# Gravitational Radiation (*).

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**Summary.** — A method is established to solve the Einstein equations for a system of freely gravitating pole particles, by successive approximations. It is shown how one can choose the solution that represents purely outgoing waves. It is then found that the fifth order correction to the acceleration involves a non-conservative term: energy is lost, by gravitational radiation, in an amount exactly equal to that predicted by the linearized theory. This can also be shown by directly computing the loss of mass of the system. We then turn to examine the validity of the linearized theory: it is shown that it cannot correctly describe the field at very large distances from the sources, but nevertheless it gives the right result for the radiated energy.

## 1. – Introduction.

Since the early years of the General Relativity Theory, it has been known that the linearized field equations have wavelike solutions, and that the corresponding energy-momentum pseudotensor of the gravitational field has components representing an energy flux (1). This result, however, was treated with caution, since the field equations are really non-linear. Later, approximation methods were discovered, which took account of the non-linearity, and gave the equation of motion of particles. The situation then became quite chaotic. Some authors (2-4) claimed that there was no radiation reaction,

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(1) L. LANDAU and E. LIFSHITZ: *The Classical Theory of Fields* (Cambridge, Mass., 1951), p. 331.

(2) L. INFELD and A. E. SCHEIDEGGER: *Can. Journ. Math.*, **3**, 195 (1951).

(3) A. E. SCHEIDEGGER: *Phys. Rev.*, **82**, 883 (1951); *Rev. Mod. Phys.*, **25**, 451 (1953).

(4) L. INFELD: *Ann. Phys.*, **6**, 341 (1959).

others (5,6) found a gravitational damping—which agreed, but only qualitatively, with the linearized theory—and still others (7,8) found a gravitational « antidamping », *i.e.* an energy gain!

One of the main causes of this trouble doubtless was the complexity of the field equations: when extremely cumbersome expressions have to be handled, the physical meaning of the various terms becomes unclear. The first step toward the solution of this problem is therefore to develop an approximation technique which minimizes the computational labour, and thus also the risk of errors. The method that was recently developed by the author seems well fitted for this purpose. It needs, however, some modifications, as will be shown in Section **4**. For the sake of completeness, let us briefly recall it.

## 2. – Approximation procedure.

One takes as field variables the contravariant densities $\mathfrak{g}^{\mu\nu}$ and one expands them into a series (*)

$$\mathfrak{g}^{\mu\nu} = \underset{0}{\mathfrak{g}}^{\mu\nu} + \underset{1}{\mathfrak{g}}^{\mu\nu} + \underset{2}{\mathfrak{g}}^{\mu\nu} + \dots ,$$

where each term is by one order of magnitude smaller than the previous one. One choose quasi-Galilean co-ordinates ($\underset{0}{\mathfrak{g}}^{\mu\nu} = \eta^{\mu\nu}$) subjected to the harmonic condition $\mathfrak{g}^{\mu\nu}{}_{,\nu} = 0$. The solution behaving as purely outgoing waves is then unique (9).

The Einstein equations how read, in natural units:

$$\nabla^2 \mathfrak{g}^{\mu\nu} - \ddot{\mathfrak{g}}^{\mu\nu} = -16\pi \mathfrak{g} \mathfrak{T}^{\mu\nu} + \Theta^{\mu\nu} , \tag{1}$$

where $\mathfrak{g} = (-\text{Det. } \mathfrak{g}^{\alpha\beta})^{\frac{1}{2}}$ and $\Theta^{\mu\nu}$ is quadratic in $(\mathfrak{g}^{\mu\nu} - \eta^{\mu\nu})$. When (1) is expanded into the various orders (in order to compute $\underset{n}{\mathfrak{g}}^{\mu\nu}$), $\Theta^{\mu\nu}$ thus depends only on $\underset{m}{\mathfrak{g}}^{\mu\nu}$, $m < n$, and is a known quantity. The compatibility of (1) with the harmonic condition $\mathfrak{g}^{\mu\nu}{}_{,\nu} = 0$ follows from the equations of motion (10), which are most conveniently written as $(\mathfrak{g}\mathfrak{T}^{\mu\nu})_{;\nu} = 0$, or

$$(\mathfrak{g}\mathfrak{T}^{\mu\nu})_{,\nu} = \mathfrak{g}\mathfrak{T}^{\alpha\beta}[\mathfrak{g}_{\alpha\gamma}\mathfrak{g}^{\mu\gamma}{}_{,\beta} + \tfrac{1}{2}\mathfrak{g}^{\mu\gamma}(\tfrac{1}{2}\mathfrak{g}_{\alpha\beta}\mathfrak{g}_{\varrho\sigma} - \mathfrak{g}_{\alpha\varrho}\mathfrak{g}_{\beta\sigma})\mathfrak{g}^{\varrho\sigma}{}_{,\gamma}] , \tag{2}$$

where $\mathfrak{g}_{\mu\nu} = \mathfrak{g}^{-1} g_{\mu\nu}$.

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(5) P. Havas: *Phys. Rev.*, **108**, 1351 (1957).
(6) A. Trautman: *Bull. Acad. Polon. Sc.*, **6**, 627 (1958).
(7) N. Hu: *Proc. Roy. Irish Acad.*, A **51**, 87 (1947).
(8) A. Peres: *Nuovo Cimento*, **11**, 644 (1959).
(*) Greek indices run from 0 to 3, Latin indices from 1 to 3.
(9) V. A. Fock: *Teoria Proctranstva, Vremeni i Tyagotenia* (Moscow, 1955), p. 441.
(10) F. Hennequin: *Thèse* (Paris, 1956).

For pole particles, localized at $\overset{A}{\xi}{}^k = \overset{A}{\xi}{}^k(t)$, one takes ([11])

$$\mathfrak{g}\mathfrak{T}^{\mu\nu} = \sum_A \overset{A}{M}\overset{A}{v}{}^\mu \overset{A}{v}{}^\nu \delta(\overrightarrow{x - \overset{A}{\xi}}) , \tag{3}$$

where $v^\mu = \mathrm{d}\xi^\mu/\mathrm{d}t$ and $M$ is a function of time. (We call $M$ the « effective gravitational mass ». It can be shown ([11]) that the « intrinsic mass » $m_0 = M\mathfrak{g}^{-1}\mathrm{d}s/\mathrm{d}t$ is constant). The equations of motion then read

$$(M\dot{v}^\mu) = Mv^\alpha v^\beta[\mathfrak{g}_{\alpha\gamma}\mathfrak{g}^{\mu\gamma}{}_{,\beta} + \tfrac{1}{2}\mathfrak{g}^{\mu\gamma}(\tfrac{1}{2}\mathfrak{g}_{\alpha\beta}\mathfrak{g}_{\varrho\sigma} - \mathfrak{g}_{\alpha\varrho}\mathfrak{g}_{\beta\sigma})\mathfrak{g}^{\varrho\sigma}{}_{,\gamma}] , \tag{4}$$

where the bracket has to be computed at the position of the particle under consideration, and all the singular terms in it have to be neglected.

