Super-Gain Antennas and Optical Resolving Power.

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Introduction.

According to ray optics there should be no limit to the resolving power of a perfect optical system, i.e. of a system that images mathematical points into mathematical points. However it is well known that wave optics modifies considerably the situation, in that it shows that the image of a point source is always constituted by a diffraction pattern of non-zero dimensions. For instance in the case of a circular pupil of diameter D, uniformly illuminated, the pattern consists of a central disk (Airy disk) of angular radius 1,22 λ/D, surrounded by many alternatively dark and bright rings. The rings are very faint, so that the resolving power depends mainly on the size of the disk. Of course, in evaluating the net resolving power one must take into account the properties of the receptor. But, no matter how fine the receptor may be, the upper limit of the resolving power cannot be far from that given by the well-known rule of Rayleigh, according to which the smallest distance between two points that can be seen as distinct is equal to the radius of the Airy disk.

There has always been much speculation as to whether this situation can be improved by a suitable departure from the condition of the uniform pupil, that is by making the complex amplitude a function of the coordinates in the plane of the pupil. It is well known to opticians, however, that every attempt to reduce the size of the disk bears as a consequence an increase in the brightness of the rings at the expense of the sharpness of details in the image. This result may be termed as classical. In recent years an extensive discussion of it has been given by LUNEBERG (1). After going through his beautiful theorems on this subject, one cannot escape the conclusion that it is impossible, for theoretical reasons, to ameliorate the performance of an optical system by means of any type of coating on the pupil, i. e. by any departure from the uniform pupil.

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(1) R. K. LUNEBERG: *Mathematical Theory of Optics* (Providence, 1944) p. 391.

In this situation any serious attempt to reduce the size of the central disk was given up and workers in the field of optics set themselves the much less ambitious task of stripping the disk of its rings. This is feasible and many authors have recently given methods for calculating the corresponding distribution on the pupil (2).

Super-gain antennas.

After the appearance of microwave techniques and their applications, a problem closely related to that of resolving power was tackled by theoreticians in their search for highly directive antenna arrays. From the mathematical point of view an antenna array is a given spatial distribution of radiating currents, i. e. of their complex amplitudes. Now it is well known that any distribution of alternating currents is equivalent to a distribution of electric dipoles; and the same is true for the pupil of an optical instrument, as we know from the theory of electromagnetic diffraction (3). Thus the mathematical formulation of the problem should be identical in the two cases.

Fortunately it appears that microwave researchers were not very much concerned, or perhaps even acquainted, with the old and well-established theorems of wave optics, according to which no material improvement over the uniform pupil should have been possible. As a result, an entirely new theory has been set up, which contains many revolutionary implications.

The pioneer work in this field was that of SCHELKUNOFF (4). It was based on the remark that, apart from a form factor, the amplitude radiated by a linear end-fire array of n elements can be represented by the polynomial

$$(1)\qquad A_0 + A_1 \exp[i\psi] + A_2 \exp[2i\psi] + \dots + A_{n-1} \exp[(n-1)i\psi] = \\ = A_0 + A_1 z + A_2 z^2 + \dots + A_{n-1} z^{n-1},$$

where $A_0, A_1 \dots$ are complex amplitudes, $z = \exp[i\psi]$, $\psi = (2\pi l/\lambda)\cos\theta - \varphi$, l being the spacing between the elements, θ the angle made by a typical direction with the line of sources, and φ a progressive phase delay. In the complex plane z varies on the unit circle; consequently of the $n-1$ zeros of the polynomial (1) only those lying on the unit circle may represent cones of silence. Further, of these cones of silence only those corresponding to $|\cos\theta| \leqslant 1$ have a physical existence, the others being imaginary.

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(2) For history and literature see: B. DOSSIER, P. BOUGHON and P. JACQUINOT: *Journ. des Rech. C.N.R.S.*, n. 11 (1950).

(3) For comprehensive discussions and literature see: G. TORALDO DI FRANCIA: *Nuovo Cimento*, **7**, 967 (1950).

(4) S. A. SCHELKUNOFF: *Bell. Syst. Techn. Journ.*, **22**, 80 (1943).

