Schrödinger flow of maps into symplectic manifolds*

DING Weiyue (丁伟岳) and WANG Youde (王友德)
(Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, China)

Received January 10, 1998

Abstract The definition of Schrödinger flow is proposed. It is indicated that the flow of ferromagnetic chain is actually Schrödinger flow of maps into S^2, and that there exists a unique local smooth solution for the initial value problem of one-dimensional Schrödinger flow of maps into Kähler manifolds. In case the targets are Kähler manifolds with constant curvature, it is proved that one-dimensional Schrödinger flow admits a unique global smooth solution.

Keywords: Schrödinger flow, Kähler manifold, conservative law, sectional curvature.

1 The definition of Schrödinger flow

Let (N, ω) be a symplectic manifold, and let J be an almost complex structure on N such that

$$h(\cdot, \cdot) := \omega(\cdot, J\cdot) \tag{1.1}$$

is a Riemannian metric. It is well known that any smooth function f on N corresponds to a Hamiltonian vector field V_f, which is given by the relation

$$df(v) = \omega(v, V_f).$$

From (1.1), one easily derives

$$V_f = J\mathrm{grad}f,$$

where $\mathrm{grad}f$ denotes the gradient of f with respect to the metric h. The flow defined by the o.d.e.

$$\dot{u} = J(u)\mathrm{grad}f(u)$$

is called the Hamiltonian flow with the Hamiltonian function f.

It is natural to generalize the notion of Hamiltonian flow to an infinite-dimensional setting. Let (M, g) be a Riemannian manifold and let $X = C^k(M, N)$ $(k \geqslant 1)$. We may consider X as a symplectic Banach manifold with the symplectic form Ω defined by

$$\Omega(u)(v, w) = \int_M \omega(u)(v, w)\mathrm{d}v_g, \quad \forall\, u \in X,\ \forall\, v,\ w \in T_uX.$$

Similarly, we define an inner product on the tangent bundle TX by

$$\langle v, w\rangle_u = \int_M h(u)(v, w)\mathrm{d}v_g, \quad \forall\, u \in X,\ \forall\, v, w \in T_uX.$$

If $F \in C^1(X, R)$ is any functional, we let T_F be the gradient of F with respect to the above inner product, i.e.

$$\mathrm{d}F(u)(v) = \langle T_F(u), v\rangle, \ \forall\, v \in T_uX.$$

* Project partially supported by the National Natural Science Foundation of China (Grant No. 19631020, 19701034).

Then one can show that the corresponding Hamiltonian vector field V_F (satisfying $\mathrm{d}F(u)(v) = \Omega(u)(v, V_F(u))$ has the expression

$$V_F(u) = J(u)T_F(u).$$

Definition. Let $E(u)$ be the energy of $u:(M, g) \to (N, h)$. The Schrödinger flow is the flow induced by the Hamiltonian vector field V_E.

Recall that

$$E(u) = \int_M e(u)\mathrm{d}v_g,$$

where in local coordinates

$$e(u) = \frac{1}{2}g^{\alpha\beta}h_{ij}(u)\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta}.$$

The L^2-gradient T_E of E is usually called the "tension field". We will follow Eells and Sampson[1] to denote the tension field of u by $\tau(u)$, i.e. $T_E(u) = \tau(u)$. In local coordinates

$$\tau^i(u) = \Delta_g u + g^{\alpha\beta}\Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta},$$

where Γ^i_{jk} is the Christoffel symbol on the target manifold N. The Schrödinger flow is thus defined by the equation

$$\frac{\partial u}{\partial t} = J(u)\tau(u). \tag{1.2}$$

Note that when $N = C$ is the complex plane, eq. (1.2) is just the linear Schrödinger equation

$$\frac{\partial u}{\partial t} = i\Delta u.$$

However, it is nonlinear in general, and a main problem is the existence of the initial value problem of (1.2), i.e. given a smooth map u_0: $M \to N$, we want to find a classical solution of (1.3) which satisfies, at $t=0$

$$u(\cdot, 0) = u_0. \tag{1.3}$$

Generally speaking, this is still an open problem.