All that remains now to do is to solve (1) at the various orders, so as to get the $\mathfrak{g}^{\mu\nu}$ which appear in the right hand side of (4). For this purpose, it is convenient to write

$$\mathfrak{g}^{\mu\nu} = \mathfrak{h}^{\mu\nu} + \mathfrak{s}^{\mu\nu} ,$$

where

$$\nabla^2\mathfrak{h}^{\mu\nu} - \ddot{\mathfrak{h}}^{\mu\nu} = -16\pi\mathfrak{g}\mathfrak{T}^{\mu\nu} \tag{5}$$

is readily solved by the use of Liénard-Wiechert potentials, and

$$\nabla^2\mathfrak{s}^{\mu\nu} - \ddot{\mathfrak{s}}^{\mu\nu} = \Theta^{\mu\nu} . \tag{6}$$

One now has to choose the expansion parameter: the most suitable one would be the reciprocal velocity of light, but it has already been taken as unity. An equivalent choice is to assume that the velocities are small quantities of the first order. Accelerations and masses are therefore of the second order. As the only dependence of the field quantities on time is through the positions and velocities of the sources, it follows that a time derivative of a field quantity is by one order of magnitude smaller than a space derivative.

One thus expands ([12])

$$\mathfrak{h}^{\mu\nu} = 4\sum_{n=0}^{\infty}\left\{\left[\frac{1}{(2n)!}\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\sum Mv^\mu v^\nu R^{2n-1}\right] + \varepsilon\left[\frac{1}{(2n+1)!}\frac{\mathrm{d}^{2n+1}}{\mathrm{d}t^{2n+1}}\sum Mv^\mu v^\nu R^{2n}\right]\right\} , \tag{7}$$

where $\varepsilon$ is an arbitrary constant (for pure retarded potentials, $\varepsilon = -1$). Here $R = \sqrt{R^k R^k}$ and $R^k = x^k - \xi^k$.

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([11]) W. Tulczyjew: *Bull. Acad. Polon. Sc.*, **5**, 279 (1957).
([12]) A. Peres: *Nuovo Cimento*, **11**, 617 (1959).

In this expression, we suppose that the co-ordinates $\xi^k$ and velocities $v^k$ are known (initial conditions) and that the masses and accelerations have to be computed

$$M = m + \underset{1}{m} + \underset{2}{m} + \dots ,$$
$$\dot{v}^k = a^k + \underset{1}{a}^k + \underset{2}{a}^k + \dots .$$

Note that $\underset{n}{m}$ and $\underset{n}{a}^k$ are small quantities of order $(n+2)$.

## 3. – Non-radiative approximations.

Let us now work step by step. The lowest order term in $\mathfrak{g}^{\mu\nu}$ is

$$\underset{2}{\mathfrak{g}}^{00} = \underset{2}{\mathfrak{h}}^{00} = 4 \sum (m/R) ,$$

Introducing this into (4) one gets the Newtonian approximation

$$\dot{m} = 0 , \qquad a^k = \frac{\partial}{\partial \xi^k} \sum{}' \frac{m}{R} .$$

Next, one has $\underset{3}{\mathfrak{g}}^{00} = \underset{3}{\mathfrak{g}}^{kl} = 0$, and $\underset{3}{\mathfrak{g}}^{0k} = \underset{3}{\mathfrak{h}}^{0k} = 4 \sum (mv^k/R)$, whence it follows that there is no first order correction to the Newtonian approximation ($\underset{1}{m} = \underset{1}{a}^k = 0$) and that the second order correction to mass is [12]

$$\underset{2}{m} = \tfrac{1}{2} mv^2 + 3m \sum{}' (m/R) .$$

Non-linear contributions to the field appear at the fourth order [12]. One has

$$\left\{\begin{aligned} \underset{4}{\mathfrak{z}}^{00} &= 7 \sum \frac{m^2}{R^2} + 14 \sum_{A,B}{}' \overset{A}{m}\overset{B}{m} S^{kk} , \\ \underset{4}{\mathfrak{z}}^{0k} &= 0 , \\ \underset{4}{\mathfrak{z}}^{kl} &= \frac{1}{2} \sum m^2 \left[\frac{\delta^{kl}}{R^2} - (\log R)_{,kl}\right] + \sum_{A,B}{}' \overset{A}{m}\overset{B}{m} (2\delta^{kl} S^{nn} - 4 S^{kl}) , \end{aligned}\right.$$

where

$$S^{kl} \equiv \frac{\partial}{\partial \overset{A}{\xi^k}} \frac{\partial}{\partial \overset{B}{\xi^l}} \log (\overset{A}{R} + \overset{B}{R} + D) ,$$

and

$$D = \sqrt{D^k D^k} , \qquad D^k = \overset{A}{\xi^k} - \overset{B}{\xi^k} .$$

This has to be added to

$$\begin{cases} \underset{4}{\mathfrak{h}}^{00} = 4\sum\left(\dfrac{\underset{2}{m}}{R} + \dfrac{m}{2}\ddot{R}\right), \\ \underset{4}{\mathfrak{h}}^{0k} = 0\,, \\ \underset{4}{\mathfrak{h}}^{kl} = 4\sum\dfrac{mv^k v^l}{R}\,. \end{cases}$$

From this, we can compute the second order correction to the acceleration $\underset{2}{a}^k$ which leads, in the two-body problem, to the perihelion advance (8).

## 4. – Radiation field.

The radiation field—*i.e.* that part of $\mathfrak{g}^{\mu\nu}$ which is proportional to $\varepsilon$—appears in the fifth order. (Terms proportional to $\varepsilon^2$ appear only in the ninth order, and we shall not have to deal with them). From (7), one has

$$\underset{5}{\mathfrak{h}}^{00} = 4\varepsilon\sum\left[\dot{\underset{2}{m}} + \frac{m}{6}\,(\dddot{R}^2)\right],$$

$$\underset{5}{\mathfrak{h}}^{kl} = 4\varepsilon\sum m(\dot{v^k v^l})\,.$$

As $\underset{5}{\Theta}^{00} = \underset{5}{\Theta}^{kl} = 0$, one is tempted to write $\underset{5}{\mathfrak{z}}^{00} = \underset{5}{\mathfrak{z}}^{kl} = 0$. This is indeed what we have done in our previous paper (and what has also been done by other authors). As a matter of fact, the situation is not so simple; as clearly stated by SCHEIDEGGER (13), the problem is not to find non-conservative equations of motion, but to show that these solutions indeed correspond to *free* particles, *i.e.* to purely outgoing waves. Unless this is proved, it may always be objected that the energy change of the system, or part of it, is due to the interaction with some external radiation field.

The difficulty is essentially due to the non-linearity of the field equations: one cannot state that a given term, in the field, is due to a certain source, because the field does not depend linearly on the sources. There is therefore a danger of introducing into the solution some external field (pure radiation) that is not caused by the sources under consideration, but that influences their motion.

