books of ref. [2, 19-23] and also the classical text [24]). In contrast, a systematic study of the variational principles in their different formulations and in connection with the simplest (elementary) systems of relativistic field theory has not been carried out. This is one of the objects of the present work in which, in particular, we shall obtain the equations of motion of those systems and the Noether currents underlying their Poincaré symmetry. This application has, apart from the illustrative character of the methods involved, the interest of the differences between the various variational principles and, in particular, the consideration of the modified Hamilton principle. In fact, we shall devote particular attention to the question of the equivalence between the ordinary and the modified Hamilton principle for the different systems. As a result, since the « Hamiltonian » we shall introduce in field theory is a Lorentz scalar one—it is not the zeroth component of a four-vector—, in the case of equivalence we shall obtain an invariant Hamiltonian formulation which, despite its simplicity, is generally overlooked in favour of the common Lagrangian one and to which it is completely equivalent.

This review, however, is not restricted to field theory. The other objective of this work is to show the parallelism which exists between field theory and classical mechanics. We believe this is best exhibited by the geometric formulation, which is specially suited to manifest the underlying structural unity of both theories and to exploit their similarities. With this aim, we have included previously a systematic review of the well-established symplectic (*) structure of (time-independent) mechanics and also described the contact structure of the mechanics of nonautonomous systems and the symplectic structure which may be constructed on the manifold of solutions. The question of the symmetries is also analysed in the case of mechanics, to show that its for-

(*) The word *symplectic* seems to have been invented by Hermann WEYL (H. WEYL: *The Classical Groups, Their Invariants and Representations* (Princeton, N. J., 1946, first edition 1939)).

mulation is conceptually similar to the case of field theory. The variational approach to mechanics, however, is only briefly discussed after the general description of the variational formalism (on the cross-sections of a fibre bundle) given in the part devoted to field theory. This incidentally shows the non-preponderant role of the variational principles in classical mechanics, a fact often obscured in the more conventional approaches.

The distribution of the topics covered by this review can be found in the index. The paper is self-contained; all the mathematical notions required for its reading beyond the notion of exterior differential or Lie derivative are included in two appendices at the end. No attempt has been made towards completeness in the bibliography; the references included are those which are most relevant to the text among those known to the authors. A very complete bibliography on some of the topics covered by this paper may be found in the second reference given in [2].

PART I

Time-independent mechanical systems: symplectic structure.

We review in the following sect. **2-4** the mechanics of autonomous systems or systems which do not depend explicitly on time. Time will thus appear in this part as a parameter and not as an independent variable contributing in one unit to the dimension of the manifold. The theory is developed in a way as to stress the similarities with the more general cases to be considered later on. As general references for this part (and also for part of the next) the reader may consult ref. [2, 2*a*, 2*b*, 19, 21, 22, 22*a*, 23].

2. – Hamiltonian formulation of classical mechanics.

a) *The Liouville and the symplectic forms.* Let M be a differentiable manifold of dimension m, $\tau(M) = (T(M), \pi_M, M)$ the tangent bundle and $\tau^*(M) = (T^*(M), \lambda_M, M)$ the cotangent bundle, dual of $\tau(M)$ (*). The manifold M, base of both vector bundles, will be locally parametrized by the co-ordinate system $(q^1, \ldots, q^m) = \{q^i\}$. The q^i thus denote the degrees of freedom of the mechanical system, that will be defined later on, for which M will be the *configuration manifold*. The tangent space $T(M)$ will be parametrized (*) by the co-ordinate system $\{q^{iT}\} \equiv \{q^i, \mathrm{d}q^i\}$, usually written by abuse of language $\{q^i, \dot{q}^i\}$. The co-ordinate system of the cotangent space $T^*(M)$—the space of

(*) The spaces and bundles considered in this section are defined in appendix A. The reader is referred to it when necessary.

1-forms α on $T(M)$—will be denoted by the applications $\{q^i, \partial/\partial q^i \equiv p_i\}$; $T^*(M)$ will thus be the *phase space* of mechanical system. These conventions for the co-ordinate systems reflect the fact that the vectors which define the basis of the vector fibre parts of $T(M)$ $[T^*(M)]$ may be taken as $\partial/\partial q^i$ $[\mathrm{d}q^i]$.

Let $T(T^*(M))$ be the tangent space to the space $T^*(M)$ of 1-forms on $T(M)$, and $\tau(T^*(M)) \equiv (T(T^*(M)), \pi_{T^*(M)}, T^*(M))$ the corresponding tangent bundle. The various spaces up to now introduced appear in the following commutative diagram:

$$\begin{array}{ccc} T(T^*(M)) & \xrightarrow[\lambda_M^{\mathrm{T}}]{} & T(M) \\ \pi_{T^*(M)}\Big\downarrow & & \Big\downarrow \pi_M \\ T^*(M) & \xrightarrow{\lambda_M} & M \end{array},$$

where λ_M^{T} is the tangent application (to λ_M) and the action of the different maps is defined in terms of the co-ordinate systems by (*)

$$\begin{aligned} &\pi_M : (q^i, \dot q^i) \in T(M) && \to q^i \in M\,, \\ &\lambda_M : (q^i, p_i) \in T^*(M) && \to q^i \in M\,, \\ &\pi_{T^*(M)} : (q^i, p_i;\, \mathrm{d}q^i, \mathrm{d}p_i) \in T(T^*(M)) && \to (q^i, p_i) \in T^*(M)\,, \\ &\lambda_M^{\mathrm{T}} : (q^i, p_i;\, \mathrm{d}q^i, \mathrm{d}p_i) \in T(T^*(M)) && \to (q^i, \mathrm{d}q^i) \in T(M)\,, \end{aligned}$$

$$i = 1, \ldots, m\,.$$

The dual of the upper line of the diagram gives

$$\begin{array}{ccc} T(T^*(M)) & \xrightarrow[\lambda_M^{\mathrm{T}}]{} & T(M) \\ \text{dual}\Big\downarrow & & \Big\downarrow \text{dual} \\ T^*(T^*(M)) & \xleftarrow{(\lambda_M^{\mathrm{T}})^*} & T^*(M) \end{array}.$$

We may use now the application $(\lambda_M^{\mathrm{T}})^* : T^*(M) \to T^*(T^*(M))$ to define a form Λ on $T^*(M)$ $\Lambda \in \Gamma(\tau^*(T^*(M)))$, the *Liouville form*, whose exterior derivative will give the *symplectic form* ω on the cotangent space $T^*(M)$.

Let x be a point of M, $\alpha_x = \alpha_i \,\mathrm{d}q^i \in T_x^*(M)$ a 1-form and

$$X_{\alpha_x} = X^i \frac{\partial}{\partial q^i} + X_i^* \frac{\partial}{\partial p_i} \tag{2a.1}$$

an element of $T(T^*(M))$ at $\alpha_x(X^i = \mathrm{d}q^i(X_{\alpha_x}),\ X_i^* = \mathrm{d}p_i(X_{\alpha_x}))$. If $\langle\, ,\rangle$ denotes the contraction between vectors and 1-forms, it is clear that

$$\langle \lambda_M^{\mathrm{T}}(X_{\alpha_x}), \alpha_x\rangle = \langle X_{\alpha_x}, (\lambda_M^{\mathrm{T}})^* \alpha_x\rangle\,. \tag{2a.2}$$

(*) In the last line $(q^i, \mathrm{d}q^i)$ really means $q^i[\lambda_M^{\mathrm{T}}(\;)] = q^i$, $\dot q^i[\lambda_M^{\mathrm{T}}(\;)] = \mathrm{d}q^i$.

The Liouville form may now be defined [19] as the 1-form Λ over $T^*(M)$ such that $\Lambda: \alpha_x \to (\lambda_M^T)^* \alpha_x$. Indeed, since (appendix A.4)

$$\lambda_M^T(X_{\alpha_x}) = X^i \frac{\partial}{\partial q^i}, \tag{2a.3}$$

we find (2a.2) equal to $X^i \alpha_i$ and consequently Λ is written as

$$\Lambda = p_i \, \mathrm{d}q^i, \tag{2a.4}$$

since $p_i(\alpha_x) = \alpha_i$. The *symplectic form* on $T^*(M)$ is now defined as

$$\omega = -\mathrm{d}\Lambda, \tag{2a.5}$$

where the minus sign is introduced for convenience. In local co-ordinates,

$$\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i. \tag{2a.6}$$

Thus ω is an exact (2a.5) and nondegenerate (rank $(\omega) = 2m$) 2-form and $T^*(M)$ is endowed by ω with a *symplectic structure.* According to Darboux's theorem [2, 19, 22, 23] any symplectic manifold (*i.e.* any $2m$-dimensional manifold with a closed (locally exact) 2-form of rank $2m$) admits local (symplectic) charts (x^i, y_i) in which $\omega = \mathrm{d}x^i \wedge \mathrm{d}y_i$. When the symplectic structure is defined on $T^*(M)$ through (2a.5), the y_i are the $p_i \equiv \partial/\partial q^i$ and ω is given by (2a.6). Thus we may take ω in the above canonical form and extend to all symplectic manifolds any local assertion proved in the canonical basis (for instance, the base of phase space of mechanics) which is invariant with respect to canonical transformations (see below).

ω^m is, but for a numerical factor, $\mathrm{d}q^1 \wedge \ldots \wedge \mathrm{d}q^m \wedge \mathrm{d}p_1 \wedge \ldots \wedge \mathrm{d}p_m$; thus ω^m is a volume form on $T^*(M)$ and, consequently, the cotangent bundle is orientable.

b) The Poisson bracket. Symplectic diffeomorphisms (*). The application $X \to i_X \omega$, where i means inner product, defines an isomorphism between $\Gamma(\tau(T^*(M)))$, the modulus of vector fields on $T^*(M)$ (cross-sections of $\tau(T^*(M))$) and $\Gamma(\tau^*(T^*(M)))$, the modulus of 1-forms (cross-sections of $\tau^*(T^*(M))$) on $T^*(M)$. In local co-ordinates, for

$$X = X^i \frac{\partial}{\partial q^i} + X_i^* \frac{\partial}{\partial p_i}, \tag{2b.1}$$

we find

$$i_X \omega \equiv (\mathrm{d}q^i \wedge \mathrm{d}p_i)(X) = X^i \, \mathrm{d}p_i - X_i^* \, \mathrm{d}q^i. \tag{2b.2}$$

(*) As an additional reference for subsect. **2**b), the reader may consult [23a].

Reciprocally, given a 1-form $\alpha = \alpha_i \, dq^i + \alpha^{*i} \, dp_i \in \Gamma\big(\tau^*(T^*(M))\big)$, the associated field X_α is given by the condition

$$i_{X_\alpha}\omega = \alpha \tag{2b.3}$$

with the result

$$X_\alpha = \alpha^{*i}\frac{\partial}{\partial q^i} - \alpha_i \frac{\partial}{\partial p_i}\,. \tag{2b.4}$$

The Poisson bracket of two Pfaff forms α and β of $\Gamma\big(\tau^*(T^*(M))\big)$ is now defined by

$$\{\alpha, \beta\} \equiv i_{[X_\beta, X_\alpha]}\omega\,, \tag{2b.5}$$

i.e. it is the one-form associated through ω with the Lie bracket of the fields X_α, X_β associated with α and β. Thus the Poisson bracket of 1-forms may be considered as the prolongation of the Lie bracket on $\Gamma\big(\tau(T^*(M))\big)$ to to $\Gamma\big(\tau^*(T^*(M))\big)$.

Let f, g now be two functions on the symplectic manifold $T^*(M)$. *The Poisson bracket of f, g is defined by*

$$\{f, g\} \equiv \omega(X_f, X_g)\,, \tag{2b.6}$$

where X_f, X_g are the vector fields associated $\big((2b.3)\big)$ with the 1-forms df, dg (*). (The function $\omega\,(X_f, X_g)$ is sometimes called the *Lagrange bracket* of X_f, X_g.) Thus

$$\{f, g\} = i_{X_g} i_{X_f}\omega = L_{X_g} f = -L_{X_f} g\,, \tag{2b.7}$$

where L is the Lie derivative. In terms of local (canonical) co-ordinates (q^i, p_i) we have, for instance,

$$X_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i} \tag{2b.8}$$

and

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\,, \qquad i = 1, \dots, m\,, \tag{2b.9}$$

which is the usual expression for the Poisson bracket in Newtonian mechanics and so definition (2b.6) has identical global properties.

A canonical transformation is defined in classical mechanics as a transfor-

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(*) Note that, with this definition, $\{df, dg\}$ as defined by (2b.5) is simply $d\{f, g\}$ as given by (2b.6). This is easily checked by using the identity on forms $i_{[X_f, X_g]} = L_{X_f} i_{X_g} - i_{X_g} L_{X_f}$.

mation which preserves the Poisson bracket. Let (S, ω) and (S', ω') be two symplectic manifolds and $f: S \to S'$ a differentiable application. Then the application f preserves the symplectic structure when $f^*\omega' = \omega$, where f^* is the application dual of f (pull-back) operating on forms. When f is a diffeomorphism of $T^*(M)$ on $T^*(M)$ such that $f^*\omega = \omega$, f is a *symplectic diffeormorphism* or *symplectomorphism* [21] and defines a *canonical transformation* of the mechanical system. The canonical transformations preserve the volume element in phase space.

c) Hamiltonian systems. Symmetries of strictly symplectic systems. A *dynamical Hamiltonian system* on $T^*(M)$ (*) (frequently denoted by $(T^*(M), \omega, X_H)$) is a vector field X_H such that $i_{X_H}\omega$ is a closed Pfaff form. More precisely, one speaks of a locally Hamiltonian system in this situation, since in general the existence of a function H such that $i_{X_H}\omega = \mathrm{d}H$ is guaranteed only locally (Poincaré lemma). When $i_{X_H}\omega$ is both closed and exact, then

$$i_{X_H}\omega = \mathrm{d}H \tag{2c.1}$$

globally, and the dynamical system is globally Hamiltonian (**). In this case the Hamiltonian vector field admits *a* Hamiltonian function H on $T^*(M)$. We shall restrict ourselves to *globally Hamiltoniam systems* henceforth.

A Hamiltonian H is thus defined as a function $H: T^*(M) \to R$. By means of canonical co-ordinates (q^i, p_i), $H = H(q^i, p_i)$ and the associated vector field is given by

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{2c.2}$$

Thus the integral curves of X_H (applications $c: I \to T^*(M)$, where I is the unit interval, such that $\mathrm{d}c/\mathrm{d}t = X_H$) are given by the solutions of the *Hamilton equations*

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}. \tag{2c.3a}$$

whose symplectic structure is clearly exhibited when they are written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q_i \\ p_i \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} \partial H/\partial q^i \\ \partial H/\partial p_i \end{pmatrix}. \tag{2c.3b}$$

(*) These definitions are trivially extended to any symplectic manifold (S^{2m}, ω) $\omega|_U = \mathrm{d}x^i \wedge \mathrm{d}y_i$ $(U \subset S^{2m})$ and are not restricted to the case $S^{2m} = T^*(M^m)$, $\omega = -\mathrm{d}\Lambda$ previously considered.

(**) When the *first* cohomology group of S^{2m} vanishes, if the 1-form $i_X\omega$ is closed it is *also* exact, and a locally Hamiltonian system is also globally Hamiltonian.

It should be noted that in the above equations dq^i/dt is not necessarily $\dot{q}^i$; this will happen in the case of regularity to be discussed later on.

Under symplectic diffeomorphisms, the symplectic form ω and consequently the Hamilton equations are preserved. In the same way, a Hamiltonian vector field preserves the symplectic form and, if $L_X\omega = 0$, X is Hamiltonian: $L_X\omega = 0 \Leftrightarrow d(i_X\omega) = 0$, since $L_X = i_X\,d + d i_X$. X is often referred to as an (infinitesimal) *canonical transformation.*

Let us now consider the symmetries of the strictly symplectic dynamical systems up to now considered, *i.e.* those defined on a symplectic manifold S^{2m} (in particular $T^*(M)$). These systems, defined by a vector field X_H of $\Gamma(\tau(S))$ $\big(\Gamma\big(\tau(T^*(M))\big)\big)$, correspond to mechanical systems whose Hamiltonian function—the « energy integral »—does not depend explicitly on time.

Let $\mathcal{G}$ be a group of diffeomorphisms of the symplectic manifold of generators given by the Hamiltonian vector fields X^a_G. Then $\mathcal{G}$ is a *symmetry of the Hamiltonian system* X_H, and the functions G^a defined by

$$i_{X^a_G}\omega = dG^a \tag{2c.4}$$

are *constants of the motion*, when

$$L_{X^a_G} H = 0 \tag{2c.5}$$

(this condition may be relaxed to $L_{X^a_G}H = \text{const}$). Clearly, (2c.4) and (2c.5) imply

$$dH(X^a_G) = 0 = (i_{X_H}\omega)(X^a_G) = \omega(X^a_G, X_H) = -(i_{X^a_G}\omega)(X_H) = -dG^a(X_H)\,, \tag{2c.6}$$

i.e. the 1-forms dG^a are first integrals of X_H (*) and consequently the G^a are constants of the motion. This result may be considered as a restricted version of the Noether theorem to be considered later on (subsect. 7c)). We finally note that (2c.6) and (2b.5) give

$$[X^a_G, X_H] = 0 \tag{2c.7}$$

and that the expression $\omega(X^a_G, X_H) = 0$ reproduces the familiar result

$$\{G^a, H\} = 0\,, \tag{2c.8}$$

indicating that the Poisson bracket of the Hamiltonian and a conserved quantity is zero.

(*) A Pfaff form α is said to be a first integral of X if $\alpha(X) \equiv \langle X, \alpha\rangle = 0$. This generalizes the following definition: a function f is a first integral of X if $X.f = L_X f = df(X) = 0$.

In the case of the symplectic mechanics the symplectic manifold is $(T^*(M), \omega)$, where ω is defined by (2a.5). Thus we find that, in general, to consider a symmetry problem the vector field has to be defined on the symplectic manifold. However, in the case of mechanics, X_G^a is usually defined as a diffeomorphism of M, not of $T^*(M)$. Thus a canonical procedure is required to lift vector fields on M to vector fields on the cotangent space $T^*(M)$. The situation is summarized by the following diagram:

$$\begin{array}{ccc} T(T^*(M)) & \xrightarrow{\lambda_M^T} & T(M) \\ \bar{X} \in \Gamma(\tau(T^*(M)))\uparrow & & \uparrow X \in \Gamma(\tau(M)) \\ T^*(M) & \xrightarrow{\lambda_M} & M \end{array},$$

where $\bar{X}$ is the lift of X to the cotangent bundle; the diagram is easily understood by recalling that a vector field on a manifold is a cross-section of the corresponding tangent bundle (appendix A). $\bar{X}$ is now uniquely determined by the following two conditions (cfr. appendix A.4):

a) $X\circ\lambda_M = \lambda_M^T\circ\bar{X}$ ($\bar{X}$ is projected onto X, commutativity of the diagram),

b) $L_{\bar{X}}\Lambda = 0$ (invariance of the canonical Liouville form).

By writing $\bar{X} = \bar{X}^i(\partial/\partial q^i) + X_i^*(\partial/\partial p_i)$, $X = X^i(\partial/\partial q^i)$, condition *a*) immediately gives $\bar{X}^i = X^i$ and condition *b*) gives

$$X_i^* = -p_j\frac{\partial X^j}{\partial q^i}, \tag{2c.9}$$

so that ((A.4.4))

$$\bar{X} = X^i\frac{\partial}{\partial q^i} - p_j\frac{\partial X^j}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{2c.10}$$

In this way, the vector field $\bar{X}$ is automatically Hamiltonian, since $L_{\bar{X}}\Lambda = 0$ implies $i_{\bar{X}}\mathrm{d}\Lambda = -\mathrm{d}(i_{\bar{X}}\Lambda) = -\mathrm{d}(\Lambda(\bar{X}))$, so that $i_{\bar{X}}\omega$ is exact. Thus the restricted above-mentioned version of the Noether theorem may be formulated for the case of mechanics $(T^*(M), \omega = -\mathrm{d}\Lambda)$ as follows:

«Let X_H be a Hamiltonian system on $T^*(M)$ and let $\mathrm{d}H = i_{X_H}\omega$. If X_s is a vector field on M such that $(\mathrm{d}H)(\bar{X}_s) = 0$, where $\bar{X}_s$ is the canonical lift of X_s to $T^*(M)$, then the 1-form $\mathrm{d}(i_{\bar{X}_s}\Lambda) = \mathrm{d}(\Lambda(\bar{X}_s))$ (or the function $\Lambda(\bar{X}_s) = G_s$) is a first integral of X_H.»

The theorem follows from the above considerations by putting $\omega = -\mathrm{d}\Lambda$ and realizing then that $i_{\bar{X}_s}\omega = \mathrm{d}G_s$ implies $\mathrm{d}(i_{\bar{X}_s}\Lambda) = \mathrm{d}G_s$. In local co-ordinates we obtain from (2c.10) and $\Lambda = p_i\mathrm{d}q^i$ the result $\Lambda(\bar{X}_s) = X_s^i p_i$.

Example. As a simple example, let us obtain in the case of a free particle the conserved quantities associated to the generators (on E_3, the Euclidean three-dimensional space) of the translations and the rotations of the Galilei group (*). These generators are written on M as

$$X_{(i)} = \delta_i^j \frac{\partial}{\partial x^j}, \qquad M_{(i)} = \varepsilon_{ij\cdot}^{\;\;k} x^j \frac{\partial}{\partial x^k}, \qquad i, j, k = 1, 2, 3 . \tag{2c.11}$$

Thus, since $X^j_{(i)} = \delta^j_i$ and $M^k_{(i)} = \varepsilon_{ij\cdot}^{\;\;k} x^j$, the conserved quantities turn out to be, as expected, the momentum $\boldsymbol{p}$ and the angular momentum $\boldsymbol{x}\wedge\boldsymbol{p}$. Note that, in evaluating the conserved quantities, only the components on M of the symmetry vector field are relevant. However, the full vector field on $T^*(M)$ is required to check whether X_s is a symmetry or not. For instance, for the rotations one gets from (2c.11) and (2c.10)

$$\bar{M}_{(i)} = \varepsilon_{ij\cdot}^{\;\;k}\left(x^j \frac{\partial}{\partial x^k} - p_k \frac{\partial}{\partial p_j}\right) \tag{2c.12}$$

and one easily verifies that $(\mathrm{d}H)\bar{M}_{(i)} = 0$, since $\mathrm{d}H = (\boldsymbol{p}/m)\,\mathrm{d}\boldsymbol{p}$.

3. – Lagrangian formulation of classical mechanics.

a) The Legendre transformation [9, 19, 2]. In the previous section we have constructed the Liouville form $\Lambda \in \Gamma\big(\tau^*(T^*(M))\big)$ in a canonical way and developed from it the Hamiltonian formalism on the cotangent space $T^*(M)$ (**). There is not, however, such a Liouville form on $T(M)$ nor there exists a canonical way to transport the symplectic structure on $T^*(M)$ to $T(M)$ because of the absence of a *canonical* isomorphism between a space $(T^*(M))$ and its dual $(T(M))$. Nevertheless, it is possible to transport the formalism to $T(M)$ when a function is defined on this space, which satisfies certain conditions. Such a function $L: T(M) \to R$, $L = L(q^i, \dot{q}^i)$ is called *Lagrangian* and the equivalence between both formulations follows when the Lagrangian is *regular*.

To see how this can be performed, we need first the concept of vertical or fibre derivative. Let M be a manifold of dimension m, $T(M)$ its tangent space, $T_x(M)$ the fibre space over a point $x \in M$ (which is itself a vector space of the same dimension m) and L a Lagrangian function on $T(M)$. The fibre derivative D^v of L_x (restriction of L to the fibre over $x \in M$), $D^v L_x$, is defined as the derivative of L restricted to the fibre at x, $T_x(M)$. By extending this

(*) In the natural realization, the Galilean boosts require the explicit presence of time and thus cannot be defined on M.
(**) As already mentioned, it is possible to develop directly a Hamiltonian formulation on any symplectic space (S^{2m}, ω).

definition of D^v to all fibres, the *vertical or fibre derivative* D_L (the derivative of L in each fibre of $T(M)$) is a map $D_L: T(M) \to T^*(M)$ such that $\forall x \in M$, $X_x \in T_x(M)$,

$$D_L: X_x \in T_x(M) \to (D^v L)(X_x) \equiv D(L|_x)(X_x) \in T_x^*(M) . \tag{3a.1}$$

Indeed $(D^v L_x)(X_x)$ is a linear application and defines a covector of $T_x^*(M)$, $(D^v L)(X_x)\cdot(X_x') \in R$. By definition, D_L is a fibre-preserving map; however, since the correspondence $X_x \to (D^v L_x)(X_x)$ is not necessarily linear, D_L is not, in general, a vector bundle mapping.

Let e (e^*) be a point of $T(M)$ $(T^*(M))$. Using, as is customary, the same symbol for the co-ordinate system as for the co-ordinates themselves, we obtain

$$e \in T(M) \to e^* = D_L e \in T^*(M) \Big/ \left\{ \begin{array}{l} q^i(e) = q^i \\ \dot{q}^i(e) = \dot{q}^i \end{array} \right\} , \quad \left\{ \begin{array}{l} q^i(e^*) = q^i \\ p_i(e^*) = \left.\dfrac{\partial L}{\partial \dot{q}^i}\right|_e \end{array} \right\} . \tag{3a.2}$$

(3a.2) defines D_L in terms of local co-ordinates. D_L will have an inverse if the Jacobian of the transformation det $(\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j)$ (the Hessian of L) is different from zero; this corresponds to the situation for which the tangent application $D_L^{\mathrm{T}}: T(T(M)) \to T(T^*(M))$ is an isomorphism. In this case D_L is frequently referred to as the *Legendre transformation* and L is said to be regular.

