The Nuclear Thomas-Fermi Model and the Optical Potential for Single-Channel Reactions.

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Summary. — A new approach to the description of the nuclear interaction of nucleons with finite even-even nuclei is outlined. The radial and energy dependence of the optical potential describing the interaction of nucleons with ^{4}He, ^{12}C and ^{16}O is extensively investigated.

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1. – Introduction.

As remarked by SINHA [1] «the early sixties in the history of the optical potential unfortunately turned out like a blind man's survey of the parameter jungle and quite a lot of the work, although useful as a straightforward guide-line for the prediction of scattering data, did not serve any fruitful purpose in furthering understanding of the underlying theoretical concepts». The situation is now only moderately improved and fundamental dark corners still exist, which need exploring. The goal of setting up a physically reliable, analytically simple and numerically manageable theoretical tool for constructing the optical potential has been missed, mainly because of the ambitious aim of developing formally rigorous treatments which, unfortunately, are not practicable. A stringent critical analysis of the optical-model predic-

(1) B. SINHA: *Phys. Rep. C*, **20**, 1 (1975), p. 3.

tions cannot be carried out because most of the available results have been obtained by means of approximate calculations which are far from being transparent and are not controllable numerically. None of the innumerable investigations on the scattering of nucleons from nuclei [2] has so far been conceived with the aim of giving a clear-cut answer to the following questions: *a*) How can an optical model be constructed with the hope of being a correct model? *b*) How can one avoid putting tremendous efforts into calculations on empirical models which have no connection with reality? *c*) How can the experimental data be used to discard erroneous models and enable the research to converge towards the correct one?

As a first step towards the goal of giving an answer to these questions, we shall outline an unconventional approach to the problem of constructing optical potential $V(r, E)$ for finite even-even nuclei. Our program is to deduce the radial and energy dependence of $V(r, E)$ consistently with the stability of the target nucleus and in such a way that the potential be exploitable for incident nucleons of whatever energy E. The potential is assumed to be the sum of a real and an imaginary part, *i.e.*

$$V(r, E) = V_{\mathrm{R}}(r, E) + iV_{\mathrm{I}}(r, E)\,; \tag{1.1}$$

both potentials $V_{\mathrm{R}}(r, E)$ and $V_{\mathrm{I}}(r, E)$ can be split into a central and a spin-orbit contribution

$$V_{\mathrm{R}}(r, E) = V_{\mathrm{R}}^{\mathrm{C}}(r, E) + V_{\mathrm{R}}^{\mathrm{SO}}(r, E)\,, \tag{1.2a}$$

$$V_{\mathrm{I}}(r, E) = V_{\mathrm{I}}^{\mathrm{C}}(r, E) + V_{\mathrm{I}}^{\mathrm{SO}}(r, E)\,. \tag{1.2b}$$

We generalize Fermi's conjecture [3] and define the spin-orbit contributions to the optical potential as Thomas terms, namely

$$V_{\mathrm{R}}^{\mathrm{SO}}(r, E) = (\lambda_{\mathrm{R}}^2/r)(\boldsymbol{\sigma}\cdot\boldsymbol{L})\{\mathrm{d}V_{\mathrm{R}}^{\mathrm{C}}(r, E)/\mathrm{d}r\}\,, \tag{1.3a}$$

$$V_{\mathrm{I}}^{\mathrm{SO}}(r, E) = (\lambda_{\mathrm{I}}^2/r)(\boldsymbol{\sigma}\cdot\boldsymbol{L})\{\mathrm{d}V_{\mathrm{I}}^{\mathrm{C}}(r, E)/\mathrm{d}r\}\,, \tag{1.3b}$$

where lengths λ_{R} and λ_{I} have to be determined by fitting the scattering data.

The only experimental data used to evaluate $V(r, E)$ are the binding energy and the r.m.s. radius of the target nucleus and the observed energy dependence of the total neutron-proton cross-section. Such minimal experimental informa-

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(2) P. E. HODGSON: *Nuclear Reactions and Nuclear Structure* (Clarendon Press, Oxford, 1971); see also P. E. HODGSON: *Annu. Rev. Nucl. Sci.*, **17**, 1 (1967); *Nuovo Cimento A*, **81**, 250 (1984).

(3) E. FERMI: *Nuovo Cimento*, **11**, 407 (1954).

tion is sufficient for predicting the radial and energy dependence of $V(r, E)$: the achievement of this goal is made possible primarily by the extension to finite even-even nuclei of the differential equation of infinite nuclear matter, discussed in a previous paper (4).

2. – Extension of the differential equation of infinite nuclear matter to finite even-even nuclei.

2·1. *The differential equation of finite nuclear matter.* – It has been shown in I, sect. **5**, that the potential energy $v_\infty(p, \varkappa)$ of a nucleon having momentum p in the infinitely extended Fermi sea obeys the hyperbolic partial differential equation

$$\left\{\frac{\partial^2}{\partial p^2} + \frac{2}{p}\frac{\partial}{\partial p} - \frac{\partial^2}{\partial \varkappa^2} + \frac{2}{\varkappa}\frac{\partial}{\partial \varkappa}\right\} v_\infty(p, \varkappa) = 0 , \tag{2.1}$$

where the limiting momentum $\varkappa$ is defined as

$$\varkappa = (3\pi^2 A \varrho_0/2)^{\frac{1}{3}} = (9\pi/8)^{\frac{1}{3}}(A^{\frac{1}{3}}/R_\infty) . \tag{2.2}$$

The total energy of the nucleus is $(p \leqslant \varkappa)$

$$W_\infty(\varkappa) = (3A/4\pi\varkappa^3)\int\{(p^2/2M) + \tfrac{1}{2} v_\infty(p, \varkappa)\}\, \mathrm{d}\boldsymbol{p} ; \tag{2.3}$$

the saturation properties of nuclear forces require that

$$\{\mathrm{d}W_\infty(\varkappa)/\mathrm{d}\varkappa\}_{\varkappa=\varkappa_\mathrm{F}} = 0 , \qquad W_\infty(\varkappa_\mathrm{F}) = b_\mathrm{V} A , \tag{2.4}$$

where b_V is the average volume energy and $\varkappa_\mathrm{F}$ is the Fermi momentum obtained by putting in eq. (2.2) $R_\infty = r_0 A^{\frac{1}{3}}$. Equation (2.1) expresses in differential form the analytical constraints of the single-particle potential energy arising from the total antisymmetry of the nuclear wave function. It is remarkable that all particular integrals $v_\infty(p, \varkappa)$, specified according to saturation prescriptions (2.4), exactly fulfil the relation

$$b_\mathrm{V} = \varepsilon_\mathrm{F} + v_\infty(\varkappa_\mathrm{F}, \varkappa_\mathrm{F}) = w_\infty(\varkappa_\mathrm{F}, \varkappa_\mathrm{F}) , \tag{2.5}$$

where $\varepsilon_\mathrm{F} = \varkappa_\mathrm{F}^2/2M$ is the Fermi energy and $w_\infty(p, \varkappa)$ is the single-particle total

(4) C. VILLI: *Nuovo Cimento* A, **74**, 37 (1983); hereafter this paper will be referred to as I.

energy. Relation (2.5) expresses the Hugenholtz and Van Hove theorem [5]: at the density minimizing the total energy of infinite nuclear matter, the average volume energy is equal to the total energy of the most energetic nucleon.

The following remarks will better explain the role of eq. (2.1) in nuclear-matter problems. Let us represent the ground state of a nucleus by a totally antisymmetrized wave function Φ, built up in determinantal form from A individual functions $\Psi_m = \psi_j \omega_\mu$ for single particles, and assume that the nucleons, described by space, spin and isobaric spin co-ordinates, interact *via* two-body forces through the potential $v(1,2) = \boldsymbol{O}_{12} f(r_{12})$, whose exchange properties and radial behaviour will be left completely unspecified. Then, if the nuclear interaction is treated as a perturbation and the $A/4$ spatial wave functions $\psi_{p_j}(\boldsymbol{r}_m) = \Omega^{-\frac{1}{2}}(\varkappa) \exp[i\boldsymbol{p}_j \cdot \boldsymbol{r}_m]$ are associated in turn with all four spin and isobaric-spin wave functions ω_μ, the expectation value $V(\varkappa)$ of the potential energy of the nucleus is given by the second term appearing in eq. (2.3), where ($p \equiv p_j$, $q \equiv p_i$)

$$v_\infty(p, \varkappa) = \sum_{i=1}^{\frac{1}{4}(A-1)} [c_1 \langle ij|f|ij\rangle - c_2 \langle ij|f|ji\rangle] , \tag{2.6a}$$

$\langle ij|f|ij\rangle$ and $\langle ij|f|ji\rangle$ being the ordinary and exchange matrix elements of $f(r_{12})$, *i.e.*

$$\langle ij|f|ij\rangle \simeq \varkappa^3 \int f(r_{12})\, \mathrm{d}\boldsymbol{r}_1\, \mathrm{d}\boldsymbol{r}_2 , \tag{2.6b}$$

$$\langle ij|f|ji\rangle \simeq \int \psi_p^*(\boldsymbol{r}_1)\, \psi_q^*(\boldsymbol{r}_2)\, f(r_{12})\, \psi_q(\boldsymbol{r}_1)\, \psi_p(\boldsymbol{r}_2)\, \mathrm{d}\boldsymbol{r}_1\, \mathrm{d}\boldsymbol{r}_2 ; \tag{2.6c}$$

the two constants c_1 and c_2 are given by

$$c_1 = \sum_{\mu,\nu=1}^{4} \langle \mu\nu|\boldsymbol{O}_{12}|\mu\nu\rangle , \qquad c_2 = \sum_{\mu,\nu=1}^{4} \langle \mu\nu|\boldsymbol{O}_{12}|\nu\mu\rangle . \tag{2.6d}$$

Equation (2.6*a*) is easily worked out and finally yields ($p \leqslant \varkappa$; $r \equiv r_{12}$)

$$v_\infty(p, \varkappa) = \varkappa^3 \int_0^\infty r^2 [c_1' - c_2'(\varkappa r)^{-1} j_0(pr)\, j_1(\varkappa r)]\, f(r)\, \mathrm{d}r , \tag{2.7a}$$

$$c_1' = 2(A-1)\, c_1/3\pi^2 A , \qquad c_2' = (A-1)\, c_2/4\pi A , \tag{2.7b}$$

$j_0(x)$ and $j_1(x)$ being spherical Bessel functions. The single-particle potential energy (2.7) is a particular integral of eq. (2.1). A different way to evaluate $v_\infty(p, \varkappa)$ is based on an ingenious modification of the Hartree-Fock theory,

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(5) N. M. HUGENHOLTZ and L. VAN HOVE: *Physica*, **24**, 363 (1958).

namely in replacing potential v by the t-matrix, and then expressing $\langle ij|t|ij\rangle$ and $\langle ij|t|ji\rangle$ in terms of nucleon-nucleon scattering amplitudes. In this case, instead of eq. (2.7), one has for $p \leqslant \varkappa$

$$v_\infty(p, \varkappa) = -\frac{4}{\pi M}\int_0^{\frac{1}{2}(\varkappa-p)} F(q) q^2 \, dq - \frac{1}{2\pi M p}\int_{\frac{1}{2}(\varkappa-p)}^{\frac{1}{2}(\varkappa+p)} F(q)[\varkappa^2 - (2q-p)^2] q \, dq \,, \tag{2.8}$$

where $F(q)$ is a complicated function of the nucleon momentum, expressed as a sum of scattering amplitudes, classified according to spin and isobaric-spin substates of the two-nucleon system ([6]). No reliable results can be obtained from eq. (2.8) because of the insufficient knowledge of the nucleon-nucleon asymptotic phase shifts. It is, nevertheless, worthwhile pointing out that the single-particle potential (2.8) is a particular integral of eq. (2.1).

It is seen that the momentum dependence of potential (2.7) is entirely brought about by the total antisymmetry of the nuclear wave function, whereas that of potential (2.8) arises from energy-momentum conservation under the constraints imposed by the Pauli principle. A different source of momentum dependence has to be searched for in the nucleon-nucleon correlations existing in nuclear matter. The presence of a nucleon at point $\boldsymbol{r}$ influences the probability of finding another nucleon at a point $\boldsymbol{r}'$ in the neighbourhood of $\boldsymbol{r}$. Although in the neighbourhood of a nucleon there may be considerable fluctuations in the density of other particles, the nuclear-matter distribution preserves its homogeneous character. This situation, however, affects the energy of a nucleon at $\boldsymbol{r}$ so that the energy operator is no longer diagonal in co-ordinate space. Consequently, the nucleon-nucleon correlations can be conceived of as being due to a nonlocal nucleon-nucleon potential $u(\boldsymbol{r}, \boldsymbol{r}')$. The Schrödinger equation is

$$-(1/2M)\nabla^2\psi(\boldsymbol{r}) + \int u(|\boldsymbol{r}-\boldsymbol{r}'|)\,\psi(\boldsymbol{r}')\,d\boldsymbol{r}' = E\psi(\boldsymbol{r}) \,. \tag{2.9}$$

Using plane waves of constant momentum p, the effective two-nucleon potential, evaluated as Fourier transform of $u(|\boldsymbol{r}-\boldsymbol{r}'|)$, reads

$$V(\boldsymbol{r}, \boldsymbol{p}) = \int u(|\boldsymbol{r}-\boldsymbol{r}'|)\exp[i\boldsymbol{p}\cdot(\boldsymbol{r}-\boldsymbol{r}')]\,d\boldsymbol{r}' \,. \tag{2.10}$$

Let us assume a separable form for $u(|\boldsymbol{r}-\boldsymbol{r}'|)$, *i.e.*

$$u(|\boldsymbol{r}-\boldsymbol{r}'|) = u_0(\tfrac{1}{2}|\boldsymbol{r}+\boldsymbol{r}'|)\,G(|\boldsymbol{r}-\boldsymbol{r}'|) \,, \tag{2.11a}$$

([6]) K. A. BRUECKNER, C. A. LEVINSON and H. M. MAHMUD: *Phys. Rev.*, **95**, 219 (1954); K. A. BRUECKNER: *Phys. Rev.*, **96**, 508 (1954); N. FUKUDA and R. G. NEWTON: *Phys. Rev.*, **103**, 1558 (1956).

where $G(|\boldsymbol{r}-\boldsymbol{r}'|)$ is the Gaussian function normalized to 1,

$$\text{(2.11}b\text{)} \qquad \begin{cases} G(|\boldsymbol{r}-\boldsymbol{r}'|) = \{1/\pi\sqrt{\pi}\, a_0^3\} \exp\left[-|\boldsymbol{r}-\boldsymbol{r}'|^2/a_0^2\right], \\ \int G(r)\, \mathrm{d}\boldsymbol{r} = 1, \end{cases}$$

a_0 being the nonlocality length; at the limit $a_0 \to 0$ $G(|\boldsymbol{r}-\boldsymbol{r}'|)$ operates under integral sign as the delta-function $\delta(\boldsymbol{r}-\boldsymbol{r}')$: in this case potential V becomes local and the momentum dependence disappears. Let us now identify V with the two-body potential used to deduce eq. (2.7). The effect of the nonlocality ($a_0 \neq 0$) modifies the momentum dependence of the exchange matrix element $\langle ij|V|ji\rangle$ and generates a p-dependence also of the ordinary matrix element $\langle ij|V|ij\rangle$. It is found that

$$\text{(2.12)} \qquad \begin{cases} v_\infty(p, \varkappa) = \varkappa^3 \int_0^\infty r^2 [c_1' g_1(r, p;\ a_0) - c_2' g_2(r, p;\ a_0)]\, \mathrm{d}r, \\ g_1(r, p;\ 0) = f(r), \qquad g_2(r, p;\ 0) = (\varkappa r)^{-1} j_0(pr)\, j_1(\varkappa r)\, f(r); \end{cases}$$

the explicit expressions of $g_i(r, p;\ a_0)$ $(i = 1, 2)$ will not be given. It is nevertheless clear that the resulting momentum dependence of $v_\infty(p, \varkappa)$ is a combined effect of the total antisymmetry of the nuclear wave function and of the nonlocal nature of the two-body potential. The single-particle potential energy (2.12) is a particular solution of eq. (2.1). In conclusion, eq. (2.1) accounts for the momentum dependence of the single-particle potential without any specification of the two-body forces involved, nor of the type of nucleon-nucleon correlations (clusters of three or more nucleons), nor of the effective role played by the intrinsic or apparent nonlocality of the two-body potential.

A more refined theory of infinitely extended nuclear matter is based on a detailed description of nucleon-nucleon scattering in the Fermi sea. The difference in behaviour inside and outside nuclear matter is due to the exclusion principle. When two nucleons of arbitrary initial momenta collide as an isolated pair, the momentum of the final state may be divided between the particles in many ways, with probabilities determined solely by the inter-nucleon potential. In nuclear matter the number of possible final states is greatly reduced since so many of the states available for isolated nucleons are already occupied by other nucleons. Hence, the scattering in nuclear matter is greatly different from that *in vacuo*. Clearly, in nuclear matter only nucleon-nucleon collisions are allowed for which the final state of both particles have momenta above the Fermi level: starting from two levels below Fermi's, such collisions would defy the conservation of energy. Hence, there is no real scattering and the asymptotic wave functions contain no phase shifts. However, the possibility of a virtual scattering state outside the Fermi sea

has the effect of distorting the wave function at distances of the order of the Fermi wave length (corresponding approximately to the value of the non-locality parameter a_0). The effect of nucleon-nucleon interactions by causing transitions from occupied states into occupied ones is to spread the momentum distribution of the single nucleons in the neighbourhood of the limiting momentum $\varkappa = \varkappa_F$, calculated at equilibrium density, namely by lowering the density in momentum space below $\varkappa_F$ and giving rise to a tail above this value. In spite of this, most of the scattering effects due to the internucleon potential are eliminated: as is well known, this is the intimate reason why shell model works. Brueckner's theory provides the mathematical tool for handling such a complicated problem. The exact solutions of the nonlinear system of equations, expressing the reaction matrix, are unknown and the approximations used to obtain numerical results give rise to complications and ambiguities which outweigh the heuristic value of the theory. Serious mathematical complexities discourage one from going into the details of the self-consistency problem. It will suffice to point out that the density dependence of the reaction matrix gives rise to the so-called re-arrangement energy, which constitutes an additional contribution to the single-particle potential (7). Consequently, the Hugenholtz and Van Hove theorem is violated. Since the validity of this theorem is strictly related to eq. (2.1), which predicts that the re-arrangement energy is equal to zero, we conclude that $v_\infty(p, \varkappa)$, calculated according to Brueckner's theory, does not fulfil eq. (2.1). So far no clear-cut computational evidence exists for sharing the opinion according to which the nonzero magnitude of the re-arrangement energy is a characteristic many-body effect which manifests itself through high-order effects in the reaction matrix. Indeed, it can be shown that the appearance of re-arrangement energies is strictly bound to the criterion adopted in the application of variational methods, which—in turn—influences the choice of the single-particle potential. One might suspect either that the re-arrangement energy is a sort of «ghost» energy created by mathematical procedures (and/or approximations) or that it would probably disappear in an exact formulation of Brueckner's theory. Anyway, eq. (2.1) seems to offer a powerful guide to an overall description of infinitely extended nuclear matter.

With this theoretical scheme in mind, we have been tempted to extend the validity of the mathematical structure of eq. (2.1) to finite even-even nuclei by replacing the limiting momentum $\varkappa$ with the function $\chi(r)$, obtained by the self-explanatory generalization of definition (2.2) to nuclei characterized by a nonuniform density distribution $\varrho(r)$ (normalized to 1), *i.e.*

$$\chi(r) = \{(3\pi^2/2) A \varrho(r)\}^{\frac{1}{3}} ; \tag{2.13}$$

(7) K. A. BRUECKNER: *Phys. Rev.*, **110**, 597 (1958).

definition (2.13) is valid only for even-even nuclei ($A = 2Z = 2N$). The new differential equation is

$$\left\{\frac{\partial^2}{\partial p^2} + \frac{2}{p}\frac{\partial}{\partial p} - \frac{\partial^2}{\partial \chi(r)^2} + \frac{2}{\chi(r)}\frac{\partial}{\partial \chi(r)}\right\} v\{p, \chi(r)\} = 0 . \tag{2.14}$$

Equation (2.14) is the spring board for a substantial improvement of the Thomas-Fermi model. Let us define $\mathrm{d}\mathcal{N}(\boldsymbol{r}, \boldsymbol{p})$ as the number of nucleons with momentum $\boldsymbol{p}$ at point $\boldsymbol{r}$ in a small volume element of phase space. Then, the basic assumption is that all states are filled equally as long as a certain momentum $\chi(r)$ is not exceeded, after which they are all empty.

Assuming $\hbar = c = 1$, one has

$$\begin{cases} \mathrm{d}\mathcal{N}(\boldsymbol{r}, \boldsymbol{p}) = 2\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{p}/(2\pi)^3 , & p \leqslant \chi(r) ; \\ \mathrm{d}\mathcal{N}(\boldsymbol{r}, \boldsymbol{p}) = 0 , & p > \chi(r) ; \end{cases} \tag{2.15}$$

factor 2 comes from the fact that the number of states is doubled by the presence of the spin. By integrating (2.15) over $\boldsymbol{p}$ one obtains eq. (2.13), which is the fundamental relationship between nuclear density $\varrho(r)$ and local maximum momentum $\chi(r)$.

The Hartree-Fock approach to finite nuclei is extremely difficult [8]. The short-range correlations have to be treated by a ladder summation through the introduction of a reaction matrix conceived of as a functional of the local density. The semi-classical approximation of such a functional in powers of the density gradient turns out to be disastrous: the shell structure of the nucleus is lost! Calculating a self-consistent single-particle potential is based upon multi-step iterations which require enormously long computer times and make a systematic investigation practically impossible. This discouraging situation stimulates one to ascertain whether eq. (2.14) is capable, at least for finite even-even nuclei, of playing the « steering » role played by eq. (2.1) for infinite nuclear matter.

2·2. *The density distribution of even-even nuclei with* $A \leqslant 16$. – Several empirical forms of density function $\varrho(r)$ have been considered in the literature [9].

([8]) P. HOHENBERG and W. KOHN: *Phys. Rev. B*, **136**, 864 (1966); L. I. SHAM and W. KOHN: *Phys. Rev.*, **145**, 561 (1966); K. A. BRUECKNER, J. L. GAMMEL and H. WEITZNER: *Phys. Rev.*, **110**, 431 (1958); K. T. R. DAVIES, S. J. KRIEGER and M. BERANGER: *Nucl. Phys.*, **84**, 545 (1966); J. P. SWENNE, A. K. KERMAN and F. VILLARS: *Phys. Rev.*, **147**, 710 (1966).

([9]) K. A. BRUECKNER, J. R. BUCHLER, S. JORNA and R. LOMBARD: *Phys. Rev.*, **171**, 1188 (1968). See also C. W. DE JAGER, H. DE VRIES and C. DE VRIES: *At. Data Nucl. Data Tables*, **14**, 479 (1974).

Neglecting the small differences between the radial distribution of protons and neutrons, the simplest analytic form of $\varrho(r)$ is

$$\varrho(r) = \sum_P \int \sum_j \Psi^*(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_A)\, \delta(\boldsymbol{r} - \boldsymbol{r}_j)\, \Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_A)\, \mathrm{d}\boldsymbol{r}_1 \ldots \mathrm{d}\boldsymbol{r}_A , \tag{2.16}$$

where $\Psi(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_A)$ is the totally antisymmetrized ground-state nuclear wave function expressed as a Slater determinant and $\sum_P$ indicates the summation over all permutations. Assuming that the ground states of the even-even nuclei with $A \leqslant 16$ can be adequately described by the lowest shell model configuration, it can be shown quite generally that $\varrho(r)$ is independent of the mode of coupling in the shell model, and depends only on the shape of the common potential well. Assuming an infinite parabolic well, density function (2.16) becomes

$$\begin{cases} \varrho(r) = \varrho_0\{1 + \eta(r/s)^2\} \exp\left[-(r/s)^2\right], \\ \varrho_0 = 2/\pi\sqrt{\pi}(2 + 3\eta)\, s^3 , \end{cases} \tag{2.17a}$$

$$\int \varrho(r)\, \mathrm{d}\boldsymbol{r} = 1 ; \tag{2.17b}$$

the r.m.s. radius is

$$R \equiv R(s, \eta) = s\{3(2 + 5\eta)/2(2 + 3\eta)\}^{\frac{1}{2}} . \tag{2.17c}$$

Information on the shape of the central potential, on the strength of the spin-orbit coupling and on the type, shape and strength of the residual two-nucleon interaction is concealed in the two unknown parameters s and η; in particular, length s is related to the curvature of the well and to the energy interval between two successive levels of the harmonic oscillator. Parameters s and η will be determined according to the stability prescriptions of the considered even-even nuclei: it will be shown that this goal can be achieved provided one also puts simultaneously into play optical-model requirements and consistency constraints arising from the theory of infinitely extended nuclear matter.

The analytic form (2.17) is used for describing the charge density of nuclei with two protons in the s-shell and $Z - 2$ protons in the p-shell [10]: in this case, using appropriate harmonic well wave functions, it is found that

$$\eta = (Z - 2)/3 . \tag{2.18}$$

Remarkable fits of the elastic-scattering data of high-energy electrons from

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[10] R. L. B. Elton: *Nuclear Sizes* (Oxford University Press, Oxford, 1961).

^{4}He, ^{12}C and ^{16}O have been obtained ([11]) using a charge distribution having the form (2.17), namely

$$\begin{cases} \varrho_{ch}(r) = (\varrho_0)_{ch}\{1 + \eta_{ch}(r/s_{ch})^2\} \exp[-(r/s_{ch})^2], \\ (\varrho_0)_{ch} = 2/\pi\sqrt{\pi}(2 + 3\eta_{ch}) s_{ch}^3, \end{cases} \tag{2.19a}$$

$$\int \varrho_{ch}(r)\, d\boldsymbol{r} = 1, \tag{2.19b}$$

$$R_{ch}^2 = s_{ch}\{3(2 + 5\eta_{ch})/2(2 + 3\eta_{ch})\}^{\frac{1}{2}}. \tag{2.19c}$$

The capability of density (2.19) is strictly dependent on the parameter

$$\eta_{ch} = \tfrac{1}{3}(Z - 2); \tag{2.20}$$

which gives $\eta_{ch} = 0$ for ^{4}He, $\eta_{ch} = \frac{4}{3}$ for ^{12}C and $\eta_{ch} = 2$ for ^{16}O. For our purposes it is essential to clarify the relation existing between nuclear density (2.17) and charge density (2.19), namely between the set of parameters (s, η) and (s_{ch}, η_{ch}). A clear-cut starting point is the following: *a*) density (2.17) expresses the radial distribution of the centre of mass of the nucleons (and—in particular—of the protons) bound in the nuclear system; *b*) the incident electron feels the combined electromagnetic effects arising both from the nonuniform centre-of-mass distribution of the protons and from the individual charge density distribution $\varrho_p(r)$ characterizing each proton. Consequently, the charge distribution $\varrho_{ch}(r)$ used in the analyses of the electron-nucleus scattering data is related to $\varrho_p(r)$ and $\varrho(r)$ through the folding integral

$$\varrho_{ch}(r) = \int \varrho_p(|\boldsymbol{r} - \boldsymbol{x}|)\, \varrho(x)\, dx. \tag{2.21}$$

Let us introduce the vector $\boldsymbol{X} = \boldsymbol{r} - \boldsymbol{x}$; taking into account that

$$|r - x| \leqslant X \leqslant r + x, \tag{2.22}$$

equation (2.21) becomes

$$\begin{cases} \varrho_{ch}(r) = \dfrac{2\pi}{r}\left[\displaystyle\int_0^r x I_1(x, r)\varrho(x)\, dx + \int_r^\infty x I_2(x, r)\varrho(x)\, dx\right], \\ I_1(x, r) = \displaystyle\int_{r-x}^{r+x} X\varrho_p(X)\, dX, \qquad I_2(x, r) = \int_{x-r}^{x+r} X\varrho_p(X)\, dX. \end{cases} \tag{2.23}$$

([11]) H. F. EHRENBERG, R. HOFSTADTER, U. MAYER-BERKHOUT, D. G. RAVENHALL and S. SOBOTKA: *Phys. Rev.*, **113**, 666 (1959). See also R. HOFSTADTER: *Annu. Rev. Nucl. Sci.*, **7**, 231 (1957).

In order to perform transparent calculations, we resort to the Gaussian proton model which, although inadequate in accounting for the electron-proton scattering data, nevertheless gives suitable qualitative results of general validity. The considered proton model is expressed by the charge density normalized to 1

$$\begin{cases} \varrho_p(r) = (1/\pi\sqrt{\pi}\, s_p^3) \exp[-(r/s_p)^2], \\ \int \varrho_p(r)\, \mathrm{d}\boldsymbol{r} = 1 ; \end{cases} \tag{2.24a}$$

the r.m.s. radius is

$$R_p = s_p \sqrt{\tfrac{3}{2}} . \tag{2.24b}$$

Equation (2.23) becomes

$$\begin{cases} \varrho_{ch}(r) = (2/\sqrt{\pi}\, s_p r)\, F(r) \exp[-(r/s_p)^2], \\ F(r) = \int_0^\infty x \sinh(2rx/s_p^2)\, \varrho(x) \exp[-(x/s_p)^2]\, \mathrm{d}x . \end{cases} \tag{2.25}$$

Density (2.19) is obtained from eq. (2.25) assuming for $\varrho(r)$ the form (2.17); the parameters characterizing $\varrho_{ch}(r)$ are found to be

$$\begin{cases} (\varrho_0)_{ch} = \{2 + 3(s_p/s_{ch})^2\eta\}/\pi\sqrt{\pi}(2 + 3\eta)\, s_{ch}^3 , \\ s_{ch}^2 = s^2 + s_p^2 , \qquad \eta_{ch} = [2(s/s_{ch})^2/\{2 + 3(s_p/s_{ch})^2\}]\eta . \end{cases} \tag{2.26}$$

It is readily ascertained that the following equality holds:

$$R_{ch}^2 = R^2 + R_p^2 . \tag{2.27}$$

The form factor corresponding to density (2.19) is

$$\mathscr{F}(q^2) = [1 - \{\eta_{ch}/2(2 + 3\eta_{ch})\} s_{ch}^2 q^2] \exp[-(s_{ch}\, q/2)^2] , \tag{2.28}$$

where $q \equiv q(E, \theta)$ is the relativistic momentum transfer. The angular position of the zero of $\mathscr{F}(q^2)$ is a surprisingly close guide to the narrow diffraction minimum of the differential electron-nucleus cross-section. The fit of the angular position provides an accurate determination of parameters s_{ch} and η_{ch}. The fit is improved by taking into account that the origin of the co-ordinates is the centre of mass of the nuclear system and not the centre of the well as is assumed in the usual shell model treatment; the correction is

$$s_{ch}^2 = \{(A-1)/A\} s^2 + s_p^2 . \tag{2.29}$$

The validity of eq. (2.20) (concerning finite-size protons) and *not* that of eq. (2.18) (concerning pointlike nucleons) possesses strong experimental evidence. It is astonishing that this circumstance has never been specifically stressed. For our purposes it is sufficient to have given

$$s < s_{\mathrm{ch}} , \qquad \eta > \eta_{\mathrm{ch}} . \tag{2.30}$$

A deeper insight into density function (2.17) will be given in subsect. **9**·2.

2·3. *An ansatz on the nuclear radius.* – In order to make as much use as possible of nuclear-matter results in describing finite nuclei, we adopt the following *ansatz*: the r.m.s. radius $R(s, \eta)$ of a nonuniform density distribution $\varrho(r)$, calculated for an even-even nucleus at the minimum of the total energy of the nucleus ground state, is equal to the length $R_\infty = r_0 A^{\frac{1}{3}}$ characterizing the infinite nuclear matter at the minimum $W_\infty(\varkappa_{\mathrm{F}}) = b_{\mathrm{v}} A$, *i.e.*

$$R(s, \eta) = r_0 A^{\frac{1}{3}} ; \tag{2.31}$$

from eq. (2.31) one expresses s as a function of η, namely

$$s = r_0 A^{\frac{1}{3}} [2(2 + 3\eta)/3(2 + 5\eta)]^{\frac{1}{2}} . \tag{2.32}$$

The empirical « nuclear radius » $R_{00} = r_{00} A^{\frac{1}{3}}$, crudely determined in the framework of phenomenological theories, is related to the r.m.s. radius $R(s, \eta)$ by the relation

$$r_{00} = (\tfrac{5}{3})^{\frac{1}{2}} r_0 . \tag{2.33}$$

Ansatz (2.31) does not contradict the approximation intrinsic to the nuclear Thomas-Fermi model, namely that the energy and density dependence is locally the same as that of a homogeneous medium in its ground state: of course, this is equivalent to assuming that the correlations between the nucleons in finite nuclei are the same as in nuclear matter. Should *ansatz* (2.31) be groundless, then the results achieved through nuclear-matter calculations ought to be considered not only heuristically meaningless but also conceptually misleading. This is, in fact, not the case! By means of the hydrodynamical effective-mass approximation, introduced in subsect. **5**·4, we shall pour into the theory of finite nuclei some crucial information obtained from the overall description of infinitely extended nuclear matter, based upon the differential equation (2.1): this will make it possible to determine the nuclear parameters s and η and then to construct the optical potential.