In practice, the difficulty appears in the following way: at each stage of the approximation procedure, one has to solve a Poisson equation, and there

(13) A. E. SCHEIDEGGER: *Phys. Rev.*, **99**, 1883 (1955).

is a considerable freedom of choice of solutions, each representing a possible motion and a gravitational field belonging thereto ([13]). Only one of these solutions behaves at infinity as purely outgoing waves ([9]); the remaining ones contain also incoming waves. It is in general difficult to determine which solution is the correct one, because the $n$-th term of a series expansion in powers of $(v/c)$ behaves in the wave zone as $R^{n-2}$, and no boundary conditions for each stage of the procedure are known.

As far as $\mathfrak{h}^{\mu\nu}$ is concerned, the difficulty can be obviated by directly taking the expansion of the Liénard-Wiechert potentials, *i.e.* eq. (7). However, it is much more difficult to determine the radiative part of $\mathfrak{g}^{\mu\nu}$ ([14]). Some information can however be obtained by the following argument:

A solution behaving at infinity like $f(t-R/c)/R$ has an expansion $f(t)/R - f'(t)/c + \dots$. It is therefore reasonable to stipulate that if a term such as $f(t)/R$ occurs at some approximation stage, one must add $-f'(t)$ — or, more generally, $\varepsilon f'(t)$ — at the next stage ([15]). (Note that this is a solution of the Laplace equation). In the case that $f(t)$ is a constant, this vanishes, but two approximations higher there will be a term $\varepsilon f\ddot{R}^2/6$. (Note that this also is solution of the Laplace equation.) For instance, $\underset{4}{\mathfrak{h}}^{kl} = 4\sum(mv^k v^l/R)$ is followed by $\underset{5}{\mathfrak{h}}^{kl} = 4\varepsilon\sum m\dot{(v^k v^l)}$, and $\underset{2}{\mathfrak{h}}^{00} = 4\sum(m/R)$ is followed, in the third place (*i.e.* in $\underset{5}{\mathfrak{h}}^{00}$) by $4\varepsilon\sum m\dddot{R}^2/6$.

Now it is easily shown that for large $R$

$$\mathfrak{g}^{kl} \to \frac{-\delta^{kl} + D^k D^l/D^2}{2DR},$$

so that

$$\underset{4}{\mathfrak{g}}^{00} \to -\frac{14}{R}\sum{}' \frac{\overset{A}{m}\overset{B}{m}}{D},$$

and

$$\underset{4}{\mathfrak{g}}^{kl} \to -\frac{2}{R}\sum{}' \overset{A}{m}\overset{B}{m}\frac{D^k D^l}{D^3}.$$

We thus have

$$\underset{5}{\mathfrak{g}}^{00} = -14\varepsilon\sum{}' \overset{A}{m}\overset{B}{m}\dot{\left(\frac{1}{D}\right)},$$

and

$$\underset{5}{\mathfrak{g}}^{kl} = -2\varepsilon\sum{}' \overset{A}{m}\overset{B}{m}\dot{\left(\frac{D^k D^l}{D^3}\right)}.$$

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([14]) I am indebted to Dr. D. W. SCIAMA for calling my attention to this point.
([15]) A. PERES: *Nuovo Cimento*, **13**, 670 (1959).

Further one has [12]

$$\underset{5}{\mathfrak{h}}^{0k} = 4 \sum \left[ \underset{2}{m} \frac{v^k}{R} + \frac{1}{2} m (v^k \ddot{R}) \right],$$

$$\underset{5}{\mathfrak{g}}^{0k} = \frac{1}{2} \sum m^2 \left[ 15 \frac{v^k}{R^2} - v^l (\log R)_{,kl} \right] + 16 \sum_{A,B}{}' \overset{A}{m}\overset{B}{m} \left( \frac{3}{4} \overset{A}{v}{}^l S^{lk} + \overset{B}{v}{}^k S^{ll} - \overset{B}{v}{}^l S^{lk} \right).$$

At large distances, this last expression behaves as

$$\underset{5}{\mathfrak{g}}^{0k} \to \frac{8}{R} \sum{}' \frac{\overset{A}{m}\overset{B}{m}}{D} \left( -\frac{3}{4} \overset{A}{v}{}^k - \overset{B}{v}{}^k + \frac{3}{4} \overset{A}{v}{}^l \frac{D^k D^l}{D^2} - \overset{B}{v}{}^l \frac{D^k D^l}{D^2} \right).$$

Taking into account that $\sum m a^k = 0$, one gets

$$\underset{6}{\mathfrak{g}}^{0k} = -2\varepsilon \sum{}' \overset{A}{m}\overset{B}{m} \left( \frac{v^l \dot{D}^k D^l}{D^3} \right),$$

which has to be added to

$$\underset{6}{\mathfrak{h}}^{0k} = 4\varepsilon \sum \left[ (\underset{2}{\dot{m}} v^k) + m \underset{2}{a}{}^k + \frac{m}{6} (v^k \dddot{R}^2) \right].$$

(It is easily verified that $\underset{5}{\mathfrak{g}}^{00}{}_{,0} + \underset{6}{\mathfrak{g}}^{0k}{}_{,k} = 0$ as it should be. The difficulties that previously arose concerning this point were due to a different choice of the radiative fields in the two expressions.)

## 5. – Lorentz-invariance.

If it can be shown that the method developed in the previous sections is Lorentz-invariant, we can be assured that it gives the correct result because of Fock's uniqueness theorem [9].

We have supposed up to now that velocities were small quantities of the first order. This is equivalent to the assumption that the reciprocal velocity of light (which has been taken here as unity) is small of the first order. As the velocity of light is Lorentz-invariant *by definition*, our approximation method should also be Lorentz-invariant. This has to be understood in the following way:

Let us perform a Lorentz transformation

$$x'^k = x^k - \frac{\beta^k t}{\sqrt{1-\beta^2}} + \left[ \frac{1}{\beta^2 \sqrt{1-\beta^2}} - \frac{1}{\beta^2} \right] \beta^k \beta^l x^l, \qquad t' = \frac{t - \beta^l x^l}{\sqrt{1-\beta^2}},$$

of all the quantities that appear in our equations. The velocity $\beta^k$ is an arbitrary constant, subject only to the limitation of being small of the first order.

In general, $\underset{n}{\mathfrak{g}}'^{\mu\nu}$ will now depend on all the $\underset{m}{\mathfrak{g}}^{\mu\nu}$, $m \leqslant n$. *We require* $\underset{n}{\mathfrak{g}}'^{\mu\nu}$ *to depend in the wame way on the transformed sources, as* $\underset{n}{\mathfrak{g}}^{\mu\nu}$ *on the original sources.*

This requirement may further limit the freedom of choice of solutions that still remains after the rules given in the previous section. In practice, it may be very difficult to apply it, because the $R^k$ are not the spatial components of a four vector (they are defined for *equal times* of the source and field point). Their transformation law is therefore very complicated: it involves not only $\beta^k$ and $v^k$, but also all higher time derivatives of $v^k$.