Other considerations on the Legendre transformation may be found in [25].

b) The symplectic form ω_L on $T(M)$. Lagrange equations. Let $L(q^i, \dot{q}^i)$ be a regular Lagrangian on the configuration space. D_L allows us to transport the symplectic form on $T^*(M)$ to $T(M)$; if D_L^* is the pull-back of the Legendre application acting on forms,

$$D_L^*: \omega = \mathrm{d}q^i \wedge \mathrm{d}p_i \to \omega_L = \mathrm{d}q^i \wedge \mathrm{d}\left(\frac{\partial L}{\partial \dot{q}^i}\right) . \tag{3b.1}$$

Thus

$$\omega_L = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j} \mathrm{d}q^i \wedge \mathrm{d}\dot{q}^j . \tag{3b.2}$$

Clearly,

$$\omega_L^m \propto \det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) \mathrm{d}q^1 \wedge \ldots \wedge \mathrm{d}q^m \wedge \mathrm{d}\dot{q}^1 \wedge \ldots \wedge \mathrm{d}\dot{q}^m \tag{3b.3}$$

is a volume form on $T(M)$ (L is regular) in the same way ω^m was for $T^*(M)$ (subsect. **2**a)).

We may now proceed along lines similar to those of subsect. **2**c). A Hamiltonian vector field X_E on $T(M)$ is a vector field such that

$$i_{X_E} \omega_L = \mathrm{d}E , \qquad X_E \in \tau(T(M)) \tag{3b.4}$$

(we consider X_E globally Hamiltonian, so that $i_{X_E}\omega$ is exact). Thus, given a function E on $T(M)$, we may associate to it a set of « Hamiltonian equations ». The question now arises of finding E from L so that (3*b*.4) leads to the Lagrange equations. This is accomplished by giving the *action* A_L of L as an intermediate step. The action A_L of L is the application $A_L\colon T(M) \to R$ defined by

$$A\colon X_x \in T_x(M) \to (D^v L)(X_x)\cdot(X_x) \in R\,. \tag{3b.5}$$

In local co-ordinates,

$$A_L\colon (q^i, \dot q^i) \to \frac{\partial L}{\partial \dot q^i}\dot q^i\,. \tag{3b.6}$$

Then the function E, the *energy* of the system described by L, is the real function on $T(M)$ given by

$$E = A_L - L = \frac{\partial L}{\partial \dot q^i}\dot q^i - L(q^i, \dot q^i)\,. \tag{3b.7}$$

It is now easy to show that $X_E = X_E^i(\partial/\partial q^i) + \dot X_E^i(\partial/\partial \dot q^i)$ generates the Lagrange equations through (3*b*.4), if E is the energy as given by (3*b*.7). As an example, we shall perform this calculation in detail. From

$$\mathrm{d}E = \frac{\partial E}{\partial q^i}\mathrm{d}q^i + \frac{\partial E}{\partial \dot q^i}\mathrm{d}\dot q^i \tag{3b.8}$$

and

$$i_{X_E}\omega_L = \frac{\partial^2 L}{\partial \dot q^i\,\partial q^j}(X_E^i\,\mathrm{d}q^j - X_E^j\,\mathrm{d}q^i) + \frac{\partial^2 L}{\partial \dot q^i\,\partial \dot q^j}(X_E^i\,\mathrm{d}\dot q^j - \dot X_E^j\,\mathrm{d}q^i) \tag{3b.9}$$

we get

$$\begin{cases} \dfrac{\partial E}{\partial q^i} = \left(\dfrac{\partial^2 L}{\partial \dot q^j\,\partial q^i} - \dfrac{\partial^2 L}{\partial \dot q^i\,\partial q^j}\right)X_E^j - \dfrac{\partial^2 L}{\partial \dot q^i\,\partial \dot q^j}\dot X_E^j\,, \\ \dfrac{\partial E}{\partial \dot q^i} = \dfrac{\partial^2 L}{\partial \dot q^i\,\partial \dot q^j}X_E^j \equiv M_{ij}X_E^j\,. \end{cases} \tag{3b.10}$$

Thus a vector field X_E which satisfies (3*b*.4) is given by

$$\begin{cases} X_E^j = M^{ji}\dfrac{\partial E}{\partial \dot q^i}\,, \\ \dot X_E^j = M^{ji}\left[-\dfrac{\partial E}{\partial q^i} + \left(\dfrac{\partial^2 L}{\partial \dot q^k\,\partial q^i} - \dfrac{\partial^2 L}{\partial \dot q^i\,\partial q^k}\right)M^{kl}\dfrac{\partial E}{\partial \dot q^l}\right], \end{cases} \tag{3b.11}$$

where M^{ji} is the inverse of M_{il}. X_E generates the following equations:

$$X_E^i = \frac{\mathrm{d}q^i}{\mathrm{d}t}, \qquad \dot{X}_E^i = \frac{\mathrm{d}\dot{q}^i}{\mathrm{d}t} \tag{3b.12}$$

and it only remains to show that (3*b*.12) are equivalent to the Lagrange equations when E is the energy of L. From

$$\frac{\partial E}{\partial \dot{q}^i} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\dot{q}^j, \qquad \frac{\partial E}{\partial q^i} = \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j}\dot{q}^j - \frac{\partial L}{\partial q^i} \tag{3b.13}$$

we get $X_E^i = \dot{q}^i = \mathrm{d}q^i/\mathrm{d}t$, which indicates that eqs. (3*c*.12) couple together (because of the regularity of L) in a single second-order set of equations which reduces to

$$\left\{\begin{array}{l} \dfrac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j}\dfrac{\mathrm{d}^2 q^k}{\mathrm{d}t^2} + \dfrac{\partial^2 L}{\partial q^k \partial \dot{q}^j}\dfrac{\mathrm{d}q^k}{\mathrm{d}t} - \dfrac{\partial L}{\partial q^j} = 0\,, \\ \textit{i.e.} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{q}^j}\right) - \dfrac{\partial L}{\partial q^j} = 0\,, \end{array}\right. \tag{3b.14}$$

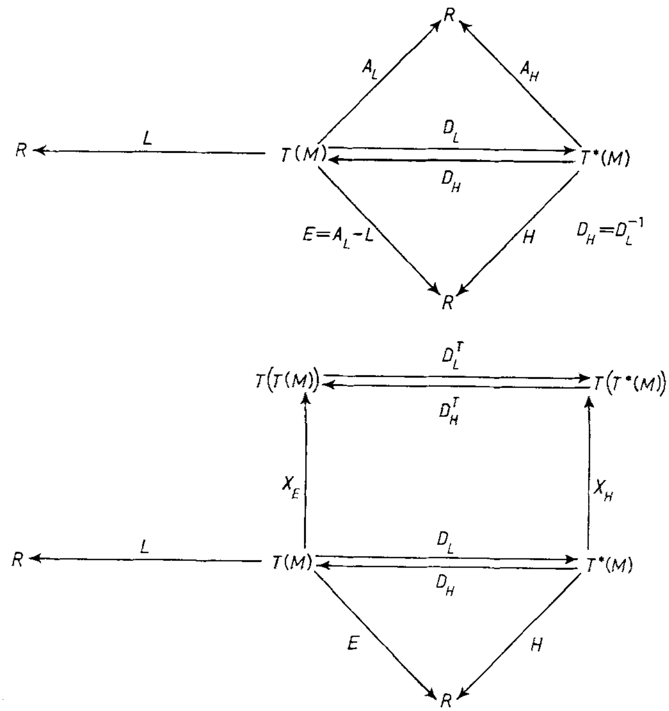
which constitute the *Lagrange equations*.

4. – Equivalence between the (regular) Hamiltonian and Lagrangian formulations.

We may now ask which is the Hamiltonian H (or the vector field X_H) which corresponds to the energy E of subsect **3***b*). Since D_L is a diffeomorphism between $T(M)$ and $T^*(M)$, there exists an application $D_L^{-1}(\equiv D_H): T^*(M) \to T(M)$. Then the (obviously *regular*) Hamiltonian H is defined by

$$H = E \circ D_L^{-1} = E \circ D_H\,. \tag{4.1}$$

Thus the function $H: T^*(M) \to R$ takes at (q^i, p_j) the value $\dot{q}^i(\partial L/\partial \dot{q}^i) - L$, as was expected. There is now a complete symmetry between both (regular) Hamiltonian and Lagrangian formalisms; L determines $D_L(= D_H^{-1})$ and H, and the regular Hamiltonian H (for which the regularity condition reads $\det(\partial^2 H/\partial p_i \partial p_j) \neq 0$) determines $D_H(= D_L^{-1})$ and L. The contents of subsect. **3***a*) through subsect. **3***c*) may be now summarized by the following commutative diagrams:



that we take from [2]. In the first of them, A_H is the *action* on $T^*(M)$ defined as

$$A_H: (q^i, p_j) \to p_i \frac{\partial H}{\partial p_i} \tag{4.2}$$

and D_H may be written as

$$D_H: (q^i, p_j) \to \left(q^i, \frac{\partial H}{\partial p_j}\right) = (q^i, \dot{q}^j) .$$

PART II

Time-dependent mechanical systems: contact structure.

5. – Introduction.

As has been shown in the previous part, one of the virtues of the symplectic formalism for the time-independent mechanical systems is that of providing a canonically defined Poisson bracket for any pair of differentiable functions on the phase space $T^*(M)$. Only the definition of the symplectic form is necessary, and $T^*(M)$ is *always* endowed with a symplectic structure. (By contrast, a symplectic structure is not granted on $T(M)$; this requires the existence of a regular Lagrangian.)

When the definition space of physical quantities is enlarged to include the time t (a situation which corresponds to the « explicit » dependence on t), the phase space is accordingly extended to $T^*(M)\times R$ (and the « configuration » space to $T(M)\times R$), and thus the symplectic structure is lost since the dimension of the space $(2m+1)$ is odd. Nevertheless, on the manifold of *solutions* of a Hamiltonian field depending on time, it is possible to define a symplectic structure in such a way that a symplectomorphism with $T^*(M)$ is obtained for each time t_0 (*). As in the case of the « time-independent » mechanics (*i.e.* mechanics of autonomous systems), the definition of a symplectic structure on the manifold of solutions on $R\times T(M)$ will require the existence of a regular Lagrangian. In any case, however, the canonical structure is again lost in the sense that the previous existence of a Hamiltonian on $T^*(M)\times R$ (or a Lagrangian on $T(M)\times R$) is required.

Let us now turn our attention to the contact structure which, for time-independent systems, replaces the symplectic structure.

6. – Contact structures.

Let $\mathscr{C}$ be a manifold of odd dimension $2m+1$. A *contact structure* is the pair $(\mathscr{C}, \omega)$ where ω is a closed *two*-form of maximal rank (which will be $2m$). An *exact contact manifold* is the pair $(\mathscr{C}, \Theta)$, where Θ is a *one*-form of constant class $2m+1$, *i.e.* such that the codimension of the characteristic space of Θ(**) is $2m+1$. It is not difficult to see that Θ defines an exact contact structure if and only if $\Theta\wedge(\mathrm{d}\Theta)^m$ is a volume form on $\mathscr{C}$ and that accordingly $\mathscr{C}$ is orientable.

The Darboux theorem of subsect. **2***a*) is easily extended to the odd-dimension case. Since the class of a form of constant class is the minimal number of functions required to express it, around each point $y\in\mathscr{C}$ there is a local

(*) In the case of mechanics, the definition of a symplectic structure on the manifold of solutions does not present special difficulties. The case of classical field theory is more difficult, however, since the equations of motion are equations on partial derivatives instead of ordinary differential equations and thus a point in « phase space » does not determine a single solution. The case of mechanics will be considered in sect. **9**.

(**) Given a form Ω, a *characteristic vector field* of Ω is a vector field X such that $i_X\Omega = i_X\mathrm{d}\Omega = 0$. The codimension of the characteristic vector space at a point y (*i.e.* codim $(\mathrm{rad}\,\Omega\cap\mathrm{rad}\,\mathrm{d}\Omega)(y)$) is the *class* of the form Ω at y. If Ω defines a contact structure, $(\mathrm{rad}\,\Omega)\cap(\mathrm{rad}\,\mathrm{d}\Theta)=0$. If X is a characteristic vector field of Ω, Ω is said to be an *absolute integral invariant* of X.

co-ordinate system (z, u^i, v_i), such that on $U \subset \mathscr{C}$, $y \in U$:

$$\Theta|_U = \mathrm{d}z + v_i\,\mathrm{d}u^i\,. \tag{6.1}$$

The 1-form Θ is called *contact 1-form*; it is clear that any contact structure $(\mathscr{C}, \omega)$ is locally exact.

An instance of contact structure is obtained when, in a time-independent (autonomous) mechanical system, the symplectic form is restricted to a hypersurface of constant energy. Here, however, we are rather more interested in the reciprocal situation, *i.e.* in extending—by adding the time—a symplectic structure to a contact structure. This process is guaranteed by the following *Proposition*:

Let (S, ω) be a symplectic manifold of dimension $2m$ and let p be the canonical projection of $R\times S \to S$ defined by $p:(z, s) \to s$. Then $(R\times S, p^*\omega \equiv \tilde{\omega})$ —where $p^*\omega$ is the pull-back of ω to $R\times S$—is a contact structure. If, in addition, $\omega = -\,\mathrm{d}\Lambda$, where Λ is a 1-form, then $(R\times S, \Theta)$, where

$$\Theta = \mathrm{d}z + p^*\Lambda\,, \tag{6.2}$$

is an exact contact structure. The contact form $\tilde{\omega}$ has as characteristic vector field $\big(i_X\tilde{\omega} = 0$ (*)$\big)$ $X = \partial/\partial z$, the generator of the displacement on R. In fact, if ω is closed and the characteristic bundle [2] $\big(\{X \in T(R\times S)\,|\,i_X\omega = 0\}\big)$ is of dimension one, ω is a contact form.

7. – Mechanics of nonautonomous systems.

a) Preliminaries. Let (S, ω) be a symplectic manifold, $H(t, s)$ a differentiable function on $R\times S$ and let $p^*\omega \equiv \tilde{\omega}$ denote the lifting to $R\times S$ of the symplectic form on S. The closed two form on $R\times S$

$$\Omega_H = \tilde{\omega} + \mathrm{d}H\wedge\mathrm{d}t \tag{7a.1}$$

defines a contact structure, because, as is immediately checked, $\mathrm{d}t\wedge\Omega_H^m$ is a volume form on $R\times S$, since ω^m is a volume form on S. Thus $(R\times S, \Omega_H)$ is a contact manifold of dimension $2m+1$.

If, in addition, $\Omega_H = -\,\mathrm{d}\Theta_H$ and $H(s, t) \neq 0$ for every pair $(t, s) \in R\times S$, then $(R\times S, \Theta_H)$, where

$$\Theta_H = p^*\Lambda - H\,\mathrm{d}t\,, \qquad \mathrm{d}\omega = -\,\mathrm{d}\Lambda\,, \tag{7a.2}$$

(*) Since $\mathrm{d}\tilde{\omega} = 0$, to define a characteristic vector field it suffices that $i_X\tilde{\omega} = 0$.

is an exact contact manifold. The contact 1-form Θ_H is usually called *Poincaré-Cartan form*; in the calculus of variations (to be considered in sect. **12**) the integral $\int\Theta_H$ is frequently called *Hilbert's invariant integral.* The following *proposition* (due to Cartan) will be useful later on:

Let (S, ω) be a symplectic manifold, and let $H(t, s)$ be a function on $R\times S$. There is a unique vector field $\tilde{X}_H \in \Gamma(\tau(R\times S))$ such that

a) $\mathrm{d}t(\tilde{X}_H) = 1$,

b) $i_{\tilde{X}_H}\Omega_H = 0$ (or $i_{\tilde{X}_H}\mathrm{d}\Theta_H = 0$ in the case $\Omega_H = -\mathrm{d}\Theta_H$ (*)).

To show that this is the case, it is sufficient to note that condition *a*) requires $\tilde{X}_H$ to be of the form

$$\tilde{X}_H = \frac{\partial}{\partial t} + X_t\,, \qquad X_t \in \Gamma(\tau(S))\,. \tag{7a.3}$$

Then, if we call $H_t = H|_{t\times S}$ (*i.e.* the restriction of H to a time t) condition *b*) reads, since $\Omega_H = \tilde{\omega} + \mathrm{d}H\wedge\mathrm{d}t$,

$$i_{\tilde{X}_H}\Omega_H = i_{X_t}\omega + \mathrm{d}H_t\,(X_t)\,\mathrm{d}t - \mathrm{d}H_t = 0\,,$$

which is zero if X_t is the Hamiltonian field of ω (associated with H_t), since then ((2*c*.1)) $i_{X_t}\omega = \mathrm{d}H_t$ and $(\mathrm{d}H)X_t = i_{X_t}i_{X_t}\omega = 0$. Thus $\tilde{X}_H$ is the sum of the generator of the time translations and of the Hamiltonian vector field corresponding to the fixed-time Hamiltonian H_t; $\tilde{X}_H$ may be called the *dynamical system associated with the contact form* $\Omega_H(\Theta_H)$.

b) *Hamilton equations.* Let us now extend the situation of subsect. **2***c*) to include the explicit dependence on time. With the same notation of sect. **2**, let M be a differentiable manifold of dimension m, $T^*(M)$ the cotangent space, Λ the Liouville form on $T^*(M)$, but let now H be a differentiable function on $R\times T^*(M)$. The pair $(R\times T^*(M), \Theta_H)$ with Θ_H having the same expression as in (7*a*.2) is clearly an *exact contact manifold associated with the Hamiltonian function H.* Let $\{t, q^i, p_i\}$ be a local co-ordinate system on $R\times T^*(M)$. (We might mention that the space $R\times T^*(M)$ obtained by adding time to the usual even-dimensional phase space is called *evolution space* [21].) Then $\tilde{X}_H$ is given by

$$\tilde{X}_H = \frac{\partial}{\partial t} + \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\right). \tag{7b.1}$$

(*) In the language of Cartan [1], this is equivalent to saying that Θ_H (Ω_H) is a *relative* (*absolute*) *integral invariant* of $\tilde{X}_H$.

The integral curves corresponding to this dynamical system are thus given by the equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}\lambda}=\frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}\lambda}=-\frac{\partial H}{\partial q^i}, \qquad \frac{\mathrm{d}t}{\mathrm{d}\lambda}=1, \tag{7b.2}$$

where λ is the parameter of the curves. The last equation allows us to identify (up to a shifted origin) λ with time and thus the first two are the familiar Hamilton equations (cf. (2c.3)). We shall call $\mathscr{U}_H$ the space of cross-sections $s\in\Gamma(R\times T^*(M)\to R)$ solutions of (7b.2).

Given $\tilde{X}_H$, it is trivial to calculate the evolution with time of any physical quantity specifed by a function F on $R\times T^*(M)$:

$$L_{\tilde{X}_H}F=\frac{\partial F}{\partial t}+\{F,H\}, \tag{7b.3}$$

where $\{\,,\}$ indicates the Poisson bracket on $T^*(M)$. F is a constant of the motion if $L_{\tilde{X}_H}F=0$; if $F\neq F(t)$, (2c.8) is recovered. In particular, if $F=H$, $L_{\tilde{X}_H}H=\partial H/\partial t$, which shows that H is not a conserved quantity if $H=H(t)$.

c) Symmetries and the Noether theorem. Let us consider a mechanical system on $R\times T^*(M)$ defined by a Hamiltonian function H. We shall write the associated Poincaré-Cartan form as $\Theta=\Lambda-H\,\mathrm{d}t$ without expliciting the subscript H and the pull-back p^*. Before formulating the Noether theorem, we prove the following simple *lemma*:

Let X be a vector field on $R\times T^*(M)$, $X\in\Gamma\big(\tau(R\times T^*(M))\big)$. Then we have

$$i_X\,\mathrm{d}\Theta|_s=0 \tag{7c.1}$$

for any section $s\in\mathscr{U}_H$, the set of solutions of the Hamiltonian field $\tilde{X}_H$ given by (7b.1).

The proof is done by direct computation. We shall make it here since expressions similar in structure to (7c.1) will appear later on (sect. **11**).

An arbitrary vector field on $R\times T^*(M)$ is of the form

$$X=X^0\frac{\partial}{\partial t}+X^i\frac{\partial}{\partial q^i}+X_i^*\frac{\partial}{\partial p_i}. \tag{7c.2}$$

From $\mathrm{d}\Theta=\mathrm{d}p_i\wedge\mathrm{d}q^i-\mathrm{d}H\wedge\mathrm{d}t$ we get $i_X\mathrm{d}\Theta=-i_X\omega-\mathrm{d}H(X)\,\mathrm{d}t+\mathrm{d}t(X)\,\mathrm{d}H_t$, *i.e.*

$$i_X\,\mathrm{d}\Theta=\left(X_i^*+X^0\frac{\partial H}{\partial q^i}\right)\mathrm{d}q^i-\left(X^i-X^0\frac{\partial H}{\partial p_i}\right)\mathrm{d}p_i-\left(\frac{\partial H}{\partial q^i}X^i+\frac{\partial H}{\partial p_i}X_i^*\right)\mathrm{d}t$$

and, restricting ourselves to cross-sections $s \in \Gamma(R\times T^*(M))$,

$$(7c.3)\qquad i_X\,\mathrm{d}\Theta|_s = \\ = \left\{X_i^*\left(\frac{\mathrm{d}q^i}{\mathrm{d}t}-\frac{\partial H}{\partial p_i}\right)+X^i\left(-\frac{\partial p_i}{\partial t}-\frac{\partial H}{\partial q^i}\right)+X^0\left(\frac{\partial H}{\partial q^i}\frac{\mathrm{d}q^i}{\mathrm{d}t}+\frac{\partial H}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t}\right)\right\}\mathrm{d}t = 0\,.$$

It is clear that $(7c.3)$ is zero for cross-sections of $\mathscr{U}_H$. Reciprocally, given an arbitrary vector field $X\in\Gamma\big(\tau(R\times T^*(M))\big)$, a cross-section s of $\Gamma(R\times T^*(M)\to R)$ satisfies $(7c.1)$ if it is a solution of the Hamilton equations, since, X_i^* and X^i being arbitrary, the coefficients in $(7c.3)$ must be zero (the coefficient of X^0 then vanishes identically). Thus the solutions of the Hamilton equations are equally determined by the sections s which satisfy $(7c.1)$ $\forall X$. This result will be also obtained from the variational principle (sect. **12**).

The *Noether theorem* may now be established precisely [10]. Let X_s be a vector field which corresponds to a symmetry of the dynamical system $(R\times T^*(M), \Theta)$, *i.e.* such that

$$(7c.4)\qquad L_{X_s}\Theta = \mathrm{d}\alpha_{X_s}\,,$$

where $\mathrm{d}\alpha_{X_s}$ is a 1-form (which may be zero) depending on X_s. Then

$$(7c.5)\qquad \mathrm{d}(i_{X_s}\Theta-\alpha_{X_s})|_s = 0 \qquad\qquad \forall s\in\mathscr{U}_H$$

and the function $(i_{X_s}\Theta-\alpha_{X_s})$ is the Noehter invariant associated with the symmetry generated by X_s. The proof is simple: $(7c.4)$ may be written as $(i_{X_s}\mathrm{d}+\mathrm{d}i_{X_s})\Theta = \mathrm{d}\alpha_{X_s}$, and, by restricting ourselves to cross-sections of $\mathscr{U}_H$, $(7c.5)$ is obtained by using $(7c.1)$.

In the particular case in which $H\neq H(t)$, $L_{X_s}\Theta = 0$ (we shall omit $\mathrm{d}\alpha_{X_s}$) gives for a field X_s on $T^*(M)$ $(X^0 = 0)$

$$(7c.6)\qquad L_{X_s}\Lambda = 0\,,\qquad L_{X_s}H = 0\,.$$

The second expression of $(7c.6)$ is simply $(2c.5)$ and the first, written as $i_{X_s}\omega = \mathrm{d}(i_{X_s}\Lambda)$, tells us that X_s is a Hamiltonian field in the sense of subsect. **2***c*). This reproduces again the results of subsect. **2***c*) $\big((2c.8)\big)$ with the conserved quantity $G_s = i_{X_s}\Lambda$.

Example. Let us consider the simplest case of a free particle of mass m, for which $H = \boldsymbol{p}^2/2m$. The generators of the Galilei boosts on $R\times T^*(M)$ are given by

$$(7c.7)\qquad X_{(i)} = t\delta_i^j\frac{\partial}{\partial q^j}+m\delta_{ij}\frac{\partial}{\partial p_j}\,.$$

Then

$$(7c.8)\qquad \begin{cases} i_{X_{(i)}}\mathrm{d}\Theta = m\delta_{ij}\,\mathrm{d}q^j - t\,\mathrm{d}p_i - p_i\,\mathrm{d}t\,, \\ i_{X_{(i)}}\Theta \;= p_i t\,, \qquad \mathrm{d}i_{X_{(i)}}\Theta = p_i\,\mathrm{d}t + t\,\mathrm{d}p_i\,, \\ L_{X_{(i)}}\Theta \;= \mathrm{d}(m\delta_{ij}q^j)\,. \end{cases}$$

Thus the associated Noether invariant is given by $i_{X_{(i)}}\Theta - m\delta_{ij}q^j = p_i t - mq_i$, *i.e.* $(\boldsymbol{p}/m)t - \boldsymbol{x}$ is a constant of the motion.

An extended note on the Galilei group. The above expression of the generators of the Galilei boosts could be obtained, for instance, from the corresponding ones on $R\times T(M)$ ((B3.21)) by means of the Legendre transformation. The reader may wonder, however, about the appearance of the mass m of the particle in (7c.7) which does not appear in the expression of the boosts on $R\times T(M)$. This is a result which may be traced to the peculiar structure of the Galilei group [26] and to the fact that, although there is a canonical way to prolong vector fields on $R\times M$ to $R\times T(M)$ ((B3.19)), this is not the case when one tries to expand their action to the evolution space $R\times T^*(M)$, which is the relevant space for the Hamiltonian formalism. In fact, it may be seen in general that, to obtain a canonical realization of a symmetry group (canonical meaning here in terms of canonical transformations, see the next section), one has to consider [27, 28] the realization of its Lie algebra in terms of Poisson brackets in which the structure constants of the Lie algebra are kept only up to some additive numerical constants. In this way, what turns out to be relevant is an extension of the Lie algebra by neutral elements (*i.e.* which commute with all others). These, which give rise to the factor system or phase exponents, determine an extension of the symmetry group by the « phase group ». The different extensions are determined by the second cohomology group; in the case of the Galilei group in which we are interested, $H^2(\mathscr{G}, U_1)$ (where U_1 is the « phase group ») turns out to be $Z_2\otimes R$ [29]. The elements of R (the cyclic group $(1, -1)$ is unimportant in our context) characterize the mass of the particle. This is the reason why the mass of the particle may be interpreted [21] as a cohomology class of the Galilei group. In contrast, the role played by the mass in the case of Poincaré is completely different: it has been known since the work of Wigner [30] that $H^2(\mathscr{P}_+^\uparrow, U_1) = Z_2$, a result which is usually formulated by saying that all the projective representations of $\mathscr{P}_+^\uparrow$ come from the representations of $\overline{\mathscr{P}}_+^\uparrow$, its universal covering group. Thus the mass of the elementary systems appears as an index partially labelling an irreducible representation of $\mathscr{P}_+^\uparrow$, not an extension of it.