2 Some known results related to the Schrödinger flow

There is an interesting equation from physics known as the equation of ferromagnetic chain, and it has the form

$$\frac{\partial u}{\partial t} = u \times \Delta u, \quad u(\cdot, 0) = u_0, \tag{2.1}$$

where u_0 is a smooth map from a Riemannian manifold (M, g) into the unit sphere S^2 in R^3, and $\times$ is the cross product for vectors in R^3. This may be considered as the simplest nonlinear example of the Schrödinger flow because eq. (2.1) can be written in the form of (1.2).

Note that the right-hand side of eq. (2.1) can be rewritten as $u \times \tau(u)$, where the tension field is given by

$$\tau(u) = \Delta u + |\nabla u|^2 u,$$

which is the tangential part of Δu. Note also

$$J(u) := u \times: T_u S^2 \to T_u S^2$$

is just the standard complex structure on S^2. Thus, we can write the equation in the form of

(1.2), and the flow of ferro-magnetic chain is actually the Schrödinger flow for $N = S^2$.

For the solvability of the initial value problem (2.1), the only result on the classical solution is the following.

Theorem 2.1[1]. *If* $M = S^1$, *then* (2.1) *has a unique, smooth, global solution*.

For dim $M \geqslant 2$, there are no results even on the short time existence of a classical solution to (2.1). However, we should mention that global weak solutions do exist, as was proved in references [2] and footnote1).

If (N, h) is a Kähler manifold, then we can choose a complex local coordinate system in N so that the operation of $J(u)$ is just a multiplication by $\mathrm{i} = \sqrt{-1}$. In such a local coordinate system, the linear part of (1.2) is just the standard Schrödinger operator. In fact problem (1.2) and (1.3) with small initial data can be written in the following form:

$$\frac{\partial u}{\partial t} = \mathrm{i}[\Delta u + \Gamma(u, \bar{u})(\mathrm{d}u, \mathrm{d}\bar{u})],$$

$$u(\cdot, 0) = u_0 : M \rightarrow C^n.$$

Such initial value problems have recently been studied by Kenig et al.[2] and Hayashi-Hirata[3]. They required $M = R^m$ and used Fourier transformation and methods in harmonic analysis to treat the problem. Under certain conditions on u_0, they can prove the short time existence and even global existenece of a unique solution. For instance, ref. [2] proved the short time existence under the condition

$$\sum_{|\alpha| \leqslant s_0} \int_{R^m} |\partial^\alpha u_0|^2 \mathrm{d}x + \sum_{|\alpha| \leqslant s_1} \int_{R^m} |\partial^\alpha u_0|^2 |x|^{s_1} \mathrm{d}x < \varepsilon,$$

where $s_0 \geqslant 3m + 9/2$ and $s_1 = 2m + 2$.

It may be possible to use similar methods to study this problem for $M = T^m$, the m-torus. One may consult Bourgain's work on nonlinear Schrödinger equations[3,4]. However, it seems difficult to apply such methods to the general case. Essentially the difficulty is that, when Fourier analysis is not available one does not know how to study in depth the linear Schrödinger operator on a compact manifold. On the other hand, the method used in ref. [1] is to approximate (2.1) by a parabolic system and apply the classical energy estimates in a clever way. This method relies more heavily on the geometric structures of the equation, and it seems more suitable for the study of the general Schrödinger flow.

In the next section we will give a generalization of Theorem 2.1. Here we point out that the uniqueness part in Theorem 2.1 actually holds for the general Schrödinger flow, more precisely we have the following.

Proposition 2.1. *Let M be a compact Riemannian manifold without boundary. Then the* C^3*-solutions to the initial value problem* (1.2) *and* (1.3) *are unique*.