Fortunately, the rules of the previous section are sufficient to give unambiguous results up to the seventh order, and we shall have no need to test the Lorentz-invariance of our formulae. We shall take it for granted, because they seem to be the only reasonable solution of Lorentz-invariant equations.

## 6. – Radiation reaction.

We can already compute the fifth order radiative correction to the mass. From (4), one has $\underset{5}{\dot{m}} = m(\frac{3}{4}\underset{5}{\mathfrak{g}}^{00} - \frac{1}{4}\underset{5}{\mathfrak{g}}^{kk})_{,0}$. It can be shown, however, that $\underset{5}{\mathfrak{g}}^{kk} = 3\underset{5}{\mathfrak{g}}^{00}$, so that $\underset{5}{m} = 0$.

The fifth order radiative correction to the acceleration is given by (4), as

$$\underset{5}{a}^k = \underset{6}{\mathfrak{g}}^{0k}{}_{,0} + \tfrac{1}{4}\,\underset{7}{\mathfrak{g}}^{\mu\mu}{}_{,k} - \tfrac{1}{2}\underset{5}{\mathfrak{g}}^{00}\underset{2}{\mathfrak{g}}^{00}{}_{,k} - \tfrac{1}{4}\underset{5}{\mathfrak{g}}^{kl}\underset{2}{\mathfrak{g}}^{00}{}_{,l} - v^k\underset{5}{\mathfrak{g}}^{kl}{}_{,0}\,,$$

where $\mathfrak{g}^{\mu\mu} = \mathfrak{g}^{00} + \mathfrak{g}^{kk}$. We thus still need $\underset{7}{\mathfrak{g}}^{\mu\mu}{}_{,k}$. One has, from (7)

$$\begin{cases} \underset{7}{\mathfrak{h}}^{00} = 4\varepsilon \sum \left[\underset{4}{\dot{m}} + \dfrac{1}{3\,!}\,\dddot{(\underset{2}{m}R^2)} + \dfrac{1}{5\,!}\,\overset{\cdot\cdot\cdot\cdot\cdot}{(mR^4)} + 2mv^k\underset{2}{a}^k - \dfrac{m}{3}\,R^k\underset{2}{\dot{\alpha}}^k\right], \\[2ex] \underset{7}{\mathfrak{h}}^{kl} = 4\varepsilon \sum \left[m(v^k\underset{2}{a}^l + v^l\underset{2}{a}^k) + (\underset{2}{\dot{m}}v^kv^l) + \dfrac{m}{3\,!}\,\dddot{(v^kv^lR^2)}\right]. \end{cases}$$

Moreover, one has, from (6)

$$\begin{cases} \nabla^2\underset{7}{\mathfrak{S}}^{00} = \underset{7}{\Theta}^{00} + \underset{5}{\ddot{\mathfrak{S}}}^{00} = \underset{5}{\mathfrak{g}}^{kl}\underset{2}{\mathfrak{g}}^{00}{}_{,kl} - 14\varepsilon \sum' \overset{A}{m}\overset{B}{m}\left(\dddot{\dfrac{1}{D}}\right), \\[2ex] \nabla^2\underset{7}{\mathfrak{S}}^{kl} = \underset{5}{\ddot{\mathfrak{S}}}^{kl} = -\,2\varepsilon \sum' \overset{A}{m}\overset{B}{m}\left(\dddot{\dfrac{D^kD^l}{D^3}}\right), \end{cases}$$

whence

$$(8)\qquad \nabla^2 \underset{7}{\mathfrak{s}}^{\mu\mu} = \underset{5}{\mathfrak{g}}^{kl} \underset{2}{\mathfrak{g}}^{00}{}_{,kl} - 16\varepsilon \sum{}' \overset{A}{m}\overset{B}{m} \left(\dddot{\frac{1}{D}}\right).$$

From the rules of Section **4**, we must also know the asymptotic behavior of $\underset{6}{\mathfrak{s}}^{\mu\mu}$ in order to solve (8). One has, from (6)

$$(9)\qquad \nabla^2 \underset{6}{\mathfrak{s}}^{\mu\mu} = \underset{6}{\Theta}^{\mu\mu} + \frac{\partial^2}{\partial t^2}(\underset{4}{\mathfrak{s}}^{00} + \underset{4}{\mathfrak{s}}^{kk}) =$$

$$= \underset{6}{\Theta}^{\mu\mu} + 8\frac{\partial^2}{\partial t^2}\left\{\sum \frac{m^2}{R^2} + \sum_{A,B}{}' \left(\frac{1}{\overset{A}{R}\overset{B}{R}} - \frac{1}{D\overset{A}{R}} - \frac{1}{D\overset{B}{R}}\right)\right\}.$$

Here, $\underset{6}{\Theta}^{\mu\mu}$ is an extremely cumbersome expression. It is sufficient for our purpose to know that it falls off, at large distances, like $R^{-4}$, and therefore its contribution to $\underset{6}{\mathfrak{s}}^{\mu\mu}$ falls off as $f(t)/R$. This produces a term like $\varepsilon f'(t)$ in $\underset{7}{\mathfrak{s}}^{\mu\mu}$, which does not concern us, however, because only the spatial derivatives of this expression are needed. The curled bracket in (9) gives a contribution to $\underset{6}{\mathfrak{s}}^{\mu\mu}$:

$$8\frac{\partial^2}{\partial t^2}\left\{\sum m^2 \log R + \sum{}' \overset{A}{m}\overset{B}{m}\left[\log(\overset{A}{R} + \overset{B}{R} + D) - \frac{\overset{A}{R} + \overset{B}{R}}{D}\right]\right\}.$$

Unless all $\dot{D} = 0$, this expression behaves at large distances like $R$, and there corresponds to it, in $\underset{7}{\mathfrak{s}}^{\mu\mu}$, a term behaving as $R^2$. This can readily be verified from the solution of (8):

$$\underset{7}{\mathfrak{s}}^{\mu\mu} = 2\left(\underset{5}{\mathfrak{g}}^{ll}\sum\frac{m}{R} - \underset{5}{\mathfrak{g}}^{kl}\sum\frac{mR^kR^l}{R^3}\right) - 4\varepsilon\sum{}'\overset{A}{m}\overset{B}{m}\left(\dddot{\frac{\overset{A}{R^2} + \overset{B}{R^2}}{D}}\right) + \varepsilon f'(t).$$

All the quantities needed to compute $\underset{5}{a}^k$ are thus known.