Finally, it might be interesting to mention that the special role played by the mass in Galilean mechanics also shows up in the Lagrangian formulation on $R\times T(M)$, although not directly, since the extension of vector fields to this space is canonically defined with independence of the given Lagrangian associated with the dynamical system. In fact, it is well known [31] that the Lagrangian of a free particle is not invariant under Galilean boosts, but that the transformed Lagrangian differs from it in a total time derivative of a certain function involving the mass, sometimes called gauge function, which in the variational approach does not alter the equations of motion. It may be then seen [32] that there is a close relation between the gauge functions and the group exponents mentioned earlier and that these gauge functions are specified by the equivalence classes of the exponents, *i.e.* by the mass m.

d) Canonical transformations. Let (S, ω) and (S', ω') be two symplectic manifolds of dimension $2m$ and let $(R\times S, \Omega_H)$, $(R\times S', \Omega_{H'})$ be the corresponding contact manifolds. A diffeomorphism $C: R\times S \to R\times S'$ is a canonical transformation

a) if the time is preserved, *i.e.* if the diagram

$$\begin{array}{ccc} R\times S & \xrightarrow{C} & R\times S' \\ p\downarrow & & \downarrow p' \\ R & \xleftrightarrow{I} & R \end{array},$$

where I is the identity on R, is commutative; *and*

b) if H and H' are functions on $R\times S$ and $R\times S'$,

$$C^* \Omega_{H'} = \Omega_H ,$$

where $\Omega_H = \tilde{\omega} + \mathrm{d}H\wedge \mathrm{d}t$ and $\Omega_{H'} = \tilde{\omega}' + \mathrm{d}H'\wedge \mathrm{d}t$.

It may be seen that, as a consequence of *b)*, $C^{\mathrm{T}}\circ \tilde{X}_H = \tilde{X}_{H'}\circ C$ and that the canonical transformations accordingly preserve the Hamiltonian form of the equations of motion. We shall not dwell any longer on the canonical transformations nor consider the Hamilton-Jacobi theory, for which the reader should refer to the extensive treatment of ref. [2] or to [22].

8. – Regular dynamical systems and contact structure on $R\times T(M)$: Lagrangian formalism.

Many of the previous considerations have been made for a general contact structure on $R\times S$, where S is a symplectic but otherwise arbitrary manifold. Thus the above study may be extended in principle to $R\times T(M)$. However, the problem which immediately arises is that there is no canonical symplectic structure on $T(M)$. The situation is analogous to that of the « time-independent » mechanics and may be solved in a similar fashion. Given a function L on $R\times T(M)$ (locally $L = L(t, q^i, \dot{q}^i)$), the explicitly time-dependent

Lagrangian function, we may define the Legendre transformation $D_L: R\times T(M) \to R\times T^*(M)$ by trivially extending the definition given in subsect. **3***a*) to the present situation to transport the Liouville form Λ and the symplectic form from $T^*(M)$ to $T(M)$. The closed two-form $\tilde{\omega}_L \equiv p^*\omega_L = p^*(\mathrm{d}[D_L^*\Lambda])$ will again require the regularity condition for L (nonzero Hessian) to define a contact structure on $R\times T(M)$.

The reasoning now proceeds along lines similar to those of subsect. **3***b*). Once the 1-form $p^*(D_L^*\Lambda)$ defining the exact contact structure has been obtained, and given a function E on $R\times T(M)$, the Poincaré-Cartan 1-form may be written as

$$\Theta_E = p^*(D_L^*\Lambda) - E\,\mathrm{d}t \tag{8.1}$$

and a « Hamiltonian » field $\tilde{X}_E$ may be obtained whose trajectories on the base manifold will be the Lagrange solutions associated with L if $E = A_L - L$, where A_L is the action of L. In local co-ordinates on $R\times T(M)$

$$p^*(D_L^*\Lambda) = \frac{\partial L}{\partial \dot{q}^i}\,\mathrm{d}q^i\,, \tag{8.2}$$

$$A_L = \frac{\partial L}{\partial \dot{q}^i}\,\mathrm{d}\dot{q}^i\,, \qquad E = \frac{\partial L}{\partial \dot{q}^i}\,\dot{q}^i - L(t, q^i, \dot{q}^i) \tag{8.3}$$

and the Poincaré-Cartan form is written as

$$\Theta_E = \frac{\partial L}{\partial \dot{q}^i}\,\theta^i + L\,\mathrm{d}t\,, \tag{8.4}$$

where $\theta^i \equiv \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t$ is called *structure form* of the bundle $R\times T(M)\to R$ (see appendix B.4).

To conclude this section we will just mention that the (regular) Hamiltonian H which defines a dynamical system with contact form Θ_H and whose trajectories correspond to those of the dynamical system defined by Θ_E is related with E through the expression

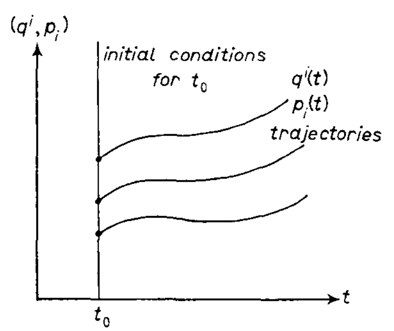
$$H = E\circ D_L^{-1} \tag{8.5}$$

similar to (4.1).

9. – Symplectic structure on the manifold of solutions in the case of the « time-dependent » mechanics.

a) *Introduction.* As has been pointed out at the beginning of this part, the necessity of introducing the time variable in the definition of the base manifold for (explicitly) time-dependent mechanical systems spoils the symplectic structure described in part I. The loss of this symplectic structure also implies the absence of the Poisson brackets which might be relevant in a possible quantization of the mechanical system. This fact alone is already a sufficient motivation to look for a way of circumventing the difficulty of the odd dimensionality of the contact manifold to reintroduce a canonical structure into the theory. In fact, looking at the manifold of solutions of the mechanical system, one observes that such a manifold is even dimensional and that, accordingly, it might possibly admit a symplectic structure. We devote this section to showing that it is indeed possible to endow the manifold $\mathcal{U}_H$ of solutions $s \in \mathcal{U}_H \subset \Gamma(R \times T^*(M))$ of the Hamiltonian system $\tilde{X}_H$ with such a structure.

The most immediate problem in pursuing this task is the definition of the tangent space in each *point* $s \in \mathcal{U}_H$. In the present case of the classical mechanics there are no special difficulties, since $\mathcal{U}_H$ is made up of solutions of an ordinary differential equation determined by the vector field $\tilde{X}_H$ on the fibre bundle $R \times T^*(M) \to R$ (or, when the Lagrangian description is used, on $R \times T(M)$). Because one is dealing with an ordinary differential equation, the manifold of solutions may be characterized by the set of initial conditions. By using canonical co-ordinates (q^i, p_i) for $T^*(M)$, the situation is intuitively depicted by the following figure:



The evolution space.

Given a dynamical system, the trajectory is uniquely determined by giving (q^i, p_i) at a certain time, and the trajectories do not intersect with each other; the set of the initial conditions is thus in one-to-one correspondence with $\mathcal{U}_H$. In the more general case of classical field theory, however, the solutions which describe the behaviour of the physical system are obtained from a system of partial derivative equations. Thus, to proceed in an analogous manner, the space $\mathcal{U}_{\mathcal{H}}$ of solutions of the field equations would have to be characterized by a space whose points (the initial conditions) would be submanifolds of dimension $m-1$ of the generalized « phase space », where m is the dimension of the manifold which plays in the variational formalism a role similar to that of time in mechanics (in the applications considered in the next part, this manifold is the Minkowski space $\mathscr{M}$; it is clear that the $\varphi(\mathbf{x})$ of field theory are the analogous to the q^i of mechanics). Thus, the process of endowing $\mathscr{U}_{\mathscr{H}}$ with a canonical structure is clearly more difficult. We shall not consider it here (see ref. [10, 11], the last given in [4], the first of [7], [33, 34] and references therein) to treat instead the case of mechanics, whose precise formulation we shall develop basing ourselves on an already existing work [10].

b) The tangent space to the manifold of solutions $\mathscr{U}_H$. The fibre space $E \xrightarrow{\pi} B$ we are dealing with in this part is not necessarily a vector bundle (*) and accordingly the structure of the space of cross-sections (see, *e.g.*, [35-37]) is complicated in general. Nevertheless, the set $\Gamma(E)$ of differentiable cross-sections of E may be endowed with a structure of a locally Banach manifold upon which the tangent space may be defined (see, *e.g.*, [37]). We shall, however, define the tangent space to the manifold of solutions following a procedure which is specially suited for our purposes. The construction is based on the following theorem [36] which is applicable to our case:

Let $E \xrightarrow{\pi} B$ be a (C^∞) differentiable fibre bundle over B. If $s \in \Gamma(E)$, the set of its (C^∞) differentiable cross-sections, the tangent space $T_s(\Gamma(E))$ at s can be identified canonically with $\Gamma(s^* T^v(E))$, where $s^* T^v(E) \to B$ is the pull-back bundle (appendix A.1) of the vertical tangent bundle $T^v(E) \xrightarrow{p^v} E$ by s. This may be described by the following diagram:

$$\begin{array}{ccc} s^* T^v(E) & \xrightarrow{S} & T^v(E) \\ {\scriptstyle s^* p^v}\downarrow & & \downarrow{\scriptstyle p^v} \\ B & \xrightarrow{s} & E \end{array},$$

where $s^* T^v(E) = \{(x, X^v_e) \in B \times T^v(E) / s(x) = e = p^v(X_e)\}$ and $S(x, X^v_e) = X^v_e$. We note that $T^v(E) \xrightarrow{p^v} E$ is the bundle tangent to the fibres of $E \xrightarrow{\pi} B$, so that its sections are vector fields of the form

$$X^v = X(x, y)\frac{\partial}{\partial y}, \tag{9b.1}$$

where (x, y) is a local co-ordinate system for E.

It is now clear that the cross-sections of $s^* T^v(E) \to B$ are in one-to-one correspondence with the sections of $T^v(E) \to E$ when one restricts oneself to $s(B) \subset E$ on which s is a bijection: given a section $\sigma : x \in s(B) \subset E \to X^v \in T^v(E)$,

(*) In the case of the time-dependent mechanics, $E = R \times M \to R$ is not necessarily vectorial, since M is not necessarily a vector space. For the case of the critical cross-sections (trajectories) obtained in a variational approach, R is restricted to the closed (compact) time interval limited by the two fixed end points.

the section $\sigma_{s^*}: x \in B \to (x, X^v) \in s^* T^v(E)$ is determined, and reciprocally. We may thus identify in what follows the tangent space to $\Gamma(E)$ at $s \in \Gamma(E)$ with the set of vertical vector fields on E restricted to $s(B)$.

The redefinition of $T_s(\Gamma(E))$ as $\Gamma(s^* T^v(E))$ allows us to parallely redefine the differential of the application

$$f^\Gamma : \Gamma(E_1) \to \Gamma(E_2) \tag{9b.2}$$

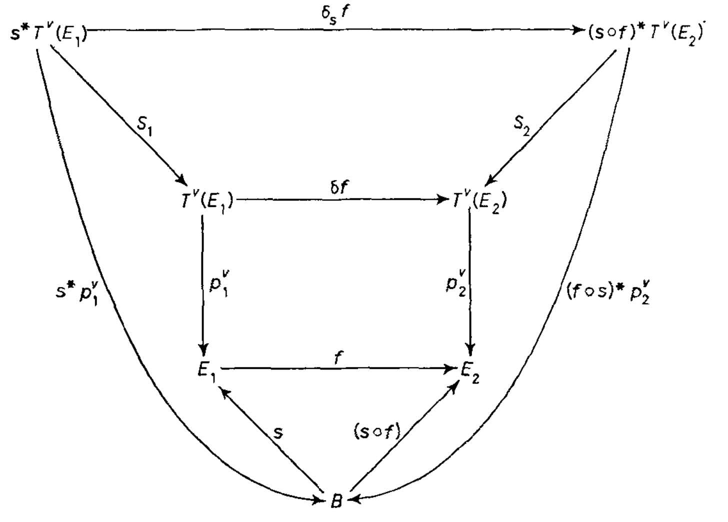
induced by a homeomorphism between two fibre bundles E_1 and E_2 over the same base B in a form which will be specially adequate for our purposes. This we shall do in several steps. First, given $f: E_1 \to E_2$, the application $\mathrm{d}f: T(E_1) \to \to T(E_2)$ defines, when restricted to $T^v(E_1)$, an application $\delta f: T^v(E_1) \to T^v(E_2)$, which is called vertical differential, $\delta f \equiv \mathrm{d}f|_{T^v(E)}$. Now, given $s \in \Gamma(E_1)$, δf induces an application $\delta_s f: s^* T^v(E_1) \to (f \circ s)^* T^v(E_2)$ in the following natural way:

$$\delta_s f : (x, X^v_{s(x)}) \to \big(x, [\delta f(X^v_{s(x)})]_{(f\circ s)(x)}\big)\,, \quad \forall \in B,\ X^v_{s(x)} \in T^v(E_1)/p^v(X^v_{s(x)}) = s(x)\,. \tag{9b.3}$$

$\delta_s f$ is called *vertical differential of f along the section s*. Finally, the differential of f^Γ at the point s is given by

$$(\mathrm{d}f^\Gamma)_s = (\delta_s f)^\Gamma\,, \tag{9b.4}$$

i.e. by the application on the space of cross-sections $\Gamma(s^* T^v(E_1))$ naturally induced by the action of $\delta_s f$ on $s^* T^v(E_1)$ (*). We may thus use the r.h.s. of (9b.4) to compute the l.h.s. The following diagram may be helpful in considering the above definitions:



(*) $\mathrm{d}f^\Gamma$ at s acts on $T_s(\Gamma(E))$ and $(\delta_s f)^\Gamma$ on $\Gamma(s^* T^v(E))$, spaces which are in a one-to-one correspondence.

Let us now return to the case of mechanics. In this case we have to construct the tangent space $T_s(\mathscr{U}_H)$ at s to the manifold of solutions $\mathscr{U}_H$ of a Hamiltonian vector field $\tilde{X}_H$ on $R\times T^*(M)$ (or $R\times T(M)$ in the dual situation). This construction has to be compatible with the fact that $\mathscr{U}_H$ is a submanifold (although not an open submanifold) of $\Gamma(E)$, where now E is the fibre bundle $E = R\times T^*(M) \to R$. Thus we define $T_s(\mathscr{U}_H)$ as the vector subspace of $T_s(\Gamma(E))$ made out of the cross-sections of $T^v(E)|_{s(R)}$ (which may be identified with those of $s^*T^v(E)$) *which commute with* $\tilde{X}_H$.

It is not difficult to convince oneself of the consistency of this definition. If $\mathscr{F}(\mathscr{U}_H)$ is the ring of functions on the manifold $\mathscr{U}_H$, $T_s(\mathscr{U}_H)$ has to be the set of the derivations of the functions $f\in\mathscr{F}(\mathscr{U}_H)$ around s. Since $\mathscr{U}_H = E/\mathscr{R}$, where $\mathscr{R}$ is the equivalence relation which places in the same class all points of E belonging to the same trajectory, $\mathscr{F}(\mathscr{U}_H)$ may be obtained from the functions of $\mathscr{F}(E)$ which are constant over each one of the trajectories, *i.e.* such that they are first integrals of $\tilde{X}_H$, $\tilde{X}_H.f = 0$ (subsect. 7*b*)). Indeed, the restriction of the function f on E (which we may take of class C^∞) to $\mathscr{U}_H$ is possible, since $\mathscr{U}_H$ can be injected into E ($\mathscr{U}_H = E/\mathscr{R}$) and f takes a constant value on each trajectory ($\mathrm{d}f$ is transverse to the flow of $\tilde{X}_H$, $\mathrm{d}f(\tilde{X}_H) \equiv \tilde{X}_H.f = 0$) so that the restriction defines a differentiable function on $\mathscr{U}_H$. Let us now impose to $Y\in\Gamma(\tau(E))$ the condition of being a derivation of $\mathscr{F}(\mathscr{U}_H)$, *i.e.* a linear application of $\mathscr{F}(\mathscr{U}_H)$ on $\mathscr{F}(\mathscr{U}_H)$ satisfying the product derivation law. $\forall f\in \mathscr{F}(\mathscr{U}_H)$, $Y.f\in\mathscr{F}(\mathscr{U}_H)$ clearly implies that $[\tilde{X}, Y] = 0$. However, Y, as such, does not satisfy the condition $Y.f = 0$, $\forall f \Rightarrow Y = 0$. This makes it necessary to take the vertical part of Y, Y^v (*); accordingly Y is a vertical vector field of $T^v(E)|_{s(R)}$, $s\in\mathscr{U}_H$.

c) *Symplectic form on* $\mathscr{U}_H$. Let Y^v_s, Z^v_s be two elements of $T_s(\mathscr{U}_H)$ and hence vertical, since it has been shown that all the fields tangent to $\mathscr{U}_H$ at s have to be vertical. The symplectic form ω at the point s is defined by

$$(\omega)_s(Y^v_s, Z^v_s) \equiv \mathrm{d}\Theta(Y^v_s, Z^v_s)\,, \tag{9c.1}$$

where $\mathrm{d}\Theta$ is locally given by $\mathrm{d}p^i\wedge\mathrm{d}q_i - \mathrm{d}H_t\wedge\mathrm{d}t$. To see that (9*c*.1) is a good definition, we have to check *a*) that the r.h.s. is not an arbitrary function but a constant for a given s and *b*) that ω establishes a one-to-one correspondence between Hamiltonian fields on $\mathscr{U}_H$ and differentials ((9*b*.4)) of functions defined on $\mathscr{U}_H$. Both conditions are fulfilled:

(*) Given $s\in\mathscr{U}_H$, there is a local co-ordinate system $\{\hat{t}, \hat{q}^i, \hat{p}_j\}$ in which s is written $\hat{q}^i = \hat{p}_i = 0$. In such a system $\tilde{X} = \partial/\partial\hat{t}$ and $\tilde{X}.f = 0 \Rightarrow f \neq f(\hat{t})$.
Thus $Y.f = 0\ \forall f (Y = Y^0(\partial/\partial\hat{t}) + Y^i(\partial/\partial\hat{q}^i) + Y^*_i(\partial/\partial\hat{p}_i))$ implies $Y = 0$ only if $Y = Y^v$.

a) To see that $d\Theta(Y_s^v, Z_s^v) \in R$, *i.e.* that $\tilde{X}.d\Theta(Y_s^v, Z_s^v) = 0$ (*), we write for the vertical field Y_s^v at s (= with support on the trajectory s defined by $s^i(t)$ and $s_i^*(t)$)

$$(9c.2)\qquad Y_s^v = Y^i(t, s^j(t), s_k^*(t))\frac{\partial}{\partial q^i} + Y_i^*(t, s^j(t), s_k^*(t))\frac{\partial}{\partial p_i},$$

since Y_s^v is a function on R (the time) valued on the fibre of $T^v(E)$, and a similar expression for Z_s. Now

$$(9c.3)\qquad d\Theta(Y_s^v, Z_s^v) = i_{Z_s} i_{Y_s} d\Theta = Y_i^* Z^i - Y^i Z_i^*,$$

so that

$$(9c.4)\qquad \tilde{X}.d\Theta(Y_s^v, Z_s^v) \equiv \left(\frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i} + X_i^*\frac{\partial}{\partial p_i}\right)F = \frac{d}{dt}F \equiv \frac{d}{dt}(Y_i^* Z^i - Y^i Z_i^*).$$

Taking into account that $[\tilde{X}, Y_s^v] = 0$, one obtains the constraints on (9c.2):

$$(9c.5)\qquad \frac{dY^i}{dt} = Y^k\frac{\partial X^i}{\partial q^k} + Y_k^*\frac{\partial X^i}{\partial p_k}, \qquad \frac{dY_i^*}{dt} = Y^k\frac{\partial X_i^*}{\partial q^k} + Y_k^*\frac{\partial X_i^*}{\partial p_k}$$

and similar ones for Z_s^v (note that dY/dt corresponds to $\partial Y/\partial t + (\partial Y/\partial s)\cdot$ $\cdot(ds/dt) + (\partial Y/\partial s^*)(ds^*/dt)$ if one keeps in mind the origin of the different dependences on t). Using them now on (9c.4), one gets $\tilde{X}.d\Theta(Y_s^v, Z_s^v) = 0$, since $X^i = \partial H/\partial p_i$, $X_i^* = -\partial H/\partial q^i$ ((7b.1)).

b) In analogy with the situation on the symplectic mechanics where $L_Y\Lambda = 0 \Rightarrow i_Y\omega = df_Y$, $f_Y = \Lambda(Y)$ (subsect. **2**c)), the following theorem now holds true:

Given a vertical field Y^v on $E = R\times T^*(M)$ such that $L_{Y^v}\Theta = 0$, its restriction Y_s^v to a section $s \in \mathscr{U}_H$ belongs to $T_s(\mathscr{U}_H)$ and verifies that

$$(9c.6)\qquad i_{Y_s^v}\omega = [d(i_{Y^v}\Theta)]_{(s)},$$

where the subscript (s) on the r.h.s. means « with *support* on the section s » (that is, the coefficients of the 1-form have support on s ((9b.4)); this should not be confused with the *restriction* $d(i_{Y^v}\Theta)|_s$ which implies putting, *e.g.*, $dq^i =$ $= (dq^i/dt)\,dt$, etc., and which is zero for $s \in \mathscr{U}_H$ ((7c.3))).

To prove (9c.6) we first find the conditions which $L_Y\Theta = 0$ imposes on an arbitrary vector field (to find the corresponding ones for Y^v it is sufficient to

(*) Since $\tilde{X}_H$ moves along trajectories, this equality implies that $d\Theta(\ )$ is a constant. We shall omit the subscript H in $\tilde{X}_H$ henceforth.

put $Y^0 = 0$). These are, from the coefficients of $\mathrm{d}t$, $\mathrm{d}q^i$, $\mathrm{d}p_i$,

$$\text{(9c.7)} \qquad \begin{cases} Y^k \dfrac{\partial H}{\partial q^k} + Y_k^* \dfrac{\partial H}{\partial p_k} + Y^0 \dfrac{\partial H}{\partial t} = p_k \dfrac{\partial Y^k}{\partial t} - H \dfrac{\partial Y^0}{\partial t}, \\ Y_k^* + p_i \dfrac{\partial Y^i}{\partial q^k} - H \dfrac{\partial Y^0}{\partial q^k} = 0, \\ p_i \dfrac{\partial Y^i}{\partial p^k} - H \dfrac{\partial Y^0}{\partial p^k} = 0. \end{cases}$$

For a vertical vector $Y_s^v \in T_s(\mathscr{U}_H)$ one finds

$$\text{(9c.8)} \qquad i_{Y_s^v}\omega = Y_i^* \,\mathrm{d}q^i - Y^i \,\mathrm{d}p_i - \left(\frac{\partial H}{\partial q^i} Y^i + \frac{\partial H}{\partial p_i} Y_i^*\right)\mathrm{d}t$$

and

$$\text{(9c.9)} \qquad [\mathrm{d}(i_Y \Theta)]_{(s)} = [\mathrm{d}(p_i Y^i)]_{(s)} =$$

$$= \left[p_i \frac{\partial Y_i}{\partial q^k}\mathrm{d}q^k + \left\{Y^k + p_i \frac{\partial Y^i}{\partial p_k}\right\}\mathrm{d}p_k + p_i \frac{\partial Y^i}{\partial t}\mathrm{d}t\right]_{(s)}.$$

By using (9c.7), (9c.8) and (9c.9) are seen to be equal (*).

The reciprocal theorem also holds true: Given a function f on $\mathscr{U}_H$, *i.e.* such that as a function on E $\tilde{X}.f = 0$, there exists a unique vertical field Y_f^v which commutes with $\tilde{X}$ (and thus $Y_{fs}^v \in T_s(\mathscr{U}_H)$) and such that (cf. subsect. **2**b))

$$\text{(9c.10)} \qquad i_{Y_{fs}^v}\omega = \mathrm{d}f.$$

Indeed, it is trivial to find from (9c.10) and the condition $\tilde{X}.f = 0$ that

$$\text{(9c.11)} \qquad Y^i = \frac{\partial f}{\partial p_i}, \qquad Y_i^* = -\frac{\partial f}{\partial q^i}$$

and to check that the vertical field Y_f^v thus determined satisfies the condition $[\tilde{X}, Y_f^v] = 0$.