A uniform array has the zeros of z equispaced on the whole unit circle; but SCHELKUNOFF was able to show that a remarkable improvement in the gain can be obtained by placing the $n-1$ zeros all inside the actual range of z, that is on the arc of the unit circle corresponding to $|\cos\theta| \leqslant 1$. This is illustrated for instance in fig. 1, where the directive properties of two six-element arrays with $l = \lambda/8$ are represented; curve A refers to a uniform array and curve B to an array with its nulls equispaced in the actual range of z. The comparison between the two curves is really astonishing.

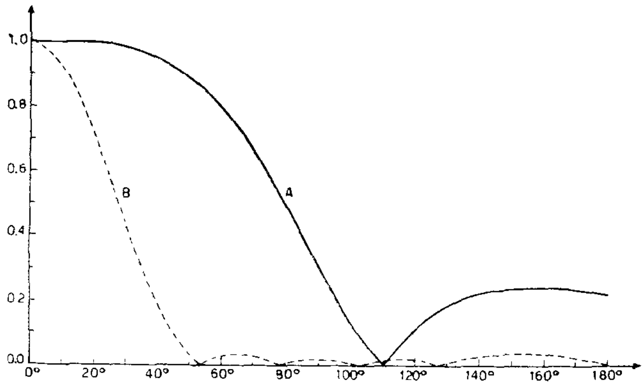

Fig. 1. – Directive properties of two six-element end-fire arrays with the spacing $l = \lambda/8$; curve A refers to a uniform array and curve B to an array with its nulls equispaced in the actual range of z. (From SCHELKUNOFF).

The revolutionary result of SCHELKUNOFF aroused great interest and people began to ask themselves whether the best distribution of currents on an antenna of given over-all length could be found out by mathematics. But very soon BOUWKAMP and DE BRUIJN (5) were able to prove for a linear antenna the amazing theorem that no such optimum distribution exists and that, as a consequence, there is no upper limit to the gain obtained. The extension of such results to two-dimensional current distributions was considered by RIBLET (6).

After the appearance of the startling paper by BOUWKAMP and de BRUIJN the question arose among workers in the antenna field of how to calculate actual current distributions yielding arbitrary high gains. A very interesting method, applying to the case of broadside arrays, was pointed out by DOLPH (7)

(5) C. J. BOUWKAMP and N. G. DE BRUIJN: *Philips Res. Rep.*, **1**, 135 (1946).
(6) H. J. RIBLET: *Proc. I.R.E.*, **36**, 620 (1948).
(7) C. L. DOLPH: *Proc. I.R.E.*, **34**, 335 (1946).

and RIBLET [8]; the latter showed that by choosing the polynomial (1) equal to a suitable combination of Tchebyscheff polynomials one could obtain as great a directivity as desired. However, a numerical application of this method, made by YARU [9] brought a disheartening result: the reactive currents required were enormous, so as to render the practical realization of the array absolutely impossible.

The role of the evanescent waves.

At this point another question seems to be unavoidable. Apart from the many difficulties of a practical nature, the theoretical existence of complex amplitude distributions yielding any desired directivity with a pupil of finite size is established beyond any doubt. What is wrong then with the old arguments of wave optics?

In order to answer this question we must first recall some results obtained by the present author in the theory of diffraction. The problems of optical diffraction can be dealt with either by means of the well-known principle of Huygens-Fresnel, or by means of a superposition of diffracted waves, whose interference on the surface of the pupil gives rise to the actual distribution of complex amplitude. The latter procedure was given the name of principle of reverse interference [10]. In the case of a plane pupil this principle can be applied by choosing as diffracted waves a set of plane waves. Each of them is characterized by its complex amplitude and by the first two direction cosines of its direction of propagation. However, on the basis of well-known properties of the Fourier transform, it is easy to show that, if the pupil has finite size, some of the diffracted waves must have at least one direction cosine greater than unity. Consequently they cannot be ordinary plane waves; they are instead evanescent waves, attenuated in the direction perpendicular to the pupil.

The existence of evanescent waves was known in the case of total reflection [11]. Their presence in diffraction phenomena was first postulated by the present author [12] and met with some scepticism, until it was revealed experimentally beyond any doubt with the aid of microwaves [13].

