RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Tomo XLIX (2000), pp. 521-526

A WEAK NONSMOOTH PALAIS-SMALE CONDITION AND COERCIVITY

NIKOLAOS C. KOUROGENIS - NIKOLAOS S. PAPAGEORGIOU

In this paper we show that a generally nonsmooth locally Lipschitz function which satisfies the nonsmooth C-condition (nonsmooth Cerami condition) and is bounded from below, is coercive. The Cerami condition is a weak form of the well-known Palais-Smale condition, which suffices to prove minimax principles.

1. Introduction.

The Palais-Smale condition ("PS-condition" for short) plays a critical role in optimization theory and in the variational analysis of elliptic boundary value problems. The PS-condition is a compactness-type condition which guarantees the convergence of minimizing sequences of a particular type, thereby proving the existence of an actual minimizer. In the smooth case the PS-condition has the following form: Let Y be a Banach space and $\phi : Y \to \mathbf{R}$ a Frechet differentiable function. We say that ϕ satisfies the PS-condition, if any sequence $\{x_n\}_{n\geq 1} \subseteq Y$ such that $\{\phi(x_n)\}_{n\geq 1}$ is bounded and $\phi'(x_n) \xrightarrow{n\to\infty} 0$, possesses a strongly convergent subsequence. This condition was extended by Chang [4] to nonsmooth, locally Lipschitz functionals $\phi : Y \to \mathbf{R}$. In this case the PS-condition has the following

1991 AMS Subject Classification: 47H99, 49J52.

Key words and phrases: Palais-Smale condition, nonsmooth Palais-Smale condition, nonsmooth C-condition, Ekeland variational prinicple, locally Lipschit function, Clarke's subdifrential, coercive function.

form: Any sequence $\{x_n\}_{n\geq 1} \subseteq Y$ such that $\{\phi(x_n)\}_{n\geq 1}$ is bounded and $m(x_n) = \min\{\|x^*\| : x^* \in \partial\phi(x_n)\} \overset{n\to\infty}{\longrightarrow} 0$, has a strongly convergent subsequence. Here by $\partial\phi(x_n)$ we denote the subdifferential at x_n in the sense of Clarke [5] of f. Recall that if $\phi \in C^1(Y, \mathbf{R})$, then $\partial\phi(x_n) = \{\phi'(x_n)\}$ for all $x \in Y$. Using this condition Chang [4] developed a critical point theory for variational problems with nonsmooth, locally Lipschitz energy functionals. A weaker form of the smooth PS-condition was introduced by Cerami [3]. In the more general nonsmooth, locally Lipschitz setting, Cerami's condition has the following form: Any sequence $\{x_n\}_{n\geq 1} \subseteq Y$ such that $\{\phi(x_n)\}_{n\geq 1}$ is bounded and $(1+\|x_n\|)m(x_n) \overset{n\to\infty}{\longrightarrow} 0$, has a strongly convergent subsequence. In what follows, we call this condition the "nonsmooth C-condition". This weaker condition can give us either via Ekeland's variational principle or via a deformation lemma, various minimax principles (see Bartolo-Benci-Fortunato [1] (smooth case) and Zhong [9], Kourogenis-Papageorgiou [7] (nonsmooth cases)).

It has been observed that in the differentiable case, the PS-condition implies coercivity for a functional which is bounded below. This was proved by Costa-Silva [6] (for a Frechet differentiable functional) and by Calcovic-Li-Willen [2] (for a Gateaux differentiable functional which is also lower semicontinuous). In this note we extend this result to nonsmooth locally Lipschitz functionals which satisfy the nonsmooth C-condition.

In the proof of our main result we will need the following recent generalization of the Ekeland variational principle, due to Zhong [9].

PROPOSITION 1. *If $h : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous nondecreasing function such that $\int_0^\infty \frac{1}{1+h(r)}dr = +\infty$, (Y, d) is a complete metric space, $x_0 \in Y$ is fixed, $\phi : Y \to \bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ is lower semicontinuous function not identically $+\infty$ which is bounded from below,*

then for any given $\lambda > 0$, $\varepsilon > 0$ and $y \in Y$ such that $\phi(y) \leq \inf_Y \phi + \varepsilon$, we can find $x \in Y$ such that

(a) $\phi(x) \leq \phi(y)$;

(b) $\phi(x) \leq \phi(u) + \dfrac{\varepsilon}{\lambda(1+h(d(x_0,x)))}d(u,x)$ *for all $u \in Y$; and*

(c) $d(x, x_0) \leq d(y, x_0) + \bar{r}$, *where $\bar{r} > 0$ is such that*

$$\int_{d(y,x_0)}^{d(y,x_0)+\bar{r}} \frac{1}{1+h(r)} dr \geq \lambda.$$

2. Main result.

The next theorem and its corollary extend the results of Costa-Silva [6] and Calkovic-Li-Willem [2] to nonsmooth, locally Lipschitz functionals which are bounded from below and satisfy only the nonsmooth C-condition.