Note. In particular, the function f on $\mathscr{U}_H$ might be $f = i_Z\Theta$, where Z is a vector field satisfying $L_Z\Theta = 0$ which, contrarily to (9c.6), is not necessarily

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(*) Note that, strictly speaking, $i_{Y^v}\omega$ defined as 1-form on $T_s(\mathscr{U}_H)$ does not have horizontal component ($\mathrm{d}t$ in (9c.8)) and that, correspondingly, $[\mathrm{d}i_{Y^v}\Theta]_{(s)}$ has to be restricted to be vertical (and thus $\mathrm{d}t$ omitted in (9c.9)). However, this precision is not relevant for the calculation.

vertical. In fact, one may check that $\tilde{X}.(i_Z\Theta) = 0$ if $L_Z\Theta = 0$, so that f is really a function on $\mathscr{U}_H$, and then evaluate from $i_{Y^v_{fs}}\omega = \mathrm{d}(i_Z\Theta)$ the vertical field associated with f, which turns out to be

$$Y^v_{fs} = (Z^v - Z^0\tilde{X}^v)_{(s)}\,. \tag{9c.12}$$

It is interesting to observe that the invariant associated to Y^v_{fs} is again given by f. This is easily seen, since $L_{Y^v_{fs}}\Theta$ results to be

$$L_{Y^v_{fs}}\Theta = L_Z\Theta - \mathrm{d}[Z^0 i_{\tilde{X}}\Theta] = \mathrm{d}[Z^0(p_iX^i - H)]\,, \tag{9c.13}$$

so that the Noether invariant associated with Y^v_{fs} is $(i_{Y^v_{fs}}\mathrm{d}\Theta|_s = 0)$

$$i_{Y^v_{fs}}\Theta + Z^0 i_{\tilde{X}}\Theta\,, \tag{9c.14}$$

which, since $i_{Y^v_{fs}}\Theta = p_iY^i$, turns out to be $p_iZ^i - HZ^0 = i_Z\Theta \equiv f$. The Noether invariant is the same as for Z and this shows the consistency of the definitions (*).

d) Poisson bracket. After having recuperated on $\mathscr{U}_H$ the symplectic correspondence between vector fields on $\mathscr{U}_H$ and 1-forms, it is possible to define the Poisson bracket between two arbitrary functions on $\mathscr{U}_H$. This is done in the usual way (cfr. subsect. **2***b*)): given two functions f and g on the manifold of solutions, the Poisson bracket is defined by

$$\{f, g\} \equiv \omega(Y^v_f, Y^v_f)\,. \tag{9d.1}$$

In local co-ordinates (10.1) takes the form

$$\{f, g\} = Y^*_{fi}Y^i_g - Y^*_{gi}Y^i_f = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i}\,, \tag{9d.2}$$

where use has been made of (9c.11).

In particular, if $f = i_Z\Theta$, $g = i_{Z'}\Theta$ with $L_Z\Theta = L_{Z'}\Theta = 0$ (Z not necessarily vertical),

$$\{f, g\} = \omega(Y^v_f, Y^v_g) = \mathrm{d}\Theta(Y^v_f, Y^v_g)\,, \tag{9d.3}$$

(*) The fact that a nonvertical component Z^0 shows up in the expression of Y^v_{fs} is not an inconvenience; the symmetries compatible with the fibration $R\times T^*(M)\to R$ will not alter the form (9c.11) (*i.e.* will not transform Z^0 into $Z'^0 + Z'^v$, for instance).

which may be shown to be equal to

$$\mathrm{d}\Theta(Z, Z') = \Theta([Z, Z']) \equiv i_{[Z',Z]}\Theta\,, \tag{9d.4}$$

which shows (cf. (2b.5)) that the algebra of the Noether invariants may be identified with the Poisson algebra on $\mathscr{U}_H$ [10].

e) A trivial example: equations of motion on the manifold of solutions of the free particle. Let $\mathscr{U}$ now be the manifold of solutions of the equations of motion for the free particle, of Hamiltonian $H = \boldsymbol{p}^2/2m$. It is obvious that $\mathscr{U}$ may be parametrized by P_i and K^i, Noether invariants associated with the translations and boosts and which determine the initial momenta and positions. In particular, as may be easily checked from the expression of the Noether invariant associated with the Galilean boosts $(x^i - (p^i/m)\,t)$, the symplectic form may be written as

$$\omega = \mathrm{d}K^i \wedge \mathrm{d}P_i\,, \tag{9e.1}$$

where the P_i, K^i themselves play the role of Darboux canonical co-ordinates; K^i, P_i are real functions on $\mathscr{U}$. The function H on $T^*(M)\times R$ is a first integral of $\tilde{X}$ and accordingly a function on $\mathscr{U}$. Let us call $\bar{H}$ to H defined as a function on $\mathscr{U}$.

Thus $\mathscr{U}$ is a symplectic manifold on which a Hamiltonian $\bar{H}$ has been given. It is then clear that the equations of motion are given by

$$i_X\omega = \mathrm{d}\bar{H} \tag{9e.2}$$

and, since $\mathrm{d}\bar{H} = (\partial\bar{H}/\partial K^i)\,\mathrm{d}K^i + (\partial\bar{H}/\partial P_i)\,\mathrm{d}P_i$, we find

$$X = \frac{P^i}{m}\frac{\partial}{\partial K^i}\,, \tag{9e.3}$$

so that the equations of motion in terms of the parameter λ read

$$\frac{\mathrm{d}K^i}{\mathrm{d}\lambda} = \frac{P^i}{m}\,, \qquad \frac{\mathrm{d}P_i}{\mathrm{d}\lambda} = 0\,, \tag{9e.4}$$

i.e. $K^i = (P^i/m)\,\lambda + q_0^i$ and $P^i = p_0^i$, where q_0^i and p_0^i are constants. The result is not surprising: the trajectory for (9e.2) goes through all the initial conditions of the trajectories of the physical motion which correspond to the same energy. Note that $i_X\omega = \mathrm{d}\bar{H}$ implies that $\bar{H}$ is a constant of the motion on $\mathscr{U}$.

PART III

The variational approach and field theory.

10. – Introduction (*).

The geometric formulation of field theory will be developed by using the variational approach, in part due to the fact that this formalism has been historically fruitful in the development of the basis of field theory and in part because it is in field theory that this approach (together with the corresponding boundary conditions) is more necessary from the physical point of view. We shall give nevertheless a general formulation for the variational principles and apply it to the case of mechanics to recover (sect. **12**), as a particular case, the equations of motion of the time-dependent mechanics which were considered in part II in connection with the contact structure.

The usual classical variational formalism tries to find the critical points (cross-sections) of a real function (the action integral) defined on a differentiable submanifold of $\Gamma(E)$, the space of cross-sections of a certain differentiable fibre bundle (E, π, M), not necessarily a vector bundle. More precisely, given a section $s_0 \in \Gamma(E)$, one defines $\Gamma_0(E)$ as the submanifold of sections $s \in \Gamma_0(E) \subset \subset \Gamma(E)$ such that they coincide with s_0 on some subset V of the boundary ∂M of M, *i.e.*

$$\Gamma_0(E) = \{s \in \Gamma(E)/s|_V = s_0|_V\} ,$$

and applies the variational formalism to select among the cross-sections of $\Gamma_0(E)$ those which are critical.

This variational problem, which selects critical sections submitted to the above type of boundary conditions, is called *Dirichlet variational problem.* Nevertheless, one may consider other types of variational problems [36] as, for instance, the following:

Free boundary problems. These arise when M is compact and without boundary; in this case the critical sections are looked for among the whole $\Gamma(E)$.

End manifold problems. Given a closed differentiable subbundle F of $E|_{\partial M}$ (the restriction of the fibre bundle E to the boundary of M), $\Gamma_F(E)$ is defined as the set of cross-sections s of $\Gamma(E)$ such that $s(\partial M) \subset F$. $\Gamma_F(E)$ is locally a closed linear subspace of $\Gamma(E)$ and thus a differentiable submanifold of $\Gamma(E)$, and the critical sections are looked for among those of $\Gamma_F(E)$.

(*) As mentioned in sect. **1**, we deal in this part with classical (as opposed to quantized) field theory.

We shall restrict ourselves in our applications to field theory to the Dirichlet variational problem considered above. In its more frequent formulation (*), $\mathscr{L}$ is defined as a real function on the bundle $J^1(E)$ of the 1-jets (appendix B) of the bundle E, and I is the functional on $\Gamma(E)$ given by

$$I(s) = \int_{j^1(s)(M)} \mathscr{L}\omega\,, \qquad s \in \Gamma(E), \tag{10.1}$$

where j^1 is the 1-jet prolongation and ω is the injection to $J^1(E)$ of the volume form on M through the projection $\pi^1: J^1(E) \to M$. The critical points are obtained from the condition

$$(\mathrm{d}I)_s = 0\,, \qquad s \in \Gamma_0(E), \tag{10.2}$$

where d is defined as in subsect. **9***b*). The explicit expression of (10.2) in a physical case determines the equations of motion of the system associated with the function $\mathscr{L}$, the Lagrangian; in field theory E will be a *vector* bundle over the Minkowski space $\mathscr{M}$ and we shall take for s_0 the cross-section zero at infinity (**).

The restriction to mechanics is slightly different as far as boundary conditions are concerned. The bundle $E = R \times M \to R$ of mechanics is not necessarily vectorial and thus there is no zero section. Moreover, the necessity of taking zero boundary conditions at infinity is not relevant, since moving particles may separate themselves arbitrarily from a given domain. As a consequence, an arbitrary closed interval $[t_1, t_2]$ of R is used, and the cross-sections are forced to take the value of a given section at its boundary. For the ordinary Hamilton variational principle this implies that the cross-sections $q(t)$ satisfy $q(t_1) = q_1$, $q(t_2) = q_2$ at the end points t_1, t_2.

11. – Variational principles on $J^1(E)$ and $J^{1*}(E)$ in field theory.

a) The Lagrangian approach to field theory. As is known (see, *e.g.*, [7, 10, 11, 18]) the formulation of the ordinary Hamilton principle starts from the definition of a function $\mathscr{L}: J^1(E) \to R$, the Lagrangian density, on the bundle

(*) Others will also be considered. For instance, the *modified Hamilton principle* will consider the functional I' defined on $\Gamma(J^1(E))$.

(**) The convenience is well known, however, of the use of cross-sections which do not belong to $\Gamma_0(E)$; this is the case of the plane-wave solution of the field equations (the mathematical cross-section corresponds to the physical field, sect. **11**). Strictly speaking, these plane waves correspond to solutions to an « end manifold variational problem » for which the physical space is reduced to a compact subset, and periodical conditions are imposed at the boundary. This is the case of the « box normalization »; the resulting discreteness of physical variables disappears when the size of the box goes to infinity.

$J^1(E)$ of the 1-jets of the vector bundle $(E, \pi, \mathscr{M})$ with base the Minkowski space $\mathscr{M}$ and fibre $V^n(R)$ or $V^n(C)$ (*). The solutions—« trajectories »—of the variational problem constitute a subset of the modulus $\Gamma(E)$ of the cross-sections of E which is composed by the solutions of the Euler-Lagrange (EL) equations (**). In this scheme, the physical fields defined by certain Lagrangians are simply cross-sections of E which satisfy the EL equations. As usual, the cross-sections of $\Gamma_0(E)$ will be taken of class C^∞ and vanishing at infinity; nevertheless, this does not imply that other cross-sections of the EL equations, such as plane waves, have to be necessarily discarded.

Let $E = V^n \times \mathscr{M}$ be parametrized by the co-ordinate system (x^μ, y^α) $(\mu = 0, 1, 2, 3;\ \alpha = 1, ..., n)$; the bundle $(J^1(E), \pi, \mathscr{M})$ will be parametrized by $(x^\mu, y^\alpha, y^\alpha_\mu)$ (appendix B.2). Given a Lagrangian density $\mathscr{L}$ on $J^1(E)$, the *Hamilton functional* on the space of cross-sections $\Gamma_0(E)$ is defined by

$$I(\psi) = \int_{j^1(\psi)(\mathscr{M})} \mathscr{L}\pi^{1*}\omega\,, \qquad \psi \in \Gamma_0(E)\,, \tag{11a.1}$$

where $j^1(\psi) \equiv \bar{\psi}^1$ is the 1-jet prolongation (appendix B) of the cross-section ψ, ω is the volume form on $\mathscr{M}$, $\omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ and again $\pi^{1*}(\omega)$ the prolongation of ω into $J^1(E)$ (which has the same expression, since $\pi^{1*}(\omega)$ only has components on $\mathscr{M}$).

Given the Hamilton functional I, the *ordinary Hamilton principle* (principle I, PI) states that the action (11*a*.1) must be extremal, *i.e.*

$$(\delta_\psi I)(X) \equiv \int_{j^1(\psi)(\mathscr{M})} L_{\bar{X}^1}(\mathscr{L}\omega) = 0\,, \qquad \forall X \in \Gamma(\tau(E))\,, \tag{11a.2}$$

where the Lie derivative is taken with respect to the 1-jet prolongation $\bar{X}^1$ of an arbitrary vector field on E, $X = X^\mu(\partial/\partial x^\mu) + X^\alpha(\partial/\partial y^\alpha)$. The role of the 1-jet prolongation is easily understood: for a section $y^\alpha(x)$, the 1-jet prolongation is given by $\big(y^\alpha(x), y^\alpha_\mu(x) = \partial_\mu y^\alpha(x)\big)$ (***) and thus this formalism accommodates the dependence of $\mathscr{L}$ on the first derivatives of the field (a dependence on higher-order derivatives will obviously require higher prolongations [18]). As for the field X, the 1-jet prolongation $\bar{X}^1$ is used to define

(*) E has the structure of direct product bundle, $E = V \times \mathscr{M}$. This is not a limitation: fibre bundles over the Minkowski space are trivial. For a classification of fibre bundles see, *e.g.*, [38].

(**) In the most typical cases (Maxwell, Klein-Gordon, Dirac, etc.), these fields provide the support space for an unitary representation of the Poincaré group.

(***) For the sake of notational simplicity, y^α and y^α_μ also denote the functions $y^\alpha(x)$, $y^\alpha_\mu(x)$ which determine a certain cross-section $\psi^1 \in \Gamma(J^1(E))$. We also note that the relevant vector fields in (11*a*.2) are vertical ((10.2), subsect. **9***b*)); the horizontal component may be kept nevertheless, since it does not produce any new condition.

its action on $J^1(E)$ and its expression is found from the condition (appendix B.3)

$$L_{\bar{X}^1}\theta^\alpha = A^\alpha_\beta \theta^\beta\,, \qquad \theta^\alpha \equiv \mathrm{d}y^\alpha - y^\alpha_\mu\,\mathrm{d}x^\mu\,; \tag{11a.3}$$

in this way $L_{\bar{X}^1}$ maps cross-sections which are 1-jet prolongations into themselves.

The fact that only the vertical part of X is relevant in (11*a*.2) allows us to write

$$\bar{X}^1 = X^\alpha \frac{\partial}{\partial y^\alpha} + \left(\frac{\partial X^\alpha}{\partial x^\mu} + \frac{\partial X^\alpha}{\partial y^\beta} y^\beta_\mu\right)\frac{\partial}{\partial y^\alpha_\mu} \equiv X^\alpha \frac{\partial}{\partial y^\alpha} + \bar{X}^\alpha_\mu \frac{\partial}{\partial y^\alpha_\mu} \tag{11a.4}$$

and that

$$L_{\bar{X}^1}(\mathscr{L}\omega) = (L_{\bar{X}^1}\mathscr{L})\omega = X^\alpha \frac{\partial \mathscr{L}}{\partial y^\alpha}\omega + \bar{X}^\alpha_\nu \frac{\partial \mathscr{L}}{\partial y^\alpha_\mu}\mathrm{d}x^\nu \wedge \theta_\mu\,, \tag{11a.5}$$

since, with $\theta_\mu \equiv (-)^\mu \mathrm{d}x^0 \wedge \ldots \wedge \widehat{\mathrm{d}x^\mu} \wedge \ldots \wedge \mathrm{d}x^3$, $\mathrm{d}x^\nu \wedge \theta_\mu = \omega\delta^\nu_\mu$. By using the definition of X^α_μ and θ^α and that $L_{\bar{X}^1}\mathrm{d}y^\beta = (\partial X^\beta/\partial y^\alpha)\,\mathrm{d}y^\alpha + (\partial X^\beta/\partial x^\mu)\,\mathrm{d}x^\mu$, (11*a*.5) reads

$$L_{\bar{X}^1}(\mathscr{L}\omega) = X^\alpha\left\{\frac{\partial \mathscr{L}}{\partial y^\alpha}\omega - \mathrm{d}\left(\frac{\partial \mathscr{L}}{\partial y^\alpha_\mu}\right)\wedge\theta_\mu\right\} + \\ + \mathrm{d}\left(\frac{\partial \mathscr{L}}{\partial y^\alpha_\mu}X^\alpha\theta_\mu\right) - \frac{\partial X^\alpha}{\partial y^\beta}\frac{\partial \mathscr{L}}{\partial y^\alpha_\mu}\theta^\beta\wedge\theta_\mu\,. \tag{11a.6}$$

If we take into account that the second term in (11*a*.6) will not contribute to the integral, that the third is zero when restricting to 1-jet prolongation cross-sections on account that $\theta^\alpha|_{\bar{\psi}^1} = 0$ and that the second contribution to the first is $(\mathrm{d}/\mathrm{d}x^\mu)(\partial \mathscr{L}/\partial y^\alpha_\mu)\cdot\omega$, (11*a*.2) will be fulfilled if

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathscr{L}}{\partial y^\alpha_\mu}\right) - \frac{\partial \mathscr{L}}{\partial y^\alpha} = 0\,. \tag{11a.7}$$

(11*a*.7) is the familiar EL equation; in it, $y^\alpha_\mu(x) = \partial_\mu y^\alpha(x)$. The space of its solutions will be called $\mathscr{U}_{\mathscr{L}}$.

The ordinary Hamilton principle starts from a functional defined on $\Gamma(E)$ ((11*a*.1)). Let us now consider the *modified Hamilton principle* (principle II, PII) whose starting point is a functional I' defined on $\Gamma(J^1(E))$. In complete analogy with the case of mechanics (cf. (8.4)) let us introduce the Poincaré-Cartan form Θ as

$$\Theta = \Omega + \Omega'\,, \qquad \Omega \equiv \mathscr{L}\omega\,, \qquad \Omega' = \theta^\alpha \wedge \Omega_\alpha\,, \tag{11a.8}$$

$$\Omega_\alpha \equiv \frac{\partial \mathscr{L}}{\partial y^\alpha_\mu}\theta_\mu\,. \tag{11a.9}$$

Note that $\Theta \neq \mathscr{L}\omega$, but that, if we restrict ourselves to 1-jet prolongation cross-sections (as in PI), $\Theta|_{\bar{\psi}^1} = \mathscr{L}\omega|_{\bar{\psi}^1}$.

The *modified Hamilton functional* I' on cross-sections of $J^1(E)$ is defined by

$$I'(\psi^1) = \int_{\psi^1(\mathscr{M})} \Theta\,, \qquad \psi^1 \in \Gamma_0(J^1(E))\,, \tag{11a.10}$$

and PII establishes that

$$(\delta_{\psi^1} I')(X^1) \equiv \int_{\psi^1(\mathscr{M})} L_{X^1}\Theta = 0\,, \qquad \forall X^1 \in \Gamma\big(\tau(J^1(E))\big)\,, \tag{11a.11}$$

which leads to the equation (*)

$$i_{X^1}\,\mathrm{d}\Theta|_{\psi^1} = 0\,, \qquad \forall X^1\,, \tag{11a.12}$$

frequently called *Euler-Cartan equation.*

Let us now evaluate (11a.12) in local co-ordinates. Θ may be written as

$$\Theta = \frac{\partial\mathscr{L}}{\partial y^\alpha_\mu}\,\mathrm{d}y^\alpha\wedge\theta_\mu + \left(\mathscr{L} - \frac{\partial\mathscr{L}}{\partial y^\alpha_\mu}\,y^\alpha_\mu\right)\omega\,, \tag{11a.13}$$

expression from which $\mathrm{d}\Theta$ is found to be

$$\begin{aligned}\mathrm{d}\Theta = &\left\{-\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathscr{L}}{\partial y^\alpha_\mu}\right) + \frac{\partial\mathscr{L}}{\partial y^\alpha} - y^\beta_\mu\frac{\partial}{\partial y^\alpha}\left(\frac{\partial\mathscr{L}}{\partial y^\beta_\mu}\right)\right\}\mathrm{d}y^\alpha\wedge\omega + \\ &+ \frac{\partial}{\partial y^\beta}\left(\frac{\partial\mathscr{L}}{\partial y^\alpha_\mu}\right)\mathrm{d}y^\beta\wedge\mathrm{d}y^\alpha\wedge\theta_\mu + \frac{\partial}{\partial y^\beta_\nu}\left(\frac{\partial\mathscr{L}}{\partial y^\alpha_\mu}\right)\mathrm{d}y^\beta_\nu\wedge\mathrm{d}y^\alpha\wedge\theta_\mu - y^\alpha_\mu\frac{\partial}{\partial y^\beta_\nu}\left(\frac{\partial\mathscr{L}}{\partial y^\alpha_\mu}\right)\mathrm{d}y^\beta_\nu\wedge\omega\,.\end{aligned} \tag{11a.14}$$

The expression for a general vector field of $\Gamma\big(\tau(J^1(E))\big)$ is

$$X^1 = X^\mu\frac{\partial}{\partial x^\mu} + X^\alpha\frac{\partial}{\partial y^\alpha} + X^\alpha_\mu\frac{\partial}{\partial y^\alpha_\mu}\,, \tag{11a.15}$$

(*) In the above expressions we have not made explicit the fact that, in general, the fibre will be of type $V^n(C)$ (the physical fields are complex in general), but this is straigthforward: if $(x^\mu, y^\alpha, y_{\alpha^*}, y^\alpha_\mu, y_{\alpha^*\mu})$ is the system of local co-ordinates for $J^1(E)$—the asterisk denotes complex conjugate—it is sufficient to define now in (11a.8)

$$\Omega' = \theta^\alpha\wedge\Omega_\alpha + \theta_{\alpha^*}\wedge\Theta^{\alpha^*}\,,$$

where

$$\theta_{\alpha^*} = \mathrm{d}y_{\alpha^*} - y_{\alpha^*\mu}\,\mathrm{d}x^\mu\,, \qquad \Omega^{\alpha^*} = \frac{\partial\mathscr{L}}{\partial y_{\alpha^*\mu}}\,\theta_\mu\,, \tag{11a.8'}$$

while leaving the rest of the expressions unaltered.

so that (11a.12) is satisfied if the equations for cross-sections

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathscr{L}}{\partial y^\alpha_\mu}\right)-\frac{\partial \mathscr{L}}{\partial y^\alpha}=0 \tag{11a.16a}$$

and

$$\frac{\partial}{\partial y^\beta_\nu}\left(\frac{\partial \mathscr{L}}{\partial y^\alpha_\mu}\right)\left[\frac{\partial y^\alpha}{\partial x^\mu}-y^\alpha_\mu\right]=0\,, \tag{11a.16b}$$

which come from the coefficients of X^α and X^β_ν, are simultaneously fulfilled. The coefficient of X^μ is zero if eqs. (11a.16) are satisfied, so that the corresponding component of X^1 might have been omitted from the beginning (verticality of X^1).

We shall call $\mathscr{U}'_{\mathscr{L}}$ the space of critical cross-sections for PII. Inspection of the above equations shows that, in general, this space is different from $\mathscr{U}_{\mathscr{L}}$. This was to be expected: no restriction to 1-jet prolongations was made in (11a.10)-(11a.12). However, in the case of regularity of the Lagrangian density, *i.e.* when the Hessian $\det(\partial\mathscr{L}/\partial y^\alpha_\mu\partial y^\beta_\nu)\neq 0$, (11$a$.16$b$) gives the 1-jet prolongation condition $y^\alpha_\mu(x)=\partial_\mu y^\alpha(x)$ and (11a.16a) gives the EL equations; in that case, $\mathscr{U}_{\mathscr{L}}=\mathscr{U}'_{\mathscr{L}}$.

b) Variational formalism on $J^{1*}(E)$ and covariant Hamiltonian formalism. The formalism on $J^1(E)$ based on Θ, although Hamiltonian in form—the above equations constitute formally a system of *first-order* differential equations—does not have a Hamiltonian aspect. The typical Hamiltonian formalism involves momenta so that the bundle $J^1(E)$ is inadequate for its formulation. It may, however, be constructed on $J^{1*}(E)$, the dual bundle of $J^1(E)\to E$: such a formalism, based on the definition of a Lorentz scalar Hamiltonian density $\mathscr{H}$ on $J^{1*}(E)$, will lead to manifestly covariant Hamilton-type equations. In the case of regularity these equations will be equivalent to the EL equations obtained through PI for the associated Lagrangian density. However, the formalism will not depend on the existence of a previous Lagrangian density. As another additional advantage, the formulation on $J^{1*}(E)$ will turn out specially suitable to characterize the symmetries of the Hamiltonian problem in field theory.

Let $(x^\mu, y^\alpha, y_{\alpha^*}, \pi^\mu_\alpha, \pi^{\alpha^*\mu})$ be a local parametrization of $J^{1*}(E)$. A Hamiltonian density is defined as a (Hermitian real) scalar function on $J^{1*}(E)$. The Poincaré-Cartan form on $J^{1*}(E)$ is now defined as (cf. (7a.2))

$$\Theta^*\equiv y^\alpha\,\mathrm{d}\pi^\mu_\alpha\wedge\theta_\mu+y_{\alpha^*}\,\mathrm{d}\pi^{\alpha^*\mu}\wedge\theta_\mu-\mathscr{H}\omega \tag{11b.1}$$

and the modified Hamilton functional as

$$I'^*(\psi^{1*})=\int_{\psi^{1*}(\mathscr{M})}\Theta^*\,,\qquad \psi^{1*}\in\Gamma_0\big(J^{1*}(E)\big). \tag{11b.2}$$

The solutions of the variational problem are the cross-sections ψ^{1*} for which

(11b.3) $$(\delta_{\psi^{1*}} I'^*)(X^{1*}) \equiv \int_{\psi^{1*}(\mathcal{M})} L_{X^{1*}} \Theta^* = 0\,, \qquad \forall X^{1*} \in \Gamma\Big(\tau\big(J^{1*}(E)\big)\Big)\,,$$

where now

(11b.4) $$X^{1*} = X^\mu \frac{\partial}{\partial x^\mu} + X^\alpha \frac{\partial}{\partial y^\alpha} + X_{\alpha*} \frac{\partial}{\partial y_{\alpha*}} + X^\mu_\alpha \frac{\partial}{\partial \pi^\mu_\alpha} + X^{\alpha*\mu} \frac{\partial}{\partial \pi^{\alpha*\mu}}\,.$$

Equation (11b.3) leads again to

(11b.5) $$i_{X^{1*}} \mathrm{d}\Theta^*|_{\psi^{1*}} = 0\,, \qquad \forall X^{1*} \in \Gamma\Big(\tau\big(J^{1*}(E)\big)\Big)\,.$$

Let us evaluate (11b.5) in local co-ordinates, which we shall do in the real case to simplify the notation.