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(8) H. J. RIBLET: *Proc. I.R.E.*, **35**, 489 (1947).

(9) N. YARU: *Proc. I.R.E.*, **39**, 1081 (1951).

(10) G. TORALDO DI FRANCIA: *Ottica*, **7**, 117 (1942).

(11) See, for instance: J. A. STRATTON: *Electromagnetic Theory* (New York, 1941), p. 499.

(12) G. TORALDO DI FRANCIA: *Ottica*, **7**, 197 (1942).

(13) M. SCHAFFNER and G. TORALDO DI FRANCIA: *Nuovo Cimento*, **6**, 125 (1949).

It is now easy to remove the apparent contradiction between the new results on super-gain antennas and the statements of conventional wave optics. It is true that, if we attempt to reduce the angular size of the central beam of diffracted waves, the amplitude of some other side waves must necessarily increase. But these waves of increased amplitude may very well be evanescent waves; they contribute nothing to the radiation pattern and correspond to a merely reactive power.

The above explanation was first pointed out by WOODWARD and LAWSON ([14]) for the case of a radiating aerial; but, of course, it applies equally well to the case of diffraction through an aperture.

Classical wave optics seems to have missed the discovery of super-directive pupils, because it overlooked the role of evanescent waves in diffraction. And this in turn was due to the use of the Huygens-Fresnel principle, which can very well be applied to the solution of diffraction problems in the radiation zone, but fails to tell us what happens near the surface of the aperture.

It is easy also to understand the significance of the already mentioned procedure of SCHELKUNOFF ([4]). A uniform array has its cones of silence equispaced in the region of radiating waves ($|\cos\theta|\leqslant 1$) as well as in the region of evanescent waves ($|\cos\theta|>1$); SCHELKUNOFF removes the null points from the region of evanescent waves, where they are of no use, and transfers them to the region of radiating waves, so that the over-all intensity is reduced, though leaving the principal maximum unaltered.

Super-resolving pupils.

Let us now try to transfer the results obtained in the field of antennas to the problem of ameliorating the resolving power of an optical system. The major difficulty we are now confronted with is the enormous size of the pupil. The size of a microwave antenna is of the order of magnitude of the wavelength, while, generally speaking, the size of an optical pupil exceeds the wavelength by many powers of ten. In this situation the requirement of keeping very low the intensity of all diffracted waves around a sharp central maximum up to the region of evanescent waves leads to unbelievably bulky calculations. Fortunately it seems unnecessary to put such a stringent condition. Removal of the luminous rings from the central maximum as far out as the region of evanescent waves would be necessary for an instrument having a 180° angular field; in practice it is sufficient to have the rings well outside the field of the instrument. This

([14]) P. M. WOODWARD and J. D. LAWSON: *Journ. I.E.E.*, **95**, 363 (1948).

is illustrated in fig. 2, where the illumination curve of an ideal diffraction pattern is drawn close to the field diaphragm of the instrument. The field diaphragm stops the luminous rings, thus leaving only the central maximum.

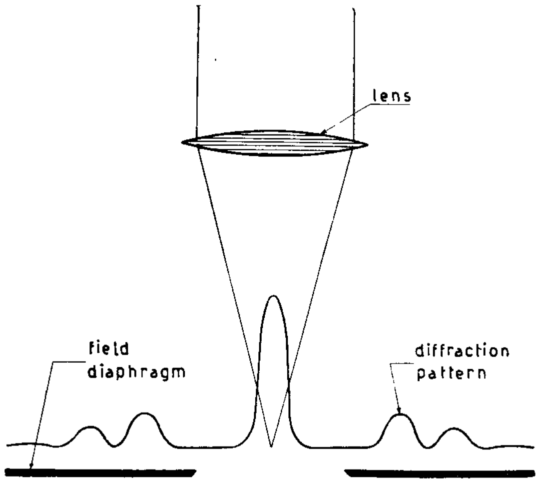

Fig. 2. – Arrangement screening out the large luminous rings of an ideal diffraction pattern.

We shall now describe a simple method for obtaining the desired diffraction pattern from a circular pupil of diameter D.