THEOREM 2. *If Y is a Banach space, $\phi : Y \to \mathbf{R}$ is locally Lipschitz and bounded below and $c = \underline{\lim}_{\|y\|\to\infty} \phi(y)$ is finite,*

then there exists a sequence $\{x_n\}_{n\geq 1} \subseteq Y$ such that $\|x_n\| \to \infty$, $\phi(x_n) \to c$ and $\|x_n\| m(x_n) \to 0$ as $n \to \infty$.

Proof. Let $\eta = \inf[\phi(y) : y \in Y]$. Because $\phi(\cdot)$ is bounded below, η is finite. By the definition of c we can find $M_n > 0$ such that for any $x \in Y$ with $\|x\| \geq M_n$, $\phi(x) \geq c - \dfrac{1}{n}$, $n \geq 1$, $M_{n+1} \geq M_n + 2$, $M_1 \geq 1$. We also can find $\{y_n\}_{n\geq 1}$ such that $\phi(y_n) < c + \dfrac{1}{n}$ and

$$\|y_n\| > \frac{M_n + 2 + \frac{1}{2}\left(\frac{1}{n}\right)^{\frac{1}{2}}}{1 - \frac{1}{2}\left(\frac{1}{n}\right)^{\frac{1}{2}}}, \ n \geq 1.$$

From the definition of $\{y_n\}_{n\geq 1}$ we have that $\|y_n\| > M_n + 2 \overset{n\to\infty}{\longrightarrow} 0$.

Now, we define $\phi_n : Y \to \mathbf{R}$, $n \geq 1$ such that $\phi_n(x) = \phi(x)$ when $\|x\| > M_n + 1$, $\phi_n(x) = \phi(x) + (c - \eta)(M_n + 1 - \|x\|)$ when $M_n < \|x\| \leq M_n + 1$ and $\phi_n(x) = \phi(x) + c - \eta$ when $\|x\| \leq M_n$. Then ϕ_n, $n \geq 1$, are locally Lipschitz and $\eta_n = \inf[\phi_n(y) : y \in Y] \geq c - \dfrac{1}{n}$.

Apply proposition 1 with $h(r) = \|y_n\|$, $x_0 = y_n$, $\varepsilon = \varepsilon_n = c + \dfrac{1}{n} - \eta_n$

and $\lambda = \lambda_n \frac{1}{2}\left(\frac{1}{n}\right)^{\frac{1}{2}}$. We can find $x_n \in Y$, $n \geq 1$, such that

$$\phi_n(x_n) \leq \phi_n(y_n) = \phi(y_n) \leq c + \frac{1}{n},$$

$$\phi_n(x_n) \leq \phi_n(u) + \left(\frac{1}{n}\right)^{\frac{1}{2}} \frac{4}{1+\|y_n\|}\|x_n - u\|$$

for all $u \in Y$ and

$$\|x_n - y_n\| \leq \left(\frac{1}{n}\right)^{\frac{1}{2}} \frac{\|y_n\|+1}{2}$$

for all $n \geq 1$.

Let $u = x_n + tv$ with $t > 0$ and $v \in Y$. Then we have

$$-\xi_n\|v\| \leq \frac{\phi_n(x_n+tv) - \phi_n(x_n)}{t}$$

where $\xi_n = \left(\frac{1}{n}\right)^{\frac{1}{2}} \frac{4}{1+\|y_n\|} \downarrow 0$ as $n \to \infty$. Letting $t \downarrow 0$ we obtain

$$-\xi_n\|v\| \leq \phi_n^0(x_n; v) \quad \text{for all} \quad n \geq 1 \quad \text{and all} \quad v \in Y.$$

