SOME GENERALIZATIONS OF KY FAN'S BEST APPROXIMATION THEOREM

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Abstract

We pressent new Ky Fan type best approximation theorems for a discontinuous multivalued map on metrizable topological vector spaces and hyperconvex spaces. In addition, fixed point results are derived for the map studied. Our work generalizes severl results in approximation theory.

Key words *best approximation, fixed point, * - nonexpansive multivalued map, almost quasi-convex function, metrizable topological vector space, hyperconvex space*

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1 Introduction

In 1969, Ky Fan proved the following best approximation result:

Theorem A([6],Theorem1). *Let C be a compact convex set in a locally convex Hausdorff topological vector space X. If $f : C \to X$ is continuous, then either f has a fixed point or there exist an $x \in C$ and a continuous seminorm p on X such that*

$$p(x - fx) = d_p(fx, C),$$

where $d_p(fx, C) = \inf\{p(fx - y) : y \in C\}$.

This well-known best approxiamation theorem due to Ky Fan plays an important role in approximation theory, fixed point theory, nonlinear analysis, game theory and minimax theorems. Among several a pplications, this result serves as an important tool in ascertaining approximate solutions of systems of equations. It has been extended in various directions by many authors (e.g. see [12] and [18]). Prolla[15] has generalized it for a pair of continuous functions on a normed space while Sehgal and Singh[16] obtained its generalization for continuous multifunctions. Fixed point theorems for multivalued maps and some other related results have been used to prove the existence of best approximation for multivalued maps (see e.g. [8,12,14,15]).

A hyperconvex space is a metric space satisfying a property about the intersection of closed balls. Recently, approximation theory in hyperconvex spaces has been the focus of several researchers. For more information about approximation theory and related concepts in hyperconvex spaces, we refer to [5,9,11,17] where further references are given.

In this paper, we employ a variant argument, namely; use Ky Fan's intersection lemma to establish approximation results for continuous maps and a discontinuous class of multivalued maps, namely, *-nonexpansive maps on compact convex and noncompact convex sets in the settings of metrizable topological vector spaces and hyperconvex spaces.

In Section 4, we establish Ky Fan type approximation results in hyperconvex spaces. Hyperconvexity facilitates in obtaining some results of Section 3 under weaker assumptions. Also, our results present multivalued analog of some well-known approximation theorems for hyperconvex spaces.

In Section 2, we recall certain technical preliminaries and establish notational conventions for the sake of completeness.

2 Preliminaries

Let X denote a topological vector space (TVS, for short). Throughout, we assume that its topology is tacitly generated by an F-norm on it; that is, there is a real-valued map, say, q on X such that

(i) $q(x) \geq 0$ and $q(x) = 0$ iff $x = 0$;

(ii) $q(x+y) \leq q(x) + q(y)$;

(iii) $q(\lambda x) \leq q(x)$ for all $x, y \in X$ and for all scalars λ with $|\lambda| \leq 1$;

(iv) if $q(x_n) \to 0$, the $q(\lambda x_n) \to 0$ for all scalars λ;

(v) if $\lambda_n \to 0$, then $q(\lambda_n x) \to 0$ for all $x \in X$, where (λ_n) is a sequence of scalars.

The formula

$$d(x,y) = q(x-y)$$

defines a metric on X. A topological vector space X is called metrizable if there is a metric on

X such that the metric topology coincides with the given topology.

A single-valued selfmap T of a metric space (X, d) is called nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. A generalization of this notion for multivalued maps has been introduced by Husain and Latif[7] as follows.

Let X be a metrizable TVS, $C \subseteq X$ and $T : C \to 2^X$ a multifunction. Then T is called *-nonexpansive (cf.[3,8,19]) if for all $x, y \in C$ and $u_x \in Tx$ satisfying $d(x, u_x) = d(x, Tx)$, there exists $u_y \in Ty$ satisfying $d(y, u_y) = d(y, Ty)$ such that

$$d(u_x, u_y) \leq d(x, y).$$

Beg, Khan and Hussain[3], Hussain and Khan[8] and Xu[19] have extensively used this concept in their investigations. Recall that x is a fixed point of T if $x \in Tx$.