Proof. We need to use an equivalent form of the Schrödinger flow equation. Note that we can always imbed N isometrically into some Euclidean space R^l so that N is considered as a submanifold of R^l. For any $y \in N$, let $P(y): R^l \rightarrow T_yN$ be the orthogonal projection onto the tangent space of N at y. Then for any C^2-map $u: M \rightarrow N \subset R^l$ we have

$$\tau(u) = P(u)\Delta u.$$

1) Hayashi, N., Hirata, H., Global existence of small solutions to nonlinear Schrödinger equations, Preprint.

Thus, eq.(1.3) becomes

$$\frac{\partial u}{\partial t} = J(u)P(u)\Delta u.$$

Assume now that $u, v: M \times [0, T] \to N \subset R^l$ are two solutions to the above equation with the same initial value at $t = 0$. Then

$$\frac{\partial(u-v)}{\partial t} = [J(u)P(u) - J(v)P(v)]\Delta u + J(v)P(v)\Delta(u-v),$$

and we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |u-v|^2$$
$$= \int_M \langle u-v, [J(u)P(u) - J(v)P(v)]\Delta u\rangle + \int_M \langle u-v, J(v)P(v)\Delta(u-v)\rangle.$$

Note that the first integral on the right-hand side is controlled by

$$C\int_M |u-v|^2,$$

where constant C depends on the C^2-norm of u. The second integral, after integration by parts, can be seen to be bounded by

$$C\int_M (|\nabla(u-v)|^2 + |\nabla(u-v)||u-v|),$$

where constant C depends on the C^1-norm of v. Therefore, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |u-v|^2 \leqslant C\left(\int_M |u-v|^2 + \int_M |\nabla(u-v)|^2\right),$$

where constant C depends on the C^2-norms of u and v. Moreover,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\nabla(u-v)|^2$$
$$= \int_M \langle \Delta(u-v), [J(u)P(u) - J(v)P(v)]\Delta u\rangle$$
$$+ \int_M \langle \Delta(u-v), J(v)P(v)\Delta(u-v)\rangle.$$

Note that the second integral on the right-hand side vanishes, because $J(u): T_uN \to T_uN$ is anti-symmetric. For the first integral, after integrating by parts we see that it is bounded by

$$C\int_M [|\nabla(u-v)|^2 + |u-v|^2] = \|u-v\|^2_{W^{1,2}},$$

where C depends again on the C^3-norms of u and v. So setting $f(t) = \|u(t) - v(t)\|^2_{W^{1,2}}$, we have

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leqslant Cf(t),$$

which gives

$$f(t) \leqslant f(0)\exp(Ct).$$

Since $f(0) = 0$ and $f(t) \geqslant 0$, we must have $f(t) \equiv 0$, i.e. $u \equiv v$. This completes the proof of Proposition 2.1.

3 Results in the one-dimensional case

In this section we generalize the 1-D result in ref.[1] to the case of general Kählerian targets.

Theorem. *If $M = S^1$ and (N, J, h) is a complete Kähler manifold, then the initial value problem of the Schrödinger flow*

$$\begin{cases} \frac{\partial u}{\partial t} = J(u)\tau(u), \\ u(\cdot, 0) = u_0 \in C^\infty(S^1, N) \end{cases} \tag{3.1}$$

has a unique smooth solution on $S^1 \times [0, T)$, for some $T \in (0, \infty]$. Moreover, if N is compact and has constant sectional curvature, then $T = \infty$, i.e. the solution is global.

Remark. If N is a Kähler manifold with constant sectional curvature, then N has to be either a closed surface or a flat complex torus of higher dimension.