3. – The momentum-density dependence of the single-particle potential energy.

3·1. *The single-particle potential deduced by series integration.* – We shall seek for solutions of eq. (2.14) by expanding $v\{p, \chi(r)\}$ in power series of p, *i.e.*

$$v\{p, \chi(r)\} = \sum_{n=0}^{\infty} \mathscr{A}_n\{\chi(r)\} p^n , \tag{3.1}$$

where $\mathscr{A}_n\{\chi(r)\}$ are unknown functions of the local maximum momentum $\chi(r)$. Following the procedure outlined in I, it is found that

$$\mathscr{A}_1\{\chi(r)\} = 0 , \tag{3.2}$$

and that only even powers of the nucleon momentum p appear in expansion (3.1). The general integrals of eq. (2.14) in N approximation read

$$v^{(N)}\{p, \chi(r)\} = \sum_{n=0}^{N} \mathscr{B}_n^{(N)}\{\chi(r)\} p^{2n} , \tag{3.3}$$

where all terms corresponding to $n > N$ are assumed to be zero, *i.e.*

$$\mathscr{B}_{N+\nu}^{(N)}\{\chi(r)\} = 0 \tag{3.4}$$

for $\nu = 1, 2, \ldots$. Functions $\mathscr{B}_n^{(N)}\{\chi(r)\}$ satisfy the recurrence relations

$$(2n+2)(2n+3)\mathscr{B}_{n+1}^{(N)}\{\chi(r)\} = \mathfrak{D}\mathscr{B}_n^{(N)}\{\chi(r)\} , \tag{3.5}$$

where $\mathfrak{D}$ is the differential operator

$$\mathfrak{D} = \partial_{\chi\chi}^2 - \{2/\chi(r)\}\partial_\chi . \tag{3.6}$$

The determination of functions $\mathscr{B}_n^{(N)}\{\chi(r)\}$ is carried out by solving the system of differential equations

$$\begin{cases} 6\mathscr{B}_1^{(N)}\{\chi(r)\} = \mathfrak{D}\mathscr{B}_0^{(N)}\{\chi(r)\} , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\ 2N(2N+1)\mathscr{B}_N^{(N)}\{\chi(r)\} = \mathfrak{D}\mathscr{B}_{N-1}^{(N)}\{\chi(r)\} , \qquad 0 = \mathfrak{D}\mathscr{B}_N^{(N)}\{\chi(r)\} . \end{cases} \tag{3.7}$$

The solutions of the system are

$$(3.8)\qquad \begin{cases} \mathscr{B}_N^{(N)}\{\chi(r)\} = \mathscr{C}_{2N+1}^{(N)}\chi^3(r) + \mathscr{C}_{2N}^{(N)}, \\ \mathscr{B}_{N-1}^{(N)}\{\chi(r)\} = N(2N+1)[\tfrac{1}{5}\mathscr{C}_{2N+1}^{(N)}\chi^5(r) - \mathscr{C}_{2N}^{(N)}\chi^3(r)] + \tfrac{1}{3}\mathscr{C}_{2N-1}^{(N)}\chi^3(r) + \mathscr{C}_{2N-2}^{(N)}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \end{cases}$$

where $\mathscr{C}_{2N+1}^{(N)}$ are $2N+1$ arbitrary integration constants. Any physically significant solution of eq. (2.14) must fulfil for $p \geqslant 0$ the condition

$$(3.9)\qquad v\{p, \chi(\infty)\} = 0\ ;$$

it follows that

$$(3.10)\qquad \mathscr{C}_{2N}^{(N)} = \mathscr{C}_{2N-2}^{(N)} = \dots = 0\ .$$

It is worthwhile to remark that function $v^{(N)}\{p, \chi(r)\}$ depends only on the odd powers of the local maximum momentum $\chi(r)$.

The $N = 0$ approximation is not significant. Function $\mathscr{B}_0^{(0)}\{\chi(r)\}$ obeys the differential equation

$$(3.11a)\qquad \mathfrak{D}\mathscr{B}_0^{(0)}\{\chi(r)\} = 0\ ;$$

the momentum dependence of the single-particle potential is lost, *i.e.*

$$(3.11b)\qquad v^{(0)}\{p, \chi(r)\} = \tfrac{1}{3}\mathscr{C}_1^{(0)}\chi^3(r) \propto \varrho(r)\ .$$

The $N = 1$ approximation is obtained by solving the system of differential equations

$$(3.12)\qquad \mathfrak{D}\mathscr{B}_0^{(1)}\{\chi(r)\} = 6\mathscr{B}_1^{(1)}\{\chi(r)\}, \qquad \mathfrak{D}\mathscr{B}_1^{(1)}\{\chi(r)\} = 0\ ;$$

it is found that

$$(3.13a)\qquad v^{(1)}\{p, \chi(r)\} = \mathscr{B}_0^{(1)}\{\chi(r)\} + \mathscr{B}_1^{(1)}\{\chi(r)\}p^2\ ,$$

$$(3.13b)\qquad \begin{cases} \mathscr{B}_0^{(1)}\{\chi(r)\} = \tfrac{3}{5}\mathscr{C}_3^{(1)}\chi^5(r) + \mathscr{C}_1^{(1)}\chi^3(r)\ , \\ \mathscr{B}_1^{(1)}\{\chi(r)\} = \mathscr{C}_3^{(1)}\chi^3(r)\ ; \end{cases}$$

note that an unessential factor $\frac{1}{3}$ has been included in the integration constants $\mathscr{C}_3^{(1)}$ and $\mathscr{C}_1^{(1)}$. Equations (3.13) express the radial effective-mass approximation;

the nucleon effective mass $M^*\{\chi(r)\}$ is readily found to be

$$M/M^*\{\chi(r)\} = 1 + 2M\mathscr{B}_1^{(1)}\{\chi(r)\} . \tag{3.14}$$

The conjectured dependence of $M^*\{\chi(r)\}$ on the square of $\chi(r)$ is wrong [12].

A more realistic mathematical model, at least for $p \leqslant \chi(0)$, is provided by the $N = 2$ approximation: we shall call it « model I ». It is based upon the system of differential equations

$$\begin{cases} \mathfrak{D}\mathscr{B}_0^{(2)}\{\chi(r)\} = 6\mathscr{B}_1^{(2)}\{\chi(r)\} , \\ \mathfrak{D}\mathscr{B}_1^{(2)}\{\chi(r)\} = 20\mathscr{B}_2^{(2)}\{\chi(r)\} , \\ \mathfrak{D}\mathscr{B}_2^{(2)}\{\chi(r)\} = 0 . \end{cases} \tag{3.15}$$

The analytic expression of the single-particle potential provided by model I is

$$v^{(2)}\{p, \chi(r)\} = \mathscr{B}_0^{(2)}\{\chi(r)\} + \mathscr{B}_1^{(2)}\{\chi(r)\}p^2 + \mathscr{B}_2^{(2)}\{\chi(r)\}p^4 , \tag{3.16a}$$

$$\begin{cases} \mathscr{B}_0^{(2)}\{\chi(r)\} = \frac{3}{7}\mathscr{C}_5^{(2)}\chi^7(r) + \frac{3}{5}\mathscr{C}_3^{(2)}\chi^5(r) + \mathscr{C}_1^{(2)}\chi^3(r) , \\ \mathscr{B}_1^{(2)}\{\chi(r)\} = 2\mathscr{C}_5^{(2)}\chi^5(r) + \mathscr{C}_3^{(2)}\chi^3(r) , \\ \mathscr{B}_2^{(2)}\{\chi(r)\} = \mathscr{C}_5^{(2)}\chi^3(r) , \end{cases} \tag{3.16b}$$

where an unessential factor $\frac{1}{3}$ has been included in the integration constants $\mathscr{C}_5^{(2)}$ and $\mathscr{C}_3^{(2)}$. Note that the effective-mass approximation is formally obtained by putting in eqs. (3.16)

$$\mathscr{C}_5^{(2)} = 0 . \tag{3.17}$$

The integration constants $\mathscr{C}_1^{(2)}$, $\mathscr{C}_3^{(2)}$ and $\mathscr{C}_5^{(2)}$ must be determined consistently with the stability prescriptions of the considered even-even nucleus.

3·2. *The single-particle potential deduced by variable separation.* – Let us now seek for solutions of eq. (2.14) using the method of variable separation. To this end we substitute in eq. (2.14) the single-particle potential energy expressed in separable form

$$v\{p, \chi(r)\} = v_1(p)v_2\{\chi(r)\} . \tag{3.18}$$

Following the procedure developed in I, sect. **6**, it is readily ascertained that

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[12] G. L. SHAW: *Ann. Phys.* (*N. Y.*), **8**, 500 (1959).

the only physically meaningful differential equations are

$$[p^{-2}\partial_p\{p^2\partial_p\} + \alpha^2]v_1(p) = 0\,, \qquad p \leqslant \chi(0)\,, \tag{3.19a}$$

$$[\chi^2(r)\partial_\chi\{\chi^{-2}(r)\partial_\chi\} + \alpha^2]v_2\{\chi(r)\} = 0\,, \qquad p \leqslant \chi(0)\,, \tag{3.19b}$$

$$[p^{-2}\partial_p\{p^2\partial_p\} - \beta^2]v_1(p) = 0\,, \qquad p \geqslant \chi(0)\,, \tag{3.20a}$$

$$[\chi^2(r)\partial_\chi\{\chi^{-2}(r)\partial_\chi\} - \beta^2]v_2\{\chi(r)\} = 0\,, \qquad p \geqslant \chi(0)\,, \tag{3.20b}$$

where α and β are real constants. The solutions of eqs. (3.19*a*) and (3.20*a*), satisfying the conditions

$$v_1(0) \text{ finite}\,, \qquad v_1(\infty) = 0\,, \tag{3.21}$$

are

$$v_1(p) = c_1\{\sin(\alpha p)/p\}\,, \qquad p \leqslant \chi(0)\,, \tag{3.22a}$$

$$v_1(p) = c_3\beta Y(\beta p)\,, \qquad p \geqslant \chi(0)\,, \tag{3.22b}$$

where $Y(\beta p) = \exp[-\beta p]/\beta p$ is a Yukawa function in momentum space and c_1 and c_3 are unknown constants. Function $v_2\{\chi(r)\}$ must fulfil the physically obvious condition

$$v_2\{\chi(\infty)\} = 0\,. \tag{3.23}$$

To solve eqs. (3.19*b*) and (3.20*b*) we perform the transformation

$$v_2\{\chi(r)\} = \chi(r)\,\theta\{\chi(r)\}\,; \tag{3.24}$$

the following differential equations are thus found:

$$[\partial^2_{\chi\chi} + \alpha^2 - 2\chi^{-2}(r)]\theta\{\chi(r)\} = 0\,, \qquad p \leqslant \chi(0)\,, \tag{3.25a}$$

$$[\partial^2_{\chi\chi} - \beta^2 - 2\chi^{-2}(r)]\theta\{\chi(r)\} = 0\,, \qquad p \geqslant \chi(0)\,. \tag{3.25b}$$

The solution of Bessel's equation (3.25*a*) is well known, the solution of eq. (3.25*b*) has been obtained by series integration. The final results are

$$\theta\{\chi(r)\} = c_2[j_0\{\alpha\chi(r)\} - \cos\{\alpha\chi(r)\}]\,, \qquad p \leqslant \chi(0)\,, \tag{3.26a}$$

$$\theta\{\chi(r)\} = c_4[\chi^2(r) + (\beta^2/10)\chi^4(r) + (\beta^4/280)\chi^6(r)]\,, \qquad p \geqslant \chi(0)\,, \tag{3.26b}$$

where $j_0(x) = \sin x/x$ is the zeroth-order spherical Bessel function and c_2 and c_4 are integration constants; function (3.26*b*) satisfies eq. (3.25*b*) up to terms in $\chi^4(r)$. In conclusion, the factorable single-particle potential (3.18)

turns out to be

$$v\{p, \chi(r)\} = C_1\alpha\{\alpha\chi(r)\}^2 j_0(\alpha p)\, j_1\{\alpha\chi(r)\}\,, \qquad p \leqslant \chi(0)\,, \tag{3.27a}$$

$$v\{p, \chi(r)\} = C_2\beta\{\chi^3(r) + (\beta^2/10)\,\chi^5(r) + (\beta^4/280)\,\chi^7(r)\}\,Y(\beta p)\,, \qquad p \geqslant \chi(0)\,, \tag{3.27b}$$

where $j_1(x) = \{\sin x/x^2\} - \{\cos x/x\}$ is the first-order spherical Bessel function, $C_1 = c_1 c_2$ and $C_2 = c_3 c_4$.

Functions (3.27) must be linked by continuity at $p = \chi(0)$ together with their first-order derivatives calculated with respect to momentum p; consequently, the two unknown parameters C_2 and β can be expressed as functions of the other two parameters C_1 and α. It is found that

$$\beta\chi(0) = -\alpha\chi(0)\,\mathrm{ctg}\,\{\alpha\chi(0)\}\,, \tag{3.28a}$$

$$C_2 = \frac{280 C_1\alpha^3 j_0\{\alpha\chi(0)\}\, j_1\{\alpha\chi(0)\}\exp[\beta\chi(0)]}{280 + 28\beta^2\chi^2(0) + \beta^4\chi^4(0)}\,. \tag{3.28b}$$

Since $\beta\chi(0) \geqslant 0$ because $v\{\infty, \chi(r)\} = 0$, parameter $\alpha\chi(0)$ has to be searched for within the interval

$$\pi/2 \leqslant \alpha\chi(0) \leqslant \pi\,. \tag{3.29}$$

For $\alpha\chi(0) = \pi/2$ ($\beta = 0$) potential (3.27) behaves for $p > \chi(0)$ as a Coulomb function in momentum space, whereas for $\alpha\chi(0) = \pi$ in ($\beta = \infty$) it appears to be a rounded-off square well momentum space. Taking into account eqs. (3.28), potential (3.27) reads

$$v\{p, \chi(r)\} = C_1\alpha\{\alpha\chi(r)\}^2 j_0(\alpha p)\, j_1\{\alpha\chi(r)\}\,, \qquad p \leqslant \chi(0)\,, \tag{3.30a}$$

$$v\{p, \chi(r)\} = v_0\{\chi(r)\}\{\exp[-\beta p]/p\}\,, \qquad p \geqslant \chi(0)\,, \tag{3.30b}$$

$$\begin{cases} v_0\{\chi(r)\} = C_1 j_0\{\alpha\chi(0)\}\, j_1\{\alpha\chi(0)\}\exp[\beta\chi(0)]\, f\{\beta\chi(r)\}\{\alpha\chi(r)\}^3\,, \\ f\{\beta\chi(r)\} = \dfrac{280 + 28\beta^2\chi^2(r) + \beta^4\chi^4(r)}{280 - 28\beta^2\chi^2(0) + \beta^4\chi^4(0)}\,. \end{cases} \tag{3.30c}$$

Since $f\{\beta\chi(r)\}$ is a slowly varying function approximately equal to 1, the radial dependence of potential (3.30) for $p > \chi(0)$ is governed by the density function $\varrho(r)$. We shall refer to the mathematical description of the single-particle potential expressed by eqs. (3.30) as « model II ».

3·3. *The « intermediate » single-particle potential.* – A mathematical description of the radial and momentum dependence of the single-particle potential, which is—so to speak—intermediate between model I and model II, can be

constructed by assuming the validity of eq. (3.16) for $p \leqslant \chi(0)$ and that of eq. (3.27*b*) for $p \geqslant \chi(0)$. This hybrid compromise between the solutions of eq. (2.14) obtained by series integration and variable separation has a remarkable heuristic value: it will be called « model III ». From continuity prescriptions it is found that

$$\beta_{\mathrm{III}}\chi(0) = -\frac{\mathscr{B}_0^{(2)}\{\chi(0)\} + 3\chi^2(0)\mathscr{B}_1^{(2)}\{\chi(0)\} + 5\chi^4(0)\mathscr{B}_2^{(2)}\{\chi(0)\}}{\mathscr{B}_0^{(2)}\{\chi(0)\} + \chi^2(0)\mathscr{B}_1^{(2)}\{\chi(0)\} + \chi^4(0)\mathscr{B}_2^{(2)}\{\chi(0)\}}, \tag{3.31a}$$

$$C_2^{\mathrm{III}} = \frac{280 v^{(2)}\{\chi(0), \chi(0)\} \exp[\beta_{\mathrm{III}}\chi(0)]}{\chi^2(0)[280 + 28\beta_{\mathrm{III}}^2\chi^2(0) + \beta_{\mathrm{III}}^4\chi^4(0)]}. \tag{3.31b}$$

In conclusion, model III is expressed by the following potential:

$$v^{(2)}\{p, \chi(r)\} = \mathscr{B}_0^{(2)}\{\chi(r)\} + \mathscr{B}_1^{(2)}\{\chi(r)\}p^2 + \mathscr{B}_2^{(2)}\{\chi(r)\}p^4, \qquad p \leqslant \chi(0), \tag{3.32a}$$

$$v_{\mathrm{III}}\{p, \chi(r)\} = v_0^{\mathrm{III}}\{\chi(r)\}\{\exp[-\beta_{\mathrm{III}}p]/p\}, \qquad p \geqslant \chi(0), \tag{3.32b}$$

$$\begin{cases} v_0^{\mathrm{III}}\{\chi(r)\} = \chi(0) v^{(2)}\{\chi(0), \chi(0)\} \exp[\beta_{\mathrm{III}}\chi(0)]\varphi\{\chi(r)\}, \\ \varphi\{\chi(r)\} = f\{\beta_{\mathrm{III}}\chi(r)\}\{\chi(r)/\chi(0)\}^3. \end{cases} \tag{3.32c}$$

Condition $\beta_{\mathrm{III}}\chi(0) \geqslant 0$ is satisfied provided

$$\mathscr{B}_0^{(2)}\{\chi(0)\} + 3\chi^2(0)\mathscr{B}_1^{(2)}\{\chi(0)\} + 5\chi^4(0)\mathscr{B}_2^{(2)}\{\chi(0)\} \geqslant 0; \tag{3.33}$$

it follows that the internal consistency of model III is ensured by the inequality

$$-\chi^7(0)\mathscr{C}_5^{(2)} < (9/40)\chi^5(0)\mathscr{C}_3^{(2)} + (7/80)\chi^3(0)\mathscr{C}_1^{(2)}. \tag{3.34}$$

The lowest approximation of model III is obtained by describing in radial effective-mass approximation the single-particle potentials for $p \leqslant \chi(0)$. The corresponding relations can be deduced from the preceding ones by putting $\mathscr{C}_5^{(2)} = 0$ and, consequently,

$$\mathscr{B}_2^{(2)}\{\chi(r)\} = 0. \tag{3.35}$$

Equations (3.31) become

$$\beta_{\mathrm{III}}\chi(0) = -\frac{\mathscr{B}_0^{(1)}\{\chi(0)\} + 3\chi^2(0)\mathscr{B}_1^{(1)}\{\chi(0)\}}{\mathscr{B}_0^{(1)}\{\chi(0)\} + \chi^2(0)\mathscr{B}_1^{(1)}\{\chi(0)\}}, \tag{3.36a}$$

$$C_2^{\mathrm{III}} = \frac{280 v^{(1)}\{\chi(0), \chi(0)\} \exp[\beta_{\mathrm{III}}\chi(0)]}{\chi^2(0)[280 + 28\beta_{\mathrm{III}}^2\chi^2(0) + \beta_{\mathrm{III}}^4\chi^4(0)]}. \tag{3.36b}$$

The consistency of the model is ensured by the inequality

$$-\mathscr{B}_0^{(1)}\{\chi(0)\} \leqslant 3\chi^2(0)\mathscr{B}_1^{(1)}\{\chi(0)\}. \tag{3.37}$$

4. – Stability prescriptions and nuclear compressibility.

4·1. *The total energy of the nucleus.* – Using the solutions of eq. (2.14) we construct the total energy of the nucleus

$$W(s, \eta) = T(s, \eta) + V(s, \eta) \tag{4.1}$$

where, neglecting the slight differences in the radial behaviour of the density distributions of protons and neutrons, the kinetic and potential energies are

$$T(s, \eta) = (1/M)\int p^2 \, \mathrm{d}\mathcal{N}(\boldsymbol{r}, \boldsymbol{p}) = (4/5\pi M)\int_0^\infty r^2 \chi^5(r) \, \mathrm{d}r \,, \tag{4.2a}$$

$$V(s, \eta) = \int v\{p, \chi(r)\} \, \mathrm{d}\mathcal{N}(\boldsymbol{r}, \boldsymbol{p}) = (4/\pi)\int_0^\infty r^2 \, \mathrm{d}r \int_0^{\chi(r)} p^2 v\{p, \chi(r)\} \, \mathrm{d}p \,. \tag{4.2b}$$

Equation (4.1) explicitly reads

$$W(s, \eta) = \frac{4}{\pi}\int_0^\infty r^2 \left[\frac{\chi^5(r)}{5M} + \int_0^{\chi(r)} p^2 v\{p, \chi(r)\} \, \mathrm{d}p\right] \mathrm{d}r \,. \tag{4.3}$$

Several results quoted in the literature have been *de facto* obtained using $W(s, \eta)$, given by eq. (4.3) with a factor $\frac{1}{2}$ in front of the integral over p. Although it may seem trivial, it has to be pointed out that this is wrong. In fact, at the limit of a uniform distribution of nuclear matter confined in a sphere of radius R_∞, from eq. (4.3) one obtains

$$W_\infty(\varkappa) = (4R_\infty^3/3\pi)\left[(\varkappa^5/5M) + \int_0^\varkappa p^2 v(p, \varkappa) \, \mathrm{d}p\right] , \tag{4.4}$$

which identifies with eq. (2.3) using relation

$$R_\infty^3 = (9\pi/8\varkappa^3)A \,, \tag{4.5}$$

derived from eq. (2.2).

The integral function

$$I(r; s, \eta) = \frac{4}{\pi}\int_r^\infty r'^2 \left[\frac{\chi^5(r')}{5M} + \int_0^{\chi(r')} p^2 v\{p, \chi(r')\} \, \mathrm{d}p\right] \mathrm{d}r' \tag{4.6}$$

expresses the fraction of the total energy of the nucleus external to a sphere

of radius r; as is obvious, one obtains

$$I(0; s, \eta) = W(s, \eta) . \tag{4.7}$$

Let us define energy

$$u(r; s, \eta) = \frac{\mathrm{d}I(r; s, \eta)}{\mathrm{d}v} = \lim_{\delta \to 0} \frac{I(r + \delta; s, \eta) - I(r; s, \eta)}{v(r + \delta) - v(r)} , \tag{4.8}$$

where number $v(r)$ is defined as

$$\begin{cases} v(r) = (8/3\pi) \int_r^\infty r'^2 \chi^3(r') \, \mathrm{d}r' , \\ v(0) = A . \end{cases} \tag{4.9}$$

Taking the limit for $\delta \to 0$, eq. (4.8) becomes

$$u(r; s, \eta) = \frac{3}{\chi^3(r)} \int_0^{\chi(r)} p^2 \left[\frac{p}{2M} + \frac{1}{2} v\{p, \chi(r)\} \right] \mathrm{d}p . \tag{4.10}$$

Function (4.10) is formally identical to $W_\infty(\varkappa)/A$, given in eq. (2.3), provided $\chi(r)$ is replaced by $\varkappa$. Thus, $u(r; s, \eta)$ is the average total energy within a sphere having in momentum space radius $\chi(r)$. It is readily ascertained that

$$I(r; s, \eta) = \int_{r \leqslant r' \leqslant \infty} \varrho(r') u(r'; s, \eta) \, \mathrm{d}\boldsymbol{r}' . \tag{4.11}$$

The extension to a finite nucleus of the definition of the single-particle total energy, discussed in I, leads to the following relation:

$$w(r) = \left[\partial_\varrho \{ \varrho(r) u(r; s, \eta) \} \right]_{\substack{s = s_0 \\ \eta = \eta_0}} , \tag{4.12}$$

where s_0 and η_0 are the values of the density distribution parameters calculated at the minimum of the total energy of the nucleus. A straightforward calculation gives

$$w(r) = \frac{\chi^2(r)}{2M} + v\{\chi(r), \chi(r)\} - \frac{1}{2\chi^2(r)} \mathcal{Q}\{\chi(r)\} , \tag{4.13}$$

where

$$\mathcal{Q}\{\chi(r)\} = \chi^2(r) v\{\chi(r), \chi(r)\} - \int_0^{\chi(r)} p^2 \partial_\chi v\{p, \chi(r)\} \, \mathrm{d}p . \tag{4.14}$$

4·2. *A theorem concerning the single-particle potential.* – We shall now prove the following theorem: for any density distribution $\varrho(r)$ function $\mathcal{Q}\{\chi(r)\}$ is identically zero in the interval $0 \leqslant r \leqslant \infty$, *i.e.*

$$\mathcal{Q}\{\chi(r)\} \equiv 0 \tag{4.15}$$

provided $v\{p, \chi(r)\}$ be a particular integral of the hyperbolic differential equation (2.14), satisfying the asymptotic condition

$$v\{\chi(\infty), \chi(\infty)\} = v(0, 0) = 0 \,. \tag{4.16}$$

To this end we evaluate the derivative $\boldsymbol{D}_\chi \equiv \mathrm{d}/\mathrm{d}\chi(r)$

$$\boldsymbol{D}_\chi[\mathcal{Q}\{\chi(r)\}/\chi^2(r)] = [\partial_p v\{p, \chi(r)\}]_{p=\chi(r)} - \int_0^{\chi(r)} \partial_\chi[\{p/\chi(r)\}^2 \partial_\chi v\{p, \chi(r)\}]\,\mathrm{d}p \,, \tag{4.17}$$

where it has been taken into account that

$$\boldsymbol{D}_\chi v\{\chi(r), \chi(r)\} = [(\partial_p + \partial_\chi) v\{p, \chi(r)\}]_{p=\chi(r)} \,. \tag{4.18}$$

Equation (2.14) can be written in the form

$$\partial_p[\{p/\chi(r)\}^2 \partial_p v\{p, \chi(r)\}] - \partial_\chi[\{p/\chi(r)\}^2 \partial_\chi v\{p, \chi(r)\}] \,. \tag{4.19}$$

Taking into account that $\partial_p v\{p, \chi(r)\}$ varies linearly with p in the neighbourhood of $p = 0$, from eqs. (4.17) and (4.19) one has

$$\partial_\chi[\mathcal{Q}\{\chi(r)\}/\chi^2(r)] = 0 \,, \tag{4.20a}$$

$$\mathcal{Q}\{\chi(r)\} = \mathscr{C}\chi^2(r) \,, \tag{4.20b}$$

where $\mathscr{C}$ is an arbitrary integration constant. Performing the transformation $\xi = p/\chi(r)$, from eqs. (4.14) and (4.20*b*) one gets

$$v\{\chi(r), \chi(r)\} - \chi(r)\int_0^1 \xi^2[\partial_\chi v\{\chi(r)\xi, \chi(r)\}]\,\mathrm{d}\xi = \mathscr{C} \,; \tag{4.21}$$

equation (4.21) shows that condition (4.16) is satisfied provided $\mathscr{C} = 0$, which proves the theorem. One can readily check that the single-particle potentials (3.13), (3.16) and (3.30*a*) possess the remarkable property (4.15).

4·3. *Stability prescriptions.* – The first of the stability prescriptions of the nuclear ground state is

$$\lim_{r\to 0}[\boldsymbol{D}_\varrho I(r;\, s,\eta)]_{\substack{s=s_0\\ \eta=\eta_0}} = 0\,, \tag{4.22}$$

where $\boldsymbol{D}_\varrho \equiv \mathrm{d}/\mathrm{d}\varrho(r)$ and s_0, η_0 are the values of s and η at equilibrium density: as is well known, eq. (4.22) physically means that stability is ensured by the zero pressure of the nuclear « liquid ». The first-order total derivative of $I(r;\, s,\eta)$, at a fixed value of r, is

$$\boldsymbol{D}_\varrho I(r;\, s,\eta) = (\partial_\varrho s)\,\partial_s I(r;\, s,\eta) + (\partial_\varrho \eta)\,\partial_\eta I(r;\, s,\eta)\,. \tag{4.23}$$

With density (2.17) we construct function

$$F = F(\varrho;\, s,\eta) = \tag{4.24}$$
$$= \varrho - \left(2/\pi\sqrt{\pi}\, s^3\right)(2+3\eta)^{-1}\{1+\eta(r/s)^2\}\exp\left[-(r/s)^2\right] = 0\,.$$

Taking into account that $\partial_\varrho s = -(\partial_\varrho F)/(\partial_s F)$ and $\partial_\varrho \eta = -(\partial_\varrho F)/(\partial_\eta F)$, eq. (4.22) becomes

$$s_0\{\partial_s W(s,\eta)\}_0 + (2+3\eta_0)\{\partial_\eta W(s,\eta)\}_0 = 0\,, \tag{4.25}$$

where the symbol $\{\ \}_0$ means that the partial derivatives of the total energy are evaluated at $s = s_0$ and $\eta = \eta_0$. One has

$$\partial_s W(s,\eta) = (8/\pi)\int_0^\infty r^2\chi^2(r)\{\partial_s\chi(r)\}w(r)\,\mathrm{d}r\,, \tag{4.26a}$$

$$\partial_\eta W(s,\eta) = (8/\pi)\int_0^\infty r^2\chi^2(r)\{\partial_\eta\chi(r)\}w(r)\,\mathrm{d}r\,, \tag{4.26b}$$

where, owing to the theorem previously proved, $w(r)$ is given by formula (4.13) with $\mathcal{Q}\{\chi(r)\} = 0$, *i.e.*

$$w(r) = \varepsilon(r) + v\{\chi(r),\chi(r)\}\,, \tag{4.27}$$

$\varepsilon(r) = \chi^2(r)/2M$ being the local Fermi energy. For density (2.17) it is found that

$$\partial_s\chi(r) = -\left[\frac{3+(5\eta-2)(r/s)^2-2\eta(r/s)^4}{3s\{1+\eta(r/s)^2\}}\right]\chi(r)\,, \tag{4.28a}$$

$$\partial_\eta\chi(r) = -\left[\frac{3-2(r/s)^2}{3(2+3\eta)\{1+\eta(r/s)^2\}}\right]\chi(r)\,. \tag{4.28b}$$

Equation (4.25) expresses the necessary condition for nuclear stability. The sufficient condition is given by

$$W(s_0, \eta_0) = b(X^A)A \,, \tag{4.29}$$

where $b(X^A)$ is the average binding energy of the considered even-even nucleus X^A, determined by spectroscopic mass measurements. This assumption implies that the minimum $W(s_0, \eta_0)$ also includes the contribution arising from the Coulomb energy. This statement requires some comments. In the idealized conception of infinite nuclear matter, the Coulomb contribution to the single-particle potential energy fulfils eq. (2.1). It has been shown that in first-order perturbation theory the single-particle potential energies are solutions of the differential equation of infinitely extended nuclear matter provided that the two-body potential $v(r)$ is a function free of discontinuities: of course, this also occurs for the Coulomb potential $v_C(r) \propto 1/r$. It follows that in the Fermi sea the Coulomb single-particle potential energy is momentum dependent: this proves that the main cause of the momentum dependence cannot be restricted only to the exchange nature of the two-body force ([13]), but—as already pointed out in subsect. **2**'1—it is essentially brought about by the total antisymmetry of the wave function describing the nuclear system. This important property is also preserved for finite nuclei by virtue of eq. (2.14).