First we stop all the light impinging on the pupil, except for a thin ring of diameter D. As is well known, if we call θ the angle made by a typical direction with the optical axis, the amplitude of the diffraction pattern, apart from an arbitrary factor, is given by

$$A(x) = J_0(x) , \tag{2}$$

where $x = \pi D \sin\theta/\lambda$. It is also well known that this diffraction pattern has a smaller central disk than that corresponding to the uniform pupil. This is shown in fig. 3, where curve A gives the diffraction pattern (squared amplitude) of the ring-shaped aperture and curve B the diffraction pattern of the uniform pupil of equal diameter. The constant factors are so adjusted that the two central maxima have the same value. Curve A has its first zero at $x = 2{,}40$ and curve B at $x = 3{,}83$. However, in agreement with the general rule of wave optics, a reduction in the size of the disk has brought about an increase in the brightness of the rings.

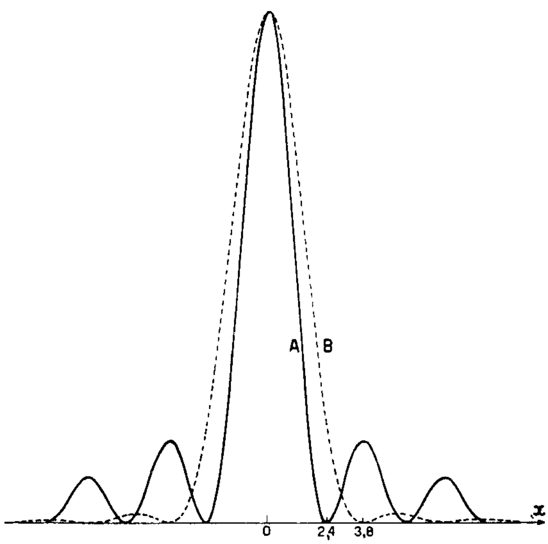

Fig. 3. – Diffraction pattern of a ring-shaped aperture (curve A) and a uniform pupil of equal diameter (curve B).

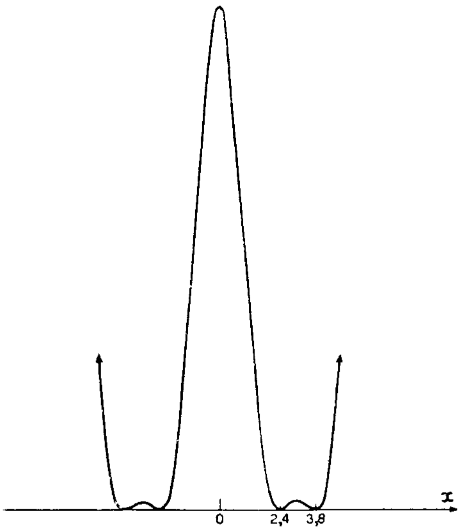

Fig. 4. – Diffraction pattern obtained with three ring apertures.

We begin now to shift the luminous rings, removing them from the vicinity of the disk. To this end we add to the thin ring aperture of diameter D on the pupil two other ring apertures of diameter $D/3$ and $2D/3$ respectively. The amplitude in the diffraction pattern will now be given by

$$(3)\qquad A(x) = = A_1 J_0\left(\frac{1}{3}\,x\right) + A_2 J_0\left(\frac{2}{3}\,x\right) + A_3 J_0(x),$$

where the factors A_1, A_2, A_3 depend on the dimensions and on the transparency of each ring. We then put the conditions $A(0) = 1$, and $A(x) = 0$ at $x = 2{,}40$ and $x = 3{,}83$. The first zero is still that of curve A of fig. 3, while the second zero is placed at the first secondary maximum of the same curve. With these conditions Eq. (3) yields a system of three equations, which can be solved with respect to A_1, A_2, A_3. Thus we find [15]:

$$A_1 = 0{,}95,\quad A_2 = -1{,}77,\quad A_3 = 1{,}82.$$

The resulting diffraction pattern is represented in fig. 4. Around the central maximum there is a large ring of nearly zero intensity.