Before proving the theorem we need to introduce our notations. For $u \in C^1(S^1 \times [0, T), N)$, let

$$u' = \frac{\partial u}{\partial s} \quad \text{and} \quad \dot{u} = \frac{\partial u}{\partial t},$$

where $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ are unit vector fields on S^1 and $[0, T)$, respectively. Both u' and $\dot{u}$ are considered as sections of the vector bundle $u^* TN$ over $S^1 \times [0, T)$. Let ∇ be the natural connection on the bundle $u^* TN$ with respect to the Riemannian metric h on TN. We will use the simplified notations

$$\nabla_s = \nabla_{\frac{\partial}{\partial s}} \quad \text{and} \quad \nabla_t = \nabla_{\frac{\partial}{\partial t}} .$$

Thus, if X is a section of $u^* TN$ we have in local coordinates

$$(\nabla_s X)^\alpha = \frac{\partial X^\alpha}{\partial s} + \Gamma^\alpha_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial s} X^\gamma,$$

and for $X = u'$ we have

$$(\nabla_t u')^\alpha = \frac{\partial^2 u^\alpha}{\partial t \partial s} + \Gamma^\alpha_{\beta\gamma}(u) \frac{\partial u^\beta}{\partial t} \frac{\partial u^\gamma}{\partial s}.$$

It is easy to see that $\nabla_s \dot{u} = \nabla_t u'$. We also note that the tension field of u in this case is just

$$\tau(u) = \nabla_s u'.$$

Since we will make use of Sobolev imbedding theorems, it is convenient to imbed isometrically the manifold N into a Euclidean space R^l for some positive integer l. Then we may consider N as a submanifold of R^l, and a map $u: S^1 \to N$ can be considered as mapping S^1 into R^l so that the Sobolev-norms of u make sense. We will denote the norm of $u \in W^{k,p}(S^1, R^l)$ by $\| u \|_{k,p}$. Note that $\| \cdot \|_{0,p}$ is just the L^p norm.

Lemma 3.1. *Let N be a complete Riemanian manifold and let S^1 be a unit circle. If $u: S^1 \to \Omega \subset N$ is a map in $W^{k,2}(S^1, R^l)$, where Ω is a compact subset of N, then for any integer $k \geqslant 1$*

$$\| u \|^2_{k,2} \leqslant 2 \| \nabla_s^{k-1} u' \|^2_{0,2} + C(k, \Omega, \| u \|_{k-1,2}).$$

Proof. The case of $k = 1$ is trivial, so we start with $k = 2$. We have

$$(\nabla_s u')^\alpha = \partial_s^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u) \partial_s u^\beta \partial_s u^\gamma.$$

It follows that

$$| \partial_s^2 u |^2 \leqslant 2 | \nabla_s u' |^2 + C(\Omega) | u' |^4$$

and

$$\| u \|^2_{2,2} \leqslant 2 \| \nabla_s u' \|^2_{0,2} + C(\Omega) \| u' \|^4_{0,4}.$$

Applying the interpolation inequality (in dimension one)

$$\| u' \|_{0,4}^{4} \leqslant \| u' \|_{0,2}^{3} \| u'' \|_{0,2},$$

we get

$$\| u \|_{2,2}^{2} \leqslant 2 \| \nabla_s u' \|_{0,2}^{2} + C(\Omega) \| u \|_{1,2}^{3} \| u \|_{2,2}, \tag{3.2}$$

which implies the lemma for $k=2$.

For $k \geqslant 2$, we note that

$$\nabla_s^{k-1} u' = \partial_s^k u + P_k(u, \partial_s u, \cdots, \partial_s^{k-2}) \partial_s^{k-1} u + Q_k(u, \partial_s u, \cdots, \partial_s^{k-2} u),$$

where P_k and Q_k are matrix and vector with polynomial components. By the imbedding theorems

$$\| u \|_{C^{k-2}} \leqslant C_k \| u \|_{k-1,2},$$

we get

$$\| u \|_{k,2} \leqslant \| \nabla_s^{k-1} u' \|_{0,2} + C(k, \Omega, \| u \|_{k-1,2}).$$

The lemma then follows easily.