4'4. *The compressibility modulus.* – The sufficient condition for nuclear stability, expressed by eq. (4.29), possesses a clear-cut physical meaning but is by no means sufficient for the existence of the minimum of the total energy. From the mathematical standpoint the sufficient condition obviously requires that the second-order derivative of the total energy with respect to density $\boldsymbol{D}^2_{\varrho\varrho}W$ should be a positive quantity. Such a requirement is always satisfied in infinite nuclear matter and by finite nuclei in the framework of the Thomas-Fermi model: this is probably the reason why so little attention is generally devoted to this quantity in connection with nuclear stability. The physical meaning of $\boldsymbol{D}^2_{\varrho\varrho}W$ can easily be disclosed by applying standard thermodynamic relations to the description of the nucleus ground state at the limit of zero temperature. Let V be the nuclear volume, P the positive pressure of the system and H the Helmholtz free energy. The isothermal compressibility C_{is} is defined as the ratio of the variation in pressure to the corresponding fractional variation of volume which it produces. The familiar relations valid for small departures from the equilibrium configuration are

$$C_{is} = -V(\partial P/\partial V) \,, \qquad P = -(\partial H/\partial V) \,. \tag{4.30}$$

([13]) H. A. BETHE: *Phys. Rev.*, **167**, 879 (1968), p. 892.

In the spirit of the Thomas-Fermi model we express C_{is} and P in terms of compressibility and respectively free energy *per* particle instead of compressibility and free energy *per* unit volume, and then we replace the partial derivative with respect to V with the total derivative with respect to density ϱ; one has

$$C_{is} = \varrho \boldsymbol{D}_\varrho P\,, \qquad P = \varrho^2 \boldsymbol{D}_\varrho (H/A)\,. \tag{4.31}$$

Since at the limit of zero temperature $\boldsymbol{D}^2_{\varrho\varrho} W = \boldsymbol{D}^2_{\varrho\varrho} H$, from eq. (4.31) one deduces that $C_{is} = \varrho C$, where

$$C = 2\varrho \boldsymbol{D}_\varrho (W/A) + \varrho^2 \boldsymbol{D}^2_{\varrho\varrho}(W/A)\,. \tag{4.32a}$$

In order to work out analytically eq. (4.32*a*), we prefer to re-write it in the form

$$C = (1/A) \lim_{r \to 0} \{2\varrho(r) \boldsymbol{D}_\varrho I(r;\, s, \eta) + \varrho^2(r) \boldsymbol{D}^2_{\varrho\varrho} I(r;\, s, \eta)\}_0\,. \tag{4.32b}$$

Owing to stability prescription (4.22), the first term in the r.h.s. of eq. (4.32*b*) is equal to zero; energies

$$C(s_0, \eta_0) = (1/A) \lim_{r \to 0} \{\varrho^2(r) \boldsymbol{D}^2_{\varrho\varrho} I(r;\, s, \eta)\}_0\,, \tag{4.33a}$$

$$K(s_0, \eta_0) = 9C(s_0, \eta_0) \tag{4.33b}$$

are conventionally denominated as nuclear compressibility and nuclear compressibility modulus, respectively. Thus the definition of quantity $K(s_0, \eta_0)$ is physically significant in so far as it is specifically related to nuclear stability. The calculation of the compressibility modulus is controversial [14]; a variety of «incompressibilities» quoted in the literature do not seem to have much to do with definition (4.33*b*).

The quantity we are interested in is

$$K(s_0, \eta_0) = \tfrac{9}{4} \lim_{r \to 0} \{\varrho^2(r) \boldsymbol{D}^2_{\varrho\varrho} I(r;\, s, \eta)\}_0\,. \tag{4.34}$$

Quantity (4.34) is generally evaluated by replacing ϱ with a length characterizing the size of the nucleus such as the r.m.s. radius [15]. We shall follow a different and more genuine procedure. The second-order total derivative of

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[14] J. P. BLAIZOT: *Phys. Rep.*, **64**, 171 (1980). See also B. K. JENNINGS and A. D. JACKSON: *Phys. Rep.*, **66**, 141 (1980).

[15] J. P. BLAIZOT, D. GOGNY and B. GRAMMATICOS: *Nucl. Phys. A*, **265**, 315 (1976); K. B. JENNINGS and A. D. JACKSON: *Nucl. Phys. A*, **342**, 23 (1980).

$I(r; s, \eta)$ with respect to density $\varrho(r)$ at a fixed value of r is

$$\boldsymbol{D}^2_{\varrho\varrho} I(r; s, \eta) = (\partial_\varrho s)\,\partial_s\{\partial_\varrho I(r; s, \eta)\} + (\partial_\varrho \eta)\,\partial_\eta\{\partial_\varrho I(r; s, \eta)\}\,, \tag{4.35a}$$

$$\boldsymbol{D}^2_{\varrho\varrho} I(r; s, \eta) = (\boldsymbol{O}_1 + \boldsymbol{O}_2)\, I(r; s, \eta)\,, \tag{4.35b}$$

where $\boldsymbol{O}_1$ and $\boldsymbol{O}_2$ are the differential operators

$$\boldsymbol{O}_1 = (\partial_\varrho s)^2 \partial^2_{ss} + (\partial_\varrho \eta)^2 \partial^2_{\eta\eta} + 2(\partial_\varrho s)(\partial_\varrho \eta)\,\partial^2_{s\eta}\,, \tag{4.36a}$$

$$\boldsymbol{O}_2 = (\partial^2_{\varrho\varrho} s)\,\partial_s + (\partial^2_{\varrho\varrho} \eta)\,\partial_\eta\,. \tag{4.36b}$$

From the theory of implicit functions one obtains

$$\partial^2_{\varrho\varrho} s = -\{(\partial_s F)^2(\partial^2_{\varrho\varrho} F) - 2(\partial_\varrho F)(\partial_s F)(\partial^2_{\varrho s} F) + (\partial_\varrho F)^2(\partial^2_{ss} F)\}/(\partial_s F)^3\,, \tag{4.37a}$$

$$\partial^2_{\varrho\varrho} \eta = -\{(\partial_\eta F)^2(\partial^2_{\varrho\varrho} F) - 2(\partial_\varrho F)(\partial_\eta F)(\partial^2_{\varrho\eta} F) + (\partial_\varrho F)^2(\partial^2_{\eta\eta} F)\}/(\partial_\eta F)^3\,, \tag{4.37b}$$

where $F = F(\varrho; s, \eta)$ is defined in eq. (4.24). It is found that at limit $r \to 0$ condition (4.25) implies

$$\{\boldsymbol{O}_2 W(s, \eta)\}_0 = 0\,. \tag{4.38}$$

The compressibility modulus (4.34) becomes

$$K(s_0, \eta_0) = (1/A)[s_0^2\{\partial^2_{ss} W(s, \eta)\}_0 + 2(2 + 3\eta_0)^2\{\partial^2_{\eta\eta} W(s, \eta)\}_0 + 2s_0(2 + 3\eta_0)\{\partial^2_{s\eta} W(s, \eta)\}_0]\,, \tag{4.39}$$

where

$$\begin{cases} \partial^2_{ss} W(s, \eta) = (8/\pi)\int_0^\infty r^2\{\omega_1(r; s, s) + \omega_2(r; s, \eta)\partial_\chi\} w(r)\,\mathrm{d}r\,, \\ \partial^2_{\eta\eta} W(s, \eta) = (8/\pi)\int_0^\infty r^2\{\omega_1(r; \eta, \eta) + \omega_2(r; \eta, \eta)\partial_\chi\} w(r)\,\mathrm{d}r\,, \\ \partial^2_{s\eta} W(s, \eta) = (8/\pi)\int_0^\infty r^2\{\omega_1(r; s, \eta) + \omega_2(r; s, \eta)\partial_\chi\} w(r)\,\mathrm{d}r\,, \end{cases} \tag{4.40}$$

$$\begin{cases} \omega_1(r; s, \eta) = \chi(r)[2\{\partial_s \chi(r)\}\{\partial_\eta \chi(r)\} + \chi(r)\{\partial^2_{s\eta} \chi(r)\}]\,, \\ \omega_2(r; s, \eta) = \chi^2(r)\{\partial_s \chi(r)\}\{\partial_\eta \chi(r)\}\,. \end{cases} \tag{4.41}$$

The preceding relations exhibit the critical dependence of the compressibility

modulus on the single-particle potential at $p = \chi(r)$; in fact,

$$K(s_0, \eta_0) = K_0(s_0, \eta_0) + K_1(s_0, \eta_0) + K_2(s_0, \eta_0) , \tag{4.42a}$$

$$\begin{cases} K_0(s_0, \eta_0) = (4/\pi A M)\int_0^\infty r^2 \Xi_0(r)\, \chi^2(r)\, \mathrm{d}r , \\ K_1(s_0, \eta_0) = (8/\pi A)\int_0^\infty r^2 \Xi_1(r)\, v\{\chi(r), \chi(r)\}\, \mathrm{d}r , \\ K_2(s_0, \eta_0) = (8/\pi A)\int_0^\infty r^2 \Xi_2(r)[\partial_\chi v\{\chi(r), \chi(r)\}]\, \mathrm{d}r , \end{cases} \tag{4.42b}$$

$$\begin{cases} \Xi_0(r) = \Xi_1(r) + \{2/\chi(r)\}\Xi_2(r) , \\ \Xi_1(r) = s_0^2 \omega_1(r; s_0, s_0) + \\ \qquad + (2 + 3\eta_0)^2 \omega_1(r; \eta_0, \eta_0) + 2s_0(2 + 3\eta_0)\omega_1(r; s_0, \eta_0) , \\ \Xi_2(r) = s_0^2 \omega_2(r; s_0, s_0) + \\ \qquad + (2 + 3\eta_0)^2 \omega_2(r; \eta_0, \eta_0) + 2s_0(2 + 3\eta_0)\omega_2(r; s_0, \eta_0) . \end{cases} \tag{4.42c}$$

Neglecting the complicated dependence on s and η of the potential parameter $C_1\alpha$ at the minimum of $W(s, \eta)$, one obtains an approximate value of the compressibility modulus by dividing by a factor 4 the result obtained from eq. (4.42) assuming $C_1\alpha$ constant.

5. – The parameters of the single-particle potential.

5·1. *Model* I *and model* III. – The total energy of the nucleus described by model I is

$$W^{(2)}(s, \eta) = \frac{4}{\pi}\int_0^\infty r^2 \chi^5(r)\left[\frac{1}{5M} + \frac{1}{3}\mathscr{C}_1^{(2)}\chi(r) + \frac{2}{5}\mathscr{C}_3^{(2)}\chi^3(r) + \frac{24}{75}\mathscr{C}_5^{(2)}\chi^5(r)\right]\mathrm{d}r . \tag{5.1}$$

This model is mathematically undetermined because dependent on five parameters $(s, \eta; \mathscr{C}_1^{(2)}, \mathscr{C}_3^{(2)}, \mathscr{C}_5^{(2)})$ bound together by the two stability equations (4.25) and (4.29). Let us, for heuristic purposes, examine the special case $\eta = 0$, which *a priori* will not be assumed to be valid for ^{4}He in spite of the fact

that $\eta_{ch}(^4\mathrm{He}) = 0$. Function $W^{(2)}(s, 0)$ becomes

$$W^{(2)}(s, 0)/A = a_0 s^{-2} + a_1 \mathscr{C}_1^{(2)} s^{-3} + a_2 \mathscr{C}_3^{(2)} s^{-5} + a_3 \mathscr{C}_5^{(2)} s^{-7} , \tag{5.2a}$$

$$\begin{cases} a_0 = (\pi/4)^{\frac{1}{3}} (\sqrt[3]{3}/5)^{\frac{1}{2}} (27/50M) A^{\frac{2}{3}} , & a_2 = (3/2\pi)^{\frac{1}{6}} (81\pi/320) A^{\frac{5}{3}} , \\ a_1 = (\pi/2)^{\frac{1}{2}} (3/8) A , & a_3 = (243\pi/32)^{\frac{1}{6}} (81\pi/250\sqrt{5}) A^{\frac{7}{3}} . \end{cases} \tag{5.2b}$$

In the considered case the stability prescriptions are expressed by equations

$$\{\mathrm{d}W(s, 0)/\mathrm{d}s\}_{s=s_0} = 0 , \qquad W(s_0, 0) = b(X^A) A , \tag{5.3a}$$

where, taking into account *ansatz* (2.31),

$$s_0 = (2/3)^{\frac{1}{3}} r_0 A^{\frac{1}{3}} . \tag{5.3b}$$

From eqs. (5.3) one obtains

$$\begin{cases} s_0^{-3} \mathscr{C}_1^{(2)} = \{5b(X^A) - 3a_0 s_0^{-2} + 7a_3 s_0^{-7} \mathscr{C}_5^{(2)}\}/2a_1 , \\ s_0^{-5} \mathscr{C}_3^{(2)} = \{a_0 s_0^{-2} - 3b(X^A) - 7a_3 s_0^{-7} \mathscr{C}_5^{(2)}\}/2a_2 . \end{cases} \tag{5.4}$$

The compressibility modulus is

$$K^{(2)}(s_0, 0) = (s_0^2/A)\{\mathrm{d}^2 W(s, 0)/\mathrm{d}s^2\}_{s=s_0} , \tag{5.5a}$$

$$K^{(2)}(s_0, 0) = 3a_0 s_0^{-2} - 15b(X^A) - 7a_3 s_0^{-7} \mathscr{C}_5^{(2)} . \tag{5.5b}$$

Model III possesses the same indeterminacy as model I because also for $\eta = 0$ one additional equation is needed in order to determine parameters $\mathscr{C}_i^{(2)}$ ($i = 1, 3, 5$). Model III, however, is useful in so far as it discloses the crucial role of parameter $\mathscr{C}_5^{(2)}$ in evaluating, through eqs. (3.31), parameters $\mathscr{C}_3$ and β_{III} which govern the momentum dependence of the external part of the single-particle potential: only $v\{p, \chi(r)\}$ for $p > \chi(0)$ comes into play in the construction of the real part of the optical potential. As is seen from eq. (5.5*b*), $\mathscr{C}_5^{(2)}$ also has an important role in determining the compressibility modulus. The complicated interlacing between nuclear stability, compressibility and optical potential, clearly exhibited by model III, also exists for model II, although masked by its mathematical features.

Only model I, developed in effective-mass approximation with $\eta = 0$, is mathematically defined. The corresponding relations can be obtained from the preceding ones assuming $\mathscr{C}_5^{(2)} = 0$ and $a_3 = 0$.

5·2. *Model* II. – Model II is characterized by six parameters, *i.e.*

$$s,\ \eta,\ C_1,\ \alpha,\ C_2,\ \beta\ ; \tag{5.6}$$

only five equations are available, namely the *ansatz* equation (2.31), the two stability equations (4.25) and (4.29) and the two continuity equations (3.28). Thus, without additional experimental information or an additional equation, the description of the nucleus ground state turns out to be parametrized as a function of one of parameters (5.6). The total energy of the nucleus is

$$W(s,\eta) = \frac{4}{\pi}\int_0^\infty r^2\left[\frac{\chi^5(r)}{5M} + \frac{(C_1\alpha)\{\alpha\chi(r)\}^4}{\alpha^3} j_1^2\{\alpha\chi(r)\}\right]\mathrm{d}r\,. \tag{5.7}$$

For our purposes it is convenient to adopt the following notation:

$$\begin{cases} \alpha\chi(r) = \Omega\psi(r/s)\,, \\ \psi(r/s) = \{1+\eta(r/s)^2\}^{\frac{1}{3}}\exp\left[-\tfrac{1}{3}(r/s)^2\right], \\ \Omega = \dfrac{\alpha}{s}\left(\dfrac{3\sqrt{\pi}A}{2+3\eta}\right)^{\frac{1}{3}}; \end{cases} \tag{5.8}$$

the Fermi energy at $r = 0$ is

$$\varepsilon_0(\eta) = \chi^2(0)/2M = \{3(3\sqrt{\pi})^{\frac{2}{3}}/4Mr_0^2\}\{(2+5\eta)/(2+3\eta)^{\frac{5}{3}}\}\,, \tag{5.9}$$

where r_0 is fixed by eq. (2.31) in accordance with the r.m.s. radius of the considered nucleus X^A.

From eq. (4.25), taking into account eqs. (4.26) and (4.28), one obtains

$$\varepsilon_0(\eta)g_1(\eta) = -(C_1\alpha)\Omega^2 G_1(\Omega,\eta)\,, \tag{5.10a}$$

$$\begin{cases} g_1(\eta) = \displaystyle\int_0^\infty x^2\{6+(5\eta-4)x^2-2\eta x^4\}(1+\eta x^2)^{\frac{1}{3}}\exp\left[-5x^2/3\right]\mathrm{d}x\,, \\ G_1(\Omega,\eta) = \\ \quad = \displaystyle\int_0^\infty x^2\{6+(5\eta-4)x^2-2\eta x^4\}(1+\eta x^2)^{\frac{1}{3}} J_0\{\Omega\psi(x)\}\exp\left[-5x^2/3\right]\mathrm{d}x\,, \end{cases} \tag{5.10b}$$

$$J_0\{\Omega\psi(x)\} = j_0\{\Omega\psi(x)\}\,j_1\{\Omega\psi(x)\}\,, \tag{5.10c}$$

where $x = r/s$. The minimum of the average total energy (5.8) is

$$g_2(\eta) + (C_1\alpha)\Omega G_2(\Omega, \eta) = b(X^A) , \tag{5.11a}$$

$$\begin{cases} g_2(\eta) = \dfrac{24\varepsilon_0(\eta)}{5(2+3\eta)\sqrt{\pi}} \displaystyle\int_0^\infty x^2(1+\eta x^2)^{5/3} \exp[-5x^2/3]\,\mathrm{d}x , \\ G_2(\Omega, \eta) = \dfrac{12}{(2+3\eta)\sqrt{\pi}} \displaystyle\int_0^\infty x^2(1+\eta x^2)^{4/3} j_1^2\{\Omega\psi(x)\} \exp[-4x^2/3]\,\mathrm{d}x . \end{cases} \tag{5.11b}$$

We now substitute in eq. (5.10a) the potential parameter $C_1\alpha$ determined from eq. (5.11a); it is found that

$$g_1(\eta) g_3(\eta) = \Omega\{G_1(\Omega, \eta)/G_2(\Omega, \eta)\} , \tag{5.12a}$$

$$g_3(\eta) = \varepsilon_0(\eta)/\{g_2(\eta) - b(X^A)\} . \tag{5.12b}$$

Equation (5.12) allows one to determine parameter Ω as a function of η consistently with the stability prescriptions. The dependence of $\Omega(\eta)$ on the mass number A is brought about only by the average binding energy $b(X^A)$ and by the parameter $r_0 = r_0(X^A)$ introduced in eq. (2.31). Of course, not only the internal part of $v\{p, \chi(r)\}$ in momentum space turns out to be parametrized as a function of η, but also the external part: indeed, eqs. (3.28) are

$$\beta\chi(0) = -\Omega(\eta) \operatorname{ctg}\{\Omega(\eta)\} = \gamma(\eta) , \tag{5.13a}$$

$$C_2(\eta) = -\sqrt[3]{\frac{3}{\pi(2+3\eta)}} \frac{140(2+5\eta) r_0^2 \varepsilon_0(\eta) g_1(\eta) J_0\{\Omega(\eta)\} \exp[\gamma(\eta)]}{\{280 + 28\gamma^2(\eta) + \gamma^4(\eta)\} G_1\{\Omega(\eta), \eta\}} . \tag{5.13b}$$

The limit for $\eta \to 0$ of the preceding relations identifies with the results otherwise obtainable directly from eqs. (5.3).

For the sake of simplicity we evaluate the s-dependence of the compressibility modulus (4.30) for a fixed value of η_0. At equilibrium density one has

$$K(s, \eta_0) = c_0(\eta_0) H_0(\eta_0) + c_1(\eta_0)\Omega^2(\eta_0)\left(\frac{s_0}{s}\right)^2 \left\{H_1(\eta_0, s) + \Omega(\eta_0) H_2(\eta_0, s)\frac{s_0}{s}\right\} , \tag{5.14a}$$

where

$$\begin{cases} c_0(\eta) = (8/3\sqrt{\pi})\{\varepsilon_0(\eta)/(2+3\eta)\} , \\ c_1(\eta) = (8/3\sqrt{\pi})\{C_1\alpha/(2+3\eta)\} , \end{cases} \tag{5.14b}$$

$$
(5.15a)\qquad \begin{cases} H_0(\eta) = \displaystyle\int_0^\infty x^2(1+\eta x^2)^{-\frac{1}{2}} h_0(x,\eta)\exp[-5x^2/3]\,\mathrm{d}x\,, \\ H_1(\eta,s) = \displaystyle\int_0^\infty x^2(1+\eta x^2)^{-\frac{1}{2}} h_1(x,\eta) J_0\left\{\Omega(\eta)\frac{s_0}{s}\psi(x)\right\}\exp[-5x^2/3]\,\mathrm{d}x\,, \\ H_2(\eta,s) = \displaystyle\int_0^\infty x^2 h_2(x,\eta) J_1\left\{\Omega(\eta)\frac{s_0}{s}\psi(x)\right\}\exp[-2x^2]\,\mathrm{d}x\,; \end{cases}
$$

$$
(5.15b)\qquad \begin{cases} J_0\{y\} \equiv j_0\{y\}\, j_1\{y\}\,, \\ J_1\{y\} \equiv j_0^2\{y\} - j_1^2\{y\}\,; \end{cases}
$$

$$
(5.16a)\qquad h_m(x,\eta) = \sum_{n=0}^{4} a_{mn}(\eta)\, x^{2n}\,,
$$

$$
(5.16b)\qquad \begin{cases} a_{00} = 54\,, & \\ a_{01}(\eta) = 222\eta - 78\,, & a_{03}(\eta) = 40\eta - 142\eta^2\,, \\ a_{02}(\eta) = 200\eta^2 - 220\eta + 20\,, & a_{04}(\eta) = 20\eta^2\,, \\ a_{10} = 36\,, & \\ a_{11}(\eta) = 162\eta - 54\,, & a_{13}(\eta) = 24\eta - 102\eta^2\,, \\ a_{12}(\eta) = 150\eta^2 - 156\eta + 12\,, & a_{14}(\eta) = 12\eta^2\,, \\ a_{20} = 9\,, & \\ a_{21}(\eta) = 30\eta - 12\,, & a_{23}(\eta) = 8\eta - 20\eta^2\,, \\ a_{22}(\eta) = 25\eta^2 - 32\eta + 4\,, & a_{24}(\eta) = 4\eta^2\,. \end{cases}
$$

5·3. *The angular-momentum distributions in the Thomas-Fermi model.* – We shall explore the possibility of determining parameter η by equating the mean value of the square of the orbital angular momentum $\langle \ell^2 \rangle_{\mathrm{TF}}$, calculated according to the Thomas-Fermi model, with $\langle \ell^2 \rangle_{\mathrm{shell}}$ predicted by the shell model, *i.e.*

$$
(5.17)\qquad \langle \ell^2 \rangle_{\mathrm{TF}} = \langle \ell^2 \rangle_{\mathrm{shell}}\,.
$$

The following considerations are restricted to even-even nuclei only. The value of $\langle \ell^2 \rangle_{\mathrm{shell}}$ is obtained from the level scheme characterizing the considered

nucleus X^A; using standard spectroscopic notation one has

$$\langle \ell^2(X^A)\rangle = \sum_{j\ell} \nu_\ell(j)\ell(\ell+1)/\sum_{j\ell}\nu_\ell(j)\,, \qquad \sum_{j\ell}\nu_\ell(j) = A/2\,. \tag{5.18}$$

For the first two closed shells one has

$$\langle \ell^2(^4\mathrm{He})\rangle_{\mathrm{shell}} = 0\,, \qquad \langle \ell^2(^{16}\mathrm{O})\rangle_{\mathrm{shell}} = \tfrac{3}{2}\,. \tag{5.19}$$

The number $\langle \ell^2\rangle_{\mathrm{TF}}$ is defined as

$$\langle \ell^2\rangle_{\mathrm{TF}} = \int_0^\infty \ell^2\pi(\ell)\,\mathrm{d}\ell \Big/ \int_0^\infty \pi(\ell)\,\mathrm{d}\ell\,, \tag{5.20}$$

where $\pi(\ell)$ is the distribution function of nucleons with orbital momentum between ℓ and $\ell + \mathrm{d}\ell$. The number of particles with angular momentum larger than ℓ is expressed by the integral function

$$\Pi(\ell) = \int_\ell^\infty \pi(\ell)\,\mathrm{d}\ell\,; \tag{5.21}$$

then, the distribution function $\pi(\ell)$ is given by

$$\pi(\ell) = -\{\mathrm{d}\Pi(\ell)/\mathrm{d}\ell\}\,. \tag{5.22}$$

Function $\Pi(\ell)$ is readily obtained from model (2.15)

$$\Pi(\ell) = \int \mathrm{d}\mathcal{N}(\boldsymbol{r}, \boldsymbol{p})\,, \tag{5.23}$$

by choosing the limits of integration over $\boldsymbol{p}$ and $\boldsymbol{r}$ properly. Let θ be the angle between $\boldsymbol{r}$ and $\boldsymbol{p}$. Since $\boldsymbol{l} = \boldsymbol{r}\times\boldsymbol{p}$, one has

$$|\cos\theta| = (r^2p^2 - \ell^2)^{\frac{1}{2}}/rp\,; \tag{5.24}$$

integrating over the angles and over $p \leqslant \chi(r)$ it is found that

$$\Pi(\ell) = (4/3\pi)\int (1/r)\{r^2\chi^2(r) - \ell^2\}^{\frac{3}{2}}\,\mathrm{d}r\,, \tag{5.25}$$

where the integration over r has to be performed over the region where the

radical is real. From eq. (5.22) one obtains [16]

$$\pi(\ell) = (4\ell/\pi)\int (1/r)\{r^2\chi^2(r) - \ell^2\}^{\frac{1}{2}}\,\mathrm{d}r\ ; \tag{5.26a}$$

note that

$$\int_0^\infty \pi(\ell)\,\mathrm{d}\ell = H(0) = A/2\ . \tag{5.26b}$$

Taking into account definition (2.13), from eqs. (5.26) one deduces that

$$\langle \ell^2 \rangle_{\mathrm{TF}} = (16/15\pi)(3\pi^2/2)^{\frac{5}{3}} A^{\frac{2}{3}} \int_0^\infty r^4 \varrho^{\frac{5}{3}}(r)\,\mathrm{d}r\ . \tag{5.27}$$

The numerical values of $\langle \ell^2 \rangle_{\mathrm{TF}}$ obtained from eq. (5.27) are very sensitive to the behaviour of the density function near the edge of the nucleus. The integral over the fourth power of r is responsible for the general tendency of the Thomas-Fermi model to predict too high average angular momenta for light elements. Equation (5.17) is strongly violated for $A = 4$: indeed, if one describes the density distribution of ^{4}He by a pure Gaussian function ($\eta = 0$), it is found that

$$\langle \ell^2(^4\mathrm{He}) \rangle_{\mathrm{TF}} = (27/125)(3/5)^{\frac{1}{2}}(36\pi)^{\frac{1}{3}}\ , \tag{5.28}$$

whereas the average shell value is zero. This suggests that eq. (5.17) should be modified as follows:

$$\langle \ell^2(A) \rangle_{\mathrm{shell}} = \langle \ell^2(A) \rangle_{\mathrm{TF}} - \langle \ell^2(^4\mathrm{He}) \rangle_{\mathrm{TF}}\ . \tag{5.29}$$

The correction brought about by the shifting term is very effective for light nuclei only and turns out to be less relevant as A increases (note that for a standard nucleus $\langle \ell^2 \rangle_{\mathrm{shell}} \propto 20$). Equation (5.29) provides a theoretical tool for correlating the density distribution parameters. In particular, for density (2.17) it is found that parameter η is determined by equation

$$\begin{cases} I(\eta) = (5/16)(\sqrt{\pi}/9)^{\frac{1}{3}} A^{-\frac{2}{3}} [\langle \ell^2(A) \rangle_{\mathrm{shell}} + \langle \ell^2(^4\mathrm{He}) \rangle_{\mathrm{TF}}]\ , \\ I(\eta) = (2 + 3\eta)^{-\frac{5}{3}} \int_0^\infty x^4 (1 + \eta x^2)^{\frac{5}{3}} \exp[-5x^2/3]\,\mathrm{d}x\ . \end{cases} \tag{5.30}$$

The method outlined above will be applied to ^{16}O in subsect. **9**·4.

([16]) E. FERMI: *Z. Phys.*, **49**, 550 (1928).

5·4. *The hydrodynamical nucleon mass approximation.* – Before concluding that the determination of η is a hopeless task for even-even nuclei with $A \leqslant 16$, we shall explore a different approach. To this end we recall that in I, sect. **3**, the following theorem has been proved: if the infinitely extended nuclear matter is conceived of as a nonviscous and incompressible fluid and the relation $R_\infty = r_0 A^{\frac{1}{3}}$ is assumed to be valid, then the effective mass of a nucleon plunged in it is exactly equal to $\frac{1}{2}$ the mass of the free nucleon. This result has been obtained according to classic hydrodynamics considerations and is brought about by the dipole-velocity potential used for describing the motion of a nucleon in the Fermi sea. It is interesting to evaluate the deviations from the predicted value $0.5M$ of the effective mass of a nucleon in a realistic liquidlike nuclear matter, *i.e.* in a peculiar fluid characterized by saturation prescriptions (2.4). With this aim in view we define the single-particle total energy as

$$w_\infty(p, \varkappa_F) = (p^2/2M) + v_\infty(p, \varkappa_F)\,, \tag{5.31}$$

where the explicit expression of $v_\infty(p, \varkappa_F)$ is given by the first of eqs. (9.4).

In the considered case the nucleon effective mass is usually defined as

$$M/M^*_\infty(p, \varkappa_F) = (M/p)\,\partial_p w_\infty(p, \varkappa_F) = 1 + (M/p)\,\partial_p v_\infty(p, \varkappa_F)\,. \tag{5.32}$$

For our purposes it is significant to evaluate ratio (5.32) at $p = \varkappa_F$. At equilibrium density it is found that

$$a_\infty(\varkappa_F) \equiv M/M^*_\infty(\varkappa_F) = 1 + (2/15\varepsilon_F)(3\varepsilon_F - 5b_V)\,\alpha_\infty \varkappa_F\,, \tag{5.33}$$

where $\alpha_\infty \varkappa_F$ is determined by solving the first of eqs. (9.6). If nuclear matter behaved as a perfect classic fluid having the peculiar property of exerting attractive forces upon the moving nucleon, in accordance with the above-mentioned theorem the predicted value of $a_\infty(\varkappa_F)$ would be

$$a_\infty(\varkappa_F) = 2\,; \tag{5.34}$$

we shall ascertain in subsect. **9·3** that eq. (5.33) gives $a_\infty(\varkappa_F) > 2$. The more general concept of mass tensor will not be considered. Equation (5.33) accounts sufficiently well for the slope of $v_\infty(p, \varkappa_F)$ at the Fermi sphere. As is obvious, from definition (5.32) one obtains the nucleon effective mass (3.14) provided $v_\infty(p, \varkappa_F)$ is described simply as a linear function of p^2.

The extension of definition (5.32) to a nucleon embedded in a finite even-even nucleus characterized by a nonuniform density distribution is

$$M/M^*\{p, \chi(r)\} = 1 + (M/p)\,\partial_p v\{p, \chi(r)\}\,, \tag{5.35}$$

where $v\{p, \chi(r)\}$ is given by eq. (3.27*a*). Equation (5.35) is not so trivial as it may appear at first; indeed, function (5.31) cannot be extended to a finite nucleus by simply replacing $\varkappa_F$ with $\chi(r)$: we only know that at $p = \chi(r)$ the single-particle total energy is expressed by function $w(r)$, defined by eq. (4.27). For these reasons we have ignored the nucleon mass tensor, which provides a more refined hydrodynamical definition of the nucleon effective mass. The internal logic of the theory, based on the differential equations (2.1) and (2.14) and to the *ansatz* (2.31) implies that at the origin the nuclear system plays the role of the nuclear matter associated to a given finite even-even nucleus. It is then tempting to assume that

$$a\{\chi(0)\} = a_\infty(\varkappa_F) , \tag{5.36}$$

where $a\{\chi(0)\} = M/M^*\{\chi(0), \chi(0)\}$. Equation (5.36) ensures a direct and very crucial link between nuclear-matter calculations and those concerning finite nuclei: it will be denominated as the « hydrodynamical mass approximation ». From eq. (5.36) it is found that parameter η fulfils equation

$$(C_1\alpha)\Omega^3 j_1^2(\Omega) = -2\{a_\infty(\varkappa_F) - 1\}\varepsilon_0(\eta) . \tag{5.37}$$

Parameter $\eta = \eta_0$ at equilibrium density will be determined in subsect. **9**'3. The optical-model prescription, introduced in the next section, implies that it must be

$$2a_\infty(\varkappa_F) - 3 \geqslant \beta\chi(0) . \tag{5.38}$$

6. – The radial and energy dependence of the real optical potential for nucleons scattered by even-even nuclei.