A multivalued function $T : C \to 2^X$ is upper semicontinuous (usc) (lower semicontinuous(lsc)) if $T^{-1}(B) = \{x \in C : Tx \cap B \neq \emptyset\}$ is closed (open) for each closed (open) subset B of X. If T is both usc and lsc, then it is continuous. We denote by $C(X)$, the family of all nonempty closed subsets of X and H denotes the Hausdorff metric on $C(X)$. A map $T : X \to C(X)$ is called H-continuous if it is continuous as a map from X into the metric space $(C(X), H)$. If T is comapct-valued, then the two notions of continuity are equivalent (see [20]).

The set of best approximations to $x \in X$ from C is a set-valued map defined as

$$P_C(x) = \{y \in C : d(x, y) = d(x, C)\}.$$

If $P_C(x) \neq \phi$ (singleton) for each $x \in X$, then C is called a proximinal (Chebyshev) set. In case $P_C(x)$ is single-valued, it is called a proximity map (or a metric projection) and is denoted by p.

Following Xu[19], we define the set (possibly empty),

$$P_T(x) = \{u_x \in Tx : d(x, u_x) = d(x, Tx)\}.$$

In general, *-nonexpansive maps are neither nonexpansive nor continuous as is clear from the following.

Example 2.1 (see also [8,Example 1.1]). Let $T : [0, 1] \to 2^{[0,1]}$ be defined by

$$Tx = \begin{cases} \{\frac{1}{2}\}, & x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1], \\ [\frac{1}{4}, \frac{3}{4}], & x = \frac{1}{2}. \end{cases}$$

Then $P_T(x) = \{\frac{1}{2}\}$ for all $x \in [0, 1]$. This implies that T is a *-nonexpansive map. Observe that

$$H(T(1/3), T(1/2)) = H(\{1/2\}, [1/4, 3/4]) = \max\{0, 1/4\} = 1/4 > 1/6 = |1/3 - 1/2|.$$

So T is not a nonexpansive multivalued map. The map is not lsc because if we take $V_{1/4}$ as a small open neighborhood of 1/4, then the set

$$T^{-1}(V_{1/4}) = \{x \in [0,1] : Tx \cap V_{1/4} \neq \phi\} = \{\frac{1}{2}\}$$

is not open. Hence T is not continuous. Note that 1/2 is a fixed point of T.

A mapping $f : C \to X$ is called selector of the map $T : C \to 2^X$ if $f(x) \in Tx$. For $x \in X$, the inward set, $I_C(x)$, of C at x is defined by $I_C(x) = \{x + r(u - x) \in X : u \in C, r > 0\}$. The closure of $I_C(x)$ is denoted by $\overline{I_C(x)}$.

For a finite subset $\{x_1, \cdots, x_n\}$ of a TVS X, we write the convex hull of $\{x_1, \cdots, x_n\}$ as

$$Co\{x_1, \cdots, x_n\} = \left\{ \sum_{i=1}^{n} \alpha_i x_i : 0 \leq \alpha_i \leq 1, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

The following result known as Ky Fan's intersection Lemma [6] is needed.

Theorem B. *Let C be a subset of a TVS X and $F : C \to 2^X$ a closed-valued map such that $Co(x_1, \cdots, x_n) \subseteq \cup_{i=1}^{n} F(x_i)$ for each finite subset $\{x_1, \cdots, x_n\}$ of C. If $F(x_0)$ is compact for at least one x_0 in C, then $\bigcap_{x \in C} F(x) \neq \phi$.*

Let C be a convex subset of metrizable TVS X and $g : C \to C$ a continuous map. Then g is said to be

(i) almost affine if

$$d(g(rx_1 + (1-r)x_2), y) \leq rd(gx_1, y) + (1-r)d(gx_2, y),$$

(ii) almost quasi-convex if

$$d(g(rx_1 + (1-r)x_2), y) \leq \max\{d(gx_1, y), d(gx_2, y)\},$$

where $x_1, x_2 \in C, y \in X$ and $0 < r < 1$.