To prove Theorem, we approximate (3.1) by the following initial value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon \tau(u) + J(u) \tau(u), \\ u(\cdot, 0) = u_0 \in C^\infty(S^1, N), \end{cases} \tag{3.3}$$

where $\varepsilon > 0$ is a small constant. It is easy to check that the equation with $\varepsilon > 0$ is a parabolic system, hence local solutions exist (cf. e.g. ref. [7]). This means that for every $\varepsilon > 0$ there exists $T_\varepsilon \in (0, \infty]$ such that (3.3) has a unique smooth solution $u_\varepsilon : S^1 \times [0, T_\varepsilon) \to N$. The crucial point now is to get the *a priori* estimates for u_ε. The following Proposition 3.1 gives the necessary estimates which hold for $\varepsilon = 0$, too.

Proposition 3.1. *Let $u = u_\varepsilon$ be a local solution of (3.3) with $\varepsilon \in [0, 1]$. There exists a constant $T > 0$ such that the solution exists on the time interval $[0, T]$ and satisfies the following estimates:*

$$\mathrm{dist}_N(u(s,t), u_0(s,t)) \leqslant 1, \quad \forall s \in S^1, \ t \in [0, T], \tag{3.4}$$

$$\| \nabla^k u'(t) \|_{0,2} \leqslant C_k, \quad \forall k \geqslant 0, \ t \in [0, T], \tag{3.5}$$

where constants C_k depend only on the initial map u_0 and the target manifold N.

Proof. We first note that (3.5) holds trivially for $k = 0$ because the energy $E(u)$ decreases along the flow (3.3). In fact we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(u(t)) &= \int_{S^1} \langle u', \nabla_s \dot{u} \rangle \mathrm{d}s \\ &= -\int_{S^1} \langle \tau(u), \dot{u} \rangle \mathrm{d}s = -\int_{S^1} \langle \tau(u), \varepsilon \tau(u) + J(u) \tau(u) \rangle \mathrm{d}s \\ &= -\varepsilon \int_{S^1} |\tau(u)|^2 \mathrm{d}s \leqslant 0, \end{aligned}$$

where we have used $\langle \cdot, \cdot \rangle$ to denote the inner product w.r.t. the metric h on N and noticed that the complex structure J is anti-symmetric w.r.t. this inner product. It follows that

$$\| u'(t) \|_{0,2}^2 = 2E(u(t)) \leqslant 2E(u_0).$$

Next, since N may not be compact we let $\Omega = \{ p \in N : \mathrm{dist}_N(p, u_0(S^1)) < 1 \}$, which is an open subset of N with compact closure $\bar{\Omega}$. Let

$$T' = \sup \{ t > 0 : u(S^1, t) \subset \Omega \}.$$

Then for $t \leqslant T'$ we have the following estimate:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^1}|\dot{u}|^2 &= \int_{S^1}\langle\dot{u},\nabla_t\dot{u}\rangle\\ &= \int_{S^1}\langle\dot{u},\nabla_t(\varepsilon\nabla_s u' + J\nabla_s u')\rangle\\ &= \varepsilon\int[\langle\dot{u},\nabla_s^2\dot{u}\rangle + \langle R(u)(u',\dot{u})u',\dot{u}\rangle]\\ &\quad + \int[\langle\dot{u},J\nabla_s^2\dot{u}\rangle + \langle R(u',\dot{u})Ju',\dot{u}\rangle]\\ &= -\varepsilon\int|\nabla_s\dot{u}|^2 + \varepsilon\int\langle R(u)(u',\dot{u})u',\dot{u}\rangle + \int\langle R(u)(u',\dot{u})Ju',\dot{u}\rangle\\ &\leqslant C(\Omega)\int|\dot{u}|^2|u'|^2.\end{aligned}$$

Here R is the Riemannian curvature of N. In the above computation we have used the fact $\nabla J = 0$ (since N is Kähler) and

$$\int_{S^1}\langle\dot{u},J\nabla_s^2\dot{u}\rangle = -\int_{S^1}\langle\nabla_s\dot{u},J\nabla_s\dot{u}\rangle = 0.$$

Since $|\dot{u}|^2 = (1+\varepsilon^2)|\tau(u)|^2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tau(u)\|_2^2 \leqslant C(\Omega)\|u'(t)\|_{C^0}^2\|\tau(u)\|_2^2. \tag{3.6}$$