We pass now to the case of four ring apertures on the pupil, with the diameters equal respectively to $D/4$, $2D/4$, $3D/4$, D. The resulting amplitude will be

$$(4)\qquad A(x) = A_1 J_0\left(\frac{1}{4}\,x\right) + + A_2 J_0\left(\frac{2}{4}\,x\right) + A_3 J_0\left(\frac{3}{4}\,x\right) + A_4 J_0(x).$$

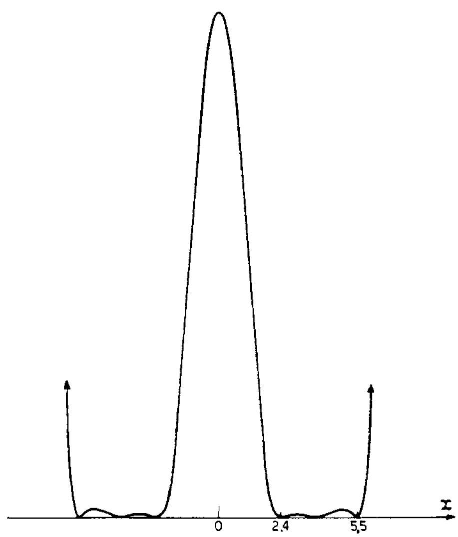

Fig. 5. – Diffraction pattern obtained with four ring apertures.

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[15] Of course, a negative coefficient means that the corresponding ring aperture has reversed phase.

We now put the conditions $A(0) = 1$, and $A(x) = 0$ at $x = 1{,}40$, $3{,}83$, $5{,}52$, the last being the second zero of $J_0(x)$. We find a system of four equations and solve it, with the result

$$A_1 = -2{,}84, \qquad A_2 = 7{,}73, \qquad A_3 = -7{,}67, \qquad A_4 = 3{,}77.$$

The corresponding diffraction pattern is represented in fig. 5. The dark ring surrounding the central disk has become larger.

Finally we try with five ring apertures, having the diameters $D/5$, $2D/5$, $3D/5$, $4D/5$, D respectively. The resulting amplitude will be

$$(5) \qquad A(x) = A_1 J_0\left(\frac{1}{5}x\right) + A_2 J_0\left(\frac{2}{5}x\right) + A_3 J_0\left(\frac{3}{5}x\right) + A_4 J_0\left(\frac{4}{5}x\right) + A_5 J_0(x)\,.$$

We put the conditions $A(0) = 1$ and $A(x) = 0$ at $x = 2{,}40$, $3{,}83$, $5{,}52$, $7{,}02$, the last value representing the third maximum of $J_0(x)$. We find the coefficients

$$(6) \qquad A_1 = 12{,}43, \quad A_2 = -34{,}11, \quad A_3 = 40{,}03, \quad A_4 = -25{,}31, \quad A_5 = 7{,}96.$$

The diffraction pattern is represented in fig. 6. These examples are sufficient for illustrating the method and for showing how it is possible, by increasing the number of ring apertures, to make the nearly dark zone surrounding the central disk as large as desired. The only drawback from the practical point of view is that the coefficients A_1, A_2 ... become larger and larger. This means that a smaller and smaller percentage of the luminous flux passing through the pupil is utilized in the central disk; the fraction of this flux, wasted in the outer rings very soon becomes enormous.

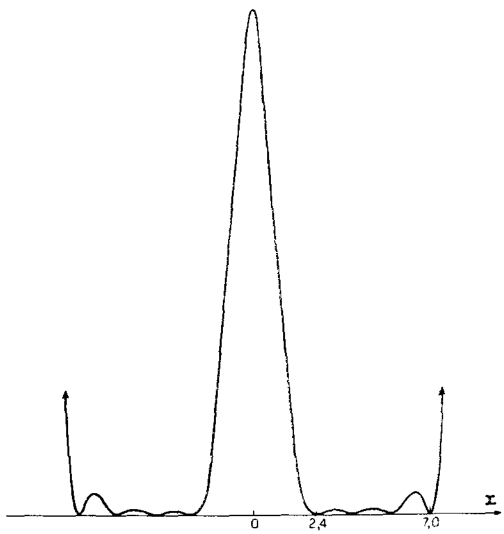

Fig. 6. – Diffraction pattern obtained with five ring apertures and coefficients (6).