6'1. *The optical transformation of model* I. – We extend to the radially dependent single-particle potentials, satisfying eq. (2.14), the definition of the real part of the central potential used in I. Consequently, $V_R^C(r, E)$ is conceived of as the optical transform of the potential energy of a nucleon incident on an even-even nucleus with an energy $E = k^2/2M$ in the laboratory system, *i.e.*

$$V_R^C(r, E) = v\{p(r, E), \chi(r)\} , \tag{6.1}$$

where $p(r, E)$ is the momentum of the incident nucleon embedded in the field of the target nucleus

$$p(r, E) = k[1 + \{-V_R^C(r, E)/E\}]^{\frac{1}{2}} . \tag{6.2}$$

According to model (2.15) it is necessarily true that

$$-V_{\mathrm{R}}^{\mathrm{C}}(r,0)\geqslant\chi^2(r)/2M=\varepsilon(r)\;; \tag{6.3}$$

inequality

$$-V_{\mathrm{R}}^{\mathrm{C}}(0,0)\geqslant\chi^2(0)/2M=\varepsilon_0(\eta) \tag{6.4}$$

has a crucial importance for the physical consistency of transform (6.1).

In radial effective-mass approximation the central potential is

$$\begin{cases} V_{\mathrm{R}}^{(1)\mathrm{C}}(r,E)=m\{\chi(r)\}\mathscr{B}_0^{(1)}\{\chi(r)\}+[1-m\{\chi(r)\}]E\,, \\ m\{\chi(r)\}=M^*\{\chi(r)\}/M\,, \qquad m\{\chi(\infty)\}=1\,. \end{cases} \tag{6.5}$$

Prescription (6.3) is satisfied provided

$$-m\{\chi(0)\}\mathscr{B}_0^{(1)}\{\chi(0)\}>\varepsilon_0(\eta)\,. \tag{6.6}$$

Substituting in eq. (6.6) the analytic expression of the nucleon effective mass given by eq. (3.14), it is found that

$$-v^{(1)}\{\chi(0),\chi(0)\}\geqslant\varepsilon_0(\eta)\,. \tag{6.7}$$

It has been pointed out that the effective-mass approximation is mathematically determined by the stability conditions only in the particular case $\eta=0$. Let us calculate $v^{(1)}\{\chi(0),\chi(0)\}$ using for $\eta=0$ the parameters $\mathscr{C}_1^{(1)}$ and $\mathscr{C}_3^{(1)}$ given by eqs. (5.4) with $\mathscr{C}_5^{(2)}$ and $a_3=0$; it is readily established that inequality (6.7) turns out to be violated. Such a drawback does not warrant any more consideration of the nucleon effective-mass approximation.

The real part of the central optical potential, deduced from model I according to transform (6.1), fulfils the second-degree equation

$$[V_{\mathrm{R}}^{(2)\mathrm{C}}(r,E)]^2-A_1(r,E)\,V_{\mathrm{R}}^{(2)\mathrm{C}}(r,E)+A_2(r,E)=0\,, \tag{6.8a}$$

$$\begin{cases} A_1(r,E)=\dfrac{1+2M\mathscr{B}_1^{(1)}\{\chi(r)\}+8M^2\mathscr{B}_2^{(2)}\{\chi(r)\}E}{4M^2\mathscr{B}_2^{(2)}\{\chi(r)\}}\,, \\ A_2(r,E)=\dfrac{\mathscr{B}_0^{(2)}\{\chi(r)\}+2M\mathscr{B}_1^{(2)}\{\chi(r)\}E+4M^2\mathscr{B}_2^{(2)}\{\chi(r)\}E^2}{4M^2\mathscr{B}_2^{(2)}\{\chi(r)\}}\,. \end{cases} \tag{6.8b}$$

It is found that

$$V_{\mathrm{R}}^{(2)\mathrm{C}}(r,E)=\tfrac{1}{2}A_1(r,E)+\tfrac{1}{2}\sqrt{A_1^2(r,E)-4A_2(r,E)}\,, \tag{6.9a}$$

where the ambiguity in sign has been eliminated by taking into account that necessarily

$$V_{\mathrm{R}}^{(2)C}(\infty, E) = 0 \,. \tag{6.9b}$$

Using eqs. (5.4) one can calculate the minimum value of $\mathscr{E}_3^{(2)}$ which for $A = 4$ satisfies equation

$$-V_{\mathrm{R}}^{(2)C}(0, 0) = \varepsilon_0(0) \,. \tag{6.10}$$

The mathematical connection between the compressibility modulus and the zero-energy limit of the real central potential at the origin, implied by model I at equilibrium density, will be examined in subsect. 9·3.

6·2. *The optical transformation of model* II. – Let us consider the optical transform of model II. The same procedure is also valid for model III. To this end, we specify eq. (6.1) using the single-particle potential (3.30*b*); one has

$$\begin{cases} V_{\mathrm{R}}^{C}(r, E)\left\{\dfrac{E - V_{\mathrm{R}}^{C}(r, E)}{\varepsilon_0(\eta)}\right\}^{\frac{1}{2}} - F_{\mathrm{R}}\{\chi(r)\} \exp\left[-\beta\chi(0)\left\{\dfrac{E - V_{\mathrm{R}}^{C}(r, E)}{\varepsilon_0(\eta)}\right\}^{\frac{1}{2}}\right], \\ F_{\mathrm{R}}\{\chi(r)\} = v_0\{\chi(r)\}/\chi(0) \,, \end{cases} \tag{6.11a}$$

$$V_{\mathrm{R}}^{C}(\infty, E) = V_{\mathrm{R}}^{C}(r, \infty) = 0 \,. \tag{6.11b}$$

The high-energy behaviour of potential (6.11) is expressed by the approximate formula

$$V_{\mathrm{R}}^{C}(r, E) \simeq F_{\mathrm{R}}\{\chi(r)\}\{\varepsilon_0(\eta)/E\}^{\frac{1}{2}} \exp\left[-\beta\chi(0)\{E/\varepsilon_0(\eta)\}^{\frac{1}{2}}\right] . \tag{6.12}$$

The low-energy behaviour of $V_{\mathrm{R}}^{C}(r, E)$ cannot be obtained using the McLaurin expansion

$$\bar{V}_{\mathrm{R}}^{C}(r, E) = V_{\mathrm{R}}^{C}(r, 0) + \sum_{n=1}^{\infty} c_n(r) E^n \,, \tag{6.13}$$

where $c_n(r) = (1/n!)\{\partial^n V_{\mathrm{R}}^{C}(r, E)/\partial E^n\}_{E=0}$. It is found that functions $c_n(r)$ for $n > 1$ diverge for $r \to \infty$, whereas for $n = 1$ $c_1(\infty)$ is a constant equal to $\frac{1}{3}$: it follows that

$$\bar{V}_{\mathrm{R}}^{C}(\infty, E) = \tfrac{1}{3} E \,, \tag{6.14}$$

a result which is clearly physically meaningless.

For $\beta = 0$ ($\Omega = \pi/2$) potential (6.11) is a root of the third-degree equation

$$[V_{\mathrm{R}}^{C}(r, E)]^3 - E[V_{\mathrm{R}}^{C}(r, E)] + \varepsilon_0(\eta)[F_{\mathrm{R}}\{\chi(r)\}]^2 = 0 \,. \tag{6.15}$$

Applying Hudde's method it is found that

$$V_{\mathrm{R}}^{\mathrm{C}}(r, E) = \frac{1}{3}E + \frac{1}{2}\sqrt[3]{-\Lambda_1(r, E)} + \sqrt{\Lambda_2(r, E)} + \frac{1}{2}\sqrt[3]{-\Lambda_1(r, E) - \sqrt{\Lambda_2(r, E)}}, \tag{6.16a}$$

$$\begin{cases} \Lambda_1(r, E) = (E/3)^3 + \{\varepsilon_0(\eta)/2\}[F_{\mathrm{R}}\{\chi(r)\}]^2, \\ \Lambda_2(r, E) = \varepsilon_0(\eta)(E/3)^3[F_{\mathrm{R}}\{\chi(r)\}]^2 + \{\varepsilon_0(\eta)/2\}^2[F_{\mathrm{R}}\{\chi(r)\}]^4; \end{cases} \tag{6.16b}$$

it is readily checked that potential (6.16) fulfils conditions (6.11*b*).

The zero-energy limit of potential (6.11) at the origin is obtained by solving the equation

$$\xi^3 = \mathfrak{N}(\eta)\exp[-\beta\chi(0)(\xi - 1)], \tag{6.17a}$$

$$\begin{cases} \xi^2 = -V_{\mathrm{R}}^{\mathrm{C}}(0, 0)/\varepsilon_0(\eta), \\ \mathfrak{N}(\eta) = -\{(C_1\alpha)/\varepsilon_0(\eta)\}\Omega^2(\eta)\, j_0\{\Omega(\eta)\}\, j_1\{\Omega(\eta)\}. \end{cases} \tag{6.17b}$$

Clearly, the optical-model prescription (6.4) implies that

$$\mathfrak{N}(\eta) \geqslant 1, \qquad \xi(\eta) \geqslant 1. \tag{6.18}$$

Thus, inequalities (6.18) are sufficient for the consistency of the optical scheme: both stability equations (5.10*a*) and (5.11*a*) are necessary to ensure that the scheme has a physical content.

According to definition (1.3*a*) the real part of the spin-orbit potential can be written in the form

$$V_{\mathrm{R}}^{\mathrm{SO}}(r, E) = 2(\lambda_{\mathrm{R}}/s_0)^2(\boldsymbol{\sigma}\cdot\boldsymbol{L})\, U_{\mathrm{R}}(r, E)\, V_{\mathrm{R}}^{\mathrm{C}}(r, E), \tag{6.19a}$$

$$U_{\mathrm{R}}(r, E) = (s_0^2/2r)\boldsymbol{D}\ln\{V_{\mathrm{R}}^{\mathrm{C}}(r, E)\}, \tag{6.19b}$$

where $\boldsymbol{D} \equiv \mathrm{d}/\mathrm{d}r$. Using the theory of implicit functions, one finds that

$$U_{\mathrm{R}}(r, E) = \frac{7\{120 + 20\beta^2\chi^2(r) + \beta^4\chi^4(r)\}D(r)}{\{280 + 28\beta^2\chi^2(r) + \beta^4\chi^4(r)\}d(r, E)}, \tag{6.20a}$$

$$\begin{cases} D(r) = (s_0^2/2r)\boldsymbol{D}\ln\{\chi(r)\} = (s_0^2/6r)\boldsymbol{D}\ln\{\varrho(r)\}, \\ d(r, E) = 1 - \frac{1}{2}\dfrac{V_{\mathrm{R}}^{\mathrm{C}}(r, E)}{E - V_{\mathrm{R}}^{\mathrm{C}}(r, E)}\left[1 + \beta\chi(0)\left\{\dfrac{E - V_{\mathrm{R}}^{\mathrm{C}}(r, E)}{\varepsilon_0(\eta)}\right\}^{\frac{1}{2}}\right]. \end{cases} \tag{6.20b}$$

The zero-energy limit at the origin of the radial spin-orbit function is

$$U_{\mathrm{R}}(0, 0) = \frac{14(\eta - 1)[120 + 20\gamma^2(\eta) + \gamma^4(\eta)]}{3\{3 + \gamma(\eta)\xi(\eta)\}[280 + 28\gamma^2(\eta) + \gamma^4(\eta)]}; \tag{6.21}$$

it is worthwhile to remark that eq. (6.21) predicts opposite signs of $V_R^{so}(0, 0)$ for even-even nuclei with $\eta < 1$ and $\eta > 1$.

6·3. *Critical remarks.* – Although potential (6.11) is generated by the separable single-particle potential (3.18), it cannot be factorized as the product of two functions $V_{R1}^c(r)$ and $V_{R2}^c(E)$ dependent only on the radial distance r and only on the incident energy E, respectively, *i.e.*

$$V_R^c(r, E) = V_{R1}^c(r)\, V_{R2}^c(E) . \tag{6.22}$$

As is shown by eq. (6.12), factorization (6.22) happens to be approximately valid only at very high energies, namely

$$\begin{cases} V_{R1}^c(r) \simeq F_R\{\chi(r)\} , \\ V_{R2}^c(E) \simeq \{\varepsilon_0(\eta)/E\}^{\frac{1}{2}} \exp\left[-\beta\chi(0)\{E/\varepsilon_0(\eta)\}^{\frac{1}{2}}\right] . \end{cases} \tag{6.23}$$

At high energies the real part of the spin-orbit potential (6.19) practically behaves like $V_R^c(r, E)$ factorized according to eqs. (6.22) and (6.23). At high energies and at large distances potentials (6.11) and (6.19) turn out to be approximately proportional to the density function, because

$$F_R\{\chi(r)\} \xrightarrow[r\to\infty]{} \varrho(r) ; \tag{6.24}$$

in such conditions the optical potential (1.2*a*) with the spin-orbit term expressed by (1.3*a*) identifies with the potential formerly introduced in optical-model calculations by FERNBACH, HECKROTTE and LEPORE [17], and later used by several other authors.

The low-energy behaviour of the real optical potential deserves some comments. The linear dependence on energy E is straightforwardly brought about by transforming the single-particle potential described in radial effective-mass approximation according to eq. (6.1). For nuclei with $A \leqslant 16$ prescription (6.4) turns out to be violated at equilibrium density: this is due to the fact that the dependence of the single-particle potential on p^2 is inadequate, in spite of the fact that it accounts for nuclear stability. Model I improves the description of the nucleus ground state by means of an additional term in p^4, which is responsible for the appearance in the central part of the optical potential of terms quadratic in the energy E.

The linear dependence on E of the central part of the potential, free from pathological features, can be obtained by Taylor's expansion of potential (6.11) in proximity of a sufficiently large energy E_0. The potential thus derived simulates (especially for $A > 16$) the behaviour obtainable in radial effective-

[17] S. FERNBACH, W. HECKROTTE and J. V. LEPORE: *Phys. Rev.*, **97**, 1059 (1955).

mass approximation: several analyses of elastic-scattering data, carried out in the past, have been spoilt by such misunderstanding brought about by weak phenomenological standpoints and by the acritical handling of fitting procedures. It is difficult to assess to what extent potential (6.11) accounts at low energies for the insufficiencies of the impulse approximation and for the neglect of multiple-scattering effects. It is encouraging to note that in our scheme neither the breakdown of the impulse approximation, nor the neglect of the presumptive role of multiple scattering influence the central part of the real optical potential because its quantitative behaviour at low energies is strongly bound to the stability requirements of the target nucleus: this is the consequence of having used the average binding energy $b(X^A)$ as input data for the determination of the parameters governing the radial and momentum dependence of $v\{p, \chi(r)\}$. The nonlocal features of the real optical potential will be examined in subsect. **9**·5. The spin-orbit part of the real optical potential might turn out to be inadequate, unless one allows for the energy dependence of length λ_R, which has to be determined by fitting the polarization data of nucleons elastically scattered by even-even nuclei.

It is worthwhile pointing out that the position of the minimum of potential (6.11) is independent of the incident energy.

In conclusion, the real optical potential (1.2*a*), constructed by means of potentials (6.11) and (6.19), correlates the description of nucleon-nucleus elastic scattering with the description of the main features of the nucleus ground state. Thus, a stringent link has been established between asymptotic phase shifts, elastic-scattering cross-sections and polarization data, and the stability conditions of the finite target nucleus. Optical-model calculations which ignore the constraints implied by such an overall interlacing are heuristically meaningless; this circumstance has been well emphasized by HODGSON [18]: « Most of the optical model analyses ... are relatively insensitive to the radial variation chosen. Provided it falls exponentially at large distances and is parametrized by a radius and a surface diffuseness, and these are optimized to fit the experimental data, almost any form may be taken for the radial variation in the nuclear interior without significantly affecting the quality of the fit ».

7. – The radial and energy dependence of the imaginary potential for nucleons scattered by finite even-even nuclei.

7·1. *The «frivolous model».* – We extend to model (2.15) the treatment used in I, sect. **10**, for the calculation of the central part of the imaginary potential. Trivial considerations, based on the continuity equation and on

[18] P. E. HODGSON: *The Optical Model of Elastic Scattering* (Clarendon Press, Oxford, 1963), p. 113.

the relation existing between the absorption probability per time unit and the mean free path of nucleons within the nuclear system, lead to the conclusion that the central part of the imaginary potential in the Thomas-Fermi scheme reads

$$V_{\mathrm{I}}^{\mathrm{c}}(r, E) = -(5A/16)\varrho(r)\, v_{\mathrm{n}}(r, E)\langle\sigma_{\mathrm{np}}\rangle \,, \tag{7.1}$$

where $v_{\mathrm{n}}(r, E)$ is the velocity of the incident nucleon in the field of the target nucleus and $\langle\sigma_{\mathrm{np}}\rangle$ is the neutron-proton cross-section locally averaged over the $A = 2Z = 2N$ nucleons of the nucleus: eq. (7.1) accounts for the implications of charge symmetry and charge independence of nuclear forces.

Let $\boldsymbol{P}(r, E)$ and $\boldsymbol{P}'(r, E)$ be the relative momenta of a two-nucleon system before and after the collision of an incident nucleon of momentum $p(r, E) > \chi(r)$ with a nucleon embedded in the nucleus with momentum $q \leqslant \chi(r)$; momentum $p(r, E)$ is expressed by the dispersive relation (6.2). The principle of local energy conservation requires that

$$P' = P(r, E) = \tfrac{1}{2}\,|\boldsymbol{p}(r, E) - \boldsymbol{q}| \,. \tag{7.2}$$

The neutron-proton cross-section σ_{np} depends on P and on the angle between $\boldsymbol{P}(r, E)$ and $\boldsymbol{P}'$, *i.e.* $\sigma_{\mathrm{np}} = \sigma_{\mathrm{np}}(P, \boldsymbol{P}\cdot\boldsymbol{P}')$. The calculation of $\langle\sigma_{\mathrm{np}}\rangle$ is performed according to the following procedure: i) quantity $\sigma_{\mathrm{np}}(P, \boldsymbol{P}\cdot\boldsymbol{P}')$ is multiplied by the flux of incoming particles of momentum $|\boldsymbol{p}(r, E) - \boldsymbol{q}|/M$; ii) the quantity thus obtained is integrated over the solid angle $\mathrm{d}\tau = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$ defined by the relative momenta $\boldsymbol{P}(r, E)$ and $\boldsymbol{P}'$, keeping momenta $p(r, E)$ and q fixed; iii) the average in momentum space gives the number of particles scattered per unit time in the whole solid angle for any value of $\boldsymbol{q}$: this result is then divided by the flux of incident particles. In conclusion, it is found that

$$\langle\sigma_{\mathrm{np}}\rangle = \frac{2}{\pi^2 p(r, E)\, n(r)} \int_0^r r'^2\,\mathrm{d}r' \int |\boldsymbol{p}(r, E) - \boldsymbol{q}|\,\sigma_{\mathrm{np}}(P, \boldsymbol{P}\cdot\boldsymbol{P}')\,\mathrm{d}\tau\,\mathrm{d}\boldsymbol{q} \,, \tag{7.3a}$$

$$n(r) = (8/3\pi)\int_0^r r'^2\chi^3(r')\,\mathrm{d}r' \,, \qquad n(\infty) = A \,. \tag{7.3b}$$

To avoid cumbersome numerical calculations, we neglect the angular dependence of the neutron-proton cross-section: this approximation alters the final results by less than one per cent. Substituting in eq. (7.3*a*) the total neutron-proton cross-section $\sigma_{\mathrm{np}}(P) \to 4\pi\sigma_{\mathrm{np}}(P, \boldsymbol{P}\cdot\boldsymbol{P}')$, one has

$$\langle\sigma_{\mathrm{np}}\rangle = \frac{1}{\pi^2 p(r, E)\, n(r)} \int_0^r r'^2\,\mathrm{d}r' \int |\boldsymbol{p}(r, E) - \boldsymbol{q}|\,\sigma_{\mathrm{np}}\{P(r', E)\}\,\mathrm{d}(\cos\theta)\,\mathrm{d}\boldsymbol{q} \,. \tag{7.4}$$

Let us define vector $\boldsymbol{Q} = \boldsymbol{p}(r, E) + \boldsymbol{q}$, which, owing to the conservation of the total momentum, turns out to be an axis of cylindrical symmetry. From the relation

$$\boldsymbol{p}' = \boldsymbol{p}(r, E) + \boldsymbol{q} - \boldsymbol{q}' = \tfrac{1}{2}\boldsymbol{Q} + \boldsymbol{P}' \tag{7.5}$$

one obtains, taking into account eq. (7.2),

$$\cos\theta = \frac{2p'^2 - p^2(r, E) - q^2}{|\boldsymbol{p}(r, E) + \boldsymbol{q}|\,|\boldsymbol{p}(r, E) - \boldsymbol{q}|}, \tag{7.6}$$

$$\int \mathrm{d}(\cos\theta) = \frac{4}{|\boldsymbol{p}(r, E) + \boldsymbol{q}|\,|\boldsymbol{p}(r, E) - \boldsymbol{q}|}\int_{p_0^2}^{p_1^2} \mathrm{d}(p'^2) = \frac{2(p_1^2 - p_0^2)}{|\boldsymbol{p}(r, E) - \boldsymbol{q}|\,|\boldsymbol{p}(r, E) + \boldsymbol{q}|}. \tag{7.7}$$

The lower limit of integration is $p_0^2 = \chi^2(r)$. The upper limit is determined by the Pauli principle and by the principle of energy conservation: in fact, the latter requires that

$$p'^2 = p^2(r, E) + q^2 - q'^2\,; \tag{7.8a}$$

the maximum of p' is obtained by putting in eq. (7.8a) $q' = \chi(r)$, which is the minimum value of q' compatible with the exclusion principle, *i.e.*

$$p_1^2 = p^2(r, E) + q^2 - \chi^2(r)\,. \tag{7.8b}$$

The inequality $p_1^2 - p_0^2 = p^2(r, E) + q^2 - 2\chi^2(r) \geqslant 0$ should hold for any q: this implies that q varies within the intervals

$$\begin{cases} \{2\chi^2(r) - p^2(r, E)\}^{\frac{1}{2}} \leqslant q \leqslant \chi(r) & \text{for } p^2(r, E) \leqslant 2\chi^2(r)\,, \\ 0 \leqslant q \leqslant \chi(r) & \text{for } p^2(r, E) \geqslant 2\chi^2(r)\,. \end{cases} \tag{7.9}$$

Equation (7.4) becomes

$$\langle\sigma_{\mathrm{np}}\rangle = \frac{2}{\pi^2 p(r, E) n(r)}\int_0^r r'^2\,\mathrm{d}r' \int \left[\frac{p^2(r, E) + q^2 - 2\chi^2(r')}{|\boldsymbol{p}(r', E) + \boldsymbol{q}|}\right] \sigma_{\mathrm{np}}\{P(r', E)\}\,\mathrm{d}\boldsymbol{q}\,. \tag{7.10}$$

Let $\mathrm{d}\boldsymbol{q} = q^2\,\mathrm{d}q\sin\Theta\,\mathrm{d}\Theta\,\mathrm{d}\Phi$ be the volume element, where Θ is identified with

the angle between $\boldsymbol{p}(r, E)$ and $\boldsymbol{q}$. Using relation

$$Q^2 = |\boldsymbol{p}(r, E) + \boldsymbol{q}|^2 = p^2(r, E) + q^2 - 2p(r, E)\, q \cos\Theta\,, \tag{7.11}$$

the relative momentum (7.2) turns out to be

$$P(r, E) = \tfrac{1}{2}\{2p^2(r, E) + 2q^2 - Q^2\}^{\frac{1}{2}}\,. \tag{7.12}$$

From eq. (7.11) one has $\sin\Theta\, \mathrm{d}\Theta = (Q/pq)\, \mathrm{d}Q$; it follows that eq. (7.10) takes the form

$$\langle\sigma_{np}\rangle = \frac{4}{\pi p^2(r, E)\varrho(r)} \int_0^r r'^2 S(r', E)\, \mathrm{d}r'\,, \tag{7.13a}$$

$$S(r, E) = \int_{q_0}^{q_1} q\{p^2(r, E) + q^2 - 2\chi^2(r)\}\, \mathfrak{S}\{p(r, E), q\}\, \mathrm{d}q\,, \tag{7.13b}$$

$$\mathfrak{S}\{p(r, E), q\} = \int_{p(r,E)\,-q}^{p(r,E)\,+q} \sigma_{np}\{P(r, E)\}\, \mathrm{d}Q\,, \tag{7.13c}$$

where the lower and upper limits of q are given in (7.9).

From the experimental total neutron-proton cross-section $\sigma_{np}(e)$, where e is the kinetic energy in the centre of mass, we derive the function

$$\bar{s}(r, E) = \frac{\mathrm{c}_1(r, E)}{\mathrm{c}_2(r, E) + e\{P(r, E)\}}\,, \tag{7.14a}$$

$$e = e\{P(r, E)\} = P^2(r, E)/M = |\boldsymbol{p}(r, E) - \boldsymbol{q}|^2/4M\,. \tag{7.14b}$$

Function (7.14a) simulates the energy dependence of the neutron-proton total cross-section and its apparent radial dependence brought about by the non-uniform density distribution $\varrho(r)$. The radial and energy behaviour of functions $\mathrm{c}_i(r, E)$ $(i = 1, 2)$ is obtained using the procedure outlined in subsect. **7**·2 and **9**·6. Substituting in eq. (7.13c) $\sigma_{np}\{P(r, E)\}$ with $\bar{s}(r, R)$, one obtains

$$\mathfrak{S}\{p(r, E), q\} = \frac{2M\mathrm{c}_1(r, E)}{\sqrt{2\{2M\mathrm{c}_2(r, E) + p^2(r, E) + q^2\}}}\, \mathfrak{Q}\{p(r, E), q\}\,, \tag{7.15a}$$

$$\mathfrak{Q}\{p(r, E), q\} = \ln \frac{4M\mathrm{c}_2(r, E) + p^2(r, E) + 3q^2 + 2q\sqrt{2\{2M\mathrm{c}_2(r, E) + p^2(r, E) + q^2}}}{4M\mathrm{c}_2(r, E) + p^2(r, E) + 3q^2 - 2q\sqrt{2\{2M\mathrm{c}_2(r, E) + p^2(r, E) + q^2}}}\,. \tag{7.15b}$$

Since $p(r, E) - q \leqslant Q \leqslant p(r, E) + q$ and $Q^2 < 2\{2Mc_2(r, E) + p^2(r, E) + q^2\}$, it follows that eq. (7.15a) can be written in the following form:

$$(7.16a)\quad \mathfrak{S}\{p(r, E), q\} = \frac{2Mc_1(r, E)}{\sqrt{2\{2Mc_2(r, E) + p^2(r, E) + q^2\}}} \sum_{n=0}^{1} (-1)^n \mathfrak{T}_n\{p(r, E), q\},$$

$$(7.16b)\quad \mathfrak{T}_n\{p(r, E), q\} = \operatorname{tgh}^{-1} \frac{p(r, E) + (-1)^n q}{\sqrt{2\{2Mc_2(r, E) + p^2(r, E) + q^2\}}}.$$

Performing the summation appearing in eq. (7.16a), the final form of function (7.13c) turns out to be

$$(7.17)\quad \mathfrak{S}\{p(r, E), q\} = -\frac{2Mc_1(r, E)}{\sqrt{2\{2Mc_2(r, E) + p^2(r, E) + q^2\}}} \operatorname{tgh}^{-1} \frac{2q\sqrt{2\{2Mc_2(r, E) + p^2(r, E) + q^2\}}}{4Mc_2(r, E) + p^2(r, E) + 3q^2}.$$

For calculational purposes it is convenient to define the following functions:

$$(7.18)\quad \begin{cases} \zeta(r, E) = \chi(r)/p(r, E), \\ \mu(r, E) = 2\zeta^2(r, E) - 1, \end{cases} \qquad \nu(r, E) = \{2/\varepsilon(r)\}\{\varepsilon(r) + c_2(r, E)\zeta(r, E)\},$$

where $\varepsilon(r) = \chi^2(r)/2M$; note that for $E \neq 0$ $\zeta(r, E)$ vanishes at infinity, whereas for $E = 0$ it diverges as $\{\chi(r)\}^{-1}$. In conclusion, the analytic form of the central part of the imaginary potential is

$$(7.19a)\quad V_{\mathrm{I}}^{\mathrm{c}}(r, E) = V_{\mathrm{I}1}(r, E) V_{\mathrm{I}2}(r, E),$$

$$(7.19b)\quad V_{\mathrm{I}1}(r, E) = -(5AM/\pi)\{\varrho(r)/\varkappa(r)\}\{E - V_{\mathrm{R}}^{\mathrm{c}}(r, E)\},$$

$$(7.19c)\quad \begin{cases} V_{\mathrm{I}2}(r, E) = \int_0^r r'^2 c_1(r', E) \mathfrak{F}_{\mathrm{I}}(r', E)\, \mathrm{d}r', \\ \mathfrak{F}_{\mathrm{I}}(r, E) = \int_{z_0}^{z_1} \mathfrak{f}_{\mathrm{I}}(z; r, E)\, \mathrm{d}z, \end{cases}$$

$$(7.19d)\quad \mathfrak{f}_{\mathrm{I}}(z; r, E) = \frac{z\{z^2 - \mu(r, E)\}}{\sqrt{2z^2 + \nu(r, E)}} \operatorname{tgh}^{-1} \frac{2z\sqrt{2z^2 + \nu(r, E)}}{3z^2 + \nu(r, E) - 1};$$

the limits of integration are

$$(7.20)\quad \begin{cases} z_0 = \sqrt{\mu(r, E)} \leqslant z \leqslant \zeta(r, E) = z_1, & \mu(r, E) \leqslant 0, \\ z_0 = 0 \leqslant z \leqslant \zeta(r, E) = z_1, & \mu(r, E) \geqslant 0. \end{cases}$$

The central part of the real optical potential appearing in eq. (7.19*b*) is given by eq. (6.11). The zero-energy limit of $V_{\mathrm{I}}^{\mathrm{C}}(r, E)$ at the origin is

$$V_{\mathrm{I}}^{\mathrm{C}}(0, 0) = (5M/4\pi^2)\, \mathfrak{c}_1(0, 0)\, \mathfrak{F}_{\mathrm{I}}(0, 0)\, V_{\mathrm{R}}^{\mathrm{C}}(0, 0)\,. \tag{7.21}$$

The model upon which the construction of potential (7.19) is based, however « frivolous » it may appear, seems nevertheless to be—about thirty years after its basic formulation—less ephemeral than many other « serious » ones.