## 7. - Radiated energy.

We define the radiated energy, per unit time, as the rate of work of the particles against their own radiation field:

$$U = -\sum m\, \underset{5}{a}^k v^k.$$

For the sake of simplicity, we shall restrict ourselves to the case of two particles revolving on circular orbits (in the Newtonian approximation) at a distance $D$ from each other. In this case, $\underset{2}{\dot{m}}$, $\underset{5}{\mathfrak{g}}^{00}$ adn $\underset{5}{\mathfrak{g}}^{kk}$ vanish. Next we choose, for convenience, a system of co-ordinates where the center of mass is at rest: $\sum mv^k = 0$. As a consequence, we can neglect all the spatial constants which appear in the first four terms of $\underset{5}{a}^k$, as they give equal contributions to the accelerations of both particles, and will therefore cancel in the computation of $U$. Thus the only relevant term of $\underset{7}{\mathfrak{h}}^{\mu\mu}$ is $4\varepsilon \sum m \overset{\cdot\cdot\cdot\cdot\cdot}{R^4}/5!$ and the only relevant term of $-\underset{6}{\mathfrak{g}}^{0k}{}_{,0}$ is $-4\varepsilon \sum m(v^k \overset{\cdot\cdot\cdot\cdot}{R^2})/3!$ Moreover it can be shown ([8]) that the contribution from $\underset{7}{\mathfrak{s}}^{\mu\mu}$ exactly cancels the fourth term of $\underset{5}{a}^k$. One thus gets

$$\underset{5}{a}^k \approx -v^l \underset{5}{\mathfrak{g}}^{kl}{}_{,0} + \varepsilon \sum m \left[\frac{2}{3}(v^k \overset{\cdot\cdot\cdot\cdot}{R^2}) + \frac{1}{30}(R^2 \overset{\cdot\cdot\cdot\cdot\cdot}{R^k})\right],$$

where the $\approx$ sign recalls that some spatial constants have been omitted, and where the right hand member has to be computed at the position of the particle under consideration. A straightforward computation gives

$$\underset{5}{a}^k \approx \frac{\varepsilon}{30} \frac{mv^k}{D^4}(m+M)(461M - 269m),$$

where $M$ is the mass of the other particle. It follows that

$$U = -\varepsilon \frac{32}{5} \frac{m^2 M^2 (m+M)}{D^5}. \tag{10}$$

For purely outgoing waves ($\varepsilon = -1$), this agrees with the result of the linearized theory ([1]). (The fact that one has previously obtained a negative radiated energy should be ascribed to the presence of incoming gravitational waves, which were absorbed by the particles).

The physical reality of this phenomenon can be confirmed by computing the rate of change of the total mass. One finds ([16])

$$\sum \underset{7}{\dot{m}} = -U,$$

as one should expect.

## 8. – Behavior of the field at very large distances.

Henceforth we take, for the sake of simplicity, $m = M$. The evolution of the system is ruled by the equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(mv^2 - \dfrac{m^2}{D}\right) = -U = -\dfrac{64}{5}\dfrac{m^5}{D^5}, \\ \dfrac{2mv^2}{D} = \dfrac{m^2}{D^2}, \end{cases}$$

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([16]) A. PERES: *Nuovo Cimento*, **13**, 439 (1959).

whence

$$D^4 = \frac{512}{5} m^3(T-t) . \tag{11}$$

$T$ is the time at which radiative capture would occur, if the last equations were rigorously correct. Actually, all the previous theory is valid only for low velocities, *i.e.* as long as $D \gg m$. For very small $D$, capture occurs even if radiative effects are discarded (17).

From (10) and (11), one gets

$$U(t) = \frac{1}{80}\left(\frac{\frac{5}{2}m}{T-t}\right)^{\frac{5}{4}}.$$

It follows that, at a distance $R$, the average energy flux and energy density of the gravitational waves are about

$$\bar{\sigma} = \frac{U(t-R)}{4\pi R^2} = \frac{1}{320\pi R^2}\left(\frac{\frac{5}{2}m}{T-t+R}\right)^{\frac{5}{4}}. \tag{12}$$

For $R \ll T-t = 5D^4/512\ \mathrm{m}^3$, this varies as $R^{-2}$, but for very large $R$,

$$R > \frac{D^4}{m^3},$$

one has

$$\bar{\sigma} \approx \frac{(\frac{5}{2}\, m)^{\frac{5}{4}}}{320\pi} R^{-13/4} .$$

As $\bar{\sigma}$ is proportional to the square of the amplitude of the radiation field, it follows that the latter falls as $R^{-13/8}$.

On the other hand, the energy density $\bar{\sigma}$ produces an additional field (18), the potential of which varies, at large distances, like $R^{-5/4}$, *i.e.* more slowly than the amplitude of the radiation field. This additional field, however, is quasi-static, as shown by (12): its time and space derivatives behave like $R^{-9/4}$, *i.e.*, they fall off faster than those of the radiation field. Therefore the energy density of this additional field can be neglected with respect to $\bar{\sigma}$.

Let us summarize: at very large distances ($R > D^4/m^3$), the field contains a static part behaving as $(m/R)$, a quasi-static part behaving as $(m/R)^{5/4}$ and a dynamic part behaving as $(m/R)^{13/8}$. (This result does not contradict a

(17) C. Darwin: *Proc. Roy. Soc.*, A **249**, 180 (1959).
(18) A. Peres and N. Rosen: *Phys. Rev.* **115**, 1085 (1959).

theorem of Papapetrou ([19]) according to which the field can be asymptotically Euclidean at infinity only if it is a static both for $t \to +\infty$ and $t \to -\infty$, because the validity of this theorem is limited to the cases where there exists some constant radius $R_0$ such that space is empty for $R > R_0$ at all times $t$. In our case, however, it is easy to see from (11) that no such $R_0$ exists.)

The fact that the energy density of the gravitational field behaves at large distances like $R^{-13/4}$ has the important consequence that the total energy of the gravitational field is bounded. (It would not be bounded if the energy density behaved like $R^{-2}$, or even $R^{-3}$.) It is therefore possible to define a total energy of the particles and the field. This is constant, because the well-known conservation theorem

$$[\mathfrak{g}(\mathfrak{T}^{\mu\nu} + \mathfrak{t}^{\mu\nu})]_{,\nu} = 0$$

can be written

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int \mathfrak{g}(\mathfrak{T}^{00} + \mathfrak{t}^{00})\,\mathrm{d}V = \int \mathfrak{g}\mathfrak{t}^{0k}\mathrm{d}S_k\,, \tag{13}$$

and the right hand member behaves like $R^{-5/4}$ for very large distances ($R > D^4/m^3$) and thus asymptotically vanishes at infinity. In fact, by considering the situation when the particles were very far apart, one gets

$$\int_{\text{all space}} \mathfrak{g}(\mathfrak{T}^{00} + \mathfrak{t}^{00})\,\mathrm{d}V = \sum m\,.$$

This fact was incorrectly interpreted by INFELD ([4]) as a proof that gravitational radiation does not exist. Actually, only the sum $\int \mathfrak{g}\mathfrak{T}^{00}\mathrm{d}V + \int \mathfrak{g}\mathfrak{t}^{00}\mathrm{d}V$ is constant, while energy is constantly pumped from the matter to the radiation field. Indeed, if we limit the domain of integration to the wave zone, *i.e.*, to distances such that

$$\left(\frac{D^3}{m}\right)^{\frac{1}{2}} \ll R \ll \frac{D^4}{m^3}\,, \tag{14}$$

both members of (13) are positive and approximately independent of $R$, as will be shown in the next section.