It is also interesting to investigate the influence on the coefficients of reducing the size of the disk. For instance let us choose for the first zero $x = 2{,}00$, instead of $x = 2{,}40$; to this end we again use five terms, as in Eq. (5), with the conditions $A(0) = 1$ and $A(x) = 0$ at $x = 2{,}0$, $3{,}5$, $5{,}0$, $6{,}0$. We obtain the coefficients

$$(7) \quad A_1 = 55{,}73, \quad A_2 = -146{,}81, \quad A_3 = 158{,}31, \quad A_4 = -89{,}42, \quad A_5 = 23{,}18.$$

The diffraction pattern is represented in fig. 7 and seems to be very promising. If we apply to it the Rayleigh rule, we find it possible to resolve a cluster of seven stars, whose mutual distances have nearly half the value required in the case of the uniform pupil. But a comparison of the coefficients (6) and (7) shows that a small reduction of the disk has been paid for with a remarkable increase in the luminous flux required to obtain $A(0) = 1$.

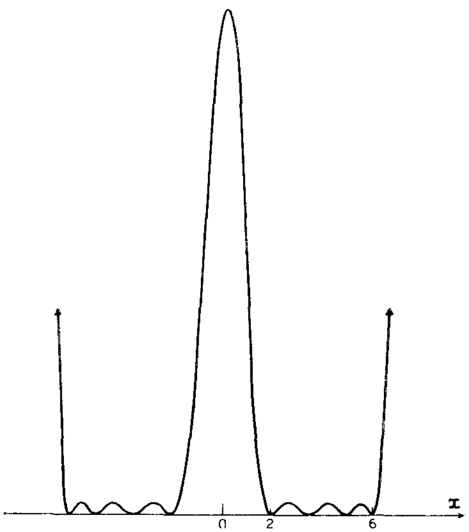

Fig. 7. – Diffraction pattern obtained with five ring apertures and coefficients (7).

Conclusion.

The preceding results show that it is necessary to give up some old ideas concerning the resolving power of an optical system.

First of all we notice that the classical limit of 1,22 λ/D, which has always been accepted as a *theoretical* limit, proves instead to be only a *practical* limit. Theoretically an optical instrument with a pupil of given size can attain as high a resolving power as desired. The only limitation, if any, is set by the amount of luminous flux that we have at our disposal. Thus once more we find an argument in favour of the energetic theory of resolving power, developed by the optical school of Arcetri, according to which it should be absurd to speak of resolving power. without specifiying the amount of energy that is available.

In the second place we find support for the views concerning resolving power that were expressed some time ago by the present author [16]. Accorindg to the modern theory of communication, the only limitation set by the wave nature of light on the performance of an optical system should concern the maximum number N of informations that the system can transmit at a time. As a crude approximation, this number is equal to the area of the pupil, divided by the square of the half wavelength, that is $N = \pi D^2/\lambda^2$. If we want all these informations to be equally distributed among all the directions of a half space (as is the case for an ideal optical system), the elementary solid angle

[16] G. TORALDO DI FRANCIA: *Atti Fond. G. Ronchi*, **6**, 73 (1951).

corresponding to each of them will be $2\pi/N = 2\lambda^2/D^2$. This gives 1,41 λ/D for the elementary plane angle, that is something very close to the classical 1,22 λ/D. But if we consent to reducing the angular field, there should be no theoretical reason preventing the attainment of a higher resolving power. This seems to be in agreement with the results contained in the present paper.

Finally we must point out a puzzling consequence of the preceding investigations. It is well known that one of the most elementary arguments used for deriving the uncertainty principle of Heisenberg is based on the limited resolving power of an optical system of finite pupil ([17]). Therefore our findings would seem to be in contradiction with the uncertainty principle. It is outside the scope of the present article to enter into this delicate question. We shall deal with it in a forthcoming paper.

Aknowledgement.

Thanks are due to A. SÁEZ of the Institute of Optics of Madrid for many interesting discussions; and to him and to Miss M. T. ZOLI of the Istituto Nazionale di Ottica of Arcetri for their intelligent aid in the computations.