According to definition (1.3*b*), the spin-orbit part of the imaginary potential turns out to be

$$V_{\mathrm{I}}^{\mathrm{SO}}(r, E) = 2(\lambda_1/s_0)^2 (\boldsymbol{\sigma}\cdot\boldsymbol{L})\, U_{\mathrm{I}}(r, E)\, V_{\mathrm{I}}^{\mathrm{C}}(r, E)\,, \tag{7.22a}$$

$$U_{\mathrm{I}}(r, E) = (s_0^2/2r)\boldsymbol{D} \ln \{V_{\mathrm{I}}^{\mathrm{C}}(r, E)\}\,. \tag{7.22b}$$

The zero-energy limit at the origin of the spin-orbit radial function is

$$U_{\mathrm{I}}(0, 0) = U_{\mathrm{R}}(0, 0) + U_0(0, 0)\,, \tag{7.23a}$$

$$U_0(0, 0) = \lim_{r\to 0} \frac{s_0^2}{2r} \left(\boldsymbol{D} \ln \{\mathfrak{c}_1(r, 0)\} + \boldsymbol{D} \ln \{\mathfrak{F}_{\mathrm{I}}(r, 0)\}\right). \tag{7.23b}$$

7·2. *Simulation of the radial dependence of the neutron-proton total cross-section.* – Accurate knowledge of the radial and energy behaviour of function $\mathfrak{s}(r, E)$, defined in eq. (7.14*a*), is crucially important in order to make the calculation of potential (7.19) reliable. We shall now outline the criterion adopted for the determination of the auxiliary functions $\mathfrak{c}_1(r, E)$ and $\mathfrak{c}_2(r, E)$. To this end we recall that the energy dependence of the neutron-proton total cross-section in Born approximation is given by the well-known formula

$$\sigma_{\mathrm{B}}(e) = a/(b + e)\,, \tag{7.24}$$

where a and b are two constants dependent on the parameters of the considered two-nucleon potential. The predicted energy behaviour of $\sigma_{\mathrm{B}}(e)$ is valid at high energies. One might attempt to reproduce the observed energy dependence of $\sigma_{\mathrm{np}}(e)$ at all energies by means of the expansion

$$\mathfrak{s}(e) = \sum_{n=0}^{\infty} (-1)^n \sigma_{\mathrm{B}}^{(n)}(e)\,, \tag{7.25}$$

where $\sigma_{\mathrm{B}}^{(n)}(e) = a_n/(b_n + e)$ are Born-like functions with unknown coefficients a_n and b_n. The use of eq. (7.25) in fitting procedures makes it necessary to over-

come several difficulties due to the large number of parameters needed. For calculational purposes it is convenient to replace eq. (7.25) with the new arbitrary function

$$\tilde{s}(e) = \sigma_{np}(0) \sum_{n=0}^{\infty} (-1)^n \{\sigma_B(e)/\sigma_{np}(0)\}^n . \tag{7.26}$$

Since for $e \neq 0$ $\sigma_B(e)/\sigma_{np}(0) < 1$, series (7.26) is rapidly convergent with increasing energy. This makes it possible to sum the series and one finds that

$$\tilde{s}(e) = \frac{\sigma_{np}(0)\sigma_B(e)}{\sigma_{np}(0) + \sigma_B(e)} = \frac{c_1}{c_2 + e} , \tag{7.27a}$$

$$c_1 = a , \qquad c_2 = \{a/\sigma_{np}(0)\} + b . \tag{7.27b}$$

Note that the Born-like function (7.27*a*) identifies with the Born formula (7.25) at the limit of high energies. The two unknown parameters $c_1 = c_1(E)$ and $c_2 = c_2(E)$ are determined by means of least-squares techniques. The same procedure, although technically far more complicated, can be used in order to determine the two unknown functions $c_1(r, E)$ and $c_2(r, E)$ appearing in eq. (7.14*a*). The limit of maximum variability of the relative momentum $P(r, E)$ is

$$P_0(r, E) = \tfrac{1}{2}\{p(r, E) - \chi(r)\} \leqslant P \leqslant \tfrac{1}{2}\{p(r, E) + \chi(r)\} = P_1(r, E) . \tag{7.28}$$

The corresponding energies in the centre-of-mass system are

$$\begin{cases} e_0(r, E) = e\{P_0(r, E)\} = \frac{1}{2}\left\{\sqrt{E} - V_R^C(r, E) - \sqrt{\varepsilon(r)}\right\}^2 , \\ e_1(r, E) = e\{P_1(r, E)\} = \frac{1}{2}\left\{\sqrt{E} - V_R^C(r, E) + \sqrt{\varepsilon(r)}\right\}^2 ; \end{cases} \tag{7.29}$$

it follows that for given values of r and E energy $e(r, E)$ varies in the interval

$$e_0(r, E) \leqslant e \leqslant e_1(r, E) . \tag{7.30}$$

The unknown functions $c_i(r, E)$ $(i = 1, 2)$ are determined by searching for the minimum of the mean-square deviation

$$\sum_{e_0 \leqslant e_i \leqslant e_1} \left[\frac{\sigma_{np}(e_i) - \tilde{s}(r, e_i)}{\Delta\sigma_{np}(e_i)}\right]^2 = \text{minimum} , \tag{7.31}$$

where $\tilde{s}(r, E)$ is expressed by eq. (7.14). More details will be given in subsect. **9**·6.

7·3. *Outline of a computational procedure.* – The calculation of integrals (7.19) is difficult and time-consuming. We shall now outline a procedure suitable for obtaining reliable numerical results. With this aim in view we define the function

$$\mathfrak{f}_0(z; r, E) = \frac{2z\sqrt{2z^2 + \nu(r, E)}}{3z^2 + \nu(r, E) - 1}, \tag{7.32}$$

and re-write eq. (7.19*d*) as

$$\mathfrak{f}_1(z; r, E) = \frac{z\{z^2 - \mu(r, E)\}}{\sqrt{2z^2 + \nu(r, E)}} \operatorname{tgh}^{-1} \mathfrak{f}_0(z; r, E) . \tag{7.33}$$

Since for $\mathfrak{f}_0^2 = \mathfrak{f}_0^2(z; r, E) < 1$ one has

$$\operatorname{tgh}^{-1} \mathfrak{f}_0 = \ln \sqrt{(1 + \mathfrak{f}_0)/(1 - \mathfrak{f}_0)} \simeq \mathfrak{f}_0 + \tfrac{1}{3} \mathfrak{f}_0^3 , \tag{7.34}$$

it follows that eq. (7.33) becomes ($\mathfrak{f}_1 \simeq \mathfrak{g}_1$)

$$\mathfrak{g}_1(z; r, E) = \frac{2z^2\{z^2 - \mu(r, E)\}}{3z^2 + \nu(r, E) - 1} + \frac{8z^4\{z^2 - \mu(r, E)\}\{2z^2 + \nu(r, E)\}}{3\{3z^2 + \nu(r, E) - 1\}^3} . \tag{7.35}$$

The approximate expression (7.35) is not valid at low incident energies and in proximity of $r = 0$. Taking into account that the interval of integration of $\mathfrak{f}_1(z; r, E)$ is smaller than 1 and decreases as r and E increase, we replace function $\mathfrak{g}_1(z; r, E)$ by the new function

$$\mathfrak{h}_1(z; r, E) = \frac{2\mathfrak{a}(z; r, E) z^2\{z^2 - \mu(r, E)\}}{3z^2 + \nu(r, E) - 1} , \tag{7.36}$$

where $\mathfrak{a}(z; r, E)$ is an unknown function ensuring the validity of approximation (7.36) also at low energies and in proximity of $r = 0$. Function $\mathfrak{a}(z; r, E) \geqslant 1$ is simply determined by solving equation

$$\mathfrak{f}_1(z; r, E) = \mathfrak{h}_1(z; r, E) . \tag{7.37}$$

From eq. (7.37) one obtains

$$\begin{cases} \mathfrak{a}(z; r, E) = \dfrac{3z^2 + \nu(r, E) - 1}{2z\sqrt{2z^2 + \nu(r, E)}} \operatorname{tgh}^{-1} \mathfrak{f}_0(z; r, E) , \\ \mathfrak{a}(0; r, E) = 1 . \end{cases} \tag{7.38}$$

Inspection of eq. (7.38) shows that an excellent approximation of $\mathfrak{a}(z; r, E)$ is

$$\mathfrak{a}(z; r, E) = \mathfrak{A}(r, E) + \mathfrak{B}(r, E)\, z \,, \tag{7.39a}$$

$$\begin{cases} \mathfrak{A}(r, E) = \dfrac{z_1 \mathfrak{a}(z_0; r, E) - z_0 \mathfrak{a}(z_1; r, E)}{z_1 - z_0} \,, \\ \mathfrak{B}(r, E) = \dfrac{\mathfrak{a}(z_1; r, E) - \mathfrak{a}(z_0; r, E)}{z_1 - z_0} \,, \end{cases} \tag{7.39b}$$

where z_0 and z_1 are given in (7.20). The integral appearing in (7.19c) becomes

$$\mathfrak{F}_1(r, E) = \int_{z_0}^{z_1} \mathfrak{h}_1(z; r, E)\, \mathrm{d}z = \tfrac{2}{9} \mathfrak{G}_1(r, E) \,, \tag{7.40a}$$

$$\mathfrak{G}_1(r, E) = \mathfrak{A}(r, E) \mathfrak{G}_{11}(r, E) + \mathfrak{B}(r, E) \mathfrak{G}_{12}(r, E) \,, \tag{7.40b}$$

$$\begin{cases} \mathfrak{G}_{11}(r, E) = \\ \quad = z_1^3 - z_0^3 - (3\mu + \nu - 1) \left[z_1 - z_0 - \sqrt{\dfrac{\nu - 1}{3}}\, \mathrm{tg}^{-1} \dfrac{(z_1 - z_0)\sqrt{3(\nu - 1)}}{3 z_0 z_1 + \nu - 1} \right] , \\ \mathfrak{G}_{12}(r, E) = \tfrac{1}{2}(z_1^2 - z_0^2) \{ \tfrac{3}{5}(z_1^2 + z_0^2) - 3\mu - \nu + 1 \} + \\ \qquad + \tfrac{1}{6}(3\mu + \nu - 1)(\nu - 1) \ln \{ (3 z_1^2 + \nu - 1)/(3 z_0^2 + \nu - 1) \} \,, \end{cases} \tag{7.40c}$$

where $\mu = \mu(r, E)$ and $\nu = \nu(r, E)$. The procedure outlined above provides a useful tool for numerical controls.

7·4. *Critical remarks.* – As is well known, the sum of a central and spin-orbit part of the Thomas type appears, within the limits of validity of the impulse approximation, to be a plausible form of the real optical potential constructed by taking into account the spin dependence of the nucleon-nucleon forces. There is no theoretical justification in assuming that this might be even approximately true for the imaginary part of the optical potential: indeed it has to be stressed that definitions (1.2b) and (1.3b) are entirely arbitrary. Nevertheless, RIESENFELD and WATSON [19] have attempted to construct the optical potential in the form (1.1), where

$$V_{\mathrm{R}}(r, E) = V_{\mathrm{R}}^{\mathrm{C}}(E)\, \varrho(r) + V_{\mathrm{R}}^{\mathrm{SO}}(E)(\boldsymbol{\sigma} \cdot \boldsymbol{L})(1/\mu^2)\, \boldsymbol{D}\{(1/r)\, \varrho(r)\} \,, \tag{7.41a}$$

$$V_{\mathrm{I}}(r, E) = V_{\mathrm{I}}^{\mathrm{C}}(E)\, \varrho(r) + V_{\mathrm{I}}^{\mathrm{SO}}(E)(\boldsymbol{\sigma} \cdot \boldsymbol{L})(1/\mu^2)\, \boldsymbol{D}\{(1/r)\, \varrho(r)\} \,, \tag{7.41b}$$

μ being the inverse Compton wave length of the pion. It should be apparent that potentials $V_{\mathrm{R}}(r, E)$ and $V_{\mathrm{I}}(r, E)$ built up in the preceding sections

[19] W. B. RIESENFELD and K. M. WATSON: *Phys. Rev.*, **102**, 1157 (1958).

are substantially different from Riesenfeld and Watson's either because the radial and energy dependence is not arbitrarily factorized as in eqs. (7.41), or because the quantitative behaviour for a given density distribution $\varrho(r)$ is governed by the stability prescriptions of the target nucleus and by the energy dependence of the neutron-proton total cross-section, without resorting to the use of asymptotic nucleon-nucleon phase shifts, whose reliability is questionable except at very low energies. It has already been pointed out that the central part of the real optical potential can be considered approximately proportional to $\varrho(r)$ only at high energies and at large distances: according to our scheme potentials (7.41) are not correct at low energies and at distances smaller than the r.m.s. radius of the target nucleus. Furthermore, the proportionality to $\varrho(r)$ of the central part of the imaginary potential is a very rough approximation of $V_{\mathrm{I}}^{\mathrm{C}}(r, E)$ expressed by eqs. (7.19): such an approximation, however, is not consistent with the shell model prescription, which will be discussed in sect. **8**. Potential (7.19) probably represents the most refined result obtainable from the so-called « frivolous model », outlined by CLEMENTEL and VILLI long ago ([20]), for finite even-even nuclei in the framework of a uniform distribution of infinitely extended nuclear matter. It is worthwhile emphasizing that potential (7.19) is strictly related to the stability conditions of the nucleus and consequently is grounded on a well-determined description of the nucleus ground state, summarized by the predicted values of the compressibility modulus. Furthermore, potential (7.19) discloses the inextricable links existing between the imaginary part of the optical potential and the real one: the former cannot be evaluated independently of the latter. It is interesting to compare the deduction of $V_{\mathrm{I}}^{\mathrm{C}}(r, E)$, expressed by eqs. (7.19), with those performed by HAYAKAWA, KAWAY and KIKUCHI ([21]), by HARADA and ODA ([22]), by GOMES ([23]), by LEMMER, MARIS and TANG ([24]), by GREENLESS, PYLE and TANG ([25]) and by SINHA and DUGGAN ([26]).

The existence of a minimum of $V_{\mathrm{I}}^{\mathrm{C}}(r, E)$ as a function of E for a given distance $r_{\mathrm{m}} = r_{\mathrm{m}}(E)$ and, *vice versa*, the existence of a minimum as a function of r for a given energy $E_{\mathrm{m}} = E_{\mathrm{m}}(r)$ are strictly dependent on the skin thickness of the target nucleus; of course, spin-orbit potential $V_{\mathrm{I}}^{\mathrm{SO}}(r, E)$ is zero at $r_{\mathrm{m}} = r_{\mathrm{m}}(E)$. The behaviour of the imaginary potential should be properly considered in choosing the more convenient energy when performing differential cross-section and polarization measurements of nucleons elastically and inelas-

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([20]) E. CLEMENTEL and C. VILLI: *Nuovo Cimento*, **1**, 176 (1955).
([21]) S. HAYAKAWA, M. KAWAY and K. KIKUCHI: *Prog. Theor. Phys.*, **13**, 415 (1955).
([22]) K. HARADA and N. ODA: *Prog. Theor. Phys.*, **21**, 260 (1959).
([23]) L. C. GOMES: *Phys. Rev.*, **116**, 1226 (1959).
([24]) R. H. LEMMER, T. A. J. MARIS and Y. C. TANG: *Nucl. Phys.*, **12**, 619 (1959).
([25]) G. W. GREENLESS, G. J. PYLE and Y. C. TANG: *Phys. Rev.*, **171**, 1115 (1968).
([26]) B. SINHA and F. DUGGAN: *Nucl. Phys. A*, **226**, 31 (1974). See also B. SINHA: *Phys. Rev. C*, **11**, 1546 (1975).

tically scattered by even-even nuclei, in order to check the quantitative predictions of the theory. As is well known, the comparison of the experimental evidence (for example for ^{4}He and ^{12}C) with the calculated results shows significant discrepancies, due to experimental difficulties and to the inadequacy of the simplifying assumptions invoked to make the scattering problem tractable. At high nucleon energies the presence or absence of a dip in the measured polarization (usually ignored because of the uncertainties in the nuclear-well shape and associated parameters) is determined in our theoretical scheme by the stability conditions of the target nucleus, which govern the quantitative behaviour of the optical potential. This is also true for the diffraction minima in the differential cross-sections. It would be interesting to ascertain if the imaginary part of the spin-orbit potential tends to fill up the troughs of the minima predicted by conventional treatments (Born and WKB approximations), which have never been observed experimentally. The influence of the imaginary spin-orbit potential on the polarization has never been investigated. In this connection it is worthwhile pointing out that, if the central part of the real optical potential is smaller or equal to the real part of the imaginary one, large polarizations can be predicted even if $V_{\mathrm{R}}^{\mathrm{so}}(r, E)$ is quite small and $V_{\mathrm{I}}^{\mathrm{so}}(r, E)$ is put equal to zero. For incident energies above meson production in nucleon-nucleon collisions, one should expect parameter λ_{I} to appear as depending also on the momentum transfer. At such energies, however, the description of the optical-model interaction outlined in this paper is not reliable.

8. – Implications of the shell model requirement fulfilled by the central part of the imaginary potential.

We shall now bring to light an important link existing between $V_{\mathrm{R}}^{\mathrm{c}}(r, E)$ and $V_{\mathrm{I}}^{\mathrm{c}}(r, E)$, which is concealed in the complicated dependence of the former on the latter. As is well known, the basic assumption of the shell model is that the nucleons are embedded in a real potential well. Then, according to model (2.15), a negative energy $w^*(r)$ must exist such that

$$V_{\mathrm{I}}^{\mathrm{c}}\{r, w^*(r)\} = 0 . \tag{8.1}$$

It is seen from eqs. (7.19) that condition (8.1) is fulfilled provided

$$z_0\{r, w^*(r)\} = z_1\{r, w^*(r)\} ; \tag{8.2}$$

it follows that $\zeta\{r, w^*(r)\} - 1$, *i.e.*

$$p\{r, w^*(r)\} - \chi(r) . \tag{8.3}$$

From eqs. (3.30*b*), (6.1) and (8.3) one obtains $V_{\mathrm{R}}^{\mathrm{c}}\{r, w^*(r)\}$ implicitly dependent on $w^*(r)$, namely

$$V_{\mathrm{R}}^{\mathrm{c}}\{r, w^*(r)\} = \beta v_0\{\chi(r)\}\, Y\{\beta\chi(r)\}\,, \tag{8.4}$$

where $Y\{\beta\chi(r)\}$ is a Yukawa function of the local maximum momentum $\chi(r)$,

$$Y\{\beta\chi(r)\} = \exp\left[-\beta\chi(r)\right]/\beta\chi(r)\,. \tag{8.5}$$

The negative energy $E \equiv w^*(r)$ fulfils equation

$$Y\{\beta\chi(r)\} = Y\left\{(\beta\chi(0)/\sqrt{\varepsilon_0(\eta_0)})[w^*(r) - V_{\mathrm{R}}^{\mathrm{c}}\{r, w^*(r)\}]^{\frac{1}{2}}\right\} \tag{8.6}$$

obtained by equating eqs. (8.4) and (6.11). A challenging task is to relate $V_{\mathrm{R}}^{\mathrm{c}}\{r, w^*(r)\}$ to the single-particle potential energy (3.27) evaluated at $p = \chi(r)$,

$$v\{\chi(r), \chi(r)\} = (C_1\alpha)\{\alpha\chi(r)\}^2 j_0\{\alpha\chi(r)\}\, j_1\{\alpha\chi(r)\}\,. \tag{8.7}$$

It is heuristically interesting to note that $v\{\chi(r), \chi(r)\}$ fulfils the nonhomogeneous differential equation

$$(\mathfrak{D} + 4\alpha^2)v\{\chi(r), \chi(r)\} = 4(C_1\alpha)\alpha^3\chi(r)\, j_1^2\{\alpha\chi(r)\}\,, \tag{8.8}$$

where the differential operator appearing in (8.8) is defined in subsect. **3**·1; the potential energy of the nucleus can be written in the form

$$V(s, \eta) = (1/\pi\alpha^2)\int_0^\infty r^2\chi^3(r)(\mathfrak{D} + 4\alpha^2)\, v\{\chi(r), \chi(r)\}\, \mathrm{d}r\,. \tag{8.9}$$

From eq. (8.4) it is ascertained that the equality holds

$$V_{\mathrm{R}}^{\mathrm{c}}\{0, w^*(0)\} = v\{\chi(0), \chi(0)\}\,. \tag{8.10}$$

Function $w^*(r)$ can be written in the form

$$w^*(r) = \varphi(r) + V_{\mathrm{R}}^{\mathrm{c}}\{r, w^*(r)\}\,, \tag{8.11}$$

where $\varphi(r)$ is a function numerically determined from eq. (8.6). For $r = 0$ it is found that

$$\varphi(0) = \varepsilon_0(\eta_0)\,; \tag{8.12}$$

in this case $w^*(0)$ is equal to $w(0)$ defined by eq. (4.27).

A priori one might be tempted to identify $w^*(r)$ with the total energy of the probe nucleon embedded in the field of the target nucleus at distance r from the origin. However, it is difficult to find convincing arguments to support this interpretation. In fact, finite nuclei do not possess the properties of nuclear matter which derive from the fact that the energy of a particle at the Fermi surface is the extra energy which would be obtained by adding one particle without changing its volume, while the average energy is the extra energy which would be obtained by adding one particle to the nuclear system without changing its density: for infinite nuclear matter these two quantities are equal when the system is in equilibrium at zero pressure. The clue to the problem lies in the boundary conditions associated to eq. (2.14), whose hyperbolic nature accounts for the propagation properties of nucleons in a highly dispersive nuclear medium. The three particular integrals $v_\infty(p, \varkappa)$ considered in subsect. **2**'1 correspond to three different specifications of the Cauchy problem associated to eq. (2.1); the boundary conditions are implicitly determined by the choice of the two-body potential and by the calculation procedure used to evaluate $v_\infty(p, \varkappa)$. Both equations (2.7) and (2.12) possess two one-parameter families of characteristic curves. The solutions of eq. (2.14) have been determined by taking into account the stability prescriptions (4.25) and (4.29) and the optical-model requirement (6.4): according to the Cauchy-Kowalewski theorem these prescriptions imply definite analytic Cauchy constraints on a non-characteristic boundary line. Through the shell model requirement (8.1), the imaginary potential leads to eq. (8.6): thus, the unknown function $w^*(r)$ has to be extracted from a characteristic line implicitly dependent on the real optical potential. This matter deserves further investigation. In the meanwhile we are unable to offer a fundamental explanation (if any) of the radial behaviour of $w^*(r)$, shown in fig. 3 for ^{4}He, ^{12}C and ^{16}O, in comparison with the corresponding $w(r)$.

9. – The optical interaction of nucleons with ^{4}He, ^{12}C and ^{16}O.

9'1. *Main features of infinitely extended nuclear matter.* – Before undertaking numerical calculations for finite even-even nuclei, it is worthwhile to summarize briefly the results obtainable in the framework of the idealized conception of infinitely extended nuclear matter. Our starting point is provided by the hyperbolic partial differential equation (2.1). Its factorable single-particle solutions are

$$v_\infty(p, \varkappa) = (C_1\alpha_\infty)(\alpha_\infty\varkappa)^2 j_0(\alpha_\infty p)\, j_1(\alpha_\infty\varkappa)\,, \qquad p \leqslant \varkappa\,, \tag{9.1a}$$

$$\begin{cases} v_\infty(p, \varkappa) = (C_1\alpha_\infty) f(\varkappa)\{\exp[-\beta_\infty p]/p\}\,, \\ f(\varkappa) = (\alpha_\infty\varkappa)^2 j_0(\alpha_\infty\varkappa)\, j_1(\alpha_\infty\varkappa)\exp[\beta_\infty\varkappa]\,, \end{cases} \qquad p \geqslant \varkappa\,. \tag{9.1b}$$

As far as the physical implications are concerned, the fundamental mathematical property of eq. (2.1) is expressed by the identity

$$\mathcal{Q}_\infty(\varkappa) = \varkappa^2 v_\infty(\varkappa, \varkappa) - \int_0^{\varkappa} p^2 \{\partial_\varkappa v_\infty(p, \varkappa)\}\, \mathrm{d}p = 0 \,. \tag{9.2}$$

The total energy (2.3) of the nuclear system is

$$W_\infty(\varkappa) = 3A\varkappa^2/2M + (3A/2)(C_1\alpha_\infty)(\alpha_\infty\varkappa)\, j_1^2(\alpha_\infty\varkappa) \,. \tag{9.3}$$

Owing to theorem (9.2), from the first of the saturation prescriptions (2.4) one obtains the separation energy relation (2.5). It follows that at equilibrium density potentials (9.1) become

$$\begin{cases} v_\infty(p, \varkappa_\mathrm{F}) = \{(b_\mathrm{V} - \varepsilon_\mathrm{F})/j_0(\alpha_\infty\varkappa_\mathrm{F})\}\, j_0(\alpha_\infty p) \,, & p \leqslant \varkappa_\mathrm{F}, \\ v_\infty(p, \varkappa_\mathrm{F}) = (b_\mathrm{V} - \varepsilon_\mathrm{F})\varkappa_\mathrm{F} \exp[\beta_\infty\varkappa_\mathrm{F}]\{\exp[-\beta_\infty p]/p\} \,, & p \geqslant \varkappa_\mathrm{F}. \end{cases} \tag{9.4}$$

The compressibility modulus turns out to be

$$\begin{cases} K_\infty(\varkappa_\mathrm{F}) = (\varkappa_\mathrm{F}^2/A)\{\mathrm{d}^2 W_\infty(\varkappa)/\mathrm{d}\varkappa^2\}_{\varkappa=\varkappa_\mathrm{F}} = 6\varepsilon_\mathrm{F} + 3(b_\mathrm{V} - \varepsilon_\mathrm{F})\, H(\alpha_\infty\varkappa_\mathrm{F}) \,, \\ H(\alpha_\infty\varkappa_\mathrm{F}) = (\alpha_\infty\varkappa_\mathrm{F})\{j_0^2(\alpha_\infty\varkappa_\mathrm{F}) - j_1^2(\alpha_\infty\varkappa_\mathrm{F})\}/j_0(\alpha_\infty\varkappa_\mathrm{F})\, j_1(\alpha_\infty\varkappa_\mathrm{F}) \,. \end{cases} \tag{9.5}$$

The parameters $\alpha_\infty\varkappa_\mathrm{F}$ and $\beta_\infty\varkappa_\mathrm{F}$ are given by the relations

$$\begin{cases} \operatorname{tg}(\alpha_\infty\varkappa_\mathrm{F}) = \dfrac{15(b_\mathrm{V} - \varepsilon_\mathrm{F})\alpha_\infty\varkappa_\mathrm{F}}{15(b_\mathrm{V} - \varepsilon_\mathrm{F}) - 2(5b_\mathrm{V} - 3\varepsilon_\mathrm{F})(\alpha_\infty\varkappa_\mathrm{F})^2} \,, \\ \beta_\infty\varkappa_\mathrm{F} = -\alpha_\beta\varkappa_\mathrm{F} \operatorname{ctg}(\alpha_\infty\varkappa_\mathrm{F}) \geqslant 0 \,; \end{cases} \tag{9.6}$$

the former of eqs. (9.6) is deduced from the second of the saturation prescriptions (2.4), the latter is deduced from the continuity relations between potentials (9.1) at $p = \varkappa_\mathrm{F}$. Note that the last of eqs. (9.6) implies that it must necessarily be

$$\pi/2 < \alpha_\infty\varkappa_\mathrm{F} < \pi \,. \tag{9.7}$$

The central part of the real optical potential is

$$\begin{cases} V_\mathrm{R}^\mathrm{C}(E, \varkappa_\mathrm{F}) \left\{\dfrac{E - V_\mathrm{R}^\mathrm{C}(E, \varkappa_\mathrm{F})}{\varepsilon_\mathrm{F}}\right\}^{\frac{1}{2}} = F_\mathrm{R}(\varkappa_\mathrm{F}) \exp\left[-\beta_\infty\varkappa_\mathrm{F}\left\{\dfrac{E - V_\mathrm{R}^\mathrm{C}(E, \varkappa_\mathrm{F})}{\varepsilon_\mathrm{F}}\right\}^{\frac{1}{2}}\right], \\ F_\mathrm{R}(\varkappa_\mathrm{F}) = (b_\mathrm{V} - \varepsilon_\mathrm{F}) \exp[\beta_\infty\varkappa_\mathrm{F}] \,. \end{cases} \tag{9.8}$$

It is seen that inequality (6.4), which for finite nuclei ensures the internal consistency of the optical scheme, is always satisfied for nuclear matter. Furthermore, in the considered case, the subject discussed in sect. **7** is simply condensed in the following equations:

$$V_{\mathrm{I}}^{\mathrm{C}}(E, \varkappa_{\mathrm{F}}) = -(5M/4\pi^2)\, \mathrm{c}_1(E)\{E - V_{\mathrm{R}}^{\mathrm{C}}(E, \varkappa_{\mathrm{F}})\} F_1(E, \varkappa_{\mathrm{F}}) , \tag{9.9a}$$

$$\begin{cases} F_1(E, \varkappa_{\mathrm{F}}) = \displaystyle\int_{z_0}^{z_1} f_1(z; E, \varkappa_{\mathrm{F}})\, \mathrm{d}z , \\ f_1(z; E, \varkappa_{\mathrm{F}}) = \dfrac{z\{z^2 - \mu(E, \varkappa_{\mathrm{F}})\}}{\sqrt{2z^2 + \nu(E, \varkappa_{\mathrm{F}})}} \operatorname{tgh}^{-1} \dfrac{2z\sqrt{2z^2 + \nu(E, \varkappa_{\mathrm{F}})}}{3z^2 + \nu(E, \varkappa_{\mathrm{F}}) - 1} , \end{cases} \tag{9.9b}$$

$$\begin{cases} \zeta(E, \varkappa_{\mathrm{F}}) = \varkappa_{\mathrm{F}}/p(E, \varkappa_{\mathrm{F}}) , \\ \mu(E, \varkappa_{\mathrm{F}}) = 2\zeta^2(E, \varkappa_{\mathrm{F}}) - 1 , \qquad \nu(E, \varkappa_{\mathrm{F}}) = (2/\varepsilon_{\mathrm{F}})\{\varepsilon_{\mathrm{F}} + \mathrm{c}_2(E)\, \zeta^2(E, \varkappa_{\mathrm{F}})\} ; \end{cases} \tag{9.9c}$$

$$\begin{cases} z_0 = \sqrt{\mu(E, \varkappa_{\mathrm{F}})} \leqslant z \leqslant \zeta(E, \varkappa_{\mathrm{F}}) = z_1 , & \mu(E, \varkappa_{\mathrm{F}}) \leqslant 0 , \\ z_0 = 0 \leqslant z \leqslant \zeta(E, \varkappa_{\mathrm{F}}) = z_1 , & \mu(E, \varkappa_{\mathrm{F}}) \geqslant 0 ; \end{cases} \tag{9.9d}$$

the meaning of the coefficients $\mathrm{c}_i(E)$ $(i = 1, 2)$ is specified in subsect. **7**'2. For infinitely extended nuclear matter the subject discussed in sect. **8** is simply condensed in the following relations:

$$V_{\mathrm{I}}^{\mathrm{C}}(b_{\mathrm{V}}, \varkappa_{\mathrm{F}}) = 0 , \qquad V_{\mathrm{R}}^{\mathrm{C}}(b_{\mathrm{V}}, \varkappa_{\mathrm{F}}) = v_\infty(\varkappa_{\mathrm{F}}, \varkappa_{\mathrm{F}}) . \tag{9.10}$$

It has been shown in I that a consistent overall description of nuclear matter can be obtained assuming the input quantities b_{V} and $r_0 = (9\pi/8)^{\frac{1}{3}}/\varkappa_{\mathrm{F}}$ localized

TABLE I. – *Numerical features of nuclear matter described by eq.* (2.1). *Quantities* ε_{F}, $v_\infty(p, \varkappa_{\mathrm{F}})$, K_∞, $V_{\mathrm{R}}^{\mathrm{C}}(0, \varkappa_{\mathrm{F}})$ *and* $V_{\mathrm{I}}^{\mathrm{C}}(0, \varkappa_{\mathrm{F}})$ *are expressed in* MeV *and the Fermi momentum* $\varkappa_{\mathrm{F}}$ *in* $(\mathrm{fm})^{-1}$ *units.*

	$b_{\mathrm{V}} = -14.0$ MeV		$b_{\mathrm{V}} = -15.0$ MeV		$b_{\mathrm{V}} = -16.0$ MeV	
r_0 in fm	1.0	1.2	1.0	1.2	1.0	1.2
$\varkappa_{\mathrm{F}}$	1.523 2	1.269 4	1.523 2	1.269 4	1.523 2	1.269 4
ε_{F}	48.098	33.401	48.098	33.401	48.098	33.401
$\alpha_\infty \varkappa_{\mathrm{F}}$	1.916 2	1.999 7	1.931 8	2.015 6	1.946 5	2.030 8
$\beta_\infty \varkappa_{\mathrm{F}}$	0.689 42	0.914 49	0.729 24	0.960 88	0.767 69	1.005 3
$v_\infty(0, \varkappa_{\mathrm{F}})$	−126.46	−104.23	−130.29	−108.08	−134.12	−111.93
$v_\infty(\varkappa_{\mathrm{F}}, \varkappa_{\mathrm{F}})$	− 62.098	− 47.401	− 63.098	− 48.401	− 64.098	− 49.401
K_∞	198.44	−175.63	207.43	184.28	216.36	192.88
$V_{\mathrm{R}}^{\mathrm{C}}(0, \varkappa_{\mathrm{F}})$	− 55.192	− 39.868	− 55.577	− 40.195	− 55.952	− 40.513
$V_{\mathrm{I}}^{\mathrm{C}}(0, \varkappa_{\mathrm{F}})$	− 0.527 08	− 0.576 11	− 0.582 98	− 0.631 45	− 0.639 46	− 0.686 7

in the intervals

$$b_v = -(15.0 \pm 1.0)\ \text{MeV}\,, \qquad r_0 = (1.10 \pm 0.10) \cdot 10^{-13}\ \text{cm}\,. \tag{9.11}$$

In view of the comparison with the results obtained for finite nuclei, we give in table I an exhaustive numerical view of infinitely extended nuclear matter pictured according to eq. (2.1).

9·2. *A mathematical method for extracting nuclear information from electron-nucleus scattering data.* – A realistic evaluation of parameter r_0, defined through *ansatz* (2.31), is of crucial importance for the physical reliability of the forthcoming calculations. Our aim is to outline an exact mathematical method for obtaining nuclear information from the analyses of electron-nucleus scattering data ([27]). The method will also bring to light the very reasons which led us to choose density (2.17) for constructing the optical potential for nucleons scattered from ^{4}He, ^{12}C and ^{16}O.