All the previous arguments are valid only in the case of retarded potentials (outgoing waves, $\varepsilon = -1$). In the case of half-retarded, half-advanced potentials (standing waves, $\varepsilon = 0$), no energy is radiated and the motion is truly periodic. The field then diverges logarithmically at spatial infinity ([18,20]).

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([19]) A. PAPAPETROU: *Ann. Phys.*, **2**, 87 (1958).
([20]) A. PAPAPETROU: *Ann. Phys.*, **20**, 399 (1957); **1**, 186 (1958).

## 9. – Validity of the linear approximation.

We already know that the linear approximation is valid neither at very small distances from the sources (of the order of the Schwarzschild radius), nor at very large distances ($R > D^4/m^3$) where the quasi-static field caused by $\bar{\sigma}$ is more important than the radiation field. Its domain of validity, if it exists, must therefore be limited to the wave zone, defined by (14).

It is therefore important to examine in detail how the linearized theory can nevertheless correctly give the radiated energy. We here follow the proof of LANDAU and LIFSHITZ [1]. The « solution » of (1) is

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + 4\int \frac{\{\mathfrak{g}\mathfrak{T}^{\mu\nu} - (1/16\pi)\Theta^{\mu\nu}\}}{r}\,\mathrm{d}V,$$

where the curled bracket has to be computed at a retarded time. Let

$$\tau^{\mu\nu} = \mathfrak{g}\mathfrak{T}^{\mu\nu} - \frac{1}{16\pi}\Theta^{\mu\nu}.$$

For large $R$, one can write

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \frac{4}{R}\left\{\int \tau^{\mu\nu}\mathrm{d}V\right\}.$$

Moreover, it follows from (1) and the harmonic condition that $\tau^{\mu\nu}{}_{,\nu} = 0$, whence

$$\tau^{00}{}_{,0} = -\tau^{0k}{}_{,k}, \qquad \tau^{k0}{}_{,0} = -\tau^{kl}{}_{,l}.$$

Let us multiply the first of these equations by $x^l$ and integrate over a large sphere of radius $R$. One gets

$$\text{(15)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}\int \tau^{00}x^l\,\mathrm{d}V = -\int \tau^{0k}{}_{,k}x^l\,\mathrm{d}V = \int \tau^{0l}\mathrm{d}V - \int (\tau^{0k}x^l)_{,k}\,\mathrm{d}V.$$

We shall now show that the last integral can be neglected. One has

$$\int (\tau^{0k}x^l)_{,k}\,\mathrm{d}V = \int \tau^{0k}x^l\,\mathrm{d}S_k.$$

Now $\tau^{0k}$ varies as $R^{-2}$ (in the wave zone) so that it seems that this expression diverges as $R$. This is not the case, however, as the following arguments shows:

it is possible to consider, instead of an infinite wave train, a short pulse of gravitational waves [21]. One can take the surface integral at a very large distance, so that the pulse has still not reached it. It then vanishes, and the volume integral vanishes also. The other integrals in eq. (15), however, remain finite.

Let us note that this argument is correct only if there is a possibility of dealing with linear wave trains, *i.e.* those that can be arbitrarily decomposed into independent pulses. In reality, there is an interaction between the various parts of a wave train, due to the non-linearity of the Einsteins equations. The previous argument is therefore valid only inasmuch one can neglect the gravitational influence of the energy density of the waves. It is therefore correct in the wave zone, but not for $R > D^4/m^3$.

Thus, in the wave zone

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \tau^{00} x^l \,\mathrm{d}V = \int \tau^{0l}\,\mathrm{d}V .$$

In a similar fashion, one can show that

$$\int \tau^{kl}\,\mathrm{d}V = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int \tau^{00} x^k x^l \,\mathrm{d}V ,$$

whence

$$\mathfrak{g}^{kl} - \eta^{kl} = \frac{2}{R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int \tau^{00} x^k x^l \,\mathrm{d}V = \frac{2}{R}\sum m\,(\ddot{\xi^k \xi^l}) . \tag{16}$$

This is indeed equal to the limit of $\underset{4}{\mathfrak{g}}^{kl}$, for large $R$, as found in Section **3**. The only difference is that the right hand member is now to be computed at time $(t - R)$, rather than $t$. (Henceforth, it will always be understood that sources have to be considered at a retarded time.)

## 10. – The wave zone.

The wave zone is characterized, in the case of outgoing waves, by

$$\frac{\partial}{\partial t} + \frac{\partial}{\partial r} \approx \frac{\partial}{r\,\partial\theta} \approx \frac{\partial}{r\,\partial\varphi} \approx 0\,(r^{-2}) .$$

Let

$$-n^k = n_k = R^k/R , \qquad -n^0 = -n_0 = 1 ,$$

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[21] D. Geissler, A. Papapetrou and H. Treder: *Ann. Phys.*, **2**, 344 (1959).

whence

$$n^\mu n_\mu = 0 . \tag{17}$$

One can write [22]

$$\mathfrak{g}^{\mu\nu}{}_{,\sigma} = a^{\mu\nu} n_\sigma / R , \tag{18}$$

where $a^{\mu\nu}$ may depend on $n_k$, but is almost independent of $R$ (except through the retardation of time).

The harmonic condition now reads

$$a^{\mu\nu} n_\nu = 0 , \tag{19}$$

whence

$$a^{0k} = a^{kl} n_l , \tag{20}$$

and

$$a^{00} = a^{0k} n_k = a^{kl} n_k n_l . \tag{21}$$

The energy-momentum pseudotensor of the gravitational field is given by [18]