([17]) See, for instance, L. I. SCHIFF: *Quantum Mechanics* (New York, 1949), p. 11.

INTERVENTI E DISCUSSIONI

— F. BRUIN:

In the example given by HEISENBERG he calculates the probability of the photon hitting the plane of observation after it has passed a slit, the total probability being one. Now if with the ring system the central disk is made smaller in diameter the probability that this disk will be hit also becomes smaller. Will this probability not become zero in the case that infinite resolution is reached?

— F. J. ZUCKER:

There are *two* practical limitations which prevent the unlimited increase of antenna gain or optical resolving power. One of them has been brought out by Dr. TORALDO in his comments on the surface waves, which create large storage fields in the vicinity of the aperture. This suggests the alternative point of view of saying that supergaining involves a high Q with the attendant high ohmic losses and narrow band width (first pointed out by L. CHU in 1947, in the *Proceedings of the I.R.E.*). As a result of this limitation, the slightest amount of supergaining (or « super-resolving ») must be paid for with a very rapid increase in *inefficiency*.

The second limitation has not yet been mentioned: it is of a statistical nature. If we try to build a « super » device, we must adhere to inhumanely close tolerances to produce the rapid and steep variations in phase and amplitude required along the aperture. This unfortunately means that supergaining must be paid for by the need to fabricate hundreds or thousands of units before *one* will be found with the desired characteristics. A general proof of this assertion will be found in the paper presented by Mr. RUZE (¹).

I would now like to comment on the question raised by Dr. TORALDO with respect to the uncertainty principle, viz., what becomes of that principle when supergaining is allowed? Let me reverse the question and ask: just why do we think there is any connection at all between the uncertainty principle and the limit of optical resolution? Many textbooks « deduce » the latter from the former, and this undoubtedly caused the query. But the « deduction » only works fortuitously, and, in my opinion, is not based on an inherent link between optics and the Heisenberg principle.

We can see this in the following way: as soon as we are told that the q-space and the p-space of the canonical variables are related via the Fourier transform, the Heisenberg principle immediately follows. It simply states that the product of the moments of inertia in the two spaces is a constant. Now it so happens that in physical optics we also have a Fourier-transform relation, namely between the $\sin\theta$ of the Fraunhofer pattern and the wave number in the aperture. Consequently there results a moment-of-inertia relation (i.e. the limit of resolution) which is formally analogous to the uncertainty principle. We have a metaphoric relation between the two rather than an intrinsic one, a circumstance already suggested by the fact that in the textbook deductions from the Heisenberg principle we may cancel out Planck's constant in the very first step, a sure indication that our end-result cannot be deeply connected with quantum-mechanical notions.

You will object and insist that Heisenberg's principle must be applicable to the photon. This is true, but then let us define the photon electromagnetically; and not from the approximate optical point of view. This is absolutely necessary in our problem, because as soon as we permit supergaining, we leave the domain of ordinary wave optics. The reason lies in the storage fields which cannot be defined in optics. To put it differently: the existence of surface waves implies that the wave number in the aperture is no longer restricted to the real domain, so that the moment-of-inertia relation changes entirely; we can limit $\sin\theta$ at will, paying for it with rapidly increasing sidelobes in the imaginary region. If we *now* wish to ask what limitations the Heisenberg principle imposes on the electromagnetically defined photon, we are led to a much more subtle problem than the relation we played with before. We are in fact in the field of quantum-electrodynamics, a different story altogether.

— P. AIGRAIN:

As regards Mr. ZUCKER's remark that the study of the light scattering experiment is not a good way to demonstrate Heisenberg's principle, and that a good demonstration of this principles rests on the properties of the wave function, as such, one can say that there are two forms of Heisenberg's principle.

1) It is impossible to construct a wave function such that $\Delta p \Delta q \leqslant \hbar/2$, the theorem referred to by Mr. ZUCKER.

(¹) In this issue, pag 364.

2) This is not a serious limitation in wave mechanics, because it is not possible to measure Δp and Δq with more precision anyway, even in the old quantum theory. The difficulty indicated by Dr. TORALDO concerns this second form.