On the other hand, since $E(u) = 1/2\|u'\|_2^2 \leqslant C$, we have

$$\|u'\|_{C^0}^2 \leqslant C(1+\|u''\|_{0,2}^2),$$

and from (3.2) we deduce

$$\|u''\|_{0,2}^2 \leqslant C(\Omega)(1+\|\tau(u)\|_{0,2}^2).$$

Hence

$$\|u'\|_{C^0}^2 \leqslant C(\Omega)(1+\|\tau(u)\|_{0,2}^2). \tag{3.7}$$

Substituting inequality (3.7) in (3.6) we see

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tau(u)\|_{0,2}^2 \leqslant C(\Omega)(1+\|\tau(u)\|_{0,2}^2)\|\tau(u)\|_{0,2}^2.$$

This ordinary differential inequality shows that for any constant $K > \|\tau(u_0)\|_{0,2}^2$, we can find $T^* = T^*(K, C(\Omega)) > 0$ such that

$$\|\tau(u(t))\|_{0,2}^2 = \|\nabla_s u'(t)\|_{0,2}^2 \leqslant K \text{ for } t\in[0,T^*]. \tag{3.8}$$

We now continue our estimates.

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^1}|\nabla_s\dot{u}|^2 &= \int_{S^1}\langle\nabla_s\dot{u},\nabla_t\nabla_s\dot{u}\rangle\\ &= \int_{S^1}[\varepsilon\langle\nabla_s\dot{u},\nabla_t\nabla_s\nabla_s u'\rangle + \langle\nabla_s\dot{u}, J\nabla_t\nabla_s\nabla_s u'\rangle].\end{aligned}$$

Changing the order of covariant derivatives of the 3rd order gives

$$\begin{aligned}\nabla_t\nabla_s\nabla_s u' &= \nabla_s\nabla_t\nabla_s u' + R(u',\dot{u})\nabla_s u'\\ &= \nabla_s[\nabla_s\nabla_t u' + R(u',\dot{u})u'] + R(u',\dot{u})\nabla_s u'\\ &= \nabla_s^3\dot{u} + \nabla_{u'}R(u',\dot{u})u' + R(\nabla_s u',\dot{u})u'\\ &\quad + R(u',\nabla_s\dot{u})u' + 2R(u',\dot{u})\nabla_s u'.\end{aligned}$$

Thus, assuming $t < T'$ so that $u(s,t)\in\Omega$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^1}|\nabla_s\dot{u}|^2$$

$$\leqslant -\varepsilon\int_{S^1}|\nabla_s^2\dot{u}|^2 + C_2(\Omega)\int_{S^1}[|\nabla_s\dot{u}|^2|u'|^2 + |\nabla_s\dot{u}|(|u'|^3|\dot{u}| + |\nabla_s u'||u'||\dot{u}|)].$$

Since we already have the C^0-control of u' by (3.7) and (3.8) provided $t\leqslant T^*$, and since by (3.3) we have

$$|\nabla_s^k\dot{u}|^2 = (1+\varepsilon^2)|\nabla_s^{k+1}u'|^2 \quad \forall k\geqslant 0,$$

we know from the above inequality that

$$\frac{d}{dt}\int_{S^1}|\nabla_s^2 u'|^2 \leqslant C_2(\Omega)\int_{S^1}(|\nabla_s^2 u'|^2 + |\nabla_s u'|^2)$$
$$\leqslant C_2(\Omega)\left(\int_{S^1}|\nabla_s^2 u'|^2 + 1\right),$$

where we have used (3.8). This implies

$$\|\nabla_s^2 u'(t)\|_{0,2}^2 \leqslant C_2(\Omega, u_0) \quad \text{for } t\leqslant T^*. \tag{3.9}$$

We note that a positive lower bound of T' can be derived from (3.9). Indeed, similar to Lemma 3.1, one can prove that

$$\|\dot{u}\|_{1,2}^2 \leqslant 2\|\nabla_s\dot{u}\|_{0,2}^2 + C(\Omega, \|\dot{u}\|_{0,2}).$$