A metric space (Y, d) is said to be hyperconvex if $\cap_\alpha B(x_\alpha, r_\alpha) \neq \phi$ for any collection $\{B(x_\alpha, r_\alpha)\}$ of all closed balls in Y for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ (see e.g.[2]). An admissible subset of a hyperconvex space Y is a set of the form $\cap_\alpha B(x_\alpha, r_\alpha)$, where $\{B(x_\alpha, r_\alpha)\}$ is a family of closed balls centered at the points $x_\alpha \in Y$ with respective radii r_α. It is well-known that an admissible subset of a hyperconvex space is itself hyperconvex (see e.g [9]). A subset E of a metric space Y is said to be externally hyperconvex (relative to Y) if for a given family $\{x_\alpha\}$ of points in Y and a family $\{r_\alpha\}$ of real numbers with $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and $d(x_\alpha, E) \leq r_\alpha$, we have $\cap_\alpha B(x_\alpha, r_\alpha) \cap E \neq \phi$. It is shown in [2] that an admissible subset of a hyperconvex space Y is externally hyperconvex relative to Y, and externally hyperconvex subsets of Y are proximinal in Y. Thus, if E is externally hyperconvex in Y and $x \in Y$, then there is $h \in E$ such that $d(x, h) = d(x, E)$. For more information on externally hyperconvex spaces, we refer to [9].

A subset E of a metric space Y is said to be weakly externally convex (relative to Y) if E is externally hyperconvex relative to $E \cup \{z\}$ for each $z \in Y$. More precisely, given any family

$\{x_\alpha\}$ of points in Y all but at most one of which lies in E, and any family $\{r_\alpha\}$ of real numbers satisfying $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$ (with $d(x_\alpha, E) \le r_\alpha$ if $x_\alpha \notin E$) implies that $\cap_\alpha B(x_\alpha, r_\alpha) \cap E \neq \phi$ (see [5] for more details).

In what follows we use $E(Y)$ to denote the family of all bounded subsets of Y which are externally hyperconvex.

3 Approximation Results

Khan, Thaheem and Hussain[10] have recently established the following pair of Prolla type approximation theorems on the basis of Ky Fan's intersection lemma (i.e. Theorem B) and the arguments used by Carbone[4]. The proofs of our results will rely on these two theorems.

Theorem C([10],Throerm3.1). *Let C be a nonempty compact convex subset of a metrizable TVS X and $g : C \to C$ a continuous almost quasi-convex onto function. If $f : C \to X$ is a continuous function, then there exists $y \in C$ such that $d(gy, fy) = d(fy, C)$.*

Theorem D([10],Theorem3.2). *Let C be a nonempty convex subset of a metrizable TVS X and $g : C \to C$ a continuous almost quasi-convex onto function. Suppose $f : C \to X$ is a continuous function. If C has a nonempty compact convex subset B such that the set*

$$D = \{y \in C : d(fy, gy) \le d(fy, gx) \text{ for all } x \in B\}$$

is compact, then there exists $y \in D$ such that $d(fy, gy) = d(fy, C)$.

The "selections" have been studied and used in a number of disciplines over the last ffty years. Recently, Agarwal and O'Regan[1], Espinola, Kirk and Lopez[5], Hussain and khan[8], Khamsi, Kirk and Yanez[9] have utilized "selections" to obtain fixed point and approximation results for multivalued maps. In general, a nonexpansive multivalued map does not admit a single-valued nonexpansive selection. However, *-nonexpansive maps and hyperconvex spaces share this property.

We establish here new Prolla type approximation results by using continuous selectors of *-nonexpansive maps on metrizable TVS and hyperconvex spaces. Our results contain as a special case the Ky Fan tye approximation results.

The following result provides a generalized Prolla type best approximation theorem for *-nonexpansive maps.

Theorem 3.1. *Let C be a nonempty compact convex subset of a uniformly convex metrizable TVS X and $g : C \to C$ a continuous almost quasi-convex onto map. If $T : C \to 2^X$ is a closed convex valued *-nonexpansive mapping, then T possesses a nonexpansive selector f such that $d(gy, fy) = d(fy, C)$ for some $y \in C$. Further,*

(i) *If $T : C \to 2^C$, then y is a coincidence point of g and T.*

(ii) *If $d(fy, pfy) = d(Ty, C)$, then $d(gy, Ty) = d(Ty, C)$, where p is the proximity map of*

X onto C.

Proof. A closed convex set in a uniformly convex metric linear space is Chebyshev, so Tx is a Chebyshev subset of X for each $x \in C$. Thus for each $x \in C$, there is unique $u_x \in Tx$ such that $\{u_x\} = P_T(x) \in Tx$. Since T is *-nonexpansive, therefore for each $x, y \in C$, we get

$$d(P_T(x), P_T(y)) = d(u_x, u_y) \leq d(x, y).$$

This implies that $P_T : C \to X$ is a nonexpansive selector of T (i.e. $P_T(x) \in Tx$). By Theorem C, there exists $y \in C$ such that

$$d(gy, P_T(y)) = d(P_T(y), C). \tag{1}$$

This proves the first part of the theorem.