$$\begin{aligned} 16\pi \mathfrak{g} \mathfrak{t}^{\mu\nu} = \mathfrak{g}^{\alpha\beta}{}_{,\gamma} \mathfrak{g}^{\pi\varrho}{}_{,\tau} [\mathfrak{g}^{\mu\nu} (\tfrac{1}{2} \delta^\gamma_\pi \delta^\tau_\alpha \mathfrak{g}_{\beta\varrho} - \tfrac{1}{4} \mathfrak{g}^{\gamma\tau} \mathfrak{g}_{\alpha\pi} \mathfrak{g}_{\beta\varrho} + \tfrac{1}{8} \mathfrak{g}^{\gamma\tau} \mathfrak{g}_{\alpha\beta} \mathfrak{g}_{\pi\varrho}) + \\ + \delta^\mu_\alpha \delta^\nu_\pi \mathfrak{g}^{\gamma\tau} \mathfrak{g}_{\beta\varrho} + \delta^\mu_\alpha \delta^\nu_\beta \delta^\gamma_\pi \delta^\tau_\varrho - \delta^\mu_\alpha \delta^\nu_\pi \delta^\gamma_\beta \delta^\tau_\varrho - \delta^\mu_\alpha \delta^\gamma_\pi \mathfrak{g}^{\nu\tau} \mathfrak{g}_{\beta\varrho} - \\ - \delta^\nu_\pi \delta^\tau_\alpha \mathfrak{g}^{\mu\gamma} \mathfrak{g}_{\beta\varrho} + \tfrac{1}{2} \mathfrak{g}^{\mu\gamma} \mathfrak{g}^{\nu\tau} \mathfrak{g}_{\alpha\pi} \mathfrak{g}_{\beta\varrho} - \tfrac{1}{4} \mathfrak{g}^{\mu\gamma} \mathfrak{g}^{\nu\tau} \mathfrak{g}_{\alpha\beta} \mathfrak{g}_{\pi\varrho}] . \end{aligned}$$

With the help of (17) and (18), one gets, in the linear approximation

$$t^{\mu\nu} = \sigma n^\mu n^\nu ,$$

where

$$\sigma = \frac{1}{64\pi R^2} (2\eta_{\alpha\pi} \eta_{\beta\varrho} - \eta_{\alpha\beta} \eta_{\pi\varrho}) a^{\alpha\beta} a^{\pi\varrho} . \tag{22}$$

Introducing

$$b^{kl} = a^{kl} - \tfrac{1}{3} \delta^{kl} a^{nn} ,$$

one gets, with the help of (20) and (21)

$$\sigma = \frac{1}{64\pi R^2} (2 b^{kl} b^{kl} - 4 b^{kp} b^{kq} n_p n_q + b^{kl} b^{pq} n_k n_l n_p n_q) .$$

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([22]) A. TRAUTMAN: *Bull. Acad. Polon. Sc.*, **6**, 407 (1958).

Moreover, one has [23]

$$\overline{n_p n_q} = \tfrac{1}{3}\,\delta_{pq}\,,$$

$$\overline{n_k\, n_l\, n_p\, n_q} = \frac{1}{15}\,(\delta_{kl}\delta_{pq} + \delta_{kp}\delta_{lq} + \delta_{kq}\delta_{lp})\,,$$

whence

$$\bar{\sigma} = \frac{1}{80\pi R^2}\, b^{kl}\, b^{kl}\,.$$

This is always a positive quantity. It follows

$$U = \int t^{0k}\,\mathrm{d}S_k = \int \sigma n_k\,\mathrm{d}S_k = 4\pi R^2 \bar{\sigma} = \frac{1}{20}\, b^{kl}\, b^{kl}\,.$$

In the case of free particles, one has, from (16) and (18)

$$a^{kl} = -2\sum m(\xi^k \dddot{\xi}^l)\,, \tag{23}$$

whence

$$U = \frac{1}{45}\left[\sum m\,(3\xi^k\xi^l - \delta^{kl}\xi^n\xi^n)^{\dddot{}}\right]^2.$$

This is the usual formula for gravitational radiation [1]. As it has been *independently* checked both for free particles (Section **7**) and for constrained motion [24], we can say that *the existence of gravitational radiation is now well established.*

From (20), (21) and (23), it follows that the form of $\mathfrak{g}^{00}$ and $\mathfrak{g}^{0k}$, in the wave zone, is [25]:

$$\mathfrak{g}^{00} = 1 + \frac{4}{R}\sum m + \frac{2}{R}\, n_k n_l \sum m\,(\xi^k \ddot{\xi}^l)\,,$$

$$\mathfrak{g}^{0k} = \frac{4}{R}\sum m v^k + \frac{2}{R}\, n_l \sum m\,(\xi^k \ddot{\xi}^l)\,.$$

When we compare these formulae with the asymptotic values of $\underset{4}{\mathfrak{g}}^{00}$ and $\underset{5}{\mathfrak{g}}^{0k}$ that are given in Section **4**, we find that they do not agree. This fact should not surprise us, because we have used in the first sections of this paper an

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([23]) L. Landau and E. Lifshitz: loc. cit., p. 206.

([24]) W. B. Bonnor: *Nature*, **181**, 1196 (1958); *Phil. Trans. Roy. Soc.*, A **251**, 233 (1959); A. Peres and N. Rosen: *Ann. Phys.* (to be published).

([25]) V. A. Fock: loc. cit., p. 417.

approximation method based on the assumption that time derivatives are much smaller than space derivatives. This is indeed correct in the vicinity of the particles, but in the wave zone, time derivatives are of the same order as radial space derivatives, and much larger than tangential space derivatives. The various orders of approximation are therefore completely different: quantities of the $n$-th order in one region contain also contributions from all lower orders in the other region, and no direct comparison can be made. The agreement for $\mathfrak{g}^{kl}$, that we have found in the previous section, should be ascribed to the fact that in both methods we were dealing with the lowest approximation to $\mathfrak{g}^{kl}$.

## 11. – Uniqueness of $\sigma$.

FOCK has shown ([9]) that the harmonic system of co-ordinates behaving at infinity as pure outgoing waves is unique (up to a Lorentz transformation). We have found, however, that the dominant term at very large distances has not the form of outgoing waves.

We now intend to show that it is sufficient to require that the system be harmonic and behave as outgoing waves in the wave zone (but not necessarily near the sources nor at infinity) in order to get a unique value for $\sigma$.

We thus assume only (18) and (19). Under an infinitesimal co-ordinate transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$, the $\mathfrak{g}^{\mu\nu}$ transform according to

$$\mathfrak{g}'^{\mu\nu} = \mathfrak{g}^{\mu\nu} + \xi^{\mu,\nu} + \xi^{\nu,\mu} - \eta^{\mu\nu}\xi^{\sigma}{}_{,\sigma} .$$

As the $\mathfrak{g}'^{\mu\nu}$ are also to behave as outgoing waves, one must have

$$\xi^{\alpha}{}_{,\beta} = \frac{c^{\alpha} n_{\beta}}{R} ,$$

where the $c^{\alpha}$ themselves satisfy

$$c^{\alpha}{}_{,\beta} = f^{\alpha} n_{\beta} .$$

Thus

$$a'^{\mu\nu} = a^{\mu\nu} + f^{\mu} n^{\nu} + f^{\nu} n^{\mu} - \eta^{\mu\nu} f^{\sigma} n_{\sigma} . \tag{24}$$

It follows that $a'^{\mu\nu} n_{\nu} = 0$, *i.e.* the new system is also harmonic.