The right-hand side of the above inequality has an upper bound given by (3.8) and (3.9). So, by the Sobolev imbedding $C^0 \to W^{1,2}$, we have

$$\|\dot{u}(t)\|_{C^0} \leqslant M$$

for some $M>0$, assuming that $t<\min\{T', T^*\}$. Thus, we have

$$\sup_{s\in S^1} d_N(u(s,t), u_0(s)) \leqslant Mt \quad \forall t\in \min\{T', T^*\}.$$

If $T' > T^*$ we get the lower bound, so we may assume $T'\leqslant T^*$. Then letting $t\to T'$ in the above inequality we see $MT'\geqslant 1$ (recalling the definition of T'). Therefore, if we set $T=\min\{1/M, T^*\}$, then (3.8) and (3.9) hold for $t\in[0,T]$. Note that T depends only on N and u_0.

To finish the proof of the *a priori* estimate (3.5), we remark that for $k\geqslant 3$ the proof is essentially the same as the proof of (3.9) and can be completed by induction on k. We leave the details to the reader. Once we get the estimate (3.5), by Lemma 3.1 we get the bounds for the Sobolv norms of the solutions. Then it is easy to see that the solution must exist on the time interval $[0,T]$. Otherwise, one can always extend the time interval of existence to cover the interval $[0,T]$ (see ref.[5]). This completes the proof of Proposition 3.1.

Remark 3.1. If N is compact, then by the above proof, the number T in Proposition 3.1 depends only on the manifold N, the energy of the initial map and the L^2 norm of the tension field of the initial map, i.e.

$$T = T(N, E(u_0), \|\tau(u_0)\|_{0,2}).$$

Proof of Theorem. By Proposition 3.1, we have the bounds for the $W^{k,2}(S^1, R^l)$-norms of $u_\varepsilon(\cdot, t)$ for all k and $t\in[0,T]$, and these bounds do not depend on $\varepsilon\in(0,1)$. Using eq. (3.3), it is easy to show that the solutions u_ε are also uniformly bounded in $W^{k,2}(S^1\times[0,t], R^l)$. Then by the standard argument, we can infer the existence of a sequence $\varepsilon_i\to 0$ such that $u_{\varepsilon_i}\to u$ in $C^\infty(S^1\times[0,T], R^l)$ and u is a solution to problem (3.1). This proves the local existence. The uniqueness follows from Proposition 2.1.

Now, let N be a closed Kähler manifold of constant sectional curvature K, and let u be the local solution of (3.1) which exists on the maximal time interval $[0,T)$. Then we know (cf. the proof of Proposition 3.1) that the energy is preserved by the solution u, i.e.

$$E(u(t)) = E(u_0) \quad \forall t \in [0, T).$$

Moreover, the following Proposition 3.2 tells us that the solution also preserves the integral

$$I(u) = \int_{S^1} |\tau(u)|^2 - \frac{K}{4}\int_{S^1} |u'|^4. \tag{3.10}$$

That is,

$$I(u(t)) = I(u_0), \quad \forall t \in (0, T).$$

From (3.2) and the inequality proceding it, we know that the above identity implies

$$\| \tau(u(t)) \|_{0,2}^2 \leqslant C(N, I(u_0)) \quad \forall t \in [0, T).$$

Now, if T is finite, then we can solve eq. (3.1) to find a local solution u_1 which satisfies the initial value condition

$$u_1(s, T - \varepsilon) = u(s, T - \varepsilon),$$

where $0 < \varepsilon < T$ is a small number. Then by Proposition. 3.1, the solution u_1 exists on the time interval $(T - \varepsilon, T - \varepsilon + \eta)$ for some constant $\eta > 0$. Since we have uniform bounds on $E(u(t))$ and $\| \tau(u(t)) \|_{0,2}$. Remark 3.1 tells us that η is independent of ε. Thus, if we choose ε sufficiently small, we have

$$T_1 = T - \varepsilon + \eta > T.$$

However, by the uniqueness result (Proposition 2.1), u_1 and u coincide on $S^1 \times [T - \varepsilon, T]$ and they together form a smooth solution of problem (3.1) on $S^1 \times [0, T_1)$. This contradicts the maximality of T, showing that we must have $T = \infty$. In other words, the solution u is global. The proof of Theorem is completed.