— R. MALVANO:

I may be wrong, anyway I think that all of the question proposed by Prof. TORALDO lies in the fact that we are only faced with a deduction from Maxwell's equations; and from Maxwell equations we cannot deduce the Heisenberg principle because this principle starts from the finite value of the Planck's constant h. In the limiting case of $h \to 0$ we have no uncertainty principle.

— P. AIGRAIN:

It looks like one should look for the answer to Dr. TORALDO's remark about Heisenberg's principle in the fact that super-directive pupils have a very high « Q » factor – that is small band width and a correspondingly long response time.

— G. TORALDO DI FRANCIA:

It had not been my intention to discuss here the question of Heisenberg principle, which I considered only as a by-product of the preceding investigation. However, the many interesting interventions on the subject induce me to express my point of view.

I quite agree with those who have remarked that the theory of optical resolving power cannot serve as a proof of the Heisenberg principle. As was pointed out, the simplest mathematical proof is that no wave function can have $\Delta p \Delta q \ll \hbar$. But let us suppose that we know the p of a particle and the p of a photon with an extremely high accuracy (no matter how uncertain their q's must consequently be). The photon is scattered by the particle and enters a super-resolving microscope. There is a very high probability that the photon is absorbed by the coating of the pupil (high reactive power); still it is not impossible that it goes into the central maximum and is revealed. What happens in this case? The Δp of the particle after the collision is practically that of the photon, the q can be read off in the microscope, and we have $\Delta p \Delta q \ll \hbar$! How are we to interpret the result of this experiment?

Something must certainly be wrong, because the uncertainty principle is beyond question to-day and a single exception would be sufficient to invalidate it. The way out of the difficulty may be looked for in more than one direction. Some possible explanations have already been mentioned in the preceding interventions. I should want to add that perhaps Kirchhoff's approximation is not sufficient for a super-resolving pupil. In that case the angular separation of two point sources would not necessarily be equal to the angular separation of their images; in other words the uncertainty in the position of a point source would not necessarily be equal to the dimensions of the diffraction pattern.

However all these explanations need to be investigated in much more detail.

— F. J. ZUCKER:

The only safe way, in my opinion, to apply the uncertainty principle to photons is to refer to quantum electrodynamics. For it is in this theory only that the attempt is made to apply the quantum principles of *particle* mechanics to the vector *fields* of electromagnetism. This leads to a quantization of the field variables, and it can be shown that, in the general case, they are then no longer simple functions of space and time but quantities which are canonically related and do not commute. I.e., if we pick any two of the canonical field variables, (electric or magnetic field strength, phase, momentum, etc.), it is no longer true that $AB - BA = 0$. The commutation function $AB - BA = C$ (where C always involves Planck's constant h) gives rise to the uncertainty relation

$$\Delta A \, \Delta B \geqslant C \, .$$

This equation, which is obeyed by the field variables in various possible combinations, has however no immediate physical significance. It does not imply, for example, that $\Delta E \, \Delta H$ are always $\geqslant C$ and therefore can never be known accurately. The reason for this is that the only E and H which can be measured are average values over a small space-time region. This is obvious if we merely consider that no matter how small the dipole length, it is still finite. To obtain relations for these average values, the uncertainty equation (in operator form) must be integrated. The result depends on whether or not the space-time regions of $\Delta A_{\text{av.}}$ and $\Delta B_{\text{av.}}$ can be connected by a light signal. If they cannot, then $\Delta A_{\text{av.}}$ and $\Delta B_{\text{av.}}$ *commute* and are thus not limited by an uncertainty relation. This will be the case, for instance, if we measure two linear components of the E or H field at the same time but in different locations; or if we measure a component of E and a component of H at the same point but not in the same moment.

The upshot is that I have not been able to find a case relevant to diffraction field probing in which quantum electrodynamics imposes a limitation on the attainable accuracy of measurement.

It seems to me therefore that the answer to Professor TORALDO di FRANCIA's question as to what becomes of the Heisenberg principle when « super-resolving » is allowed should be: Nothing, since the only correct, quantum-electrodynamical version of this principle imposes no relevant restrictions on resolving power to begin with.