Let us now go back to eq. (22) which defines $\sigma$. A straightforward computation shows that

$$\sigma' = \frac{1}{64\pi R^2} (2\eta_{\alpha\pi}\eta_{\beta\varrho} - \eta_{\alpha\beta}\eta_{\pi\varrho}) a'^{\alpha\beta} a'^{\pi\varrho} ,$$

is exactly equal to $\sigma$. This could be foreseen by noting that (24) can be written as

$$a'^{\mu\nu} = a^{\mu\nu} + c^{\mu,\nu} + c^{\nu,\mu} - \eta^{\mu\nu} c^{\sigma}{}_{,\sigma} ,$$

*i.e.* $a_{\mu\nu}$ behaves as a tensor density would under a transformation $x'^{\mu} = x^{\mu} + c^{\mu}$.

One may also define $\sigma$ with the help of the Einstein pseudotensor [26]

$$\mathfrak{t}^{\mu}_{\nu} = \frac{1}{16\pi}\left(\delta^{\mu}_{\nu}\mathfrak{L} - \mathfrak{g}^{\alpha\beta}{}_{,\nu}\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}{}_{,\mu}}\right),$$

where

$$\mathfrak{L} = \mathfrak{g}^{\alpha\beta}{}_{,\gamma}\,\mathfrak{g}^{\pi\varrho}{}_{,\tau}\left(\frac{1}{2}\,\delta^{\gamma}_{\varrho}\delta^{\tau}_{\beta}\,\mathfrak{g}_{\alpha\pi} - \frac{1}{4}\,\mathfrak{g}^{\gamma\tau}\,\mathfrak{g}_{\alpha\pi}\,\mathfrak{g}_{\beta\varrho} + \frac{1}{8}\,\mathfrak{g}^{\gamma\tau}\,\mathfrak{g}_{\alpha\beta}\,\mathfrak{g}_{\pi\varrho}\right).$$

Now

$$\frac{\partial\mathfrak{L}}{\partial\mathfrak{g}^{\alpha\beta}{}_{,\mu}} = -\frac{1}{2}\,\mathfrak{g}^{\pi\varrho}{}_{,\tau}\left[\mathfrak{g}^{\mu\tau}\left(\mathfrak{g}_{\alpha\pi}\,\mathfrak{g}_{\beta\varrho} - \frac{1}{2}\,\mathfrak{g}_{\alpha\beta}\,\mathfrak{g}_{\pi\varrho}\right) - \delta^{\tau}_{\alpha}\delta^{\mu}_{\pi}\,\mathfrak{g}_{\beta\varrho} - \delta^{\tau}_{\beta}\delta^{\mu}_{\pi}\,\mathfrak{g}_{\alpha\varrho}\right],$$

whence

$$\mathfrak{t}^{\mu}_{\nu} = \frac{1}{16\pi}\,\mathfrak{g}^{\alpha\beta}{}_{,\gamma}\,\mathfrak{g}^{\pi\varrho}{}_{,\tau}\left[\delta^{\mu}_{\nu}\left(\frac{1}{2}\,\delta^{\gamma}_{\varrho}\delta^{\tau}_{\beta}\,\mathfrak{g}_{\alpha\pi} - \frac{1}{4}\,\mathfrak{g}^{\gamma\tau}\,\mathfrak{g}_{\alpha\pi}\,\mathfrak{g}_{\beta\varrho} + \frac{1}{8}\,\mathfrak{g}^{\gamma\tau}\,\mathfrak{g}_{\alpha\beta}\,\mathfrak{g}_{\pi\varrho}\right) + \right.$$
$$\left. + \frac{1}{2}\,\delta^{\gamma}_{\nu}\left(\mathfrak{g}^{\mu\tau}\,\mathfrak{g}_{\alpha\pi}\,\mathfrak{g}_{\beta\varrho} - \frac{1}{2}\,\mathfrak{g}^{\mu\tau}\,\mathfrak{g}_{\alpha\beta}\,\mathfrak{g}_{\pi\varrho} - \delta^{\tau}_{\alpha}\delta^{\mu}_{\pi}\,\mathfrak{g}_{\beta\varrho} - \delta^{\tau}_{\beta}\delta^{\mu}_{\pi}\,\mathfrak{g}_{\alpha\varrho}\right)\right].$$

With the help of (17) and (18), one gets, in the linear approximation

$$\mathfrak{t}^{\mu}_{\nu} = \frac{1}{64\pi R^2}\,n^{\mu} n_{\nu}(2\eta_{\alpha\pi}\eta_{\beta\varrho} - \eta_{\alpha\beta}\eta_{\pi\varrho})\,a^{\alpha\beta}a^{\pi\varrho}.$$

We thus get exactly the same result as with the $t^{\mu\nu}$ of Landau and Lifshitz.

At last, let us note that a very simple *exact* formula for $\sigma$ can be obtained from

$$16\pi\mathfrak{g}\mathfrak{t}^{00} = (\mathfrak{g}^{\alpha\beta}\mathfrak{g}^{00} - \mathfrak{g}^{0\alpha}\mathfrak{g}^{0\beta})_{,\alpha\beta},$$

which holds *in vacuo* ($\mathfrak{T}^{\mu 0} = 0$). This can be written

(25) $$16\pi\mathfrak{g}\,\mathfrak{t}^{00} = \left[(-g)\,g^{00}\left(g^{kl} - \frac{g^{0k}g^{0l}}{g^{00}}\right)\right]_{,kl}.$$

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([26]) R. C. TOLMAN: *Relativity Thermodynamics and Cosmology* (Oxford, 1934), p. 224.
([27]) P. A. M. DIRAC: *Proc. Roy. Soc.*, A **246**, 333 (1958).

However

$$\left(g^{kl}-\frac{g^{0k}g^{0l}}{g^{00}}\right)g_{lm}=\delta^l_m\,,$$

and

$$(-g)g^{00}=-\operatorname{Det}(g_{kl})\,.$$

Therefore the square bracket in (25) is simply minus the co-factor of $g_{kl}$ in the determinant of the $g_{mn}$:

$$16\pi\mathfrak{g}t^{00}=-\tfrac{1}{2}\eta^{ijk}\eta^{lmn}(g_{jm}g_{kn})_{,il}\,.$$

One thus sees that only the six $g_{kl}$ give a contribution to the energy density of the gravitational field. This is related to the fact that they are canonical variables, while the four $g_{0\mu}$ are not [27].

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## RIASSUNTO (*)

Si espone un metodo ad approssimazioni successive per risolvere le equazioni di Einstein per un sistema di particelle polari che gravitano liberamente. Si dimostra che si può scegliere la soluzione che rappresenta soltanto onde uscenti. Si trova che la correzione di 5° ordine all'accelerazione comporta un termine non conservativo: l'energia si perde, per radiazione gravitazionale, in quantità esattamente uguale a quella predetta della teoria linearizzata. Questo può essere dimostrato calcolando la perdita di massa del sistema. Poi si passa ad esaminare la validità della teoria linearizzata: si dimostra che non può descrivere correttamente il campo a distanze molto grandi dall'origine, ma ciononostante da un risultato corretto per l'energia irradiata.

(*) *Traduzione a cura della Redazione.*