Here are two intermediate steps:

(11b.6) $$\mathrm{d}\Theta^* = \mathrm{d}\pi^\mu_\alpha \wedge \mathrm{d}y^\alpha \wedge \theta_\mu - \left(\frac{\partial \mathcal{H}}{\partial y^\alpha}\mathrm{d}y^\alpha + \frac{\partial \mathcal{H}}{\partial \pi^\mu_\alpha}\mathrm{d}\pi^\mu_\alpha\right)\wedge \omega\,,$$

(11b.7) $$i_{X^{1*}} \mathrm{d}\Theta^* = X^\mu_\alpha \mathrm{d}y^\alpha \wedge \theta_\mu - X^\alpha \mathrm{d}\pi^\mu_\alpha \wedge \theta_\mu + \mathrm{d}\pi^\mu_\alpha \wedge \mathrm{d}y^\alpha \wedge i_{X^1}(\theta_\mu) - \\ - \left(\frac{\partial \mathcal{H}}{\partial y^\alpha} X^\alpha + \frac{\partial \mathcal{H}}{\partial \pi^\mu_\alpha} X^\mu_\alpha\right)\omega + \left(\frac{\partial \mathcal{H}}{\partial y^\alpha}\mathrm{d}y^\alpha + \frac{\partial \mathcal{H}}{\partial \pi^\mu_\alpha}\mathrm{d}\pi^\mu_\alpha\right)\wedge i_{X^1}\omega\,.$$

Restricting to cross-sections and using $i_{X^1}(\omega) = X^\nu \theta_\nu$ and

(11b.8) $$\mathrm{d}x^\eta \wedge i_{X^1}(\theta_\mu) = X^\eta \theta_\mu - \delta^\eta_\mu X^\sigma \theta_\sigma\,,$$

one finds that (11b.5) is fulfilled when the coefficients of X^μ_α and X^α are zero,

(11b.9) $$\frac{\partial y^\alpha}{\partial x^\mu} = \frac{\partial \mathcal{H}}{\partial \pi^\mu_\alpha}\,, \qquad -\frac{\partial \pi^\mu_\alpha}{\partial x^\mu} = \frac{\partial \mathcal{H}}{\partial y^\alpha}\,,$$

i.e. if the Hamilton-type equations (11b.9) are satisfied (again the coefficient of X^μ is zero in that case). The space of solutions of (11b.9) will be called $\mathcal{U}'_{\mathcal{H}}$.

The equivalence of (11b.9) and (11a.7) for regular Lagrangians may now be easily established. By defining the Legendre transformation in the usual way, $D_L: e^1 \in J^1(E) \to D_L(e^1) = e^{1*} \in J^{1*}(E)$ $\big(D_L^{-1}: J^{1*}(E) \to J^1(E)\big)$,

(11b.10) $$x^\mu(e^{1*}) = x^\mu(e^1)\,, \qquad y^\alpha(e^{1*}) = y^\alpha(e^{1*})\,, \qquad \pi^\mu_\alpha(e^{1*}) = \left.\frac{\partial \mathcal{L}}{\partial y^\alpha_\mu}\right|_{e^1}$$

(and analogous expressions for those co-ordinates with *), then the scalar function

$$\mathscr{H} = \pi^\mu_\alpha y^\alpha_\mu + \pi^{\alpha^*\mu} y_{\alpha^*\mu} - \mathscr{L} \tag{11b.11}$$

is defined on $J^{1*}(E)$. This definition for the Hamiltonian density associated to $\mathscr{L}$ is consistent with (11b.1), where $y^\alpha \, d\pi^\mu_\alpha \wedge \theta_\mu$ is the equivalent to the Liouville form of mechanics, and with (11a.8), (11a.9). One checks immediately that the insertion of $\mathscr{H}$ in (11b.9) reproduces (11a.7) plus the 1-jet prolongation condition. Thus the regularity condition makes PII equivalent to PI by restricting cross-sections $\psi^1 \in \mathscr{U}'_{\mathscr{L}}$ to be of the $\bar{\psi}^1$ type (in the Hamiltonian-Lagrangian approach, PII on $J^1(E)$) or by establishing a one-to-one correspondence between cross-sections $\bar{\psi}^1 \in \mathscr{U}_{\mathscr{L}}$ and $\psi^{1*} \in \mathscr{U}'_{\mathscr{H}}$ (in the strictly Hamiltonian formalism, PII on $J^{1*}(E)$). In this case there is a one-to-one correspondence between $\mathscr{U}_{\mathscr{L}}$, $\mathscr{U}'_{\mathscr{L}}$, $\mathscr{U}'_{\mathscr{H}}$.

12. – Variational approach to classical mechanics.

For the sake of brevity we restrict ourselves to PII in Hamiltonian form. In that case, $J^{1*}(E)$ is the bundle $R \times T^*(M) \to R$, and, as mentioned in the introduction to this part, $\Gamma_0(J^{1*}(E))$ is the submanifold of cross-sections which take fixed values at the boundary of the closed interval $[t_1, t_2]$. The differential $\delta_s I'$ has now to be zero on the vector fields tangent to $J^{1*}(E)$ with support on $[t_1, t_2]$. Under these conditions, the critical points (cross-sections) of $\Gamma_0(J^{1*}(E))$ satisfy (cf. (11b.3))

$$\int_{s^{1*}(T)} L_{X^{1*}} \Theta = 0\,, \qquad \forall X^{1*} \in \Gamma\big(\tau(R \times T^*(M))\big)\,, \tag{12.1}$$

where $\Theta = \Lambda - H\, dt$ (sect. 7). Due to the boundary conditions which impose fixed values for the cross-sections at t_1 and t_2, the vector fields of (12.1) are zero at these points. Thus the term

$$\int_{s^{1*}(T)} d i_{X^{1*}} \Theta \tag{12.2}$$

disappears after integration and the other contribution of $L_{X^{1*}}$ gives

$$i_{X^{1*}} d\Theta|_{s^{1*}(T)} = 0\,, \qquad \forall X^{1*}. \tag{12.3}$$

The reader will recognize in (12.3) eq. (7c.1) which determines the cross-sections which belong to $\mathscr{U}_H$ (*). Indeed, since the radical of $\mathrm{d}\Theta$ is of dimension one (Θ is a contact form), (12.3) indicates that the trajectories of $\mathscr{U}_H$ are integrals of $\tilde{X}_H$, the generator of the modulus rad $(\mathrm{d}\Theta)$ $(\mathrm{d}\Theta(X^{1*}, \tilde{X}_H) = 0,\ \forall X^{1*})$.

13. – A first application to relativistic fields.

Let us now apply the previous formalisms to several simple examples extracted from the relativistic field theory of free elementary particles. We recall that $\mathscr{M}$ is the Minkowski space and $x^\mu = (x^0, x^1, x^2, x^3)$; it is clear that the dimension m of the fibre space (indices $\alpha, \beta, \ldots$) will depend on the spin of the field. The transition to the usual physical notation is made by identifying the cross-section of E with the definition of the field in each point of the Minkowski space. Because of translation invariance, Lagrangian and Hamiltonian densities will not depend on x^μ.

a) *The Klein-Gordon field.* The fibre space is of dimension one. The usual KG Lagrangian density $(\partial^\mu \phi^*)(\partial_\mu \phi) - m^2\phi^* \phi$ is, in the formalism on $J^1(E)$, given by

$$\mathscr{L}^1_{\mathrm{KG}} = y^*_\mu y_\nu \eta^{\mu\nu} - m^2 y^* y \,, \tag{13a.1}$$

where $\eta^{\mu\nu}$ is the Minkowski metric tensor $(\eta = \mathrm{diag}\,(+, -, -, -))$ and m the mass of the field. PI leads to the familiar KG equation, $(\partial^\mu\partial_\mu + m^2) y(x) = 0$. It is clear that $\mathscr{L}$ is regular; thus PII (eqs. (11*a*.16)) gives

$$\frac{\mathrm{d}}{\mathrm{d}x_\mu} y_\mu + m^2 y = 0 \,, \qquad y_\mu = \partial_\mu y \,, \tag{13a.2}$$

which reproduce the equation of the KG field.

As $\mathscr{L}$ is regular, it is possible to define an equivalent invariant Hamiltonian formulation. From (13*a*.1) and (11*b*.11) one obtains

$$\mathscr{H}^{1*}_{\mathrm{KG}} = \pi^{*\mu}\pi^\nu \eta_{\mu\nu} + m^2 y^* y \tag{13a.3}$$

and now (11*b*.9) gives the definition of the momenta $(\pi^\mu = \partial^\mu y^*,\ \pi^{*\mu} = \partial^\mu y)$ and again $(\Box + m^2) y = 0$.

As is well known, a certain amount of arbitrariness is involved in the definition of Lagrangians, since the addition of a total derivative $(d_\mu F)$ does not alter the EL equations of motion. It is not difficult to accommodate this

(*) And again that the action of the horizontal component is trivial in the sense that its coefficient is zero for critical sections (verticality of X^{1*}).

situation in the fibre bundle formulation. Consider, for instance, the Lagrangian density on $J^1(J^1(E))$ (the co-ordinates being in general $(x^\mu, y^\alpha, y_{\alpha^*}, y^\alpha_\mu, y_{\alpha^*\mu};$ $g^\alpha_{,\nu}, g_{\alpha^*,\nu}, g^\alpha_{\mu,\nu}, g_{\alpha^*\mu,\nu}))$ for the KG field

$$\mathscr{L}^{1,1}_{\mathrm{KG}} = -y^* g^\mu_{,\mu} - g^{*\mu}_{,\mu} y - y^{*\mu} y_\mu - m^2 y^* y \,. \tag{13a.4}$$

Since substituting $J^1(E)$ for E amounts to increasing the number of variables, PI clearly gives for Lagrangians on $J^1(J^1(E))$

$$\frac{\partial \mathscr{L}^{1,1}}{\partial y^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^\nu}\left(\frac{\partial \mathscr{L}^{1,1}}{\partial g^\alpha_{,\nu}}\right) = 0\,, \qquad \frac{\partial \mathscr{L}^{1,1}}{\partial y^\alpha_\mu} - \frac{\mathrm{d}}{\mathrm{d}x^\nu}\left(\frac{\partial \mathscr{L}^{1,1}}{\partial g^\alpha_{\mu,\nu}}\right) = 0 \tag{13a.5}$$

(*with* the 1-jet prolongation conditions $g^\alpha_{,\nu} = \partial_\nu y^\alpha$, $g^\alpha_{\mu,\nu} = \partial_\nu y^\alpha_\mu$) plus the corresponding equations for $(y_{\alpha^*}, g_{\alpha^*,\nu})$ and $(y_{\alpha^*\mu}, g_{\alpha^*\mu,\nu})$. In the case of the Lagrangian (13a.4), (13a.5) gives $g^{*\mu}_{,\mu} + m^2 y^* = 0$, $y^{*\mu} - \partial^\mu y^* = 0$, which combined reproduce the KG equation. Alternatively, one could have thought of a Lagrangian on $J^2(E)$ (the bundle of the 2-jets of E of co-ordinates $(x^\mu, y^\alpha,$ $y_{\alpha^*}, y^\alpha_\mu, y_{\alpha^*\mu}, y^\alpha_{\mu\nu}, y_{\alpha^*\mu\nu}))$ as, for instance,

$$\mathscr{L}^2_{\mathrm{KG}} = -\eta^{\mu\nu}(y^* y_{\mu\nu} + y^*_{\mu\nu} y + y^*_\mu y_\nu) - m^2 y^* y \,. \tag{13a.6}$$

In this case (11a.7) is no longer valid and it is necessary to substitute j^2 (the 2-jet prolongation) for j^1 in the reasoning leading from PI to the EL equations. For a Lagrangian on $J^2(E)$ these read

$$\frac{\mathrm{d}^2}{\mathrm{d}x^\mu\,\mathrm{d}x^\nu}\left(\frac{\partial \mathscr{L}^2}{\partial y^\alpha_{\mu\nu}}\right) - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathscr{L}^2}{\partial y^\alpha_\mu}\right) + \frac{\partial \mathscr{L}^2}{\partial y} = 0 \tag{13a.7}$$

with the 2-jet prolongation conditions $y^\alpha_{\mu\nu} = \partial_\mu y^\alpha_\nu$, $y^\alpha_\nu = \partial_\nu y^\alpha$. Substituting now (13a.6) into (13a.7), one obtains again the KG equation. It is to be noted in passing that the Lagrangians (13a.4) and (13a.6) provide an example of equivalent formalisms on $J^1(J^1(E))$ and $J^2(E)$, the equivalence being implemented by identifying $g_{,\nu}$, $g_{\mu,\nu}$ with y_ν, $y_{\mu\nu}$ [18].

b) The Proca field. In this case, the fibre index is also a Lorentz index. To avoid confusion with the base indices we shall use brackets to indicate the former. With $(x^\mu; y^{(\mu)}, y^*_{(\mu)}; y^{(\mu)}{}_{,\nu}, y^*_{(\mu)\nu})$ as co-ordinate system for $J^1(E)$, $\mathscr{L}$ takes the form

$$\mathscr{L}^1_{\mathrm{P}} = y^{*(\mu)}_\mu y_\nu{}^{(\nu)} - y^{*(\mu)}_\nu y_{\nu'}{}^{(\mu')} \eta_{\mu\mu'} \eta^{\nu\nu'} + m^2 y^{*(\mu)} y^{(\nu)} \eta_{\mu\nu} \,. \tag{13b.1}$$

PI gives for (13b.1) the Proca equations (in (11a.7) $y^{(\mu)}_\nu = \partial_\nu y^{(\mu)}$, etc.)

$$[\eta_{\mu\nu}(\Box + m^2) - \partial_\nu \partial_\mu] y^{(\mu)} = 0 \,, \tag{13b.2}$$

which, because $m \neq 0$, already include the subsidiary condition $\partial_\mu y^{(\mu)} = 0$ which eliminates the spin-zero part. As $\mathscr{L}$ is regular, the same result is obtained from PII and again a scalar Hamiltonan formulation becomes possible. With $(x^\mu, y^{(\mu)}, y^{*(\mu)}; \pi_{(\mu)}{}^\nu, \pi^{*(\mu)\nu})$ as a co-ordinate system for $J^{1*}(E)$ we find from (11*b*.10)

$$\pi^{*(\mu)\nu} = -y^{(\mu)\nu} + \eta^{\mu\nu} y^{(\lambda)}{}_\lambda , \qquad y^{(\mu)\nu} = \tfrac{1}{3}\pi^{*(\lambda)}{}_\lambda \eta^{\mu\nu} - \pi^{*(\mu)\nu} \tag{13b.3}$$

and from (11*b*.11)

$$\mathscr{H}^{1*}_{\mathrm{P}} = \tfrac{1}{3}\pi^{*}_{(\mu)}{}^{\mu}\pi_{(\nu)}{}^{\nu} - \pi^{*}_{(\mu)}{}^{\nu}\pi^{(\mu)}{}_{\nu} - m^2 y^{*(\mu)} y_{(\mu)} . \tag{13b.4}$$

The Hamilton-like equations (11*b*.9) read in this case

$$\frac{\partial \mathscr{H}}{\partial \pi_{(\mu)}{}^\nu} = \frac{\partial y^{(\mu)}}{\partial x^\nu} , \qquad \frac{\partial \mathscr{H}}{\partial y^{(\mu)}} = -\frac{\partial \pi_{(\mu)}{}^\nu}{\partial x^\nu} \tag{13b.5}$$

(and similar ones for π^*, y^*). It is now immediate to check that the first defines the momenta in terms of $y^{*(\mu)}$ (cf. (13*b*.3)),

$$\partial^\nu y^{(\mu)} = \tfrac{1}{3}\pi^{*(\lambda)}{}_\lambda \eta^{\mu\nu} - \pi^{*(\mu)\nu} , \tag{13b.6}$$

and that combined with the second reproduces (13*b*.2).

As in the case of the KG field, it is also possible here to define Lagrangian densities on $J^1(J^1(E))$, $J^2(E)$ leading to (13*b*.2) through PI. We shall not consider this nor the case of the Maxwell field, whose treatment, apart from the fact that the subsidiary condition is not obtained from the Lagrangian ($m = 0$) and the associated feature of gauge invariance, may be performed along similar lines to that of the Proca field.

c) *The Dirac field.* For the Dirac field E is the spinorial bundle associated to the representation $D^{\frac{1}{2},0} \oplus D^{0,\frac{1}{2}}$ of the group $SL_{2,C}$ of co-ordinates $(x^\mu, y^\alpha, y_{\alpha^*})$, $\alpha = 1, 2, 3, 4$. The simple Dirac Lagrangian on $J^1(E)$ is given by

$$\mathscr{L}^1_{\mathrm{D}} = i y_{\alpha^*}(\gamma^0\gamma^\mu)^\alpha_\beta y^\beta{}_\mu - m y_{\alpha^*}(\gamma^0)^\alpha_\beta y^\beta ; \tag{13c.1}$$

our γ-matrices are such that $\gamma^0 = \gamma^{0\dagger}$, $\gamma^i = -\gamma^{i\dagger}$. PI and (13*c*.1) immediately give the Dirac equation. However, since $\mathscr{L}$ is not regular (the Hessian vanishes identically), PII gives different results. From (11*a*.16*a*) and the corresponding adjoint equation (coming from the coefficient of X_{α^*}) we obtain

$$m y_{\beta^*}(\gamma^0)^\beta_\alpha + i\partial_\mu y_{\beta^*}(\gamma^0\gamma^\mu)^\beta_\alpha = 0 , \qquad (i\gamma^\mu\partial_\mu - m)^\alpha_\beta y^\beta = 0 , \tag{13c.2}$$

i.e. the adjoint and the normal Dirac equation. The other two equations ((11*a*.16*b*) and its associated one) are identically, zero; accordingly, no constraint is provided for the $y^\alpha{}_\mu$ and $\mathscr{U}_{\mathscr{L}} \subset \mathscr{U}'_{\mathscr{L}}$. Imposing on the $y^\alpha{}_\mu$ the condition $y^\alpha{}_\mu = \partial_\mu y^\alpha$ is equivalent to transforming PI into PII; when this is done, $\mathscr{U}_{\mathscr{L}} = \mathscr{U}'_{\mathscr{L}}$. This picture is not altered when the full Dirac Lagrangian (Hermitian and invariant under particle-antiparticle conjugation in quantum field theory) is used instead of (13*c*.2).

This result was to be expected. PII is of Hamiltonian type, *i.e.* leads to first-order differential equations. However, as the Dirac equation is already of first order, PII cannot simultaneously determine y^α and $y^\alpha{}_\mu$. Indeed, it is not possible to construct directly from the Lagrangian $\mathscr{L}^1_{\mathrm{D}}$ a scalar Hamiltonian formulation based on PII and leading to the Dirac equation. This is because the Legendre transformation (11*b*.10) is not a diffeomorphism for $\mathscr{L}_{\mathrm{D}}$; the variables in $\mathscr{H}(y^\alpha, y_{\alpha^*}, \pi^\mu_\alpha, \pi^{\mu\alpha^*})$ are not all independent and there is no equivalence between (11*b*.9) and (11*a*.7). In fact, this situation is already present in the ordinary Hamiltonian formulation of the Dirac theory, where H is the generator of the time translations and leads to the equations of motion. Nevertheless, it is still possible to write a scalar Hamiltonian $\mathscr{H}$ dealing with all variables as independent and to use the Lagrange multipliers to impose the required constraints. Indeed, the Hamiltonian density on $J^{1*}(E)$

$$\mathscr{H}_{\mathrm{D}} = y_{\alpha^*}(m\gamma^0)^\alpha_\beta y^\beta + \lambda^\beta_\mu\left(\pi^\mu_\beta - \frac{i}{2}\, y_{\alpha^*}(\gamma^0\gamma^\mu)^\alpha_\beta\right) + \lambda'_{\mu\beta}\left(\pi^{\mu\beta^*} + \frac{i}{2}\,(\gamma^0\gamma^\mu)^\beta_\alpha y^\alpha\right) \tag{13c.3}$$

(whose first term is what would be obtained directly from $\mathscr{L}^1_{\mathrm{D}}$ through (11*b*.11) and whose second and third terms include the constraints on the momenta and the Lagrange multipliers) leads through (11*b*.9) to the Dirac equation. However, it is not clear what physical meaning—if any—could be associated to the λ's.

d) *The Rarita-Schwinger spin-$\frac{3}{2}$ field.* $J^1(E)$ will be now the vector-spinor vector bundle of co-ordinates $(x^\mu, y^{(\mu\alpha)}, y_{(\mu\alpha)^*}; y^{(\mu\alpha)}{}_\nu, y_{(\mu\alpha)^*\nu})$; fibre indices $(\mu = 0, 1, 2, 3; \alpha = 1, 2, 3, 4)$ are inside brackets. The simple Lagrangian on $J^1(E)$ is given by

$$\begin{aligned}\mathscr{L}_{\mathrm{RS}} = &-iy_{(\mu\alpha)^*}(\gamma^0\gamma^\sigma)^\alpha_\beta y^{(\mu\beta)}{}_\sigma + my_{(\mu\alpha)^*}(\gamma^0)^\alpha_\beta y^{(\mu\beta)} + \\ &+ \frac{i}{3} y_{(\nu\alpha)^*}\eta^{\nu\mu}\eta^{\sigma\eta}[(\gamma_\mu)^\alpha_\beta y^{(\beta)}_{(\eta)\sigma} + (\gamma_\sigma)^\alpha_\beta y^{(\beta)}_{(\eta)\mu}] - \frac{i}{3} y_{(\nu\alpha)^*}(\gamma^\nu\gamma^\sigma\gamma^\mu)^\alpha_\beta y^{(\beta)}_{(\mu)\sigma} - \frac{1}{3} y_{(\nu\alpha)^*}(m\gamma^\nu\gamma^\mu)^\alpha_\beta y^{(\beta)}_{(\mu)}\,.\end{aligned} \tag{13d.1}$$

PI leads to the Rarita-Schwinger equations $(y^{(\mu\alpha)}{}_\nu = \partial_\nu y^{(\mu\alpha)})$

$$\begin{aligned}(-i\gamma^\sigma\partial_\sigma + m)^\alpha_\beta y^{(\mu\beta)} &+ \tfrac{1}{3}\,(i\gamma^\mu\partial_\eta + i\gamma_\eta\partial^\mu)^\alpha_\beta y^{(\eta\beta)} - \\ &- \tfrac{1}{3}\,(\gamma^\mu(i\gamma^\sigma\partial_\sigma + m)\gamma_\nu)^\alpha_\beta y^{(\nu\beta)} = 0\,,\end{aligned} \tag{13d.2}$$

which include the subsidiary conditions $\partial_\mu y^{(\mu\alpha)} = 0$, $\gamma_\mu y^{(\mu\alpha)} = 0$, which restrict the vector part to spin one and the product with the spinor part to spin $\frac{3}{2}$.

The equations for PII read in this case (coefficients of the components $X_{(\mu\alpha)^*}$ and $X_{(\mu\alpha)^*\nu}$)

$$\frac{\partial\mathscr{L}}{\partial y_{(\mu\alpha)^*}}-\frac{\mathrm{d}}{\mathrm{d}x^\sigma}\left(\frac{\partial\mathscr{L}}{\partial y_{(\mu\alpha)^*\sigma}}\right)+\frac{\partial}{\partial y_{(\mu\alpha)^*}}\left(\frac{\partial\mathscr{L}}{\partial y_{(\varrho\beta)^*\sigma}}\right)\left[\frac{\partial y_{(\varrho\beta)^*}}{\partial x^\sigma}-y_{(\varrho\beta)^*\sigma}\right]+ \\ +\frac{\partial}{\partial y_{(\mu\alpha)^*}}\left(\frac{\partial\mathscr{L}}{\partial y^{(\varrho\beta)}{}_\sigma}\right)\left[\frac{\partial y^{(\varrho\beta)}}{\partial x^\sigma}-y^{(\varrho\beta)}{}_\sigma\right]=0\,, \tag{13d.3}$$

$$\frac{\partial}{\partial y_{(\mu\alpha)^*\nu}}\left(\frac{\partial\mathscr{L}}{\partial y_{(\varrho\beta)^*\mu}}\right)\left[\frac{\partial y_{(\varrho\beta)^*}}{\partial x^\mu}-y_{(\varrho\beta)^*\mu}\right]+\frac{\partial}{\partial y_{(\mu\alpha)^*\nu}}\left(\frac{\partial\mathscr{L}}{\partial y^{(\varrho\beta)}{}_\mu}\right)\left[\frac{\partial y^{(\varrho\beta)}}{\partial x^\mu}-y^{(\varrho\beta)}{}_\mu\right]=0\,. \tag{13d.4}$$

Since the Hessian of $\mathscr{L}_{\mathrm{RS}}$ is again identically zero and has analogous characteristics as $\mathscr{L}_{\mathrm{D}}$, the remarks made in subsect. **13**c) also apply here. In particular, and as in the Dirac case, the above equations reproduce (13d.2) and leave $y^{(\mu\alpha)}{}_\nu$ unconstrained. We shall not comment on this any further and shall only mention that the case of spin $l+\frac{1}{2}$ is easily incorporated into the scheme by taking

$$\mathscr{L}_{\mathrm{RS}}=-iy_{(\mu_1\mu_2\dots\mu_l\alpha)^*}(\gamma^0\gamma^\sigma)^\alpha_\beta y^{(\mu_1\dots\mu_l\beta)}_\sigma+my_{(\mu_1\dots\mu_l\alpha)^*}(\gamma^0)^\alpha_\beta y^{(\mu_1\dots\mu_l\beta)}+ \\ +\frac{i}{3}y_{(\mu\mu_2\dots\mu_l\alpha)}[(\gamma^\mu)^\alpha_\beta y^{(\mu'\mu_2\dots\mu_l\beta)}_{\mu'}+(\gamma_{\mu'})^\alpha_\beta y^{(\mu'\mu_2\dots\mu_l\beta)\mu}]- \\ -\frac{i}{3}y_{(\mu\mu_2\dots\mu_l\alpha)^*}(\gamma^\mu\gamma^\sigma\gamma_{\mu'})^\alpha_\beta y^{(\mu'\mu_2\dots\mu_l\beta)}_\sigma-\frac{1}{3}y_{(\mu\mu_2\dots\mu_l\alpha)^*}(m\gamma^\mu\gamma_{\mu'})^\alpha_\beta y^{(\mu'\mu_2\dots\mu_l\beta)}\,, \tag{13d.5}$$

which leads to an equation which is identical to (13d.2) but for the $l-1$ additional vector indices in the fibre part of the field; the irreducibility is obtained because of the symmetry under permutations of the indices $\mu_1, \dots, \mu_l$.