To prove (i), we observe that $d(P_T(y), C) = 0$ implies that $gy = P_T(y) \in Ty$ as desired.

To prove (ii), we note that equation (1) and the assumption that $d(fy, pfy) = d(Ty, C)$ imply

$$d(gy, Ty) \leq d(gy, fy) = d(fy, C) = d(fy, pfy) = d(Ty, C) \leq d(gy, Ty).$$

Therefore, $d(gy, Ty) = d(Ty, C)$.

Next, we obtain a version of Theorem 3.1 without the compactness of C.

Theorem 3.2. *Let C be a nonempty convex subset of a uniformly convex metrizable TVS $X, g : C \to C$ continuous almost quasi-convex onto function and $T : C \to 2^X$ a closed convex valued *-nonexpansive mapping. assume that C has a nonempty compact convex subset B such that*

(i) *$D = \{z \in C : d(Tz, gz) \leq d(Tz, gx)$ for all $x \in B$ is compact.*

(ii) *for each $z \in C, d(u_z, gx) \leq d(Tz, gx)$, where $z \in C$ and satisfies*

$$d(u_z, gz) \leq d(u_z, gx)$$

for each $x \in C$. Here u_z denotes the unique best approximation of z from Tz.

Then there exists $y \in D$ such that $d(gy, Ty) = d(Ty, C)$.

Proof. As in the proof of Theorem 3.1, $P_T : C \to X$ is a nonexpansive selector of T. Define

$$E = \{y \in C : d(P_T(y), gy) \leq d(P_T(y), gx) \text{ for each } x \in B\}.$$

As both P_T and g are continuous, so E is a closed subset of C. Let $y \in E$. Then for each $x \in B$, we have (by (i))

$$d(Ty, gy) \leq d(P_Ty, gy) \leq d(P_Ty, gx) = d(u_y, gx) \leq d(Ty, gx).$$

This implies that $y \in D$. Thus P_T satisfies all the conditions of Theorem D and hence there exists $y \in D$ such that

$$d(gy, P_Ty) = d(P_Ty, C). \tag{2}$$

From (2) and the hypotheses $d(P_T z, gx) \leq d(Tz, gx)$, we get the inequality

$$d(gy, Ty) = d(gy, P_T y) = d(P_T y, C) \leq d(P_T y, gx) \leq d(Ty, gx)$$

for all $x \in C$.

As g is onto, so $d(gy, Ty) = d(Ty, C)$.

Remark 3.3. (i) In case the map T is H-continuous instead of being *-nonexpansive, the conclusions of Theorems 3.1 and 3.2 hold (the similar proofs carry over).

(ii) If we consider $T : C \to 2^C$ in Theorem 3.2, then y becomes a coincidence point of g and T.

(iii) All the results obtained so far hold good when X is a Fréchet space.

4 Approximation in Hyperconvex Spaces

We begin with an analog of Theorem 3.1 under weaker conditions. This also gives a multi-valued extension of results of Sine[17,Corollary12] and Espinola, Kirk and López[5,Theorem5.4].

Theorem 4.1. *Let C be a nonempty compact convex subset of a hyperconvex metrizable TVS $X, g : C \to C$ a continuous almost quasi-convex onto map and $T : C \to 2^X$. Suppose that either of the following conditions* (a), (b) *and* (c) *holds:*

(a) *T is *-nonexpansive and for each $x \in C, Tx$ is externally hyperconvex.*

(b) *T is continuous and Tx is bounded and externally hyperconvex for each $x \in C$.*

(c) *X has unique metric segments and T is closed-valued *-nonexpansive.*

Then T possesses a continuous selector f such that $d(gy, fy) = d(fy, C)$ for some $y \in C$. Furhter,

(i) *if $T : C \to 2^C$, then y is a coincidence point of g and T;*

(ii) *if $d(fy, pfy) = d(Ty, C)$, then $d(gy, Ty) = d(Ty, C)$, where p is the proximity map of X onto C.*

Proof. (a) Each Tx being nonempty externally hyperconvex is proximinal, therefore $P_T(x)$ is nonempty for each $x \in C$ and $P_T(x) = B(x, r) \cap Tx$, where

$$r = d(x, Tx).$$

$P_T(x)$ being the intersection of admissible and externally hyperconvex sets is externally hyperconvex for each $x \in C$[9,Lemma2]. So, $P_T : C \to E(X)$ is nonexpansive by the *-nonexpasive axiom of T. Thus by [9, Corollary 1], P_T has a nonexpansive selector $f : C \to X$ which is also a selector of T. By Theorem C, there exists y in C such that $d(gy, fy) = d(fy, C)$. This proves the first part of the result.