But $\|x_n\| \geq \|y_n\| - \|x_n - y_n\| \geq \|y_n\| - \left(\frac{1}{n}\right)^{\frac{1}{2}} \frac{\|y_n\|+1}{2} \geq M_n + 2$ (from the choice of y_n). So $\phi_n^0(x_n; v) = \phi^0(x_n; v)$ for all $v \in Y$. Let $\psi_n(v) = \frac{1}{\xi_n}\phi^0(x_n; v)$. Then $\psi(\cdot)$ is sublinear, continuous with $\psi(0) = 0$ and $-\|v\| \leq \psi_n(v)$ for all $v \in Y$. Invoking lemma 1.3 of Szulkin [8], we can find $y_n^* \in Y^*$, $n \geq 1$, such that $\|y_n^*\| \leq 1$ and $(y_n^*, v) \leq \psi_n(v)$ for all $n \geq 1$ and all $v \in Y$. Then if $x_n^* = \xi_n y_n^*$, we have

$$(x_n^*, v) \leq \phi^0(x_n, v) \quad \text{for all } n \geq 1 \text{ and all } v \in Y,$$

$$\Rightarrow x_n^* \in \partial\phi(x_n) \quad \text{for all} \quad n \geq 1.$$

Hence $m(x_n) \leq \|x_n^*\| \leq \xi_n \Rightarrow (1+\|y_n\|)m(x_n) \leq 4\left(\frac{1}{n}\right)^{\frac{1}{2}} \overset{n\to\infty}{\longrightarrow} 0.$

On the other hand $\|x_n\| \leq \|y_n\| + \|x_n - y_n\| \leq \|y_n\| + \left(\frac{1}{n}\right)^{\frac{1}{2}} \frac{\|y_n\|+1}{2}$. So, $\|x_n\| m(x_n) \overset{n\to\infty}{\longrightarrow} 0$.

Finally, $\|x_n\| \geq M_n + 2 \overset{n\to\infty}{\longrightarrow} \infty \Rightarrow \phi(x_n) \overset{n\to\infty}{\longrightarrow} c$. Therefore $\{x_n\}_{n\geq 1} \subseteq Y$ is the desider sequence. ■

An immediate consequence of this proposition is the following corollary.

COROLLARY 3. *If Y is a Banach space, $\phi : Y \to \mathbf{R}$ is locally Lipschitz, bounded below and satisfies the nonsmooth C-condition,*

then $\phi(\cdot)$ is coercive.

Now we are ready to prove our last theorem.

THEOREM 4. *If Y is a Banach space, $\phi : Y \to \mathbf{R}$ is locally Lipschitz, bounded below and satisfies the nonsmooth C-condition,*

then $\phi(\cdot)$ also satisfies the nonsmooth PS-condition.

Proof. Let $\phi : Y \to \mathbf{R}$ a locally Lipschitz, bounded below functional which satisfies the non-smooth C-condition. Let also $\{x_n\}_{n\geq 1} \subseteq Y$ along which ϕ is bounded and $m(x_n) \overset{n\to\infty}{\longrightarrow} 0$. It is easy to see that $\{x_n\}_{n\geq 1}$ is bounded because of the coercivity of $\phi(\cdot)$ (see Corollary 3). So, indeed $(1 + \|x_n\|)m(x_n) \overset{n\to\infty}{\longrightarrow} 0$ and as $\phi(\cdot)$ satisfies the nonsmooth C-condition we can extract from $\{x_n\}_{n\geq 1}$ a strongly convergent subsequence. ■

Remark. From the last theorem it follows trivially that the Min-Max theorems for locally Lipschitz and bounded below functionals which satisfy the PS-condition are expanded to the case of C-condition.

REFERENCES

[1] Bartolo P., Benci V., Fortunato D., *Abstract critical point theorems and applications to some nonlinear problems with"strong resonance" at infinity*, Nonl. Anal.-TMA, **9** (1983), 981-10122.

[2] Calcovic L., Li S., Willem M., *A note on Palais-Smale condition and coercivity*, Dif. Integral Eqns, **3** (1990), 799-800

[3] Cerami G., *Un criterio di esistenza per i punti critici su varietà illimitate*, Rend. Istituto Lombardo Sci. Let., **112** (1978), 332-336.

[4] Chang K.-C., *Variational methods for non-differentiable functionals and their applications to partial differential equations,* J. Math. Anal. Appl., **80** (1981), 102-129.

[5] Clarke F. H., *Optimization and Nonsmooth Analysis*, WIley, New York (1983).

[6] Costa D., Silva E. A., de B., *The Palais-Smale condition versus coercivity*, Nonl. Anal.-TMA, **16** (1991), 371-381.

[7] Kourogenis N. C., Papageorgiou N. S., *Nontrivial solutions for nonlinear resonant elliptic problems with discontinuities,* Colloq. Math; to appear.

[8] Szulkin A., *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems,* Ann. Inst. H. Poincaré, **3** (1986), 77-109.

[9] Zhong C. K., *On Ekeland's variational principle and a minimax theorem,* J. Math. Anal. Appl., **205** (1997), 239-250.

Pervenuto il 28 aprile 1999.

National Technical University
Department of Mathematics
Zografou Campus
Athens 157 80, Greece