14. – Symmetries in the modified Hamilton formulation.

As has been mentioned in sect. **1**, the use of the Cartan form is specially useful in studying the symmetries of a system and in formulating the associated Noether theorem [10, 11, 18]. In this section it is our intention to show how this process, which leads to the Noether currents, may be performed for the Hamiltonian-like formalism (PII).

Let us recall first the well-known Noether theorem for the ordinary Hamilton variational problem (PI) defined by a Lagrangian $\mathscr{L}$. A vector field $X\in\Gamma(\tau(E))$ is a symmetry of the system described by $\mathscr{L}$ if

$$L_{\overline{X}^1}(\mathscr{L}\omega)-\mathrm{d}\Delta|_{\overline{\Psi}^1}=0\,,\qquad \forall\overline{\Psi}^1, \tag{14.1a}$$

where $\Delta = \Delta^\mu \theta_\mu$ is a three-form which does not depend on $y^\alpha{}_\mu$. In particular, Δ may be zero as is the case for the Poincaré generators. If we put $\Delta = 0$, (14.1*a*) gives for a symmetry X

$$\mathrm{d}(i_{\bar{X}^1}\Theta)|_{\bar{\Psi}^1} = 0\,, \qquad \forall \Psi^1 \in \mathscr{U}_{\mathscr{L}}; \tag{14.1b}$$

note that in (14.1*a*) the restriction is made on cross-sections which are 1-jet prolongations and that in (14.1*b*) on cross-sections which are also solutions of the variational problem.

To extend the definition of a symmetry to PII on $J^1(E)$, we make use of the Cartan form Θ. $X^1 \in \Gamma\big(\tau(J^1(E))\big)$ is a symmetry if

$$L_{X^1}\Theta = 0 \tag{14.2a}$$

(a more general definition as in (14.1*a*) is possible, but we shall not consider it). This definition is consistent with the one given for PI; indeed, in the case of regularity in which $\mathscr{U}_{\mathscr{L}} = \mathscr{U}'_{\mathscr{L}}$, we have for fields on E $\big($(B.3.15)$\big)$ whose action on $J^1(E)$ is defined through their 1-jet prolongation

$$L_{\bar{X}^1}\Theta|_{\bar{\Psi}^1} = L_{\bar{X}^1}(\mathscr{L}\omega)|_{\bar{\Psi}^1}\,, \tag{14.3}$$

which is easily checked by writing Θ in terms of $\mathscr{L}$. Using the Cartan formula for the Lie derivative and taking into account (11*a*.12), one obtains from (14.2*a*)

$$\mathrm{d}(i_{X^1}\Theta)|_{\Psi^1} = 0\,, \qquad \forall \Psi \in \mathscr{U}'_{\mathscr{L}}, \tag{14.2b}$$

from which the conservation of the current associated with X^1 may be derived: on cross-sections of $\mathscr{U}'_{\mathscr{L}}$ and with $j \equiv *\, i_{X^1}\Theta$, we get $\delta j = 0$, where $*$ is the Hodge operator and δ the exterior codifferential. In terms of local co-ordinates, this reads $\partial^\mu j_\mu = 0$ with

$$\begin{aligned} -j^\mu = \mathscr{L}X^\mu + (X^\alpha - \partial_\nu y^\alpha X^\nu)\frac{\partial\mathscr{L}}{\partial y^\alpha{}_\mu} + (X_{\alpha*} - \partial_\nu y_{\alpha*} X^\nu)\frac{\partial\mathscr{L}}{\partial y_{\alpha*\mu}} + {}\\ + (\partial_\nu y^\alpha - y^\alpha{}_\nu)X^\mu\frac{\partial\mathscr{L}}{\partial y^\alpha{}_\nu} + (\partial_\nu y_{\alpha*} - y_{\alpha*\nu})X^\mu\frac{\partial\mathscr{L}}{\partial y_{\alpha*\nu}}\,. \end{aligned} \tag{14.2c}$$

In the case of regularity, the two last terms do not appear and the first line of (14.2*c*) reproduces the usual expression for the current $\big($which may be directly obtained from (14.1*b*)$\big)$.

The elegance of this formulation lies in the fact that again the same definition may be used for PII on $J^{1*}(E)$: $X^{1*} \in \Gamma\big(\tau(J^{1*}(E))\big)$ is a symmetry when (cf. (7*c*.4))

$$L_{X^{1*}}\Theta^* = 0\,, \tag{14.4a}$$

where now Θ^* is given by (11*b*.1). This definition is consistent with those given above, since for the case of regularity

$$(14.5)\qquad L_{X^{1*}}\Theta^*|_{j^1(\Psi)} = L_{\bar{X}^1}(\mathscr{L}\omega)|_{j^1(\Psi)}\,,$$

which is obtained by writing Θ in terms of $\mathscr{L}$ and taking into account that the resulting expression is defined on $J^{1*}(E)$ despite the presence of $\mathscr{L}$. We note in passing that an expression similar to (14.5) also holds in the case of analytical mechanics (*). From (14.4*a*) we get (cf. (7*c*.5))

$$(14.4b)\qquad \mathrm{d}(i_{X^{1*}}\Theta^*)|_{\Psi^{1*}} = 0\,, \qquad \forall\Psi^{1*}\in\mathscr{U}'_{\mathscr{H}};$$

in this case

$$(14.4c)\qquad i_{X^{1*}}\Theta^*|_{\Psi^{1*}} = \left\{\left(\pi^\nu_\alpha\frac{\partial y^\alpha}{\partial x^\nu} + \pi^{\nu\alpha^*}\frac{\partial y_{\alpha^*}}{\partial x^\nu} - \mathscr{H}\right)X^\mu + \right.$$
$$\left. + \pi^\mu_\alpha\left(X^\alpha - \frac{\partial y^\alpha}{\partial x^\nu}X^\nu\right) + \left(X_{\alpha^*} - \frac{\partial y_{\alpha^*}}{\partial x^\nu}X^\nu\right)\pi^{\mu\alpha^*}\right\}\theta_\mu$$

and $-j^\mu$ is the term within curly brackets. When $\mathscr{H}$ comes from a regular Lagrangian density (14.4*c*) gives—as it should—the same current as (14.2*c*), where $\Psi^1\in\mathscr{U}'_{\mathscr{L}}$, which is, in turn, the same result we would have obtained from PI and definition (14.1*a*). The formalism presented here allows, however, for a more general treatment of the definition of a symmetry when PII is used for systems defined through densities on $J^1(E)$ and $J^{1*}(E)$.

The previous expressions (14.2*c*) ((14.4*c*)) for PII solve the problem of finding the conserved current associated with the symmetry defined by a certain field X^1 (X^{1*}) on $J^1(E)$ ($J^{1*}(E)$). Note that, although only their components on E are necessary in these expressions, the complete vector field appears in (14.2*a*) ((14.4*a*)). In many cases the action of the transformation which might generate a symmetry is only given on E. This is the case of the generators of space-time symmetries which initially are only defined through their action on the basis and on the fibre space and which will be considered in the next section. As already mentioned, the canonical procedure for extending

(*) With the definition $\Theta_H = p_i\,\mathrm{d}q^i - H\,\mathrm{d}t$ (subsect. 7*a*)) and the prolongations of $X = X^i(\partial/\partial q^i)$ given by

$$X^{\mathrm{T}} = X^i\frac{\partial}{\partial q^i} + \dot{q}^j\frac{\partial X^i}{\partial q^j}\frac{\partial}{\partial \dot{q}^i}\,, \qquad X^* = X^i\frac{\partial}{\partial q^i} - p_j\frac{\partial X^j}{\partial q^i}\frac{\partial}{\partial p_i}$$

(see appendix A.4) the formula which corresponds to (14.5) reads

$$L_{X^*}\Theta_H = L_{X^{\mathrm{T}}}(L\,\mathrm{d}t)\,.$$

their action to $J^1(E)$ is the 1-jet prolongation (appendix B); since the extended vector fields transform 1-jet cross-sections onto themselves, this definition includes the situation for PI ((14.1a), (14.1b)) as a particular case. In terms of $X^\mu(x, y^\beta)$, $X^\alpha(x, y^\beta)$ the condition of stability of the Pfaffian system under $\bar{X}^1$ gives for $\bar{X}^\alpha_\mu$ ((B.3.4))

$$\bar{X}^\alpha_\mu = \frac{\partial X^\alpha}{\partial x^\mu} - y^\alpha{}_\nu \frac{\partial X^\nu}{\partial x^\mu} + y^\beta{}_\mu \frac{\partial X^\alpha}{\partial y^\beta} - y_{\beta^*\mu} \frac{\partial X^\alpha}{\partial y_{\beta^*}} - y^\beta{}_\mu \frac{\partial X^\nu}{\partial y^\beta} y^\alpha{}_\nu - y_{\beta^*\mu} \frac{\partial X^\nu}{\partial y_{\beta^*}} y^\alpha{}_\nu \tag{14.6}$$

and a parallel expression for $\bar{X}_{\alpha^*\mu}$.

For the symmetries of the system defined on $J^{1*}(E)$, the complete expression of the field on $J^{1*}(E)$ is required. This is because PII on $J^{1*}(E)$ is a Hamiltonian principle in the sense that it leads to first-order equations in the fields and the momenta; an analogous situation may appear when considering PII on $J^1(E)$. However, for a regular Hamiltonian on $J^{1*}(E)$ PII is equivalent to PI when applied to the corresponding Lagrangian. In this case the notion of 1-jet prolongation may be transported from $J^1(E)$ to $J^{1*}(E)$ through the Legendre transformation, the structure forms being now written as

$$\theta^{*\alpha} = \mathrm{d}y^\alpha - \frac{\partial \mathscr{H}}{\partial \pi^\mu_\alpha} \mathrm{d}x^\mu \tag{14.7}$$

and then (see appendix B.4) $\bar{X}^{1*} = X + \bar{X}^{*\sigma}_\beta(\partial/\partial\pi^\sigma_\beta)$, where

$$\bar{X}^{*\sigma}_\beta = \left[\frac{\partial^2 \mathscr{H}}{\partial\pi^\mu_\beta \partial\pi^\sigma_\alpha}\right]^{-1} \left(\frac{\partial X^\alpha}{\partial x^\mu} - \frac{\partial \mathscr{H}}{\partial \pi^\nu_\alpha}\frac{\partial X^\nu}{\partial x^\mu} + \frac{\partial \mathscr{H}}{\partial \pi^\mu_\gamma}\frac{\partial X^\alpha}{\partial y^\gamma} - \frac{\partial \mathscr{H}}{\partial \pi^\mu_\gamma}\frac{\partial X^\nu}{\partial y^\gamma}\frac{\partial \mathscr{H}}{\partial \pi^\nu_\alpha}\right). \tag{14.8}$$

15. – Application: space-time symmetries and Noether currents.

We now apply the theory of the previous section to exhibit the well-known relativistic invariance of the systems of sect. **13**. Since the final results—the Poincaré generators—are quite familiar, we shall restrict ourselves to giving the more relevant formulae to illustrate how the theory works.

a) Klein-Gordon field. The vector fields on the bundle E of the Poincaré group $\mathscr{P}^\uparrow_+$ are given by

$$P_\mu = \delta^\nu_\mu \partial_\nu , \qquad M_{\mu\nu} = \delta^{\varepsilon\sigma}_{\mu\nu} x_\varepsilon \partial_\sigma , \qquad \delta^{\varepsilon\sigma}_{\mu\nu} \equiv \eta^\varepsilon_\mu \eta^\sigma_\nu - \eta^\varepsilon_\nu \eta^\sigma_\mu . \tag{15a.1}$$

On $J^1(E)$, P_μ is given by the same expression, but $M_{\mu\nu}$ is now written as

$$M_{\mu\nu} = \delta^{\varepsilon\sigma}_{\mu\nu} x_\varepsilon \partial_\sigma + \delta_{\mu\nu,\sigma}{}^{\varepsilon\bullet} y_\varepsilon \frac{\partial}{\partial y_\sigma} + y^*_\varepsilon \delta_{\mu\nu,\sigma}{}^{\varepsilon\bullet} \frac{\partial}{\partial y^*_\sigma} , \tag{15a.2}$$

where the last two terms correspond to $\bar{X}_\sigma$ ((14.6)). It is now a simple task to prove that $L_{\bar{X}^1}\mathscr{L}\omega = 0 = L_{\bar{X}^1}\Theta$ on cross-sections $\bar{\Psi}^1$ so that the currents are given by (14.2c), where the fibre components X^α, $X_{\alpha*}$ are absent since the group acts trivially on the fibre part. Thus for the conserved current associated with P_μ and $M_{\mu\nu}$ we obtain, respectively,

$$j^\xi_{(\mu)} = -\delta^\xi_\mu(y^*_\nu y^\nu - m^2 y^* y) + y^*_\mu y^\xi + y^{*\xi} y_\mu \,, \tag{15a.3}$$

$$j^\xi_{(\mu\nu)} = -\delta^{\varepsilon\xi}_{\mu\nu} x_\varepsilon (y^*_\sigma y^\sigma - m^2 y^* y) + \delta^{\varepsilon\sigma}_{\mu\nu} x_\varepsilon y_\sigma y^{*\xi} + \delta^{\varepsilon\sigma}_{\mu\nu} x_\varepsilon y^*_\sigma y^\xi \tag{15a.4}$$

(in fact, $j^\xi_{(\mu)}$ is the energy-momentum tensor), which lead to the usual expressions for the charge densities ($\xi = 0$) in terms of the KG field by putting $y \equiv \phi$, $y_\sigma = \partial_\sigma \phi$. For instance the Hamiltonian density is given by the familiar expression

$$\mathscr{H} = \dot{\phi}^* \dot{\phi} + \nabla\phi^* \cdot \nabla\phi + m^2 \phi^* \phi \tag{15a.5}$$

and, in terms of $j^0_{(\mu)}$, (15a.4) reads $j^0_{(\mu\nu)} = x_\mu j^0_{(\nu)} - x_\nu j^0_{(\mu)}$.

The invariance of the KG system when formulated on $J^{1*}_*(E)$ now follows from the fact that $L_{X^1*}\Theta^* = 0$, and the currents from the scalar Hamiltonian may be obtained from (14.4c). However, because the system satisfies the regularity condition, both formalisms are equivalent and the same currents are obtained. We remark that, although the scalar $\mathscr{H}^{1*}$ cannot be identified with the energy, the usual Hamiltonian (15a.5) is, of course, obtained as the generator of time displacements.

Before concluding with the KG field, let us make a few comments on the other two formalisms which were mentioned in sect. **13**. It is simple to show that the Lagrangian (13a.4) which led through PI to the KG equation also leads, as it should, to the appropriate Poincaré currents. Being defined on $J^1(J^1(E))$ it is clear that the Noehter currents are given in this formalism by the expression (cf. (14.2c))

$$-j^\mu = X^\mu \mathscr{L} + (X^\alpha - g^\alpha_{,\sigma} X^\sigma)\frac{\partial \mathscr{L}}{\partial g^\alpha_{,\mu}} + (X_{\alpha*} - g_{\alpha*,\sigma} X^\sigma)\frac{\partial \mathscr{L}}{\partial g_{\alpha*,\mu}} + (X^\alpha_\nu - g^\alpha_{\nu,\sigma} X^\sigma)\frac{\partial \mathscr{L}}{\partial g^\alpha_{\nu,\mu}} + (X_{\alpha*\nu} - g_{\alpha*\nu,\sigma} X^\sigma)\frac{\partial \mathscr{L}}{\partial g_{\alpha*\nu,\mu}}\,, \tag{15a.6}$$

which is particularized to the KG case by putting $X^\alpha = 0$ and X_ν instead of X^α_ν; it may be checked that (15a.6) and (13a.4) give again (15a.3) and (15a.4). Analogous considerations may be carried out for the Lagrangian (13a.6) on $J^2(E)$. It may be shown [18] that in that formalism the current is given by the general formula

$$-j^\mu = X^\mu \mathscr{L}^2 + (X^\alpha - y^\alpha_\nu X^\nu)\left[\frac{\partial \mathscr{L}^2}{\partial y_\mu} - \frac{\mathrm{d}}{\mathrm{d}x^\sigma}\frac{\partial \mathscr{L}^2}{\partial y_{\sigma\mu}}\right] + (\bar{X}^\alpha_\nu - y^\alpha_{\nu\sigma} X^\sigma)\frac{\partial \mathscr{L}^2}{\partial y^\alpha_{\mu\nu}} \tag{15a.7}$$

on the solutions of (13a.7); note that the 1-jet prolongation of the field X now enters into the expression for j^μ. Again, by using (14.6) and restricting (15a.7) to the KG case, the Poincaré generators are recovered.

b) The Proca field. On E, $P^\mu = \eta^{\mu\nu}\partial_\nu$ as before and

(15b.1)
$$M^{\mu\nu} = \delta^{\mu\nu,\ \sigma}_{\ \ \ \varepsilon\bullet} x^\varepsilon \partial_\sigma + \delta^{\mu\nu,\varepsilon}_{\ \ \ \bullet\sigma} y_{(\varepsilon)} \frac{\partial}{\partial y_{(\sigma)}} + y^*_{(\varepsilon)} \delta^{\mu\nu,\varepsilon}_{\ \ \ \bullet\sigma} \frac{\partial}{\partial y^*_{(\sigma)}} .$$

If we start from the Lagrangian (13b.1) on $J^1(E)$, the expression of the currents is given by (14.2c), where the fibre index α has been replaced by (ϱ). The translation current is then given by

(15b.2)
$$j^{(\mu)\xi} = -\eta^{\mu\xi}\mathscr{L} - y^{*(\nu)\xi} y_{(\nu)}{}^\mu - y^{*}_{(\nu)}{}^{\mu} y^{(\nu)\xi}$$

(the terms $y^{*(\lambda)}{}_\lambda \eta^{\xi\nu} y_{(\nu)}{}^\mu$ and $y^{*\mu}_{(\nu)} \eta^{\xi\nu} y^{(\lambda)}{}_\lambda$ do not contribute) and the Lorentz current by

(15b.3)
$$j^{(\mu\nu)\xi} = x^\mu j^{(\nu)\xi} - x^\nu j^{(\mu)\xi} - y^{*}_{(\sigma)}{}^{\xi} (\Sigma^{\mu\nu})^\sigma_{\ \bullet\varepsilon} y^{(\varepsilon)} + y^*_{(\sigma)} (\Sigma^{\mu\nu})^\sigma_{\ \bullet\varepsilon} y^{(\varepsilon)\xi}$$

with $(\Sigma^{\mu\nu})_{\sigma\varepsilon} = \delta^{\mu\nu}_{\sigma\varepsilon}$. The same results are obtained from the scalar Hamiltonian density $\mathscr{H}^*_{\rm P}$ of (13b.4) and (14.4c); in particular, the usual Hamiltonian

(15b.4)
$$\mathscr{H} = -[(\nabla\varphi^*_\nu)(\nabla\varphi^\nu) + (\partial_0\varphi^*_\nu)(\partial_0\varphi^\nu) + m^2\varphi^\nu\varphi_\nu]$$

may be recovered from $j^{(0)0}$ by identifying $y^{(\mu)}(x)$ with the Proca field $\varphi^\mu(x)$.

c) The Dirac field. The action of $\mathscr{P}^\uparrow_+$ on the Dirac bundle E is given by $(\Sigma^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu],\ \Sigma^{jk} = \frac{1}{2}\sigma^i)$

(15c.1)
$$\begin{cases} P^\mu = \eta^{\mu\nu}\partial_\nu\,, \\ J^i = \delta^{jk}_{\varepsilon\sigma} x^\varepsilon \partial^\sigma + i(\Sigma^{jk})^\alpha_\beta y^\beta \dfrac{\partial}{\partial y^\alpha} - iy_{\beta\bullet}(\Sigma^{jk})^\beta_\alpha \dfrac{\partial}{\partial y_{\alpha\bullet}}\,, \quad i, j, k = 1, 2, 3 \text{ cyclic}\,, \\ N^i = \delta^{0i}_{\varepsilon\sigma} x^\varepsilon \partial^\sigma + i(\Sigma^{0i})^\alpha_\beta y^\beta \dfrac{\partial}{\partial y^\alpha} + iy_{\beta\bullet}(\Sigma^{0i})^\beta_\alpha \dfrac{\partial}{\partial y_{\alpha\bullet}}\,. \end{cases}$$

The 1-jet extensions, with the additional components

(15c.2)
$$\begin{cases} \bar{J}^{i1} = J^i - [\delta^{jk\ \ \nu}_{\ \ ,\mu\bullet} y^\alpha{}_\nu - i(\Sigma^{jk})^\alpha_\beta y^\beta{}_\mu] \dfrac{\partial}{\partial y^\alpha{}_\mu} - [y_{\alpha\bullet\nu}\delta^{jk\ \ \nu}_{\ \ ,\mu\bullet} + iy_{\beta\bullet\mu}(\Sigma^{jk})^\beta_\alpha] \dfrac{\partial}{\partial y_{\alpha\bullet\mu}}\,, \\ \bar{N}^{i1} = N^i - [\delta^{0i\ \ \nu}_{\ \ ,\mu\bullet} y^\alpha{}_\nu - i(\Sigma^{0i})^\alpha_\beta] \dfrac{\partial}{\partial y^\alpha{}_\mu} - [y_{\alpha\bullet\nu}\delta^{0i\ \ \nu}_{\ \ ,\mu\bullet} - iy_{\beta\bullet\mu}(\Sigma^{0i})^\beta_\alpha] \dfrac{\partial}{\partial y_{\alpha\bullet\mu}} \end{cases}$$

for the vector fields of the rotations and the boosts ($\bar{P}^1_\mu = P_\mu$), act on $J^1(E)$. The invariance of the Lagrangian (13c.1) may now be verified by evaluating $L_{\bar{X}^1}(\mathscr{L}\omega)|_{\bar{\psi}^1}$. As in the previous cases, this is done from $L_{\bar{X}^1}(\mathscr{L}\omega) = (L_{\bar{X}^1}\mathscr{L})\cdot \cdot\omega + \mathscr{L}(L_{\bar{X}^1}\omega)$; $L_{\bar{X}^1}\mathscr{L}|_{\bar{\psi}^1} = 0$, as seen by direct computation, and $L_{\bar{X}^1}\omega = = (\partial^\mu X_\mu)\,\omega = 0$, since $\partial^\mu X_\mu$ is zero for all Poincaré generators (in evaluating $L_{\bar{X}^1}\mathscr{L}$ the relation $\gamma^\varepsilon\,\delta^{\mu\nu}{}_{,\varepsilon\bullet}{}^{\sigma} = \frac{1}{2}[\gamma^\mu\gamma^\nu, \gamma^\sigma]$ is useful). The currents are then given by (PI)

$$(15c.3)\qquad \begin{cases} j^{(\mu)\xi} = iy_{\alpha\bullet}(\gamma^0\gamma^\xi)^\alpha_\beta\,\partial^\mu y^\beta\,, \\ j^{(\mu\nu)\xi} = \{iy_{\alpha\bullet}(\gamma^0\gamma^\xi)^\alpha_\beta\}\big(-i(\Sigma^{\mu\nu})^\beta_\gamma y^\gamma + \delta^{\mu\nu}{}_{,\varepsilon\bullet}{}^{\sigma}x^\varepsilon\,\partial_\sigma y^\beta\big)\,, \end{cases}$$

which take the customary form by putting $y^\alpha(x) \equiv \Psi(x)$. PII leads to the currents of (14.2c). However, these are not the usual currents because $\mathscr{U}'_{\mathscr{L}} \neq \mathscr{U}_{\mathscr{L}}$. Nevertheless, when $y^\alpha{}_\nu$—unconstrained by the EL equations for PII—is restricted to $y^\alpha{}_\nu = \partial_\nu y^\alpha$, the formulae (15c.3) are recovered.

d) Rarita-Schwinger field. The same remarks made for the Dirac case apply also here. We shall only give the formulae for the sake of completeness: on E,

$$(15d.1)\qquad M^{\mu\nu} = \delta^{\mu\nu}{}_{,\varepsilon\bullet}{}^{\sigma}x^\varepsilon\frac{\partial}{\partial x^\sigma} + \delta^{\mu\nu}{}_{,\varepsilon\bullet}{}^{\sigma}y^{(\alpha\varepsilon)}\frac{\partial}{\partial y^{(\alpha\sigma)}} + i(\Sigma^{\mu\nu})^\alpha_\beta\,y^{(\beta\sigma)}\frac{\partial}{\partial y^{(\alpha\sigma)}}$$

plus conjugate terms, and the analogous to (15c.3) are

$$(15d.2)\qquad \begin{cases} j^{(\mu)\xi} = -iy_{(\alpha\sigma)\bullet}(\gamma^0\gamma^\xi)^\alpha_\beta\,\partial_\mu y^{(\beta\sigma)}\,, \\ j^{(\mu\nu)\xi} = -\{iy_{(\alpha\sigma)\bullet}(\gamma^0\gamma^\xi)^\alpha_\beta\}\big(\delta^{\mu\nu}{}_{,\varepsilon\bullet}{}^{\eta}x^\varepsilon\,\partial_\eta y^{(\beta\sigma)} - \delta^{\mu\nu}{}_{,\varepsilon\bullet}{}^{\sigma}y^{(\beta\varepsilon)} - i(\Sigma^{\mu\nu})^\beta_\gamma\,y^{(\gamma\sigma)}\big)\,, \end{cases}$$

where we have omitted the conjugate terms. As usual, $j^{(\mu\nu)\xi}$ may be written in the form

$$(15d.3)\qquad j^{(\mu\nu)\xi} = x^\mu j^{(\nu)\xi} - x^\nu j^{(\mu)\xi} + \pi^\xi_{(\beta\sigma)}(\Sigma^{\mu\nu})^\sigma_{\bullet\varepsilon}\,y^{(\beta\varepsilon)} + \pi^\xi_{(\beta\sigma)}(-i\Sigma^{\mu\nu})^\beta_\gamma\,y^{(\gamma\sigma)}\,,$$

where $\pi^\xi_{(\beta\sigma)} \equiv -iy_{(\alpha\sigma)\bullet}(\gamma^0\gamma^\xi)^\alpha_\beta$ and the two Σ's have been defined in subsect. **15b**) and **15c**).