Now we prove the following proposition.

Proposition 3.2. *Let N be a surface of constant curvature K. Then for any solution u of* (3.1),

$$\frac{\mathrm{d}}{\mathrm{d}t} I(u(t)) = 0.$$

Proof. Since u satisfies eq. (3.1) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^1} |\tau(u)|^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^1} |\dot{u}|^2 \int_{S^1} \langle \dot{u}, \nabla_t (J\nabla_s u') \rangle$$

$$= \int_{S^1} \langle \dot{u}, J\nabla_s \dot{u} \rangle + \int_{S^1} \langle \dot{u}, JR(u', \dot{u})u' \rangle.$$

The first integral on the right side can be seen to vanish, because after integrating by parts the integrand becomes $\langle \nabla_s \dot{u}, J\nabla_s \dot{u} \rangle = 0$. Since N has constant curvature K, we have

$$\langle R(U, V)W, Z \rangle = K(\langle U, W \rangle \langle V, Z \rangle - \langle U, V \rangle \langle W, Z \rangle).$$

Applying the formula to the integrand of the second integral on the right side we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^1} |\tau(u)|^2 = K\int_{S^1} \langle u', \dot{u} \rangle \langle u', J\dot{u} \rangle. \tag{3.11}$$

On the other hand, we have

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^1} |u'|^4 = \int_{S^1} |u'|^2 \langle u', \nabla_t u' \rangle = \int_{S^1} |u'|^2 \langle u', \nabla_s \dot{u} \rangle$$

$$= -\int_{S^1} (|u'|^2 \langle \nabla_s u', \dot{u} \rangle + 2\langle u', \dot{u} \rangle \langle u', \nabla_s u' \rangle)$$

$$= -\int_{S^1} (|u'|^2 \langle \tau(u), J\tau(u) \rangle + 2\langle u', \dot{u} \rangle \langle u', (-J\dot{u}) \rangle)$$

$$= 2\int_{S^1} \langle u', \dot{u} \rangle \langle u', J\dot{u} \rangle .$$

Combining this with (3.11) we get

$$\frac{d}{dt}\int_{S^1} |\tau(u)|^2 = \frac{K}{4} \quad \frac{d}{dt}\int_{S^1} |u'|^4 .$$

This finishes the proof of Proposition 3.2.

4 Final remarks

For the existence of a classical solution of the general Schrödinger flow, there are more questions than results. Even in the 1-D case, if N is only almost Kähler (i. e. J is not a complex structure) we are unable to prove the local existence.

On the other hand, the example of the constant curvature case shows that conservation laws are very useful in obtaining global existence. It is therefore desirable to find new conservation laws for the Schrödinger flow, at least when N is a nice symmetric space such as CP^n.

References

1 Zhou., Y., Guo, B., Tan, S., Existence and uniqueness of smooth solution for system of ferromagnetic chain, *Science in China* Ser. A, 1991, 34: 257.

2 Kenig, C. E., Ponce G., Vega, L., Small solutions to nonlinear Schrödinger equations, *Anal. Nonlinéare*, 1993, 10(3): 255.

3 Bourgain, J., Exponential sums and nonlinear Schrödinger equations, *Geom. Funct. Anal.*, 1993, 3: 157.

4 Bourgain, J., Fourier transfrm restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, I. Schrödinger equations, *Geom. Funct. Anal.*, 1993, 3: 107.

5 Amann, H., Quasilinear parabolic systems under nonlinear boundary conditions, *Arch. Rat. Mech. Anal.*, 1986, 92: 153.