The proof for (i) is simple and we omit it.

To prove (ii), we note that the equality $d(gy, fy) = d(fy, C)$ and the hypotheses imply

$$d(gy,Ty) \leq d(gy,fy) = d(fy,C) = d(fy,pfy) = d(Ty,C) \leq d(gy,Ty).$$

(b) The selection theorem of [9, Theorem 1] implies that T has a continuous selection $f : C \to X$. Then, by Theorem C, there exists $y \in C$ such that $d(gy, fy) = d(fy, C)$ and following the arguments similar to those in (a), we get the proof for (b).

(c) We observe that a hyperconvex metric space with unique metric segments is a complete **R**-tree [see 11, Theorem 3.2]. Further, a closed subtree of a complete **R**-tree is Chebyshev [see 11, p. 70-71]. Thus, $P_T(x)$ in Tx is unique fo each x in C and hence $P_T : C \to X$ is a nonexpansive selector of T. So, the result follows from (a).

The following theorem is a multifunction analog of the results of Sine[17] and Espinola, Kirk and López [5] for hyperconvex normed spaces.

Theorem 4.2. *Suppose that C is a nonempty convex and weakly externally hyperconvex subset of a hyperconvex normed space X and $T : C \to 2^X$ satisfies either of the conditions* (a),(b) *and* (c) *of Theorem 4.1. Let M and K be compact subsets of C with M being convex. If for each x in $C \setminus K, x \notin P_M(Tx)$, then T possesses a continuous selector f such that $d(y, fy) = d(fy, \overline{I_C(y)})$ for some $y \in K$. If, in addition, $d(fy,pfy) = d(Ty, \overline{I_C(y)})$, where p is the proximity map of X onto C, then $d(y,Ty) = d(Ty, \overline{I_C(y)})$.*

Proof. As in the proof of Theorem 4.1, T has a continuous selector f in all the cases (a), (b) and (c). Thus, $P_M(fx) \subseteq P_M(Tx)$ for each $x \in C$. So, by assumption $x \notin P_M(f(x))$ for each $x \in C \setminus K$. Thus, by Theorem 1(i) of Park[13], there is y in K such that $d(y, fy) = d(fy, \overline{(I_C(y))})$. Since C being weakly externally hyperconvex is proximinal, therefore pfy is a nonempty subset of C. By hypothesis, $d(fy,pfy) = d(Ty, \overline{(I_C(y))})$, and hence we get

$$d(y,Ty) \leq d(y,fy) = d(fy, \overline{(I_C(y))}) \leq d(fy,C) \leq d(fy,pfy) = d(Ty, \overline{(I_C(y))}) \leq d(y,Ty).$$

Finally, since compact hyperconvex subspaces have the fixed point property for continuous single-valued mappings[5], the selection theorem[9,Theorem1] and Theorem 4.2 of [5] yield the following best approximation result for compact weakly externally hyperconvex set in hyperconvex metric spaces which is, in fact, a multifunction analog of Theorem 5.4 of [5].

Theorem 4.3. *Suppose that C is a nonempty compact weakly externally hyperconvex subset of a hyperconvex space X and T satisfies either of the conditions* (a),(b) *and* (c) *of Theorem 4.1. Then T possesses a continuous selector f such that $d(y, fy) = d(fy, C)$ for some $y \in C$. If, in addition, $d(fy,pfy) = d(fy, C)$, then*

$$d(y,Ty) = d(Ty,C),$$

where p is the proximity map of X onto C.

Remark 4.4. If we consider $T : C \to 2^C$ in Theorem 4.3, then we obtain the following fixed point result (Corollary 4.5) for *-nonexpansive and continuous maps which extends several well known results such as Theorem 3.2 of [7], Theorem 2 of [19], Corollary 4 of [9] and Corollaries 3.3 and 3.4 of [11].

Corollary 4.5. *Suppose that C is a nonempty compact weakly externally hyperconvex subset of a hyperconvex space X and $T : C \to 2^C$ satisfies either of the conditions* (a),(b) *and* (c) *of Theorem* 4.1. *Then T has a fixed point.*

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