e) Conformal symmetry of a massless fermion field. As we have seen previously, all Poincaré generators have in common the fact that the component acting on the fibre, $X^\alpha(\partial/\partial y^\alpha)$, does not depend on x^μ ($X^\alpha \neq X^\alpha(x^\mu)$). As a final illustration of the theory we shall now consider an example which is free of such a restriction, namely the conformal invariance of a massless fermion field. The vector fields of $\tau(E)$ associated with the conformal group are written in the form [39]

$$(15e.1)\quad \begin{cases} P_\mu = \delta^\nu_\mu \dfrac{\partial}{\partial x^\nu}\,, \\ M_{\mu\nu} = \delta^{\varepsilon\sigma}_{\mu\nu} x_\varepsilon \dfrac{\partial}{\partial x^\sigma} + i(\Sigma_{\mu\nu})^\alpha_\beta y^\beta \dfrac{\partial}{\partial y^\alpha} - i y_{\alpha*}(\Sigma^\dagger_{\mu\nu})^\beta_\alpha \dfrac{\partial}{\partial y_{\alpha*}}\,, \\ K_\mu = (2x_\mu x^\nu - x^2\delta^\nu_\mu)\dfrac{\partial}{\partial x^\nu} + 2[ix^\nu(\Sigma_{\mu\nu})^\alpha_\beta y^\beta + l x_\mu y^\alpha]\dfrac{\partial}{\partial y^\alpha} + \\ \qquad + 2[-ix^\nu y_{\beta*}(\Sigma^\dagger_{\mu\nu})^\beta_\alpha + l x_\mu y_{\alpha*}]\dfrac{\partial}{\partial y_{\alpha*}}\,, \\ D = x^\nu \dfrac{\partial}{\partial x^\nu} + l y^\alpha \dfrac{\partial}{\partial y^\alpha} + l y_{\alpha*}\dfrac{\partial}{\partial y_{\alpha*}}\,, \end{cases}$$

where l is the dimension of the field in terms of powers of length ($-\frac{3}{2}$ in this case). (15e.1) shows that the components on y^α of the vector fields corresponding to the conformal transformations depend on x^μ; in addition, we have now $\partial^\mu X_\mu \neq 0$ for K_μ and D. The 1-jet prolongations are determined by (14.6) with the result

$$(15e.2)\quad \begin{cases} \bar{P}^1_\mu = P_\mu\,, \\ \bar{M}^1_{\mu\nu} = M_{\mu\nu} + [i(\Sigma_{\mu\nu})^\alpha_\beta y^\beta{}_\sigma - \delta_{\mu\nu,\sigma}{}^{\bullet\varepsilon} y^\alpha{}_\varepsilon]\dfrac{\partial}{\partial y^\alpha{}_\sigma} - \\ \qquad - [i y_{\beta*\sigma}(\Sigma^\dagger_{\mu\nu})^\beta_\alpha + y_{\alpha*\varepsilon}\delta_{\mu\nu,\sigma}{}^{\bullet\varepsilon}]\dfrac{\partial}{\partial y_{\alpha*\sigma}}\,, \\ \bar{K}^1_\mu = K_\mu + 2\{i(\Sigma_{\mu\nu})^\alpha_\beta y^\beta + l\eta_{\mu\sigma} y^\alpha + [x_\sigma \delta^\nu_\mu - x_\mu \delta^\nu_\sigma - x^\nu \eta_{\mu\sigma}] y^\alpha{}_\nu + \\ \qquad + [ix^\nu(\Sigma_{\mu\nu})^\alpha_\beta + l x_\mu \delta^\alpha_\beta] y^\beta{}_\sigma\}\dfrac{\partial}{\partial y^\alpha{}_\sigma} + \\ \qquad + 2\{-i y_{\beta*}(\Sigma^\dagger_{\mu\nu})^\beta_\alpha + l\eta_{\mu\sigma} y_{\alpha*} + y_{\alpha*\nu}[x_\sigma\delta^\nu_\mu - x_\mu\delta^\nu_\sigma - x^\nu\eta_{\mu\sigma}] + \\ \qquad + y_{\beta*\sigma}[-ix^\nu(\Sigma^\dagger_{\mu\nu})^\beta_\alpha + l x_\mu \delta^\alpha_\beta]\}\dfrac{\partial}{\partial y_{\alpha*\sigma}}\,, \\ \bar{D}^1 = D + (l-1) y^\alpha{}_\sigma \dfrac{\partial}{\partial y^\alpha_\sigma} + (l-1) y_{\alpha*\sigma}\dfrac{\partial}{\partial y_{\alpha*\sigma}}\,. \end{cases}$$

The conformal invariance of the « fermion field » determined (PI) by the Lagrangian density

$$(15e.3)\qquad \mathscr{L} = i y_{\alpha*}(\gamma^0\gamma^\varepsilon)^\alpha_\beta y^\beta_\varepsilon$$

is checked by calculating (14.3) and noting that now the term $\mathscr{L}(L_{\bar{X}^1}\omega)$ contributes due to the fact that $\partial_\nu X^\nu_{(K_\mu)} = 8x_\mu$ and $\partial_\nu X^\nu_{(D)} = 4$. The currents are obtained from the first three terms of (14.2c) (PI); explicitly (and on $\bar{\Psi}^1 \in \mathscr{U}_{\mathscr{L}}$) the conformal and dilatation currents are given by

$$(15e.4)\quad \begin{cases} j^\xi_{(\mu)} = i y_{\beta*}(\gamma^0\gamma^\xi)^\beta_\alpha\{(2x_\mu x^\nu - x^2\delta^\nu_\mu)\partial_\nu - 2(ix^\nu\Sigma_{\mu\nu} + l x_\mu)\} y^\alpha\,, \\ j^\xi = y_{\beta*}(\gamma^0\gamma^\xi)^\beta_\alpha\{ix^\varepsilon\partial_\varepsilon - il\} y^\alpha\,, \end{cases}$$

the Poincaré currents again having the form (15c.3).

In concluding we could mention that on some occasions it may be interesting to discuss the invariance of a system by considering the current which —conserved or not—may be defined from $d(i_{X^1}\Theta)|_{\Psi^1}$, since the other term coming from $L_{X^1}\Theta$ is zero on account of the EL equations. In this way one may, for instance, discuss the relative implications of conformal and dilatation invariance of a given Lagrangian along the lines of the first two references of [39].

* * *

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APPENDIX A

Vector bundles.

In this appendix we restrict ourselves to introducing the most relevant concepts concerning vector bundles which are necessary for the main text. A more systematic study of fibre bundles may be found in ref. [37, 38, 40, 41] and, in the context of their applications to physical problems, in ref. [2, 7, 19, 42-44, 48].

A.1. *Locally trivial fibre bundles.* – Let E, F, M be topological spaces. A *locally trivial fibre bundle* of fibre F and base M is a triplet $\eta = (E, \pi, M)$ (frequently denoted $E \xrightarrow{\pi} M$), where π is a continuous application of E onto M (the *projection*), which satisfies the following condition:

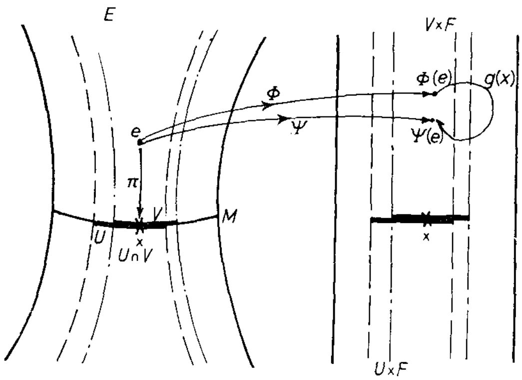
For every $x \in M$, there exists an open set $U \in M$ including x and a homeomorphism $\Phi: \pi^{-1}(U) \to U \times F$ such that $p \circ \Phi = \pi$, where p is the projection of $U \times F$ onto U. E is called the *total space* of the fibre bundle η; $\pi^{-1}(x)$ is the *fibre over x*, and $\pi^{-1}(\pi(e))$, $e \in E$, the *fibre through the point e*.

The pair (U, Φ) is called a *local chart* of η and constitutes a *trivialization* of the restriction $\eta|_U$. Given two local charts (U, Φ), (V, Ψ) such that $U \cap V \neq \{\emptyset\}$, one has for $x \in U \cap V$, $f \in F$, $\Psi \circ \Phi^{-1}(x, f) = (x, g(x)f)$, where g is an application of $U \cap V$ into the group of homeomorphisms of F. (See the figure where both the fibre and base spaces have been taken of dimension one.)

The simplest example of fibre bundle is the Cartesian product $(M \times F, \pi, M)$, where π is the canonical projection. Such a bundle is called *trivial* and admits a global chart (global co-ordinate system). Any bundle isomorphic (see below) to a direct-product bundle is also trivial.

A *cross-section* Ψ of $E \xrightarrow{\pi} M$ over $N \subset M$ is a continuous application $\Psi: N \to E$ such that $\pi \circ \Psi$ is the identity on N. The set of cross-sections will be denoted by $\Gamma(N)$.

Let $\eta = (E, \pi, M)$ and $\eta' = (E', \pi', M')$ be two locally trivial fibre bundles



with fibres F and F'. A *homomorphism* of η into η' is a pair of continuous maps (H, h), $H: E \to E'$, $h: M \to M'$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{H} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{h} & M' \end{array}$$

is commutative. The local representation of (H, h) is obtained by taking local charts (U, φ) $[\varphi: \pi^{-1}(U) \to U \times F,\ U \subset M]$ of η, $(U', \varphi')[\varphi': \pi'^{-1}(U') \to \to U' \times F',\ U' \subset M']$ of η' such that $h(U) \cap U' \neq \{\emptyset\}$. Then, for $(x, f) \in \in \big(U \cap h^{-1}(U')\big) \times F$,

$$\varphi' H \varphi^{-1}: (x, f) \to \big(h(x), l(x) f\big)\,,$$

where $l(x)$ $\big($with $x \in U \cap h^{-1}(U')\big)$ is a continuous application of the fibre F on F'. Two bundles are *isomorphic* when the horizontal arrows of the above diagram may be inverted.

Let $\eta = (E', \pi', M')$ be a locally trivial vector bundle and g a continuous application of the manifold M over M'. Then there is a fibre bundle $\eta = (E, \pi, M)$ called *reciprocal image* of η (also called *pull-back* or *induced bundle*) and an homomorphism H which makes the following diagram commutative:

$$\begin{array}{ccc} E & \xrightarrow{H} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{g} & M' \end{array}.$$

The construction is simple: it is sufficient to define E as the subset of $M \times E'$ composed by the pairs (x, e'), $x \in M$, $e' \in E'$, such that $g(x) = \pi'(e')$ and H as the application $H: (x, e') \in E \to e'$. E is frequently noted $M \times_{M'} E'$ (which indicates that it is the part of the direct product compatible with the condition imposed by the existence of the homomorphism g and the application π' over M') and π is called the pull-back of π' by g.

A.2. *Vector bundles.* – The vector bundle structure is obtained by imposing to the fibre F the condition of being a vector space $V^n(R)$ (or $V^n(C)$) of dimension n. Then the functions g which determine the change of local charts take their values $g(x)$ on the linear group of the fibre space, $GL(n)$. Thus, a vector bundle is essentially a manifold with a vector space attached to each of its points. More precisely, a vector bundle is defined as follows:

Let $\eta = (E, \pi, M)$ be a locally trivial fibre bundle of fibre given by a vector space F of dimension n. A structure of vector bundle is determined on η by a family $\hat{\mathscr{A}} = \{(U_\alpha, \varphi_\alpha)\}$ of local charts which satisfy the following conditions:

a) $\{U_\alpha\}$ is an open cover of M.

b) For every pair (α, β) for which $U_\alpha \cap U_\beta \neq \{\emptyset\}$,

$$\varphi_\beta \varphi_\alpha^{-1}(x, f) = \big(x, g_{\beta\alpha}(x) f\big) \qquad [(x, f) \in (U_\alpha \cap U_\beta) \times F] ,$$

where $g_{\beta\alpha}$ is a continuous application of $U_\beta \cap U_\alpha$ on $GL(n)$, *i.e.* $\varphi_\beta \varphi_\alpha^{-1}$ is a local vector bundle isomorphism.

c) The family $\hat{\mathscr{A}}$ is maximal, *i.e.*, if $\mathscr{A}'$ is a family of charts of η which satisfies *a*), *b*) and includes $\hat{\mathscr{A}}$, then $\mathscr{A}' = \hat{\mathscr{A}}$. $\hat{\mathscr{A}}$ is called the *atlas* of η (which is sometimes written as $\eta = (E, \pi, M; \hat{\mathscr{A}})$) and its elements are the vector charts of η.

The functions $g_{\beta\alpha}$ determine the changes of charts and are accordingly called *transition functions*; they satisfy the compatibility condition

$$g_{\gamma\beta}(x) g_{\beta\alpha}(x) = g_{\gamma\alpha}(x) , \qquad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma . \tag{A.2.1}$$

In other words, $(U_\alpha, g_{\beta\alpha})$ is a cocycle on M with values in $GL(n)$ subordinated to $\{U_\alpha\}$. Reciprocally, it can be shown that, if $\{U_\alpha\}$ is an open covering of M, F a vector space of finite dimension and $g_{\beta\alpha}: U_\beta \cap U_\alpha \to GL(F)$ a family of continuous mappings on $U_\beta \cap U_\alpha$ satisfying (A.2.1), there exists a vector bundle of fibre F, $\eta = (E, \pi, M)$, for which the g's are transition functions. Moreover, η is unique but for equivalences (*).

All the other properties of fibre bundles are easily transported to vector bundles. In particular, a differentiable vector bundle structure is obtained when the base M is a differentiable manifold and the changes of charts are given by differentiable transition functions. As a final comment, let us say that the *zero section* of E is the base M; the name is given because $x \in M$ is the zero element of the vector space $\pi^{-1}(x)$ (the fibre over x).

A vector bundle with a one-dimensional vector space as fibre is called *line bundle.*

A.3. *Tangent* $[T(M)]$ *and cotangent* $[T^*(M)]$ *space of a manifold* M. *The tangent* (*differential*) *application.* – Let M be a differentiable manifold of di-

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(*) Two locally trivial fibre bundles $\eta = (E, \pi, M)$ and $\eta' = (E', \pi', M)$ with the same base M are said to be *equivalent* if there exists an isomorphism $(H, h): \eta \to \eta'$ for which h is the identity on M.

mension m and let x be a point of M. A (differentiable) curve c at $x \in M$ is a (differentiable) application $c: I \to M$, $I \subset R$ with $0 \in I$ and $c(0) = x$.

Let c_1 and c_2 be two curves at x and let (U, φ) be a local chart with $x \in U$. c_1 and c_2 are said to be tangent at x if and only if $\varphi \circ c_1$ and $\varphi \circ c_2$ are tangent at $\varphi(x) \in R^m$ (in the usual sense of tangency in R^m). It is simple to see that the notion of tangency does not depend on the chosen local chart, provided they are compatible as charts of a differentiable manifold. In the same form it is clear that the tangency is preserved by a differentiable application f of M into another differentiable manifold N: if c_1 and c_2 are tangent at $x \in M$, $f(c_1)$ and $f(c_2)$ are also tangent at $f(x) \in N$.

It is evident that the tangency at $x \in M$ is an equivalence relation among the curves passing through x. The tangent space $T_x(M)$ is the space of the equivalence classes $\{c\}_x$. $T_x(M)$, endowed with the structure of vector space through the usual vector structure on R^m, is then called tangent vector space to M at x. The *tangent space* to M is defined as $T(M) = \bigcup_{x \in M} T_x(M)$. By taking the dual of $T_x(M)$, the cotangent vector space $T^*_x(M)$ at x is obtained; the *cotangent space* $T^*(M)$ is defined analogously as $T^*(M) = \bigcup_{x \in M} T^*_x(M)$. Both manifolds $T(M)$ and $T^*(M)$ have dimension $2m$ if M has dimension m.

Let $f: M \to N$ be a differentiable application and $\{c\}_x$ a class of tangent curves at $x \in M$. The *tangent application* to f is the application $f^{\mathrm{T}}: T(M) \to T(N)$ defined by

$$f^{\mathrm{T}}: \{c\}_x \to \{f \circ c\}_{f(x)} . \tag{A.3.1}$$

The definition is obviously class independent (see above). If g is another application $g: N \to K$,

$$(g \circ f)^{\mathrm{T}} = g^{\mathrm{T}} \circ f^{\mathrm{T}} \tag{A.3.2}$$

(functorial property) and if f is a diffeomorphism, $(f^{-1})^{\mathrm{T}} = (f^{\mathrm{T}})^{-1}$. The application f^* dual of f^{T} (or *pull-back* of f^{T}) is called the cotangent application.

Local representation. Let (U, φ) be a local chart of M, $x \in U \subset M$. A class $\{c\}_x$ of local curves at x is written in R^m as $\{\varphi \circ c\}_{\varphi(x)}$. The line in R^m $(\varphi \circ c)_{\varphi(x)} = \varphi(x) + \boldsymbol{e}t$, $t \in (0, 1)$, where $\boldsymbol{e}$ is the m-dimensional vector $\mathrm{D}(\varphi \circ c)(t = 0) \cdot \mathbf{1}$ ($\mathbf{1}$ is the natural basis in R), may be taken as a representative of the class $\{\varphi \circ c\}_{\varphi(x)}$. $\mathrm{D}(\varphi \circ c)$ is the usual Jacobian matrix; given a co-ordinate system $q^i = x^i \circ \varphi$ on M, $i = 1, \ldots, m$,

$$\mathrm{D}(q^i \circ c)(0) \cdot \mathbf{1} = \left.\frac{\mathrm{d}q^i(c(t))}{\mathrm{d}t}\right|_{t=0} .$$

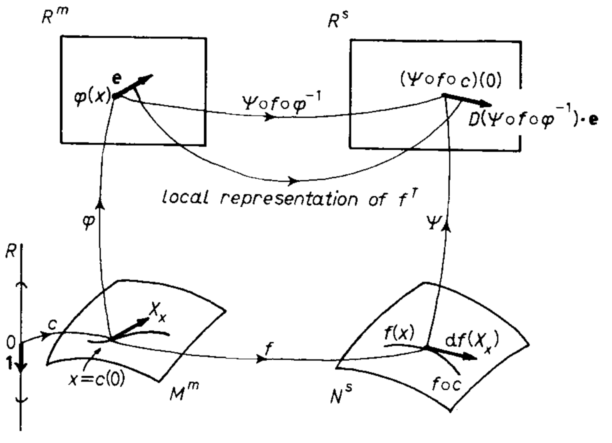
Now let $f: M \to N$ be a differentiable application and (V, ψ) a local chart around $f(x)$. The local representation of f^{T} is now given by

$$f^{\mathrm{T}}: (x, X_x) \to \big(f(x), \mathrm{D}(\psi \circ f \circ c)(0) \cdot \mathbf{1}\big) \equiv \big(f(x), (\mathrm{D}f)(x) \cdot X_x\big) , \tag{A.3.3}$$

since, by the chain rule,

$$\mathrm{D}(\psi \circ f \circ c)(0) \cdot \mathbf{1} = \mathrm{D}(\psi \circ f \circ \varphi^{-1} \circ \varphi \circ c)(0) \cdot \mathbf{1} = \\ = [\mathrm{D}(\psi \circ f \circ \varphi^{-1})](\varphi \circ c)(0) \cdot \mathrm{D}(\varphi \circ c)(0) \cdot \mathbf{1} \equiv (\mathrm{D}f)(x) \cdot X_x .$$

The second component of the tangent application, $D(f)(x)\cdot X_x$ defines the *differential of f* through $df(X_x) = (Df)(x)\cdot X_x$, where X_x is a vector field (appendix A.4) at x (*). The above relations are shown pictorially in the following diagram:



The two components of the tangent application.

$((\psi o f o \varphi^{-1}) : R^m \to R^s)$. By taking co-ordinates $\psi o f o \varphi^{-1} \to f^j(q^i)$, $i = 1, ..., m$; $j = 1, ..., s$ and Df is the Jacobian of the transformation, $\partial f^j/\partial q^i$.

A.4. *Tangent and cotangent vector bundles. Tangent and cotangent groups.* — The triplet $\tau(M) = (T(M), \pi_M, M)$, where the projection $\pi_M : T(M) \to M$ is defined by $\pi(\{c\}_x) = x$, together with the atlas $\hat{\mathscr{A}} = \{T(U_\alpha), \varphi_\alpha^T\}$, where $\{(U_\alpha, \varphi_\alpha)\}$ gives a differentiable structure to M, has the structure of a differentiable vector bundle with fibre R^m. Thus $\tau(M)$ is a vector bundle called *tangent bundle* to M. Substituting $T^*(M)$ for $T(M)$ and φ_α^* for φ_α^T, one obtains the *cotangent bundle* to M, $\tau^*(M) = (T^*(M), \lambda_M, M)$.

It is clear that the changes of charts of $\tau(M)$ are given by the differential (in the usual sense in R^m) of the changes of charts $(\varphi_\beta o \varphi_\alpha^{-1})$ of the manifold M. This allows for the following equivalent definition of $\tau(M)$: let M be a differentiable manifold of dimension m defined by its maximal atlas $\hat{\mathscr{A}} = \{(U_\alpha, \varphi_\alpha)\}$. $\tau(M)$ is the differentiable vector bundle of base M and fibre R^m defined by the cocycle $(U_\alpha, D(\varphi_\beta o \varphi_\alpha^{-1}))$.

A *vector field* X on a differentiable manifold M is a differentiable cross-section of $\tau(M)$; a *field of* 1-*forms* on M is a differentiable cross-section of $\tau^*(M)$.

Local co-ordinates. Let (U, φ) be a local chart of M and let $\{x^i\}$, $i = 1, ..., m$, be a co-ordinate system on R^m. The set of functions $\{q^i = x^i o \varphi\}$ constitutes

(*) When f is an application of M^m on R, df is the ordinary exterior derivative of the function f. Strictly speaking, in the above paragraph the tangent vector X_x should be replaced by its representation $\boldsymbol{e}$ in R^m; we have kept X_x following a common practice. The same can be said of the local representations of x and $f(x)$, which more precisely should be written as $\varphi(x)$, $f_{\varphi\psi}(\varphi(x))$.

a co-ordinate system for $U \subset M$. A chart of $\tau(M)$ on the open set U will be given by $\{T(U), \varphi^{\mathrm{T}}\}$ and thus a co-ordinate system on $T(U)$ will be given by $\{q^{i\mathrm{T}}\} \equiv \{q^i, \mathrm{d}q^i\}$; the applications $\mathrm{d}q^i$ are also frequently denoted by $\dot{q}^i$. In the cotangent space $T^*(U)$ we shall take as a co-ordinate system $\{q^i, p_i \equiv \partial/\partial q^i\}$, where $\partial/\partial q^i$ is the dual system of $\mathrm{d}q^i$; the notation $\partial/\partial q^i$ corresponds to the fact that the set of fields on U is in one-to-one correspondence with the set of derivations of the algebra of the local differentiable functions on U.

A vector field [1-form] on U will be locally written as $X = X^i(q^j)(\partial/\partial q^i)$ $[\alpha = \alpha_i(q^j)\,\mathrm{d}q^i]$, where $X^i[\alpha_i]$ are m differentiable functions on $U \subset M$.

Tangent and cotangent groups. Many of the group of transformations which are relevant in mechanics are defined only on the base space M (through their action on the co-ordinates q^i) and not on the co-ordinates *and* velocities or on the co-ordinates *and* momenta which are usually the (configuration and phase) spaces of definition of physical quantities. Thus the problem arises of extending the action of a group on M to $T(M)$ or $T^*(M)$ in a natural way. The solution is simple: if $\Phi_t: M \to M$, $t \in R$, is a one-parameter group of diffeomorphisms of M with parameter t, the tangent application Φ_t^{T} for each t is a one-parameter group of diffeomorphisms of $T(M)$ whose restriction to M is Φ_t. Thus we may extend to $T(M)$ the vector field X which generates Φ_t.

Let $X = X^i(q^j)(\partial/\partial q^i)$ be the restriction of an arbitrary vector field on M to $U \subset M$. In U, X give rise to a one-parameter group, $\Phi_t: M \to M$, $\Phi_t: x \to \Phi_t(x)$, $\Phi_0(x) = x$, of which X is the infinitesimal generator, *i.e.*

$$X^i = \frac{\mathrm{d}}{\mathrm{d}t}\,\Phi_t^i(x)\bigg|_{t=0}. \tag{A.4.1}$$

The tangent application Φ_t^{T} defines the following action on $T(U)$ (eq. (A.3.3)):

$$\Phi_t^{\mathrm{T}}: (x, X_x) \to (\Phi_t(x), \mathrm{D}(\Phi_t)(x)\cdot X_x) = \left(\Phi_t(x), \frac{\partial \Phi_t}{\partial q^j}\,\dot{q}^j(X_x)\right), \tag{A.4.2}$$

where $(q^i, \dot{q}^j)$ is the co-ordinate system of $T(U)$ and X_x an arbitrary vector of $T_x(U)$. Now the generator of Φ_t^{T}, or *prolongated vector field* X^{T}, is given by

$$X^{\mathrm{T}} = \frac{\mathrm{d}}{\mathrm{d}t}\,\Phi_t^{\mathrm{T}}\bigg|_{t=0} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\Phi_t^i, \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \Phi_t^i}{\partial q^j}\,\dot{q}^j\right)\bigg|_{t=0} = \left(X^i, \dot{q}^j\frac{\partial X^i}{\partial q^j}\right)$$

and thus

$$X^{\mathrm{T}} = X^i\frac{\partial}{\partial q^i} + \dot{q}^j\frac{\partial X^i}{\partial q^j}\frac{\partial}{\partial \dot{q}^i}, \qquad X^i = X^i(q^j). \tag{A.4.3}$$

To prolongate a vector field X on M to $T^*(M)$, one proceeds in an analogous way by constructing the one-parameter cotangent group Φ_t^* to evaluate the generator as before. The result is

$$X^* = X^i\frac{\partial}{\partial q^i} - p_j\frac{\partial X^j}{\partial q^i}\frac{\partial}{\partial p_i}, \tag{A.4.4}$$

as was derived by using the Liouville form in subsect. **2**c) (eq. (2c.10)). Both procedures for obtaining X^* are equivalent, since, given a one-parameter group Φ_t of diffeomorphisms of M, the cotangent group Φ_t^* preserves the Liouville form, $\Phi_t^*\Lambda = \Lambda$.