We define as nuclear density $\varrho_N(r)$ the distribution of the centre of mass of the protons bound in the nuclear system. Consequently, the « observed » charge distribution $\varrho_{ch}(r)$, involved in the analyses of electron-nucleus scattering data, is

$$\varrho_{ch}(r) = \int \varrho_p(|\boldsymbol{r} - \boldsymbol{x}|)\, \varrho_N(x)\, \mathrm{d}\boldsymbol{x}\,, \tag{9.12}$$

where $\varrho_p(r)$ is the proton charge density. Equation (9.12) shows in three-dimensional space that the unknown function $\varrho_N(r)$ obeys one of Fredholm's equations of the first kind, the symmetric kernel being expressed by the proton density. Standard procedures seem inadequate to give exact solutions of eq. (9.12). To circumvent mathematical deadlocks we start from the equivalent equation

$$\varrho_{ch}(r) = -(1/4\pi)\, \nabla^2 V_{eN}(r)\,, \tag{9.13}$$

where $\nabla^2 = \boldsymbol{D}^2 + (2/r)\boldsymbol{D}$ is the radial part of the Laplacian operator and $V_{eN}(r)$ is the electron-nucleus potential energy

$$V_{eN}(r) = \int v_{ep}(|\boldsymbol{r} - \boldsymbol{x}|)\, \varrho_N(x)\, \mathrm{d}\boldsymbol{x}\,, \tag{9.14}$$

$v_{ep}(r)$ being the electron-proton potential for unit charge. Introducing the vector $\boldsymbol{X} = \boldsymbol{r} - \boldsymbol{x}$ and taking into account inequality (2.22), potential $V_{eN}(r)$

([27]) Recent information on nuclear-structure studies using electron scattering can be found in the reviews C. CIOFI DEGLI ATTI: *Prog. Part. Nucl. Phys.*, **3**, 163 (1980); J. HEISENBERG: *Nucl. Phys. A*, **396**, 391c (1983); B. FOIS: *Electron scattering and nuclear structure*, in *Proceedings of the International Conference on Nuclear Physics, Florence, 1983*, Vol. **2**, Invited Papers (Bologna, 1984), p. 221-246.

becomes

$$V_{eN}(r) = (2\pi/r)\Big[\int_0^r xI_1(x,r)\,\varrho_N(x)\,dx + \int_r^\infty xI_2(x,r)\,\varrho_N(x)\,dx\Big], \tag{9.15a}$$

$$I_1(x,r) = \int_{r-x}^{r+x} X\varrho_p(X)\,dX\,, \qquad I_2(x,r) = \int_{x-r}^{x+r} X\varrho_p(X)\,dX\,. \tag{9.15b}$$

Thus, from Poisson's equation (9.13) one obtains the new integral equation

$$\varrho_{ch}(r) = \mathscr{C}(0)\,\varrho_N(r) + (1/2r)\int_0^\infty xK(x,r)\,\varrho_N(x)\,dx\,, \tag{9.16a}$$

$$\mathscr{C}(r-x) = \tfrac{1}{2}\{(r-x)\,v_{ep}(r-x) + (x-r)\,v_{ep}(x-r)\}\,, \tag{9.16b}$$

$$\begin{cases} K(x,r) = \boldsymbol{D}\{(r-x)\,v_{ep}(r-x) - (r+x)\,v_{ep}(r+x)\}\,, & x \leqslant r\,, \\ K(x,r) = \boldsymbol{D}\{(r+x)\,v_{ep}(r+x) + (x-r)\,v_{ep}(x-r)\}\,, & x \geqslant r\,. \end{cases} \tag{9.16c}$$

At this stage we need to specify the proton density $\varrho_p(\boldsymbol{r})$. We assume the analytic expression

$$\begin{cases} \varrho_p(\boldsymbol{r}) = (\mu_p^2/4\pi)(a_p - 2b_p + b_p\mu_p r)\,Y(\mu_p r) + (1-a_p)\,\delta(\boldsymbol{r})\,, \\ Y(\mu_p r) = \exp[-\mu_p r]/\mu_p r\,, \end{cases} \tag{9.17a}$$

$$\int \varrho_p(\boldsymbol{r})\,d\boldsymbol{r} = 1\,; \tag{9.17b}$$

the r.m.s. radius is ($s_p = 1/\mu_p$)

$$R_p = s_p\sqrt{6(a_p + 2b_p)}\,. \tag{9.17c}$$

Parameters a_p, b_p and s_p are determined by fitting high-energy electron-proton scattering data. Density (9.17) underlies a variety of proton models which have not been fully explored. For $a_p = b_p = 0$ or, alternatively, for $a_p = 0$ and $s_p = 0$ ($\mu_p = \infty$) one has a pointlike model: in this case the density is expressed by the delta-function and from eq. (9.12) one has $\varrho_{ch}(r) = \varrho_N(r)$. Of course, the same result is obtained from eq. (9.16a) because $v_{ep}(r) = 1/r$ implies $\mathscr{C}(0) = 1$ and $K(x,r) = 0$ for $x \leqslant r$ and $x \geqslant r$. Three models have been successfully applied to fit the data: Drell's model ($a_p = 1$, $b_p = 0$), Hofstadter's model ($a_p = 1$, $b_p = \frac{1}{2}$) and the model developed by CLEMENTEL and VILLI ($a_p = \frac{6}{5}$, $b_p = 0$) ([28]). We recall that the Gaussian model (2.24)

([28]) Details on proton models can be found in the review book by R. HOFSTADTER: *Electron Scattering and Nuclear and Nucleon Structure* (W. A. Benjamin, Inc., New York, N.Y., 1963).

has been chosen only for illustrative purposes, because it allows one to perform the folding integral (2.21) exactly.

The electron-proton potential generated by density (9.17) reads

$$v_{ep}(r) = 1/r - a_p \mu_p Y(\mu_p r) - b_p \mu_p \exp[-\mu_p r] . \tag{9.18}$$

Substituting (9.18) into kernel (9.16*c*), one has

$$\begin{cases} K(x, r) = \\ \quad = [(a_p - b_p + b_p \mu_p r) \sinh(\mu_p x) - b_p \mu_p x \cosh(\mu_p x)] \exp[-\mu_p r] , & x \leqslant r , \\ K(x, r) = \\ \quad = [(a_p - b_p + b_p \mu_p x) \sinh(\mu_p r) - b_p \mu_p r \cosh(\mu_p r)] \exp[-\mu_p x] , & x \geqslant r . \end{cases} \tag{9.19}$$

The apparently hopeless task of solving eq. (9.16*a*) characterized by kernel (9.19) can be achieved by virtue of the transformations

$$\varrho_{ch}(r) = f_{ch}(r)/r , \qquad \varrho_N(r) = f_N(r)/r , \tag{9.20}$$

where functions $f_{ch}(r)$ and $f_N(r)$ must vanish at the origin linearly with r and fulfil the prescription

$$\int \varrho_{ch}(r)\, \mathrm{d}\boldsymbol{r} = 4\pi \int_0^\infty r f_N(r)\, \mathrm{d}r = 1 . \tag{9.21}$$

Consequently, from eq. (9.16*a*) one obtains

$$f_{ch}(r) = (1 - a_p) f_N(r) + (a_p - 2b_p) \mu_p \varphi(r) + 2 b_p \mu_p \psi(r) , \tag{9.22a}$$

$$\varphi(r) = \int_0^\infty \Phi(x, r) f_N(x)\, \mathrm{d}x , \qquad \psi(r) = \int_0^\infty \Psi(x, r) f_N(x)\, \mathrm{d}x , \tag{9.22b}$$

$$\begin{cases} \Phi(x, r) = \sinh(\mu_p x) \exp[-\mu_p r] , & x \leqslant r , \\ \Phi(x, r) = \sinh(\mu_p r) \exp[-\mu_p x] , & x \geqslant r , \end{cases} \tag{9.23a}$$

$$\begin{cases} \Psi(x, r) = \tfrac{1}{2}[(1 + \mu_p r) \sinh(\mu_p x) - \mu_p x \cosh(\mu_p x)] \exp[-\mu_p r], & x \leqslant r , \\ \Psi(x, r) = \tfrac{1}{2}[(1 + \mu_p x) \sinh(\mu_p r) - \mu_p r \cosh(\mu_p r)] \exp[-\mu_p x], & x \geqslant r . \end{cases} \tag{9.23b}$$

From eq. (9.22*a*), taking into account eq. (9.21), one deduces that functions

$\varphi(r)$ and $\psi(r)$ must fulfil the condition

$$4\pi\int_0^\infty r\varphi(r)\,\mathrm{d}r = 4\pi\int_0^\infty r\psi(r)\,\mathrm{d}r = s_\mathrm{p}\,. \tag{9.24}$$

The mathematical properties of kernels $\Phi(x, r)$ and $\Psi(x, r)$ are discussed in a preceding paper ([29]). Transformations (9.20) disclose that the following equations hold:

$$(s_\mathrm{p}^2\boldsymbol{D}^2 - 1)\varphi(r) = -s_\mathrm{p} f_\mathrm{N}(r)\,, \tag{9.25a}$$

$$(s_\mathrm{p}^2\boldsymbol{D}^2 - 1)\psi(r) = -\varphi(r)\,, \tag{9.25b}$$

$$(s_\mathrm{p}^4\boldsymbol{D}^4 - 2s_\mathrm{p}^2\boldsymbol{D}^2 + 1)\psi(r) = s_\mathrm{p} f_\mathrm{N}(r)\,. \tag{9.25c}$$

Equations (9.25*a*) and (9.25*b*) transform the integral equation (9.22*a*) into the nonhomogeneous fourth-order differential equation

$$\boldsymbol{O}_\mathrm{M}\psi(r) = s_\mathrm{p} f_\mathrm{ch}(r)\,, \tag{9.26a}$$

where $\boldsymbol{O}_\mathrm{M}$ is the model operator associated to the proton density (9.17),

$$\boldsymbol{O}_\mathrm{M} = (1 - a_\mathrm{p})s_\mathrm{p}^4\boldsymbol{D}^4 + (a_\mathrm{p} + 2b_\mathrm{p} - 2)s_\mathrm{p}^2\boldsymbol{D}^2 + 1\,. \tag{9.26b}$$

Thus, the problem is solved: once $\psi(r)$ is determined by solving eqs. (9.26) in agreement with prescription (9.24), eq. (9.25*b*) gives $\varphi(r)$ and, finally, eq. (9.25*a*) expresses $f_\mathrm{N}(r)$ as a linear differential transform of $f_\mathrm{ch}(r)$.

Let us restrict ourselves to Drell's model (model D) and to Hofstadter's model (model H); the Clementel-Villi model requires rather complicated treatment, which is beyond the scope of this paper. In the former case ($a_\mathrm{p} = 1$, $b_\mathrm{p} = 0$) one has

$$\boldsymbol{O}_\mathrm{D} = -s_\mathrm{p}^2\boldsymbol{D}^2 + 1\,; \tag{9.27}$$

then, from eqs. (9.26*a*) and (9.25*b*) one obtains $\varphi(r) = s_\mathrm{p} f_\mathrm{ch}(r)$ and eq. (9.25*a*) gives

$$f_\mathrm{N}^{(\mathrm{D})}(r) = (1 - s_\mathrm{p}^2\boldsymbol{D}^2)f_\mathrm{ch}(r)\,. \tag{9.28}$$

([29]) C. VILLI: *Atti e Memorie dell'Accademia Patavina di Scienze, Lettere ed Arti*, Vol. **90**, Parte II (1977-1978), p. 125.

In the latter case ($a_p = 1$, $2b_p = 1$) the model operator is $\boldsymbol{O}_H = 1$, *i.e.* $\psi(r) = s_p f_{ch}(r)$; from eq. (9.25*c*) one deduces that

$$f_N^{(H)}(r) = (1 - 2s_p^2 \boldsymbol{D}^2 + s_p^4 \boldsymbol{D}^4) f_{ch}(r) . \tag{9.29}$$

Assuming $\varrho_{ch}(r)$ expressed by the analytic form (2.19) it is found that

$$\begin{cases} \varrho_N(r) = (\varrho_0)_{ch} \left[\sum_{i=0}^{n} \eta_i (r/s_{ch})^{2i} \right] \exp\left[-(r/s_{ch})^2 \right] , \\ (\varrho_0)_{ch} = 2/\pi\sqrt{\pi}(2 + 3\eta_{ch}) s_{ch}^3 , \end{cases} \tag{9.30a}$$

$$\int \varrho_N(r)\, \mathrm{d}\boldsymbol{r} = 1 , \tag{9.30b}$$

where $n = 2$ for model D and $n = 3$ for model H. The parameters of the nuclear density function are found to be

$$\begin{cases} \eta_0^D = 1 - 6(\eta_{ch} - 1)\xi_D^2 , & \eta_0^H = 1 - 12(\eta_{ch} - 1)\xi_H^2 - \\ & \qquad - 60(2\eta_{ch} - 1)\xi_H^4 , \\ \eta_1^D = \eta_{ch} + 2(7\eta_{ch} - 2)\xi_D^2 , & \eta_1^H = \eta_{ch} + 4(7\eta_{ch} - 2)\xi_H^2 + \\ & \qquad + 20(15\eta_{ch} - 4)\xi_H^4 , \\ \eta_2^D = -4\eta_{ch}\xi_D^2 , & \eta_2^H = -8\eta_{ch}\xi_H^2 - 16(9\eta_{ch} - 1)\xi_H^4 , \\ \eta_3^D = 0 , & \eta_3^H = 16\eta_{ch}\xi_H^4 , \end{cases} \tag{9.31}$$

where $\xi = s_p/s_{ch}$. It is readily ascertained by successive partial integrations that both eqs. (9.28) and (9.29) lead to relation (2.27) among the r.m.s. radii R_{ch}, R_N and R_p of $\varrho_{ch}(r)$, $\varrho_N(r)$ and $\varrho_p(r)$, *i.e.*

$$R_N^2 = R_{ch}^2 - R_p^2 . \tag{9.32}$$

The exact relation (9.32) will be used to determine r_0, according to *ansatz* (2.31).

The electron-nucleus input parameters are listed in table II*a*. The experimetal value of the proton r.m.s. radius is $R_p = 0.77 \cdot 10^{-13}$ cm; length s_p, calculated according to eq. (9.17*c*) for the considered proton models, turns out to be

$$s_p^D = R_p/\sqrt{6} = 0.31 \cdot 10^{-13} \text{ cm} , \qquad s_p^H = R_p/\sqrt{12} = 0.22 \cdot 10^{-13} \text{ cm} ; \tag{9.33}$$

the drawback of the Gaussian model (2.24) is mainly due to the large value $s_p^G = 0.63 \cdot 10^{-13}$ cm.

TABLE IIa. – *Parameters characterizing the* 1p*-shell charge distribution determined from the analysis of high-energy electron scattering data. Lengths are given in* fm *units.*

	^{4}He	^{12}C	^{16}O
s_{ch}	1.31	1.64	1.77
η_{ch}	0	4/3	2
R_{ch}	1.61	2.40	2.64

TABLE IIb. – *Parameters characterizing nuclear density distributions, calculated using the values of* s_{ch} *and* η_{ch} *given in table* IIa. *Lengths are given in* fm *units.*

η	^{4}He		^{12}C		^{16}O		s	^{12}C	^{16}O
	D	H	D	H	D	H	η		
η_0	1.33	1.38	0.93	0.90	0.82	0.80	s_D	1.56	1.68
η_1	−0.22	−0.30	1.86	1.96	2.73	2.84	η_D	2.12	2.38
η_2	0	0.01	−0.19	−0.24	−0.24	−0.30	s_H	1.59	1.70
η_3	0	0	0	∼0	0	∼0.01	η_H	2.17	2.42

The parameters of density (9.30), evaluated according to eqs. (9.31), are given in table II*b*. The value $\eta_1 < 0$ for ^{4}He and $\eta_2 < 0$ for ^{12}C and ^{16}O are unsatisfactory: the second- and fourth-order radial derivatives probably amplify the inadequacies concealed in the analytic form of $\varrho_{ch}(r)$ and $\varrho_p(r)$. However, since the η's have been calculated using an exact procedure, we believe that the predicted behaviour of $\varrho_N(r)$ is substantially correct. This suggests squeezing the overall information therein contained into the analytically simpler density function $\varrho(r)$, expressed by eq. (2.17). This goal can be reached by seeking for the minimum of the mean-square distance between $\varrho_N(r)$ and $\varrho(r)$, *i.e.*

$$\begin{cases} M(s,\eta) = \int [\varrho_N(r) - \varrho(r)]^2 \, \mathrm{d}r \,, \\ \partial_s M(s,\eta) = \partial_\eta M(s,\eta) = 0 \,. \end{cases} \tag{9.34}$$

The nuclear parameters s and η are determined by solving the system of equations

$$\begin{cases} \int_0^\infty x^2(3-2x^2)\varrho_N(sx)\exp[-x^2]\,\mathrm{d}x = \sqrt{\pi}/2(3/64)(4+\eta)\varrho_0(s,\eta) \,, \\ \int_0^\infty x^4(5-2x^2)\varrho_N(sx)\exp[-x^2]\,\mathrm{d}x = \sqrt{\pi}/2(15/256)(4+3\eta)\varrho_0(s,\eta) \,, \end{cases} \tag{9.35a}$$

$$\varrho_0(s,\eta) = 2/\pi\sqrt{\pi}(2+3\eta)s^3 \,. \tag{9.35b}$$

It is worthwhile anticipating that the values (s_D, η_D) and (s_H, η_H), given in table II*b*, are in excellent agreement with those determined in subsect. **9**'4, exclusively from nuclear stability and optical-model prescriptions. This is not true for ^{4}He. In conclusion, the choice of the density function (2.17) seems to be well justified at least for ^{12}C and ^{16}O.

9'3. *Numerical results obtained from model* I *and model* III. – The input data used in the following calculations are listed in table III ([30]). The r.m.s. radii $R(X^A)$ have been evaluated according to eq. (9.32), where $R_p = 0.77 \cdot$

TABLE III. – *Input data. Lengths are expressed in* fm *units, energies in* MeV.

X^A	^{4}He	^{12}C	^{16}O
$R(X^A)$	1.41	2.27	2.52
$r_0(X^A)$	0.89	0.99	1.00
r_{00}	1.15	1.28	1.30
$b(X^A)$	−7.07	−7.67	−7.98

$\cdot 10^{-13}$ cm ([31]); the auxiliary length $r_0(X^A)$ has been deduced from *ansatz* (2.31) and the length r_{00} from eq. (2.33). The average binding energy $b(X^A)$ has been taken from spectroscopic mass measurements ([32]).

We begin by examining model I which should be considered simply as a useful theoretical tool suitable for revealing the links existing among nuclear stability, compressibility and the central part of the real optical potential. The lowest indeterminacy of the model is obtained by putting $\eta_{ch} = 0$; this restricts the paradigm to ^{4}He. For $A = 4$ one has

$$(9.36) \qquad s_0 = 1.1512 \cdot 10^{-13}\,\text{cm}\,, \qquad s_0\chi(0) = 2.1991\,, \qquad \varepsilon_0(0) = 74.91\,\text{MeV}\,.$$

The numerical values of parameters (5.2*b*) are

$$(9.37) \qquad \begin{cases} a_0 = 27.9740 \cdot 10^{-26}\,\text{MeV}\cdot\text{cm}^2\,, \\ a_1 = 1.8799\,, \qquad a_2 = 7.0865\,, \qquad a_3 = 19.6175\,. \end{cases}$$

([30]) The calculations are performed assuming $M = \frac{1}{2}(M_p + M_n) = 938.9263$ MeV; in the system of units $\hbar = c = 1$ one has $1/M = 41.4594 \cdot 10^{-26}$ MeV·cm^2.

([31]) The numerical values of R given in table III agree to a few percent with recent evaluations; see J. S. MCCARTHY, I. SICK and R. R. WHITNEY: *Phys. Rev. C*, **15**, 1396 (1977) for ^{4}He; W. REUTER, G. FRICKE, K. MERLE and H. MISKA: *Phys. Rev. C*, **26**, 806 (1982); L. S. CARDMAN, J. W. LIGHTBODY jr., S. PENNER, S. P. FIVOZINSKY, X. K. MARUYAMA, W. P. TROWER and S. E. WILLIAMSON: *Phys. Lett. B*, **91**, 203 (1980) for ^{12}C; I. SICK and J. S. MCCARTHY: *Nucl. Phys. A*, **150**, 631 (1970) for ^{16}O.

([32]) S. FIARMAN and W. E. MAYERHOF: *Nucl. Phys. A*, **206**, 1 (1973); A. H. WAPSTRA and N. B. GOVE: *Nucl. Data Tables*, **9**, 267 (1971).

It is convenient to write eqs. (5.4) and (5.5) in the form

$$\begin{cases} s_0^{-3}\mathscr{C}_1^{(2)} = s_0^{-3}\mathscr{C}_1^{(1)} + (7a_3/2a_1)\, s_0^{-7}\mathscr{C}_5^{(2)}, \\ s_0^{-5}\mathscr{C}_3^{(2)} = s_0^{-5}\mathscr{C}_3^{(1)} - (7a_3/2a_2)\, s_0^{-7}\mathscr{C}_5^{(2)}, \end{cases} \tag{9.38a}$$

$$K^{(2)}(s_0, 0) = K^{(1)}(s_0, 0) - 7a_3\, s_0^{-7}\mathscr{C}_5^{(2)}. \tag{9.38b}$$

The radial effective-mass approximation is obtained by putting $\mathscr{C}_5^{(2)} = 0$. It is found that

$$\begin{cases} s_0^{-3}\mathscr{C}_1^{(1)} = -26.24 \text{ MeV}, & \mathscr{B}_0^{(1)}\{\chi(0)\} = -187.10 \text{ MeV}, \\ s_0^{-5}\mathscr{C}_3^{(1)} = 2.98 \text{ MeV}, & s_0^{-2}\mathscr{B}_0^{(1)}\{\chi(0)\} = 31.69 \text{ MeV}, \\ m\{\chi(0)\} = 0.3283, & K^{(1)}(s_0, 0) = 169.37 \text{ MeV}. \end{cases} \tag{9.39}$$

This approximation provides a rough description of the nucleus ground state; nevertheless, it conceals a serious drawback because it violates the optical prescription (6.4) expressed by inequalities (6.6) and (6.7): indeed, it is found that

$$\begin{cases} -V_R^{(1)C}(0, 0) = 61.42 \text{ MeV} < \varepsilon_0(0) = 74.91 \text{ MeV}, \\ -v^{(1)}\{\chi(0), \chi(0)\} = -33.84 \text{ MeV} < \varepsilon_0(0). \end{cases} \tag{9.40}$$

Inequalities (9.40) can be properly adjusted only at the expense of the correct r.m.s. radius of ^{4}He. It follows that the linear dependence on energy of potential (6.5) is fallacious even at very low incident energies.

Model I with $\mathscr{C}_5^{(2)} \neq 0$ possesses, at least in principle, the capability of overcoming the failure of the radial effective-mass approximation. Equations (9.38a) ensure that any value of $\mathscr{C}_5^{(2)}$, either positive or negative, is consistent with nuclear stability. It is seen that $\mathscr{C}_5^{(2)} > 0$ depresses the single-particle potential at $p = \chi(0)$ and reduces the compressibility modulus below the value predicted by the radial effective-mass approximation. Since the results expressed by (9.40) arise from the fact that $v^{(1)}\{\chi(0), \chi(0)\}$ is too small, model I is physically significant provided $\mathscr{C}_5^{(2)} < 0$. The maximum value of $-s_0^{-7}\mathscr{C}_5^{(2)}$ denoted as $\mathscr{X}_1$, can be determined by solving equation

$$\varepsilon_0^2(0) + A_1(0, 0)\,\varepsilon_0(0) + A_2(0, 0) = 0 \tag{9.41}$$

obtained from eqs. (6.8) and (6.10). It is found that $\mathscr{X}_1 = 0.0925$ MeV. The value of $s_0^{-7}\mathscr{C}_5^{(2)} > -\mathscr{X}_1$ is provided by the hydrodynamic nucleon mass approximation expressed by eq. (5.33). Assuming $b_v = -15.0$ MeV, one has

$a_\infty(\varkappa_F) = 2.0136$; in the considered case one deduces from eq. (5.32) that

$$m^{(2)}\{\chi(0)\} = 1 + 2M\mathscr{B}_1^{(2)}\{\chi(0)\} + 4M\mathscr{B}_2^{(2)}\{\chi(0)\}\chi^2(0) . \tag{9.42}$$

Then, from equation $a_\infty^{-1}(\varkappa_F) = m^{(2)}\{\chi(0)\}$ it is found that

$$s_0^{-7}\mathscr{C}_5^{(2)} = -0.1554 \text{ MeV} .$$

In conclusion, model I is characterized by the quantities

$$\begin{cases} \mathscr{B}_0^{(2)}\{\chi(0)\} = -217.55 \text{ MeV} , & K^{(2)}(s_0, 0) = 190.71 \text{ MeV} , \\ s_0^{-2}\mathscr{B}_1^{(2)}\{\chi(0)\} = 31.72 \text{ MeV} , & v^{(2)}\{\chi(0), \chi(0)\} = -102.72 \text{ MeV} , \\ s_0^{-4}\mathscr{B}_2^{(2)}\{\chi(0)\} = -1.65 \text{ MeV} , & V_R^{(2)C}(0, 0) = -89.62 \text{ MeV} , \end{cases} \tag{9.43}$$

where $V_R^{(2)C}(0, 0)$ has been evaluated using eq. (6.9).

The maximum value $\mathfrak{X}_{III}$ of $-s_0^{-7}\mathscr{C}_5^{(2)}$, consistent with model III, is obtained by solving equation

$$v^{(2)}\{\chi(0), \chi(0)\} = -\varepsilon_0(0) ; \tag{9.44}$$

it is found that $\mathfrak{X}_{III} = \mathfrak{X}_I$. Let us assume that $v^{(2)}\{p, \chi(r)\}$ for $p \leqslant \chi(0)$ be the same as in model I. All inequalities indicated in subsect. **3**'3 are verified; in particular, $\beta_{III}\chi(0) > 0$. The zero-energy limit of the real part of the central optical potential at the origin has to be evaluated by solving eq. (6.17*a*). The results are

$$\beta_{III}\chi(0) = 0.4769 , \qquad V_R^{(III)C}(0, 0) \simeq V_R^{(2)C}(0, 0) . \tag{9.45}$$

The interlacing of nuclear stability, compressibility and the central part of the optical potential, clearly exhibited by models I and III, is fully accounted for by model II, although it appears somewhat obscured by the very analytical treatment it requires.

9'4. *The nucleus ground state according to model* II. – The η-dependence of the most significant quantities characterizing model II is shown in table IV: all quantities are evaluated at the minimum of the total energy. We recall that $\mathfrak{N}(\eta)$, defined by eq. (6.17*b*), is the consistency indicator of the optical model constructed according to eqs. (6.11) and (6.19). It is seen that requirement $\mathfrak{N}(\eta) \geqslant 1$ implies values of η which are larger than those predicted by the shell model for the proton configurations of ^{4}He, ^{12}C and ^{16}O. Equation (5.12) is used to determine Ω once η has been calculated according to eq. (5.36); finally, length s is calculated according to *ansatz* (2.31). The hydro-

TABLE IV. – *η-dependence of the quantities $\Omega(\eta) \equiv \alpha\chi(0)$, $\mathfrak{N}(\eta)$, $\beta\chi(0)$ and (in* MeV*) $\varepsilon_0(\eta)$, $C_1\alpha$ and $V_R^C(0, 0)$ at the minimum of the average total energy of the nucleus. The calculations have been performed using as input parameters the values of $r_0(X^A)$ and $b(X^A)$ listed in table* III.

η	^{4}He					
	$\Omega(\eta)$	$\mathfrak{N}(\eta)$	$\beta\chi(0)$	$\varepsilon_0(\eta)$	$C_1\alpha$	$V_R^C(0, 0)$
0	2.366	0.709	2.411	75.22	— 75.38	— 66.11
1.0	2.139	0.954	1.366	57.17	— 69.42	— 55.95
1.5	2.036	1.065	1.021	50.11	— 67.25	— 51.69
2.0	1.943	1.158	0.759	44.78	— 66.01	— 48.39
2.5	1.862	1.233	0.559	40.63	— 65.26	— 45.69
3.0	1.791	1.294	0.402	37.31	— 64.79	— 43.38

η	^{12}C					
	$\Omega(\eta)$	$\mathfrak{N}(\eta)$	$\beta\chi(0)$	$\varepsilon_0(\eta)$	$C_1\alpha$	$V_R^C(0, 0)$
0	2.457	0.674	3.008	60.54	— 62.53	— 52.98
1.0	2.218	0.964	1.677	46.01	— 57.76	— 45.29
1.5	2.111	1.097	1.264	40.32	— 56.02	— 42.11
2.0	2.015	1.210	0.958	36.04	— 55.02	— 39.66
2.5	1.931	1.304	0.727	32.70	— 54.42	— 37.66
3.0	1.857	1.380	0.547	30.03	— 54.04	— 35.95

η	^{16}O					
	$\Omega(\eta)$	$\mathfrak{N}(\eta)$	$\beta\chi(0)$	$\varepsilon_0(\eta)$	$C_1\alpha$	$V_R^C(0, 0)$
0	2.475	0.667	3.142	59.42	— 61.87	— 51.97
1.0	2.233	0.966	1.743	45.16	— 57.18	— 44.52
1.5	2.125	1.104	1.315	39.58	— 55.47	— 41.43
2.0	2.029	1.222	1.000	35.37	— 54.49	— 39.08
2.5	1.944	1.319	0.761	32.10	— 53.90	— 37.15
3.0	1.870	1.399	0.577	29.474	— 53.53	— 35.50

dynamical mass approximation is consistent with prescription $\mathfrak{N}(\eta) \geqslant 1$: this is an indirect test of the reliability of the overall description of infinitely extended nuclear matter based on eq. (2.1). The numerical values thus selected of the quantities describing the nucleus ground state are given in table V. The ratio $\Delta r/R$, where Δr is the nuclear skin thickness defined as the distance at which $\varrho(r)$ falls from $0.9\varrho_0$ to $0.1\varrho_0$, turns out to be a decreasing function of the mass number, whereas the ratio $r_{\frac{1}{2}}/R$ between the half-density radius and the r.m.s. radius increases slowly with A; the alpha-particle does not appear to be «almost all surface» as happens in the case $\eta = 0$ ($\Delta r/R \simeq 1$), which is forbidden by prescription $\mathfrak{N}(\eta) \geqslant 1$. The values of η_0 for ^{12}C and ^{16}O are in surprisingly good agreement with those extracted in subsect. **9**'2 from electron-nucleus scattering data.

TABLE V. – *Numerical values of the quantities characterizing the nucleus ground state selected from those listed in table* IV *by means of eq.* (5.36). Δr *is the skin thickness defined as the distance in which* $\varrho(r)$ *falls from* $0.9\varrho_0$ *to* $0.1\varrho_0$; $r_{\frac{1}{2}}$ *is the half-density radius. Lengths are expressed in* fm *units, energies in* MeV.