APPENDIX B

Jet bundles $J^r(E)$.

We develop in this appendix the notions of the jet bundles required for the main text. The concept of jet bundles was introduced by EHRESMANN [45]; the interested reader may also consult, *e.g.*, ref. [7] to which we conform here most of our notation, and ref. [36, 46]. We shall consider only *trivial* bundles when dealing with jet bundles to avoid introducing a connection [7, 10, 11] on the fibre space E which would complicate unnecessarily the notation. In the general case, we should say (see below) « that Ψ and all *covariant* derivatives vanish » instead of simply saying « derivatives ». Because of these simplifying reasons, most of that follows will be written in local co-ordinates.

Despite the fact that only first-order jets are used in most of the main text, we shall here define the r-jets with generality, since this does not offer additional difficulties. r-th order jets are used in the context of generalized field theory, where Lagrangian densities depend on higher-order derivatives; see, *e.g.*, [18].

In what follows, the fibre of the bundle—which will be a vector space—will be on the real field R. The generalization to the complex field is trivial and directly taken care of in the main text.

B.1. *The bundle $J^r(E)$ of the r-jets of the fibre space $\eta = (E, \pi, M)$.* – Let $\eta = (E, \pi, M)$ be a differentiable vector bundle on M with fibre $V^m(R)$. E will denote henceforth both the total space and the fibre bundle itself. Let $\Gamma(E)$ be the set of all differentiable cross-sections Ψ of the fibre E (we shall take Ψ of class C^∞). $\Gamma(E)$ has a structure of $\mathscr{F}(M)$—modulus on the algebra $\mathscr{F}(M)$ of the (C^∞) differentiable functions on M. E will be assumed to be a trivial vector bundle ($E = M \times R^m$; it will be parametrized by the co-ordinate system $\{x^\mu, y^\alpha\}$, $\mu = 1, \ldots, n$, $\alpha = 1, \ldots, m$), so that $\Gamma(E)$ will always admit a basis and consequently will be a free modulus of dimension m.

Let x be an arbitrary point of M and r a positive integer. Let Γ_x^r be the submodulus of $\Gamma(E)$ composed by the cross-sections Ψ which are zero at x up to the r-th order derivative, *i.e.* such that $\partial_{\mu_1 \ldots \mu_q}\Psi(x) = 0$ for $q = 0, 1, \ldots, r$. The quotient

$$\Gamma(E)/\Gamma_x^r \equiv J_x^r(E)\,, \tag{B.1.1}$$

whose elements are the classes composed by cross-sections which take themselves and their derivatives up to order r the same values at x, is a vector space over R. The equivalence class of a certain cross-section Ψ is called the *r-jet of Ψ* at x; the point x is called the *source* of the jet and the point $\Psi(x) \in E$ the *target* of the jet. Intuitively one may describe an equivalence class as including all the functions $\Psi: M \to E$ whose Taylor development coincides up to the r-th term. The union

$$\bigcup_x J^r_x(E) \equiv J^r(E)\,, \tag{B.1.2}$$

with the target projection $\pi^r: J^r(E) \to M$ defined by $\pi^r: J^r_x(E) \to x$, has a vector bundle structure over M, and will be called the vector *bundle* $(J^r(E), \pi^r, M)$ *of the r-jets* of E. As a particular case, $J^0(E) = E$.

For convenience, we give now another definition of the bundle $J^r(E)$ which is more general, since it does not require a vector structure for E [7]. Let (E, π, M) be a differentiable fibre bundle over M and $\Gamma(E)$ the space of its differentiable cross-sections. On $\Gamma(E)\times M$ we may define the following equivalence relation:

$$\forall \Psi, \Psi' \in \Gamma(E); \qquad \forall x, x' \in M \qquad (\Psi, x) \overset{\mathscr{R}^r}{\sim} (\psi', x')\,, \tag{B.1.3}$$

if

a) $x = x'$,

b) Ψ coincides with Ψ' up to order r,

$$\partial_{\mu_1 \ldots \mu_q} \Psi(x) = \partial_{\mu_1 \ldots \mu_q} \Psi'(x)\,, \qquad q = 0, 1, \ldots, r\,.$$

The quotient set, $\Gamma(E)\times M/\mathscr{R}^r \equiv J^r(E)$, is the space of the r-jets of E. The projection $(\Psi, x) \to x$ of $\Gamma(E)\times M \to M$ goes to the quotient and thus defines a projection $\pi^r: J^r(E) \to M$ which gives to $J^r(E)$ the structure of fibre bundle over M. The fibre over x, $(\pi^r)^{-1}(x)$, may be identified with the space $J^r_x(E)$ previously defined in the case that E be a vector bundle. Note finally that, since two cross-sections coinciding up to order r also coincide for $r' \leqslant r$, one may define the map

$$\pi^{r,r'}: J^r(E) \to J^{r'}(E)\,, \tag{B.1.4}$$

which intuitively corresponds to retaining the first r' terms of the Taylor development initially given up to order r.

Given a section $\Psi \in \Gamma(E)$ and a point $x \in M$, the pair $(\Psi, x) \in \Gamma(E)\times M$ belongs to one of the classes of $\Gamma(E)\times M/\mathscr{R}^r$, precisely the one characterized by the r first derivatives of the section Ψ taken at x. The application

$$j^r: \Gamma(E) \to \Gamma\big(J^r(E)\big)\,, \qquad j^r: \Psi \to j^r(\Psi) \equiv \overline{\Psi}^r\,, \tag{B.1.5}$$

where $j^r(\Psi)(x)$ is the class defined by the first r derivatives of Ψ at x, is called *r-jet prolongation* (*or extension*) *of* Ψ and is written as $j^r(\Psi)$ or simply $\overline{\Psi}^r$ (in general, jet prolongations will be denoted by a bar).

As a particularly simple example of a 1-jet bundle, let us consider now the case for which $E = R\times M \xrightarrow{\pi} R$, where M is a differentiable manifold. $\Gamma(E)$ is then the set of curves of the manifold M, and the equivalence relation (B.1.3) for $r = 1$ is written as (with $\Psi \equiv c$, $t \in R$)

$$(c, t) \overset{\mathscr{R}^1}{\sim} (c', t') \qquad \forall c, c' \in \Gamma(E) \text{ and } t, t' \in R\,,$$

if $a)$ $t = t'$ and $b)$ $\mathrm{d}c/\mathrm{d}t = \mathrm{d}c'/\mathrm{d}t'$, *i.e.* $\mathscr{R}^1$ places in the same class all the curves with the same tangent at t. The tangent thus characterizes the equivalence class, and $\Gamma(E)\times R/\mathscr{R}^1$ is nothing but the vertical tangent space to E, $T^v(E) = R\times T(M)$, where $T(M)$ is defined as in appendix A.3. We then have $J^1(E) = R\times T(M)$, and this will be the space of definition of (time dependent) Lagrangians (sect. **8**).

B.2. *Co-ordinate system on* $J^r(E)$. – Let $U\subset M$, $\pi^{-1}(U)\approx U\times R^m$ and let $\{x^\mu, y^\alpha\}$ be a co-ordinate system for $\pi^{-1}(U)\subset E$. For the set of local sections $\Gamma(\pi^{-1}(U))$ the equivalence relation (B.1.3) may be expressed in the form

$$(\Psi, x)\overset{\mathscr{R}^r}{\sim}(\Psi', x') \tag{B.2.1}$$

if $x = x'$ and

$$\left\{\begin{aligned} & y^\alpha(\Psi)(x) = y^\alpha(\Psi')(x)\,,\\ & \partial_\mu y^\alpha(\Psi)(x) = \partial_\mu y^\alpha(\Psi')(x)\,,\\ & \cdots\cdots\cdots\cdots\cdots\cdots\\ & \partial_{\mu_1\ldots\mu_r} y^\alpha(\Psi)(x) = \partial_{\mu_1\ldots\mu_r} y^\alpha(\Psi')(x)\,. \end{aligned}\right. \tag{B.2.2}$$

The functions on $\Gamma(\pi^{-1}(U))\times U$ defined by

$$\left\{\begin{aligned} & y^\alpha:(\Psi, x) \to y^\alpha(\Psi)(x)\,,\\ & y^\alpha_\mu:(\Psi, x) \to \partial_\mu y^\alpha(\Psi)(x)\,,\\ & \cdots\cdots\cdots\cdots\cdots\cdots\\ & y^\alpha_{\mu_1\ldots\mu_r}:(\Psi, x) \to \partial_{\mu_1\ldots\mu_r} y^\alpha(\Psi)(x)\,, \end{aligned}\right. \tag{B.2.3}$$

are stable through the equivalence relation and define on the quotient set functions on $J^r(\pi^{-1}(U))$. Thus the set of functions $\{x^\mu, y^\alpha, y^\alpha_{\mu_1}, y^\alpha_{\mu_1\mu_2}, \ldots, y^\alpha_{\mu_1\ldots\mu_r}\}$, symmetric in the lower indices, constitutes a co-ordinate system on $J^r(\pi^{-1}(U))$ (as customary, we will not often distinguish between y^α, $y^\alpha(\Psi)$ or Ψ^α; no confusion should arise from that). In this co-ordinate system the projection (B.1.4) is written as

$$\pi^{r,r'}:(x^\mu, y^\alpha, y^\alpha_{\mu_1}, \ldots, y^\alpha_{\mu_1\ldots\mu_r}) \to (x^\mu, y^\alpha, y^\alpha_{\mu_1}, \ldots, y^\alpha_{\mu_1\ldots\mu_{r'}}) \tag{B.2.4}$$

and the jet prolongation (B.1.5)

$$j^r(\Psi^\alpha) = (y^\alpha, \partial_{\mu_1} y^\alpha, \ldots, \partial_{\mu_1\ldots\mu_r} y^\alpha)\,. \tag{B.2.5}$$

Note that *not* all sections $\Psi^r\in\Gamma(J^r(E))$ are r-jet prolongations of some section $\Psi\in\Gamma(E)$ and that, in general, $y^\alpha_{\mu_1\ldots\mu_q}\neq\partial_{\mu_1\ldots\mu_q}y^\alpha$. In contrast, given a point $(x^\mu, y^\alpha, y^\alpha_{\mu_1}, \ldots, y^\alpha_{\mu_1\ldots\mu_r})\in J^r(E)$, there is always a *local* section Ψ such that, at x,

$$y^\alpha_{\mu_1\ldots\mu_q}(x) = \partial_{\mu_1\ldots\mu_q} y^\alpha(\Psi)(x)\,, \qquad q = 0, 1, \ldots, r\,, \tag{B.2.6}$$

as may be verified by writing Ψ as a formal Taylor series around x with coefficients $(y^\alpha, y^\alpha_{\mu_1}, \ldots, y^\alpha_{\mu_1\ldots\mu_r})(x)$ [46, 47].

It is simple, from (B.2.6), to characterize the r-jet prolongation of $\Psi \in \Gamma(E)$. Let us consider the set of structure 1-forms on $J^r(E)$, $\{\theta^\alpha_{\mu_1 \dots \mu_s}\}$, $\alpha = 1, \dots, m$, $s = 0, 1, \dots, r-1$, defined by the following local expression:

$$\theta^\alpha_{\mu_1 \dots \mu_s} = \mathrm{d}y^\alpha_{\mu_1 \dots \mu_s} - y^\alpha_{\mu_1 \dots \mu_s \mu}\, \mathrm{d}x^\mu . \tag{B.2.7}$$

Then the r-jet extension $j^r(\Psi) \equiv \overline{\Psi}^r$ of $\Psi \in \Gamma(E)$ is the only cross-section of $J^r(E)$ such that j^r is an injection of $\Gamma(E)$ into $\Gamma(J^r(E))$ and

$$\theta^\alpha_{\mu_1 \dots \mu_s}|_{\overline{\Psi}^r} = 0 . \tag{B.2.8}$$

Thus $j^r(\Gamma(E))$ appears as a relative integral of the noncompletely integrable Pfaffian system $\{\theta^\alpha_{\mu_1 \dots \mu_s}\}$. We shall see in the next section that this characterization is specially useful in defining the jet prolongation of vector fields.

B.3. *The r-jet prolongation to $\Gamma(\tau(J^r(E)))$ of a vector field of $\Gamma(\tau(E))$.* – The r-jet prolongation $j^r(X) \equiv \overline{X}^r$ of a vector field X on E may be obtained by considering its action as a one-parameter group on E (and on M in particular) and on $\Gamma(E)$. Given the action of the group on $\Gamma(E) \times M$ and by taking the quotient by $\mathscr{R}^r$, the generator of the group on $J^r(E)$ is $j^r(X)$. However, we may take advantage of (B.2.8) and define $\overline{X}^r$ as follows.

Given a vector field X on E, its r-jet prolongation by j^r is the only vector field on $J^r(E)$ such that $\overline{X}^r$ is an *infinitesimal contact transformation* (*i.c.t.*), *i.e.* such that

$$L_{\overline{X}^r} \theta^\alpha_{\mu_1 \dots \mu_s} = \sum_{s' \leqslant s} A^{\alpha \nu_1 \dots \nu_{s'}}_{\beta \mu_1 \dots \mu_s} \theta^\beta_{\nu_1 \dots \nu_{s'}} , \qquad s = 0, 1, \dots, r-1 . \tag{B.3.1}$$

In this way, the stability of the Pfaffian system $\{\theta^\alpha_{\mu_1 \dots \mu_s}\}$ under $\overline{X}^r$ guarantees that r-jet prolongations are mapped onto r-jet prolongations.

Condition (B.3.1) allows us to calculate explicitly $\overline{X}^r$ as well as $A^{\alpha \nu_1 \dots \nu_{s'}}_{\beta \mu_1 \dots \mu_s}$. For instance, for the 1-jet prolongation we write

$$X = X^\mu(x^\nu, y^\beta) \frac{\partial}{\partial x^\mu} + X^\alpha(x^\nu, y^\beta) \frac{\partial}{\partial y^\alpha} , \qquad \overline{X}^1 = X + \overline{X}^\alpha_\mu \frac{\partial}{\partial y^\alpha_\mu} \tag{B.3.2}$$

and obtain

$$A^\alpha_\beta = \frac{\partial X^\alpha}{\partial y^\beta} - \frac{\partial X^\nu}{\partial y^\beta} y^\alpha_\nu , \tag{B.3.3}$$

$$\overline{X}^\alpha_\mu = \frac{\partial X^\alpha}{\partial x^\mu} - y^\alpha_\nu \frac{\partial X^\nu}{\partial x^\mu} + y^\beta_\mu \frac{\partial X^\alpha}{\partial y^\beta} - y^\beta_\mu \frac{\partial X^\nu}{\partial y^\beta} y^\alpha_\nu . \tag{B.3.4}$$

For the 2-jet prolongation we put

$$\overline{X}^2 = X + \overline{X}^\alpha_\mu \frac{\partial}{\partial y^\alpha_\mu} + \overline{X}^\alpha_{\mu\nu} \frac{\partial}{\partial y^\alpha_{\mu\nu}} \tag{B.3.5}$$

and similarly obtain [18], besides (B.3.3) and (B.3.4),

$$A^\alpha_{\beta\mu} = \frac{\partial \overline{X}^\alpha}{\partial y^\beta} - y^\alpha_{\mu\eta} \frac{\partial X^\eta}{\partial y^\beta} , \qquad A^{\alpha\nu}_{\beta\mu} = \frac{\partial \overline{X}^\alpha_\mu}{\partial y^\beta_\nu} , \tag{B.3.6}$$

together with

(B.3.7) $$\bar{X}^{\alpha}_{\mu\nu} = \frac{\partial \bar{X}^{\alpha}_{\mu}}{\partial x^{\nu}} - y^{\alpha}_{\mu\eta}\frac{\partial X^{\eta}}{\partial x^{\nu}} + A^{\alpha}_{\beta\mu}y^{\beta}_{\nu} + A^{\alpha\eta}_{\beta\mu}y^{\beta}_{\eta\nu}\,.$$

The above procedures of evaluating the jet prolongation are not the only ones. For example, one may take advantage of the duality existing between the Pfaff systems and the associated differential systems. For instance, the system of fields $P^{\perp}$ orthogonal to the Pfaffian system $P = \{\theta^{\alpha} \equiv \mathrm{d}y^{\alpha} - y^{\alpha}_{\mu}\,\mathrm{d}x^{\mu}\}$ is clearly generated by the fields Y_{μ} and Z^{α}_{μ} given by

(B.3.8) $$Y_{\mu} = \frac{\partial}{\partial x^{\mu}} + y^{\alpha}_{\mu}\frac{\partial}{\partial y^{\alpha}}\,, \qquad Z^{\alpha}_{\mu} = \frac{\partial}{\partial y^{\alpha}_{\mu}}\,,$$

since $\theta^{\alpha}(\lambda^{\mu}Y_{\mu} + \lambda^{\mu}_{\beta}Z^{\beta}_{\mu}) = 0$. Thus, as it is evident that $[\bar{X}^{1}, Z] \in P^{\perp}$ (X^{μ} and X^{α} do not depend on y^{α}_{μ}), the 1-jet prolongation is given by the condition

(B.3.9) $$[\bar{X}^{1}, Y_{\mu}] = A^{\nu}_{\mu}Y_{\nu} + F^{\alpha}_{\mu\nu}\frac{\partial}{\partial y^{\alpha}_{\nu}}\,.$$

Indeed, (B.3.9) is sufficient to determine, besides A^{ν}_{μ} and $F^{\alpha}_{\mu\nu}$, the expressions (B.3.2)-(B.3.4) for $\bar{X}^{1}$. That this should be the case is easily shown:

(B.3.10) $$(L_{\bar{X}^{1}}\theta^{\alpha})(Y_{\mu}) = \mathrm{d}\theta^{\alpha}(\bar{X}^{1}, Y_{\mu}) + (\mathrm{d}i_{\bar{X}^{1}}\theta^{\alpha})(Y_{\mu})\,.$$

By using the identity

(B.3.11) $$\mathrm{d}\Omega(X, Y) = L_{X}\Omega(Y) - L_{Y}\Omega(X) - \Omega([X, Y])\,,$$

valid for any 1-form Ω, the first term of the r.h.s. of (B.3.10) gives

(B.3.12) $$\mathrm{d}\theta^{\alpha}(\bar{X}^{1}, Y_{\mu}) = -\bar{X}^{\alpha}_{\mu} - \theta^{\alpha}([\bar{X}^{1}, Y_{\mu}])$$

and, since the second is $\bar{X}^{\alpha}_{\mu}$, we get

(B.3.13) $$(L_{\bar{X}^{1}}\theta^{\alpha})(Y_{\mu}) = -\theta^{\alpha}([\bar{X}^{1}, Y_{\mu}])$$

and thus $L_{\bar{X}^{1}}\theta^{\alpha} = A^{\alpha}_{\beta}\theta^{\beta}$ and (B.3.9) imply each other.

The above procedures of introducing the 1-jet prolongation of vector fields on E are of mathematical character. We can give, in addition, one more way of defining $\bar{X}^{1}$ which is directly relevant to the symmetry problem considered in sect. **14**. Consider the Lie derivative of the Poincaré-Cartan form (11*a*.8) and (11*a*.9):

(B.3.14) $$L_{X^{1}}\Theta = \left(L_{X^{1}}\frac{\partial \mathscr{L}}{\partial y^{\alpha}_{\mu}}\right)\theta^{\alpha}\wedge\theta_{\mu} + \frac{\partial \mathscr{L}}{\partial y^{\alpha}_{\mu}}(L_{X^{1}}\theta^{\alpha})\wedge\theta_{\mu} + \frac{\partial \mathscr{L}}{\partial y^{\alpha}_{\mu}}\theta^{\alpha}\wedge(L_{X^{1}}\theta_{\mu}) + L_{X^{1}}(\mathscr{L}\omega)\,.$$

When the Lagrangian density is regular, both formalism on $J^1(E)$ (PI for $\mathscr{L}$, PII for Θ) give the same results (subsect. **11***a*)). It is then natural to require

$$L_{\bar{X}^1}\Theta|_{\bar{\varphi}^1} = L_{\bar{X}^1}(\mathscr{L}\omega)|_{\bar{\varphi}^1} \tag{B.3.15}$$

for the 1-jet prolongation $\bar{X}^1$ on $J^1(E)$ of vector fields X on E. In fact, it is clear, by using (B.3.2), that the second term on the r.h.s. of (B.3.14)—which has to be zero, since the first and the third do not contribute to (B.3.15) because of (B.2.8)—again reproduces (B.3.4).

As a final remark on j^1, we note that j^1 is an isomorphism between the algebras defined by the fields on E and their images on $J^1(E)$, since j^1 is an injection and

$$j^1([X, Y]) = [j^1(X), j^1(Y)] , \tag{B.3.16}$$

as may be checked by direct computation. This important property guarantees that the structural relations between generators on E are not lost in the process of extending their action to $J^1(E)$.

As an application of the above general procedure let us consider as in appendix B.1 the particular case for which E is the trivial bundle $E = M\times \times R \xrightarrow{\pi} R$, where M is a differentiable manifold of dimension n. A vector field on E is written as

$$X = X_t\frac{\partial}{\partial t} + X^i\frac{\partial}{\partial q^i} , \qquad i = 1, ..., n . \tag{B.3.17}$$

The Pfaffian system is defined by $(\theta|_\varphi = 0 \Rightarrow \dot{q}^i = \mathrm{d}q^i/\mathrm{d}t)$

$$P = \{\theta^i = \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t\} , \qquad i = 1, ..., n , \tag{B.3.18}$$

and the 1-jet prolongation of X is given by (cf. (B.3.4))

$$\bar{X}^1 = X + \left(\frac{\partial X^i}{\partial t} - \dot{q}^i\frac{\partial X_t}{\partial t} + \dot{q}^j\frac{\partial X^i}{\partial q^j} - \dot{q}^j\frac{\partial X_t}{\partial q^j}\dot{q}^i\right)\frac{\partial}{\partial \dot{q}^i} . \tag{B.3.19}$$

When the field X is a vertical field ($X_t = 0$) and « independent of time » (which is the case of the mechanics « independent of time », sect. **2** and **3**) $\bar{X}^1$ coincides with the generator (A.4.3) of the tangent group on $T(M)$ associated with the group on M generated by X.

As an example, we may evaluate the expression on $R\times T(M)$ of the Galilei boosts. This is obtained by applying (B.3.19) to their expression on $R\times M$

$$X_{(i)} = t\delta^j_{(i)}\frac{\partial}{\partial q^j} , \tag{B.3.20}$$

which leads immediately to the result

$$\bar{X}^1_{(i)} = t\delta^j_{(i)}\frac{\partial}{\partial q^j} + \delta^j_{(i)}\frac{\partial}{\partial \dot{q}^j} . \tag{B.3.21}$$

B.4. *The prolongation to $\Gamma(\tau(J^{1*}(E)))$ of a vector field of $\Gamma(\tau(E))$.* – On $J^{1*}(E)$ there is no canonical way of establishing a notion equivalent to the 1-jet prolongation on $J^1(E)$. However, once a diffeomorphism between both spaces is established, we may associate to $y^\alpha_\mu = \partial_\mu y^\alpha$ the corresponding one in $J^{1*}(E)$. For regular Lagrangians, the Legendre transformation is such a bijection and it is possible to associate to $\partial_\mu y^\alpha$ the corresponding momentum π^μ_α ((11*b*.10)). Thus it is natural to define as the j^{1*} prolongation of the field X of (B.3.2) the vector field on $J^{1*}(E)$

$$\bar{X}^{1*} = X + \bar{X}^{*\mu}_\alpha \frac{\partial}{\partial \pi^\mu_\alpha}, \tag{B.4.1}$$

which is determined by the condition

$$L_{\bar{X}^{1*}} \theta^{*\alpha} = A^\alpha_\beta \theta^{*\beta}, \tag{B.4.2}$$

where the $\theta^{*\alpha}$ are defined by (14.7). As was to be expected, A^α_β is given by

$$A^\alpha_\beta = \frac{\partial X^\alpha}{\partial y^\beta} - \frac{\partial \mathscr{H}}{\partial \pi^\mu_\alpha}\frac{\partial X^\mu}{\partial y^\beta} \tag{B.4.3}$$

and

$$\bar{X}^{*\mu}_\alpha = \frac{\partial^2 \mathscr{L}}{\partial y^\alpha_\mu \partial y^\beta_\sigma}\left\{\frac{\partial X^\beta}{\partial x^\sigma} - \frac{\partial \mathscr{H}}{\partial \pi^\nu_\beta}\frac{\partial X^\nu}{\partial x^\sigma} + \frac{\partial \mathscr{H}}{\partial \pi^\sigma_\gamma}\left(\frac{\partial X^\beta}{\partial y^\gamma} - \frac{\partial \mathscr{H}}{\partial \pi^\nu_\beta}\frac{\partial X^\nu}{\partial y^\gamma}\right)\right\}, \tag{B.4.4}$$

i.e. the field on $J^{1*}(E)$ is defined as the field transformed by the derivative of the application D_L (which acts trivially on the components X^μ and X^α).

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