X^A	^{4}He	^{12}C	^{16}O	X^A	^{4}He	^{12}C	^{16}O
s_0	0.951	1.511	1.672	$\alpha\chi(0)$	2.012	1.987	1.964
η_0	1.620	2.156	2.377	$\beta\chi(0)$	0.951	0.879	0.814
$\Delta r/R$	0.684	0.603	0.580	$\varepsilon_0(\eta_0)$	48.69	34.91	32.83
$r_{\frac{1}{2}}/R$	1.002	1.066	1.087	$C_{1\alpha}$	– 66.89	– 54.80	– 54.02
$\alpha\{\chi(0)\}$	2.062	2.167	2.177	K	100.12	100.34	103.10

The surface $W = W(s, \eta)$ exhibits a deep and narrow stability valley. No minimum exists in the (W, s_0)-plane: the minimum and the profile of the valley is seen in the (W, η_0)-plane. The s-dependence of $W(s, \eta_0)/A$ and $K(s, \eta_0)$, calculated using, respectively, eq. (5.7) and eq. (5.14), is plotted in fig. 1 and fig. 2, respectively: the scarce stability of the calculated compressibility modulus against variations of length s is clearly brought into view. It is found $K(s_0, \eta_0) \simeq 0.5K_\infty$, K_∞ being given in table I. It is not possible to obtain an immediate, significant comparison of the values of the compressibility modulus (at least for ^{16}O), given in table V, with those evaluated in studying the coupling of surface and bulk vibrations in the nuclear breathing mode. The attempt to construct the compressibility modulus as a sum of different contributions as in the case of the semi-empirical mass formula ([33])

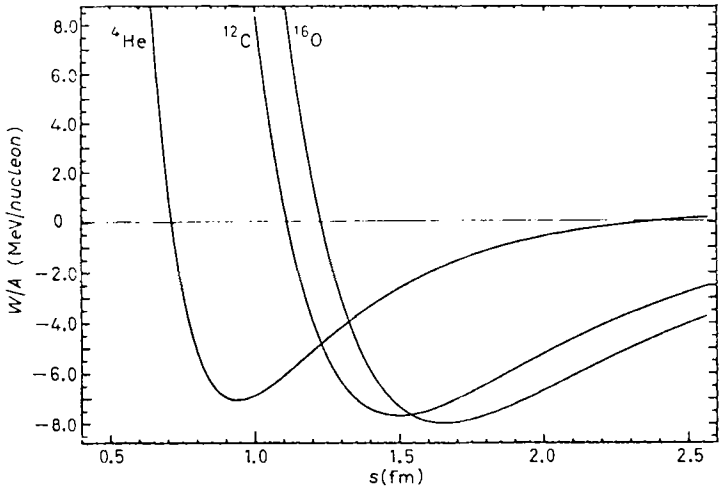

Fig. 1. – Plot of the s-dependence of the average total energy in proximity of the minimum for ^{4}He, ^{12}C and ^{16}O, calculated using eq. (5.7).

([33]) J. TREINER, H. KRIVINE, O. BOHIGAS and J. MARTORELL: *Nucl. Phys. A*, **371**, 253 (1981).

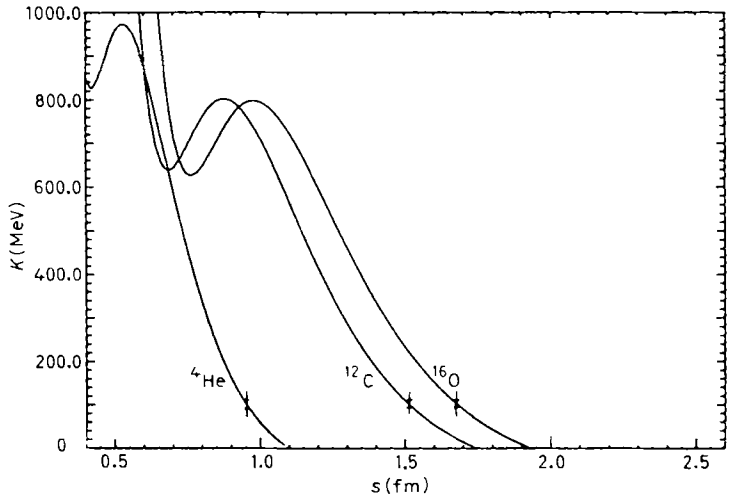

Fig. 2. – Plot of the s-dependence of the compressibility modulus K in proximity of the minimum (indicated by an arrow) for ^{4}He, ^{12}C and ^{16}O, calculated using eq. (5.14).

is objectionable. It is surprising that the description of collective 0^+ excitations in nuclei should not be based on a clear-cut definition of the related adiabatic compressibility modulus K_{ad}. Trivial thermodynamical arguments show that the adiabatic compressibility modulus is larger than the isothermic one. Consequently, the only conclusion we can draw is that

$$K_{is} = K(s_0, \eta_0) < K_{ad} . \tag{9.46}$$

It would be highly desirable to compare carefully the theoretical perspectives disclosed by eq. (2.14) with the conceptual content of standard Hartree-Fock calculations concerning ^{4}He, ^{12}C and ^{16}O ([34]).

Let us determine the value of η using the criterion based on the angular-momentum distributions in the Thomas-Fermi model, outlined in subsect. **5**·3. Equations (5.30a) reads

$$I\{\eta(X^A)\} = c_0\{c_1 + \langle \ell^2(X^A)\rangle_{\text{shell}}\}A^{-\frac{2}{3}} , \tag{9.47a}$$

$$\begin{cases} c_0 = (5/16)(\sqrt{\pi}/9)^{\frac{2}{3}} = 0.1818 , \\ c_1 = (27/125)(3/5)^{\frac{2}{3}}(36\pi)^{\frac{1}{3}} = 0.8091 ; \end{cases} \tag{9.47b}$$

the numerical values of integral (5.30b) are given in table VI. We restrict ourselves to the magic nucleus ^{16}O. In this case, parameter $\eta(^{16}\text{O})$ is determined by solving the equation

$$I\{\eta(^{16}\text{O})\} = 0.0661 . \tag{9.48}$$

([34]) D. M. Brink and E. Boeker: *Nucl. Phys. A*, **91**, 1 (1967).

The solution is $\eta(^{16}O) = 2.5$; the value $\eta = \eta_0 = 2.377$, evaluated from stability prescriptions (table V), can be obtained from eqs. (9.47) reducing the value of the arbitrary constant c_1 by less than 0.6% ($c_1' = 0.8044$). Replacing c_1 with c_1', one can expect the suggested criterion to provide a reliable tool for establishing an additional relation between the density function param-

TABLE VI. – *Numerical values of the integral $I(\eta)$ defined by eq. (5.30b).*

η	0	1.0	1.5	2.0	2.5	3.0
$I(\eta)$	0.0584	0.0620	0.0644	0.0654	0.0661	0.0667

eters of the successive even-even magic nucleus ^{40}Ca. It is worthwhile pointing out that the disturbingly slow dependence of $I(\eta)$ on η is entirely bound up with the density function (2.17) and will not be brought about by density functions suitable to describe nuclei with $A > 16$, because characterized by smaller skin thickness and steeper profiles.

The radial dependence of function $w^*(r)$ is determined by substituting in eq. (8.6) potential $V_R^C\{r, w^*(r)\}$ given by eq. (8.4). The energy $w^*(r) < 0$ is the solution of equation

$$(9.49a) \qquad \sqrt{\Phi\{w^*(r)\}} = \sqrt{\varepsilon(r)} \exp\left[-\beta\sqrt{2M}\{\sqrt{\Phi\{w^*(r)\}} - \sqrt{\varepsilon(r)}\}\right],$$

$$(9.49b) \qquad \begin{cases} \varepsilon_0(\eta_0)\Phi\{w^*(r)\} = w^*(r) - v_0\{\chi(r)\} \exp[-\beta\chi(r)], \\ \Phi\{w^*(0)\} = 1. \end{cases}$$

The radial behaviour of $w^*(r)$ at equilibrium density is plotted in fig. 3.

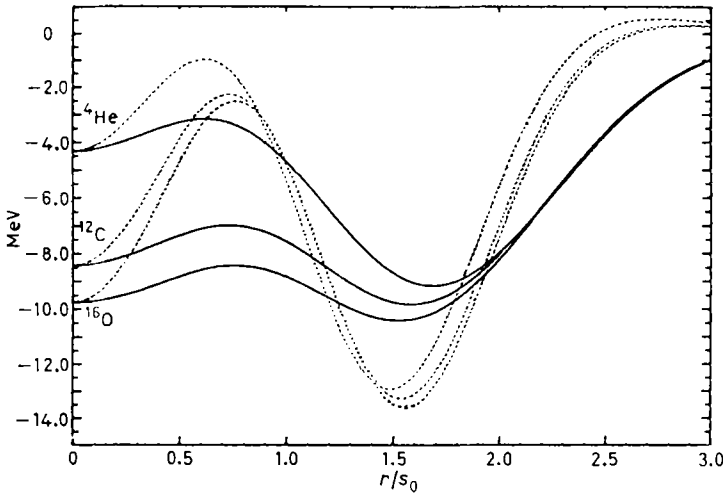

Fig. 3. – Radial behaviour of $w^*(r)$ (solid lines) and of $w(r)$ (dashed lines) at equilibrium density for ^{4}He, ^{12}C and ^{16}O.

9·5. *The real optical potential.* – The radial and energy dependence of the central part of the real optical potential $V_R^C(r, E)$ and the real spin-orbit function $W_R(r, E) = U_R V_R^C$ for nucleons elastically scattered by ^{4}He, ^{12}C and ^{16}O are plotted in fig. 4*a*), *b*), 6*a*), *b*), and 8*a*), *b*) respectively. More detailed information for selected energies is plotted in fig. 5*a*), *b*), 7*a*), *b*) and 9*a*), *b*).

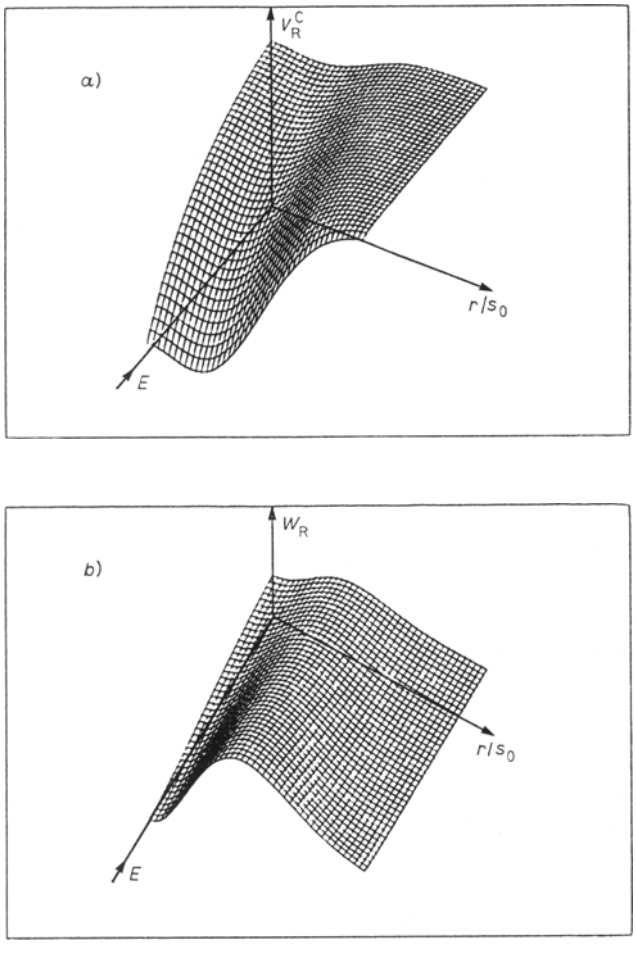

Fig. 4. – Radial and energy dependence of the real potential for ^{4}He: *a*) central term V_R^C and *b*) spin-orbit term $W_R = U_R V_R^C$. The mesh steps are $\Delta E = 5$ MeV and $\Delta(r/s_0) = 0.075$.

The potential is not capable of reproducing the « experimental » behaviour of the *S*-wave neutron-alpha phase shift as a function of energy. The hard-sphere scattering, introduced *ad hoc* in order to describe the repulsive n-α inter-

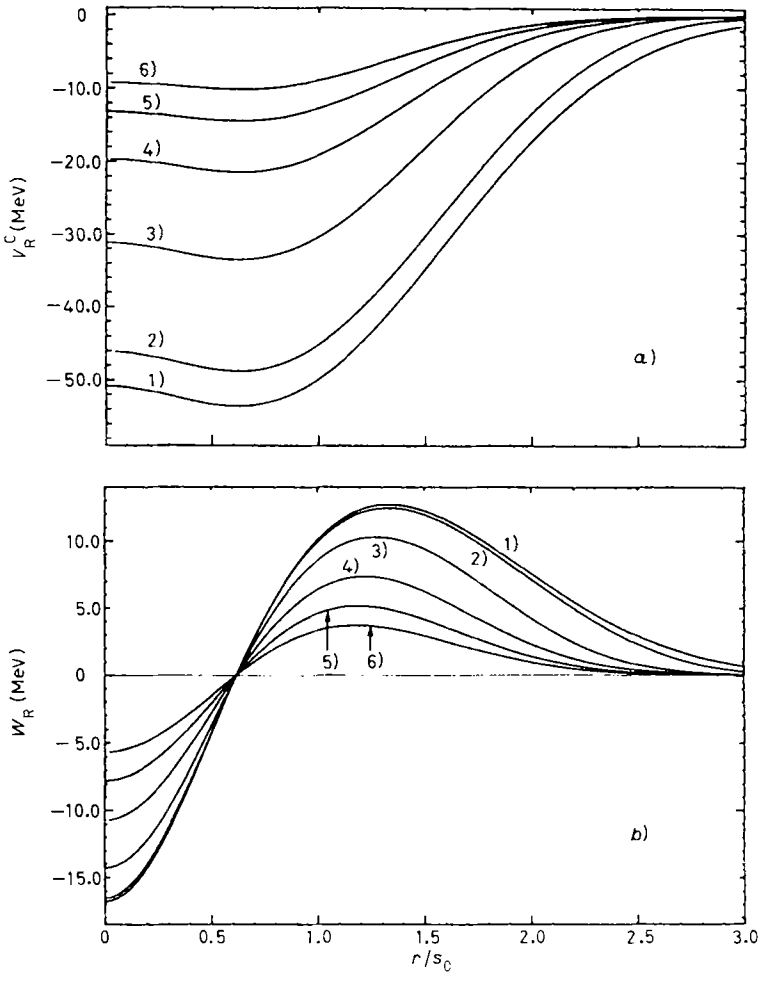

Fig. 5. – Radial dependence of the real potential for ^{4}He for selected energies: *a*) central term V_R^C and *b*) spin-orbit term $W_R = U_R l_R^C$. The resolution is $\Delta(r/s_0) = 0.03$. 1) $E = 0$ MeV, 2) $E = 10$ MeV, 3) $E = 50$ MeV, 4) $E = 100$ MeV, 5) $E = 150$ MeV, 6) $E = 200$ MeV.

action in the S-state ([35]), is not compatible with the single-particle solutions of eq. (2.14), which have to be thought of as originated by two-body potentials free from discontinuities. Thus, the nucleon-nucleon repulsion at short distances described by a core of radius r_c does not come into play. Using standard many-body techniques one can show that the density dependence

([35]) W. A. PEARCE and P. SWAN: *Nucl. Phys.*, **78**, 433 (1966). See also P. E. HODGSON: *Philos. Mag. Suppl. Adv. Phys.*, **7**, 25 (1958).

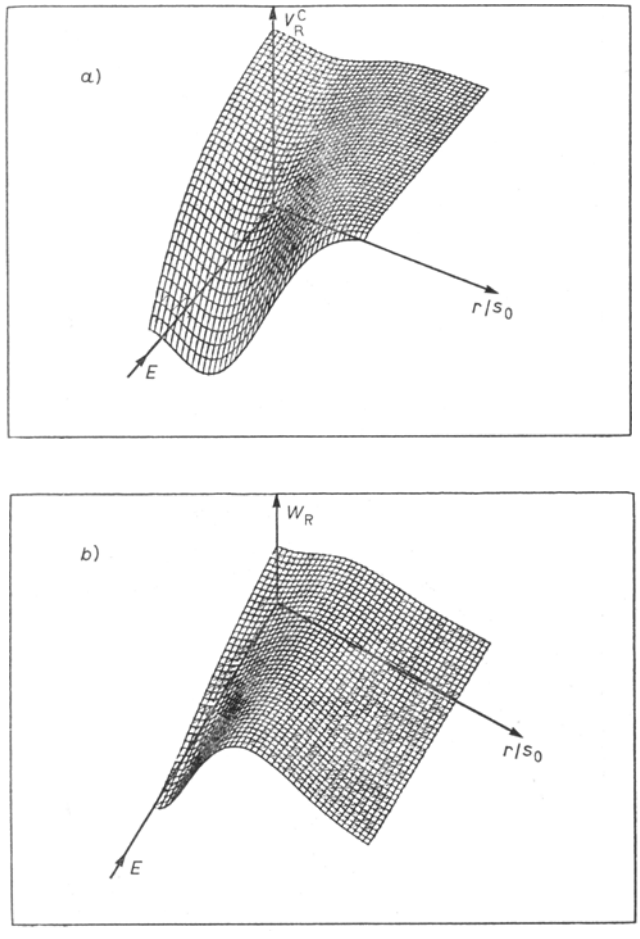

Fig. 6. – Radial and energy dependence of the real potential for ^{12}C: *a*) central term V_R^C and *b*) spin-orbit term $W_R = U_R V_R^C$. The mesh steps are $\Delta E = 5$ MeV and $\Delta(r/s_0) = 0.075$.

of the reaction matrix is almost entirely due to the repulsive core ([36]). Since energy $\mathcal{Q}\{\chi(r)\}$, defined in subsect. **4**·1, plays the role of the re-arrangement energy in Brueckner's theory, the theorem proved in subsect. **4**·2 clearly indicates that the shift in the single-particle potential energy is probably a spurious core effect and as such incompatible with eq. (2.14). The single-particle solutions of eq. (2.14) can be related to nucleon-nucleon repulsive

([36]) K. A. BRUECKNER, J. L. GAMMEL and H. WEITZNER: *Phys. Rev.*, **110**, 431 (1958); K. A. BRUECKNER, A. M. LOCKETT and M. ROTENBERG: *Phys. Rev.*, **121**, 255 (1961).

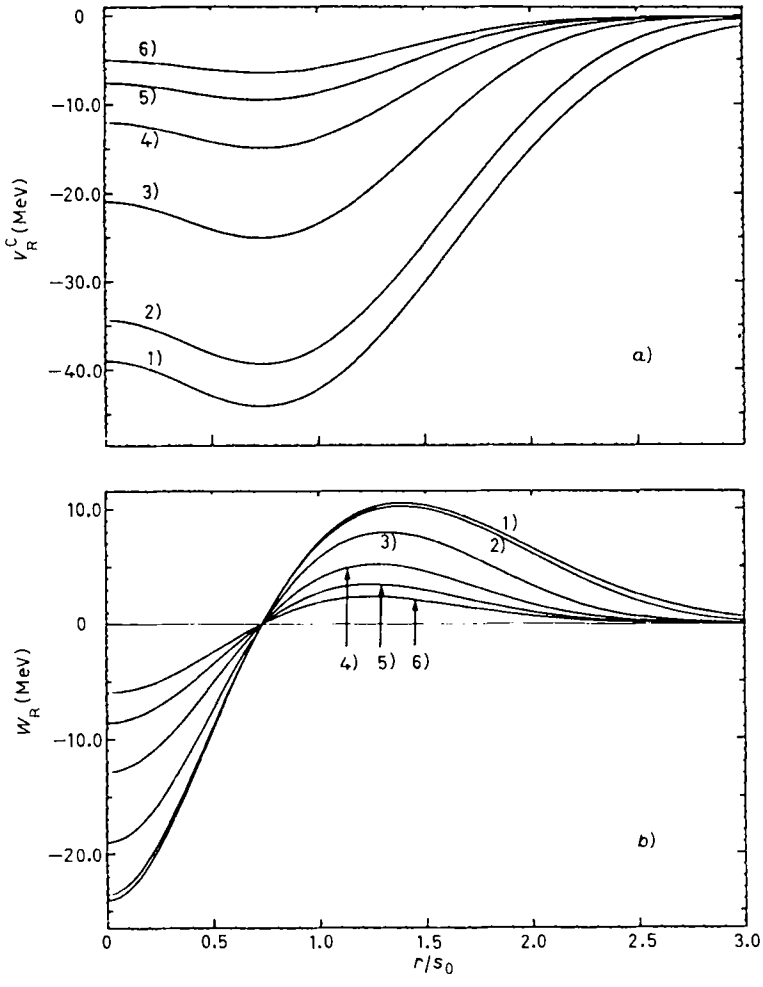

Fig. 7. – Radial dependence of the real potential for ^{12}C for selected energies: *a*) central term V_R^C and *b*) spin-orbit term $W_R = U_R V_R^C$. The resolution is $\Delta(r/s_0) = 0.03$. 1) $E = 0$ MeV, 2) $E = 10$ MeV, 3) $E = 50$ MeV, 4) $E = 100$ MeV, 5) $E = 150$ MeV, 6) $E = 200$ MeV.

interactions at short distances provided that the one-pion exchange potential is reformulated in order to overcome the serious drawback of the PS-PV meson theory of nuclear forces, based on the unrealistic description of nucleons as pointlike sources of the meson field ([37]). An investigation devoted to the search for solutions of eq. (2.14) suitable to describe the nucleon-nucleus repulsion in the optical-model scheme is in progress.

([37]) C. VILLI: *Nuovo Cimento A*, **67**, 178 (1982).

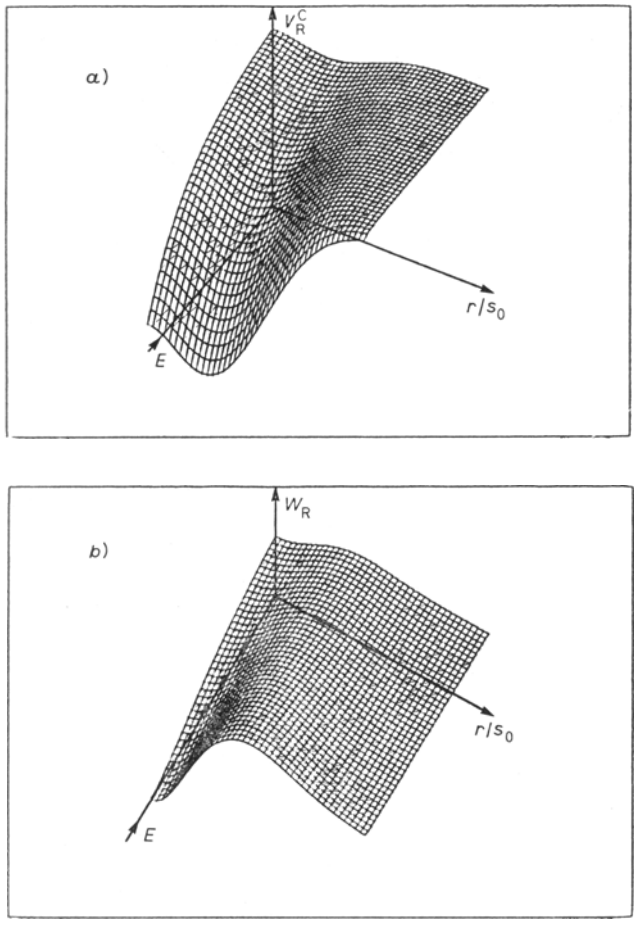

Fig. 8. – Radial and energy dependence of the real potential for ^{16}O: *a*) central term V_R^C and *b*) spin-orbit term $W_R = U_R V_R^C$. The mesh steps are $\Delta E = 5$ MeV and $\Delta(r/s_0) = 0.075$.

Is the real optical potential (1.2*a*), constructed as a sum of potentials (6.11) and (6.19), a local or a nonlocal potential? To give an answer to this question we recall that in subsect. **2**'1 it was pointed out that the momentum dependence of the single-particle potential in nuclear matter arises primarily from the total antisymmetry of the nuclear wave function and only partially from the nonlocality of the two-body potential. Let a_0 be the nonlocality length already introduced in subsect. **2**'1. A theoretical scheme is, at least in principle, worth of consideration if, at the limit $a_0 \to 0$, the disappearance of the momentum dependence brought about by the nonlocality does not cancel the momentum dependence generated by the totally antisymmetric nuclear wave function: should this occur, the foundations of quantum mechanics would

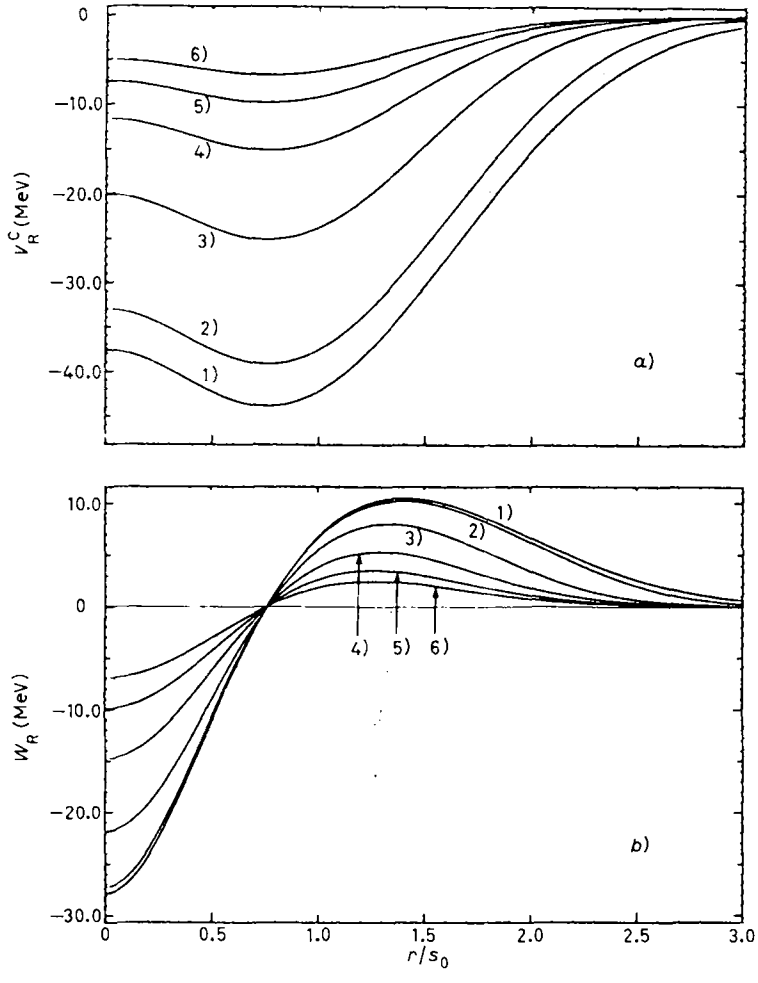

Fig. 9. – Radial dependence of the real potential for ^{16}O for selected energies: *a*) central term V_R^C and *b*) spin-orbit term $W_R = U_R V_R^C$. The resolution is $\Delta(r/s_0) = 0.03$. 1) $E = 0$ MeV, 2) $E = 10$ MeV, 3) $E = 50$ MeV, 4) $E = 100$ MeV, 5) $E = 150$ MeV, 6) $E = 200$ MeV.

be violated! Of course, this fundamental aspect of the problem is also valid for finite nuclei. The wave function of the incident nucleon, in interaction with the A nucleons of the target nucleus, undergoes the global antisymmetrization process performed over $A+1$ nucleons. It should be evident that neglect of the requirement that the wave function should be antisymmetrized with respect to the exchange of the incident nucleon with a nucleon in the target nucleus is a very misleading « approximation » ([38]). Thus, the source of

([38]) S. D. DRELL: *Phys. Rev.*, **100**, 97 (1955). See also ref. ([25]).

the energy dependence of the real optical potential has to be identified in the momentum dependence of the single-particle potential energy, the former being deduced from the latter by means of transformation (6.1): in this way one can picture the dispersive properties of finite nuclear matter as they are explored by the probe nucleon with energy E. Equation (2.14) is the only available theoretical tool capable of providing single-particle potential energies which can be easily submitted to optical transformation, consistently with the stability prescriptions of the target nucleus. Many-body techniques, apparently more refined, fail to reach this goal. According to the Hartree-Fock method, the expectation value of the total energy of the nucleus is constructed using an antisymmetrized trial nuclear wave function of shell model type. The correct wave function is determined by variational procedures by minimizing the expectation value of the total energy. The necessary condition for the minimum is that the normalization of the single-particle wave functions should be unchanged: this introduces a Lagrange multiplier, which is arbitrarily identified with the single-particle energy. The determination of the Lagrange multiplier implies the solution of a complicated self-consistency problem, which requires numerical work too massive even for large computing machines. The need to keep the calculations within reasonable bounds forces one to resort to approximations, which are responsible for the unsatisfactory results so far obtained. In conclusion, the optical transformation (6.1) is practicable only by virtue of eq. (2.14).

The conventional way of approaching the problem of the construction of real optical potential is based on the Schrödinger equation expressed in integro-differential form

$$\{(1/2M)\nabla^2 + E\}\psi(\boldsymbol{r}) = \int U(\boldsymbol{r}, \boldsymbol{r}')\psi(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' , \tag{9.50}$$

where $U(\boldsymbol{r}, \boldsymbol{r}')$ is the nonlocal optical potential and, using standard notations,

$$\psi(\boldsymbol{r}') = \sum_{LM} \psi_L(r')\, i^L\, Y_L^M(\theta', \varphi') . \tag{9.51a}$$

Since the nonlocal effects are expected to manifest themselves in proximity of point $\boldsymbol{r}$, we expand $\psi_L(r')$ in Taylor series about r,

$$\psi_L^{(N)}(r') = \sum_{n=0}^{N} \{(r' - r)^n / n!\} \{(\mathrm{d}/\mathrm{d}r')^n \psi_L(r')\}_{r' \to r} . \tag{9.51b}$$

Taking into account that the classical equivalent of the radial derivative is proportional to the momentum k of the free nucleon and that the potential must satisfy the requirement of invariance under space reflection, from eqs. (9.50)

and (9.51*b*) it is found that

$$\{(1/2M)\nabla^2 + E\}\psi^{(N)}(\boldsymbol{r}) = V^{(N)}(r, E)\psi^{(N)}(\boldsymbol{r}) , \tag{9.52a}$$

$$\begin{cases} V^{(N)}(r, E) = \sum_{n=0}^{N} W_n(r) k^{2n} , \\ W_n(r) = \{1/(2n)!\}\int U(\boldsymbol{r}, \boldsymbol{r}')(r' - r)^{2n} \, \mathrm{d}\boldsymbol{r}' . \end{cases} \tag{9.52b}$$

For $N = 1$ $V^{(1)}(r, E)$ turns out to be linearly dependent on energy E: it is formally similar to the optical potential (6.5) expressed in radial effective-mass approximation; in the considered case the nucleon effective mass $M^*(r)$ is

$$M/M^*(r) = 1 + (M/2)\int U(\boldsymbol{r}, \boldsymbol{r}')(r' - r)^2 \, \mathrm{d}\boldsymbol{r}' . \tag{9.53}$$

For $N > 1$ $V^{(N)}(r, E)$ is a rough approximation of the real optical potential derived from the single-particle solutions obtained by series integration of eq. (2.14). In particular, for $N = 2$ potential $V^{(2)}(r, E)$ is formally similar to the optical potential deduced from model I and then expanded up to terms in E^2 in McLaurin series. The energy dependence of $V^{(N)}(r, E)$ is entirely due to the nonlocality of potential $U(r, r')$. Suppose that $U(r, r')$ be expressed in separable form, *i.e.*

$$U(\boldsymbol{r}, \boldsymbol{r}') = U_0\{\tfrac{1}{2}|\boldsymbol{r} + \boldsymbol{r}'|\} G(|\boldsymbol{r} - \boldsymbol{r}'|) , \tag{9.54}$$

where $G(|\boldsymbol{r} - \boldsymbol{r}'|)$ is the nonlocality function defined by eqs. (2.11). At the limit $a_0 \to 0$, $G(|\boldsymbol{r} - \boldsymbol{r}'|) \to \delta(\boldsymbol{r} - \boldsymbol{r}')$ and from eqs. (9.52) and (9.54) one deduces that

$$\lim_{a_0 \to 0} V^{(N)}(r, E) = U_0(r) , \qquad \lim_{a_0 \to 0} M^*(r) = M , \tag{9.55}$$

i.e. the optical potential becomes energy independent as a consequence of the switching-off of the nonlocality and the nucleon effective mass identifies with the mass of the free nucleon (39). Limits (9.55) disclose a serious drawback to eq. (9.50) as it stands: indeed, the energy dependence of $V^{(N)}(r, E)$ must survive at the limit $a_0 \to 0$ because the disappearance of the nonlocality can in no way suppress the momentum dependence of the single-particle potential energy arising from the total antisymmetry of the nuclear wave function. Thus, the nonlocal potential $U(r, r')$ must possess a more intimate energy dependence, *i.e.*

$$U(\boldsymbol{r}, \boldsymbol{r}') = U(\boldsymbol{r}, \boldsymbol{r}'; E) , \qquad U_0\{\tfrac{1}{2}|\boldsymbol{r} + \boldsymbol{r}'|\} = U_0\{\tfrac{1}{2}|\boldsymbol{r} + \boldsymbol{r}'|; E\} . \tag{9.56}$$

(39) H. FESHBACH: *Annu. Rev. Nucl. Sci.*, **8**, 49 (1958), formula (31).

The source of the nonlocality of the optical potential has to be sought for in the intrinsic nonlocality of the two-body potential, originally brought to light by nonadiabatic treatments of the meson theory of nuclear forces, and in the nonlocality induced by many-body effects on the nucleon-nucleon scattering in finite (and infinite) nuclear matter. Equation (9.50) should be written in the form

$$\{(1/2M)\nabla^2 + E\}\psi(\boldsymbol{r}) = \int U(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{r}'')\psi(\boldsymbol{r}', \boldsymbol{r}'')\,\mathrm{d}\boldsymbol{r}'\,\mathrm{d}\boldsymbol{r}'' . \tag{9.57}$$

However, potential $U(\boldsymbol{r}, \boldsymbol{r}', \boldsymbol{r}'')$ characterized by a double nonlocality should be energy dependent, because otherwise the optical-model scheme based on eq. (9.57) would imply the violation of the Pauli principle.

The problem of finding a nonlocal potential with an energy-independent real part capable of fitting the scattering data at all energies appears to be somewhat misleading: indeed, it implies that also the antisymmetry effects can be accounted for by the nonlocality. This puzzling consequence does not even come into play in the case of the imaginary potential, because the origin of its energy dependence is such that it can in no way be absorbed into a presumptive nonlocality. In the light of the above considerations, it is interesting to examine critically the nonlocal potentials constructed by FRAHN and LEMMER ([40]), PEREY and BUCK ([41]) and by ENGELBRECHT and FIEDELDEY ([42]).

In conclusion, potential (1.2*a*), constructed as a sum of potentials (6.11) and (6.19), cannot be classified *sic et simpliciter* as local or nonlocal: it is a real optical potential which has inherited, through transformation (6.1) and consistently with the stability prescriptions of the target nucleus, the fully physical content concealed in eq. (2.14).

9·6. *The imaginary optical potential.* – A critical point in the computation of the imaginary optical potential (7.19) is represented by the evaluation of integral $V_{I2}(r, E)$, which presents a twofold problem. First of all, the summability of the integrand must be ensured and this requires function $c_1(r, E)$ to obey sufficient regularity conditions. A second problem is a uniformly accurate computation of the integral as a function of r and E.

A sufficient condition of regularity for $c_1(r, E)$ is obviously given by its continuity as function of r. A minimization procedure for the discrete sum (7.31) is very unlikely to guarantee continuous $c_i(r, E)$ $(i = 1, 2)$ and for this reason we substitute (7.31) with the continuous expression

$$\mathfrak{M}(c_1, c_2) = \int_{e_0}^{e_1} [\sigma_{nD}(e) - \hat{s}(e)]^2\,\mathrm{d}e = \text{minimum} , \tag{9.58}$$

([40]) W. E. FRAHN and R. H. LEMMER: *Nuovo Cimento*, **5**, 1564 (1957).
([41]) F. PEREY and B. BUCK: *Nucl. Phys.*, **32**, 353 (1962).
([42]) C. A. ENGELBRECHT and H. FIEDELDEY: *Ann. Phys.* (*N. Y.*), **42**, 267 (1967).

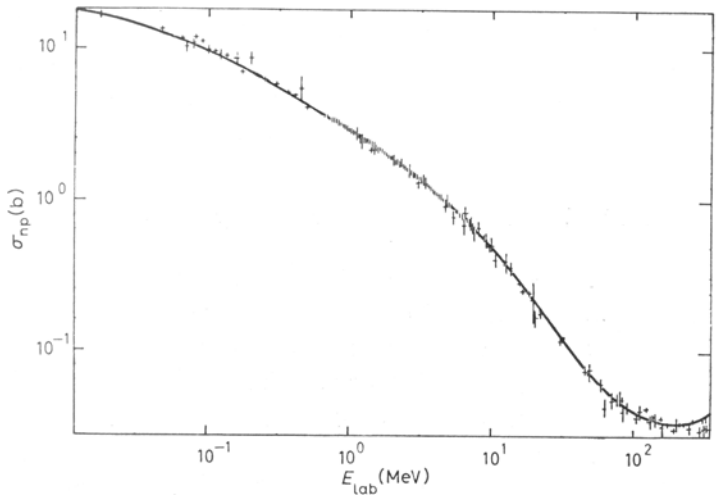

Fig. 10. – Experimental data of the neutron-proton total cross-section σ_{np} ([43]). The solid line shows the fit obtained by eq. (9.59).

where $\hat{s}(e)$ is expressed by eq. (7.14) and $\sigma_{np}(e)$ is a continuous parametrization of the experimental neutron-proton cross-section in the energy range of interest $(e_0)_{min} \leqslant e \leqslant (e_1)_{max}$. Figure 10 shows the experimental data from $e = 1.18$·

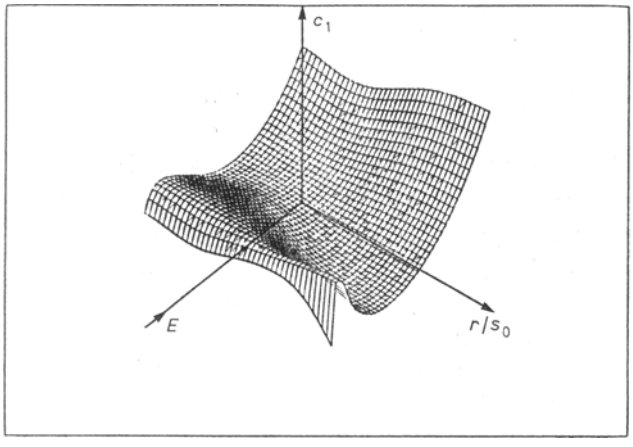

Fig. 11. – Radial and energy dependence of coefficient $c_1(r, E)$ for ^{12}C. Mesh sizes as in fig. 4.

([43]) The experimental data have been compiled on the basis of information from the NEA Data Bank and the High-Energy Reaction Analysis Group of CERN.

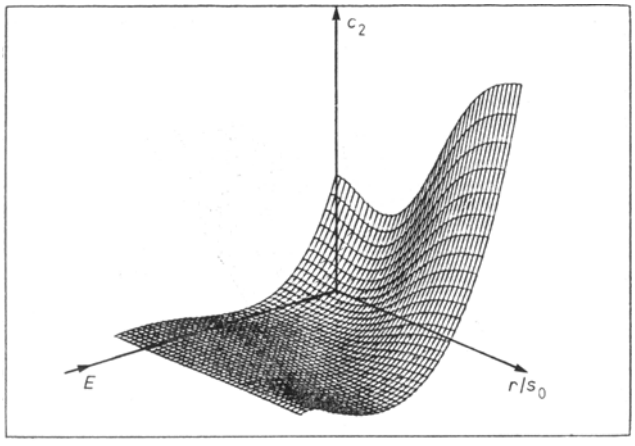

Fig. 12. – Radial and energy dependence of coefficient $c_2(r, E)$ for ^{12}C. Mesh sizes as in fig. 4.

$\cdot 10^{-2}$ MeV to $e = 342.5$ MeV ([43]) with a continuous fit given by the function

$$\sigma_{np}(e) = \pi\left(\frac{\sigma_1}{\sigma_2 e + (\sigma_3 + \sigma_4 e + \sigma_5 e^2)^2} + \sigma_7 e + \frac{\sigma_6}{(\sigma_8 + \sigma_9 e)^2}\right), \tag{9.59}$$

which has been inspired by a similar expression suggested by BAWE *et al.* ([44]).

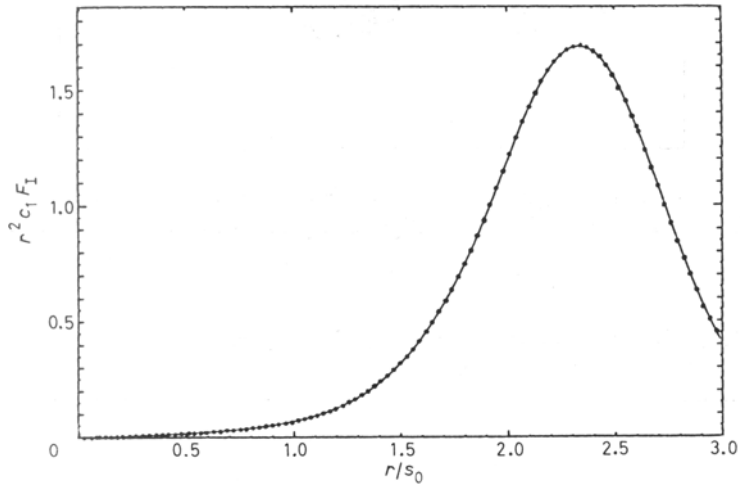

Fig. 13. – Radial behaviour of the integrand function of (7.19) for ^{12}C at $E = 10$ MeV. Dots represent values computed directly, while the solid line shows the result of the interpolation by means of Padé approximants.

([44]) S. J. BAWE jr., E. HADDAD, J. E. PERRY jr. and R. K. SMITH: *Rev. Sci. Instrum.*, **28**, 997 (1957).

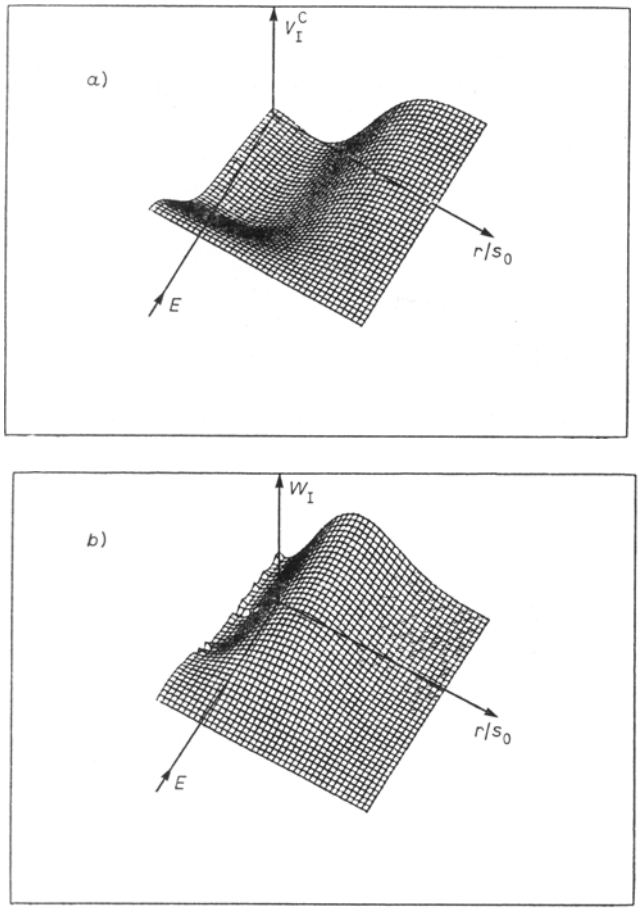

Fig. 14. – Radial and energy dependence of the imaginary potential for ^{4}He: *a*) central part V_I^C, and *b*) spin-orbit term $W_I = U_I V_I^C$. Mesh steps as in fig. 4.

TABLE VII. – *Numerical values of the coefficients of* $\sigma_{np}(e)$ *defined by* (9.59) *which fit best the experimental data.*

	$e \leqslant 9$ MeV	$e \geqslant 9$ MeV
σ_1	2.584 85	$6.834\,817 \cdot 10^{-2}$
σ_2	1.741 943	0
σ_3	−1.635 243	−3.0
σ_4	0.100 389 5	$1.671\,789 \cdot 10^{-3}$
σ_5	0	0
σ_6	0.453 006 1	1.211 405 7
σ_7	1.029 514	0.120 895 3
σ_8	0.284 709 7	1.119 203
σ_9	0.426 820 9	0.149 606 3

A best-fit procedure on the experimental data gives the values for coefficients σ_i of (9.59) summarized in table VII.

The necessary conditions for the minimum (9.58) are

$$\frac{\partial \mathfrak{M}(c_1, c_2)}{\partial c_i} = 0 , \qquad i = 1, 2 . \tag{9.60}$$

For fixed values of r and E, the unknown parameters $c_i \equiv c_i(r, E)$ at the

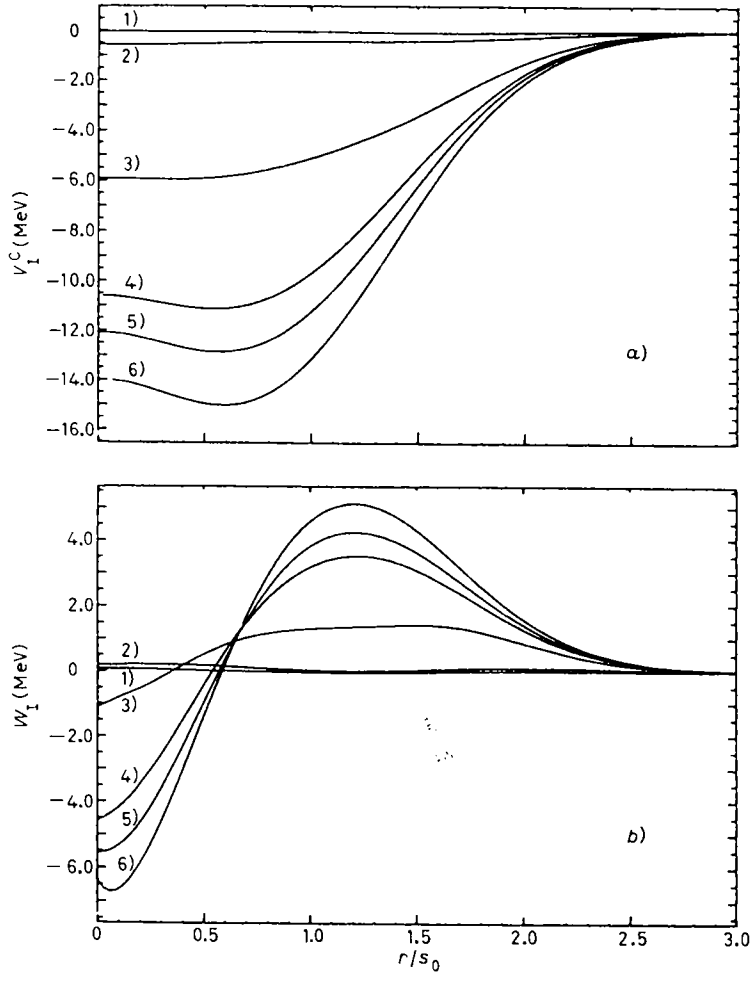

Fig. 15. – Radial dependence of the imaginary potential for ^{4}He for selected energies: *a*) central term V_I^C and *b*) spin-orbit term $W_I = U_I V_I^C$. Resolution as in fig. 5. 1) $E = 0$ MeV, 2) $E = 10$ MeV, 3) $E = 50$ MeV, 4) $E = 100$ MeV, 5) $E = 150$ MeV, 6) $E = 200$ MeV.

minimum of $\mathfrak{M}(c_1, c_2)$ fulfil equations

$$(9.61a) \qquad \begin{cases} c_1 = \dfrac{(c_2 + e_0)(c_2 - e_1)F_1(c_2)}{e_1 - e_0}, \\ F_2(c_2) = \dfrac{(2c_2 + e_0 + e_1)F_1(c_2)}{2(c_2 + e_0)(c_2 - e_1)}, \end{cases}$$

where

$$(9.61b) \qquad F_m(c_2) = \int_{e_0}^{e_1} \frac{\sigma_{np}(e)\,de}{(c_2 + e)^m}, \qquad m = 1, 2.$$

Equations (9.61) give solutions c_i continuous in r and E through the continuous

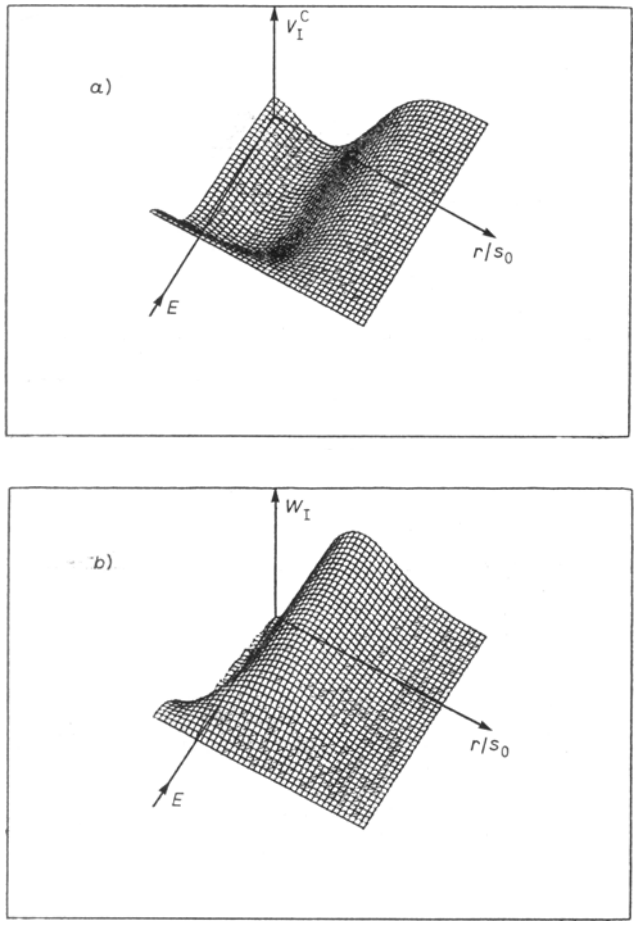

Fig. 16. – Radial and energy dependence of the imaginary potential for ^{12}C: *a*) central part V_I^C, and *b*) spin-orbit term $W_I = U_I V_I^C$. Mesh steps as in fig. 4.

functions $e_0(r, E)$ and $e_1(r, E)$. Figures 11 and 12 show the E- and r-dependence of c_1 and c_2, respectively, in the case of ^{12}C; similar results have been obtained for ^4He and ^{16}O too.

In order to compute integral (7.19*c*) with a uniform accuracy as a function of r and E, we have chosen a procedure based on Padé approximants of type II ([45]). After having computed the integrand function at the points of

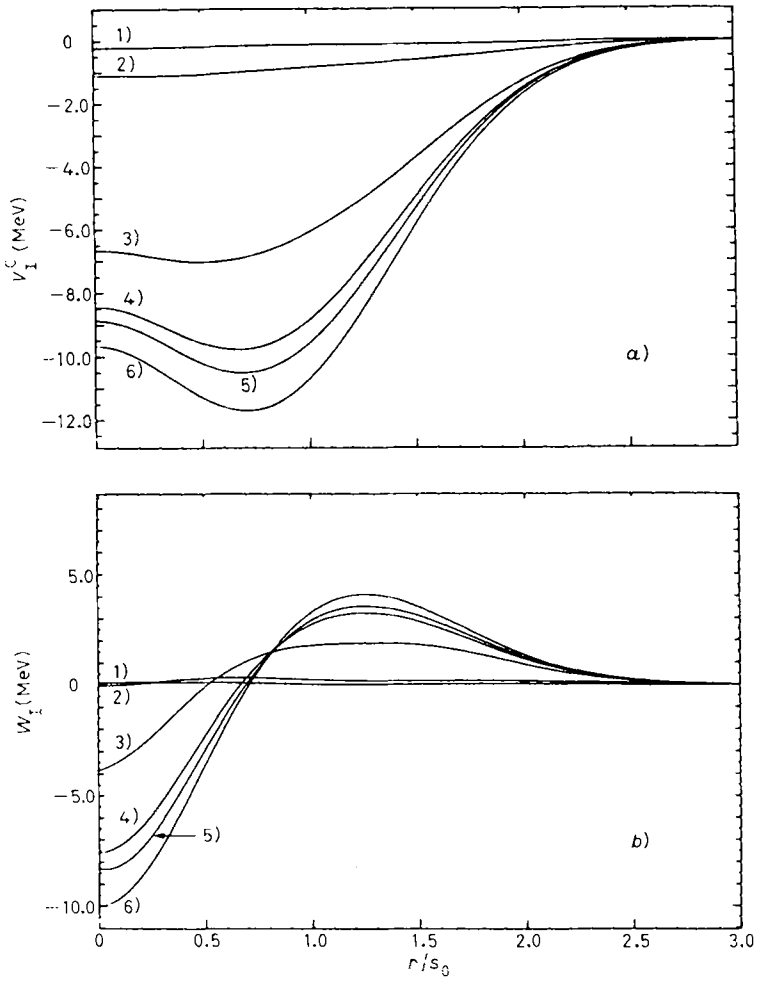

Fig. 17. – Radial dependence of the imaginary potential for ^{12}C for selected energies: *a*) central term V_I^C and *b*) spin-orbit term $W_I = U_I V_I^C$: Resolution in as fig. 5. 1) $E = 0$ MeV, 2) $E = 10$ MeV, 3) $E = 50$ MeV, 4) $E = 100$ MeV, 5) $E = 150$ MeV, 6) $E = 200$ MeV.

([45]) A. GENZ: *The ε-algorithm and some other applications of Padé approximants in numerical analysis*, in *Padé Approximants*, edited by P. P. GRAVES MORRIS (Bristol, 1972), p. 112-125.

interest on the r-axis for a given energy E, we look for a Cauchy interpolation by a polinomial rational function

$$\mathfrak{f}(r) = r^2 \mathfrak{c}_1(r, E)\mathfrak{F}_I(r, E) = \frac{\mathfrak{P}(r)}{\mathfrak{Q}(r)} . \tag{9.62}$$

In order to find the coefficients of the polynomials $\mathfrak{P}(r)$ and $\mathfrak{Q}(r)$ use is made of the ε-algorithm ([45]) in order to accelerate the convergence of the continued-fraction method. Figure 13 shows a typical case of the behaviour of the interpolating Padé approximant against the values of the interpolated points.

A similar approach based on Padé approximants has also been used to compute the limit for $r \to 0$ of the imaginary spin-orbit potential $U_I^{so}(r, E)$ (7.22*b*).

The radial and energy dependence of the central part of the imaginary potential $V_I^c(r, E)$ and the imaginary spin-orbit function $W_I(r, E) = U_I V_I^c$ for

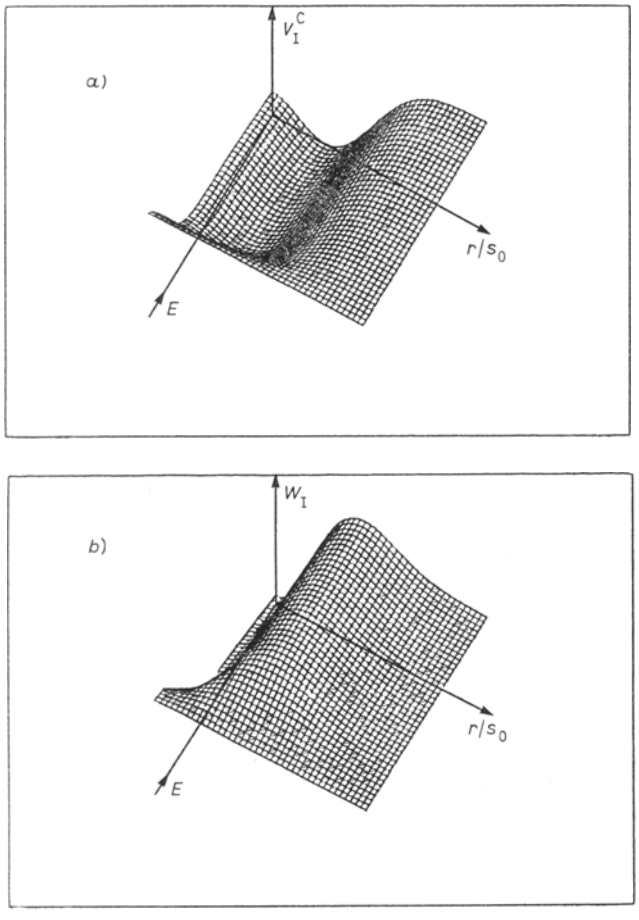

Fig. 18. – Radial and energy dependence of the imaginary potential for ^{16}O: *a*) central part V_I^c, and *b*) spin-orbit term $W_I = U_I V_I^c$. Mesh steps as in fig. 4.

nucleons elastically scattered by ^{4}He, ^{12}C and ^{16}O are plotted in fig. 14*a*), *b*), 16*a*), *b*) and 18*a*), *b*), respectively: more detailed information for selected energies is plotted in fig. 15*a*), *b*), 17*a*), *b*) and 19*a*), *b*).

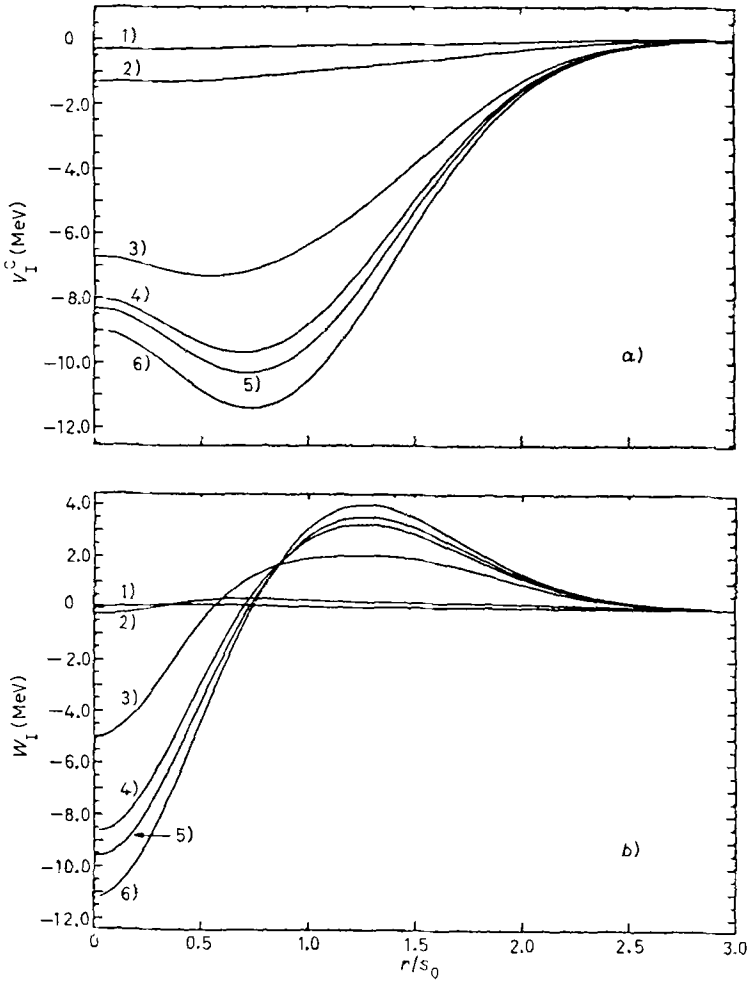

Fig. 19. – Radial dependence of the imaginary potential for ^{16}O for selected energies: *a*) central term V_I^C and *b*) spin-orbit term $W_I = U_I V_I^C$. Resolution as in fig. 5. 1) $E = 0$ MeV, 2) $E = 10$ MeV, 3) $E = 50$ MeV, 4) $E = 100$ MeV, 5) $E = 150$ MeV, 6) $E = 200$ MeV.

10. – Concluding remarks.

The basic idea followed in the construction of the optical potential for single-channel reactions is well summarized by the factorable single-particle potential expressed by eqs. (3.30): the former concerns the description of an even-even target nucleus, while the latter generates, through transformation

(6.1), the real optical potential. The linking in momentum space of the internal solution with the external one at $p = \chi(0)$, together with their first-order derivatives, allows one to transfer to the description of the nucleon-nucleus scattering, namely for $p > \chi(0)$, the parameters governing the momentum dependence of the single-particle potential for $p \leqslant \chi(0)$. Besides the continuity conditions fulfilled by the internal and external solutions of eq. (2.14), the five energy-independent parameters s_0, η_0, C_1, α, β are determined by taking into account *a*) the stability prescriptions of the target nucleus expressed by eqs. (4.22) and (4.29), *b*) the r.m.s. radius deduced from electron-nucleus scattering experiments (subsect. **9**'2) and then used according to *ansatz* (2.31), and *c*) the overall description of infinite nuclear matter through the hydrodynamical nucleon mass approximation, introduced in subsect. **5**'4: in conclusion, all loose ends of the problems are tied up and there is no room for empirical adjustments of the optical parameters, except for the lengths λ_R and λ_I. The radial and energy dependence of the real optical potential is strongly bound to the description of the nucleus ground state: indeed, eq. (3.28*a*) shows that small variations of length α imply large variations of β. The complicated dependence of the imaginary potential on $V_R^C(r, E)$ also closely relates $V_I^C(r, E)$ to the nucleus ground state. The theoretical scheme outlined in this paper makes it possible to disclose the single-particle potential underlying current optical-model analyses: it is found that the phenomenological arbitrariness generally conceals amazing pictures of monster-target nuclei, which do not exist. In conclusion, we have set up an overall theoretical scheme which is exactly the opposite of the elusive optical models referred to by HODGSON, as mentioned at the end of sect. **6**.

The total and differential cross-sections for the scattering of nucleons from nuclei, together with the strength functions, depend on the form of the optical potential in a very complicated manner. The capability of the potential to fit the observed energy dependence only provides a necessary condition for testing its reliability. The sufficient condition is given by polarization data. Still, the optical potential turns out to be not unambiguously determined by fits which generally are not quantitatively defined. An important goal to pursue is to disentangle from the observed energy dependence the competing contributions arising from the indistinguishability of the incident nucleon and from intrinsic and induced nonlocalities, the former being brought out by the presumptive nonlocality of the two-body forces and the latter by many-body effects. The current distinction between local and nonlocal potentials sounds somewhat quodlibetic. In fact, phenomenological potentials quoted in the literature as local might conceal nonlocalities and *vice versa*: the latter aspect of the problem is conceptually embarrasing as it implies that the antisymmetry effects of the nuclear wave function are conceived of as equivalent to an effective nonlocality, a conclusion which must clearly be rejected. A careful examination of the influence of the variation of the form of the optical potential

on the shift and damping of the diffraction pattern might be illuminating: in fact, one should expect that, for a given incident energy, a « pure » locality tends to shift the pattern at large angles, whereas a « pure » nonlocality shifts the pattern at smaller angles and tends to weaken both maxima and minima. The comparison of the predicted value of the zero-energy limit at the origin of the central part of the real optical potential with the values extrapolated from the analysis of the data is significant only if the effects arising from compound elastic scattering are carefully accounted for: it is worthwhile to point out that faults in fitting procedures might simulate spurious surface effects and anomalous values of the nucleon effective mass. The quality of the fit and the numerical values of the optical parameters extracted from the analysis of the data are also influenced by the analytic form of density function $\varrho(r)$. In particular, this is true for the Saxon-Woods distribution which makes the spin-orbit parts of the potential divergent at $r = 0$ and therefore is responsible for undue violence on the optical-model information derived from the overall fit of the data. For this reason ^{40}Ca will be examined in a subsequent paper, assuming a Gaussian taper distribution which gives finite values for the spin-orbit potentials at the origin. The choice of $\varrho(r)$ given by eq. (2.17) is motivated by the purpose of finding out to what extent the basic assumptions of the shell and Thomas-Fermi models can be used for building up a unified theoretical description of even-even nuclei with $A \leqslant 16$ and the optical interaction of nucleons with them. Further investigations in this direction are in progress.

The heuristic role played by the hyperbolic partial differential equations (2.1) and (2.14) is remarkable. The mathematical structure of these equations outweighs the details concerning the two-body forces both in the case of infinite nuclear matter and in the case of finite nuclei. The complicated interlacing of the single-particle solutions of eqs. (2.1) and (2.14) with the associate Cauchy problems is the very cause of the puzzling attitude to lay all the blame for the unsatisfactory results obtained by conventional treatments either on the choice of the two-body potential or on the many-body description of the nucleon propagation in the nuclear medium. Equations (2.1) and (2.14) provide a theoretical tool for expressing analytically the single-particle potential energy without resorting to any preconceived choice of the nucleon-nucleon potential: in this regard we have endeavoured to be faithful to Newton's statement « hypotheses non fingo ».

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For the realization of the plots the graphical packages MIZARTEK, developed in Milan by Dr. A. GIORGILLI, and HBOOK/HPLOT, developed at CERN, have been used.

● RIASSUNTO

Si delinea una nuova descrizione dell'interazione di nucleoni con nuclei pari-pari e si esamina in dettaglio la dipendenza radiale ed energetica del potenziale ottico che descrive l'interazione di nucleoni con ^{4}He, ^{12}C e ^{16}O.

Ядерная модель Томаса-Ферми и оптический потенциал для одно-канальных реакций.

Резюме (*). — Развивается новый подход к описанию ядерного взаимодействия нуклонов с конечными четно-четными ядрами. Исследуются радиальная и энергетическая зависимости оптического потенциала, описывающего взаимодействия нуклонов с ^{4}He, ^{12}C и ^{16}O.

(*) *Переведено редакцией.*