A NOTE ON MULTILINEAR SINGULAR INTEGRALS WITH ROUGH KERNEL*

Lan Jiacheng

(*Lishui Teachers College, China*)

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Abstract

In this paper, the author gives the weighted weak Lipschitz boundedness with power weight for rough multilinear integral operators. A simple way is obtained that is closely linked with a class of rough fractional integral operators.

Key words *multilinear operator, Lipschitz space, power weight, fractional integral*

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1 Introduction

Suppose that

$$\Omega \in L^s(S^{n-1})(s \geq \frac{n}{n-\beta})$$

is homogeneous of degree zero on R^n, and A is a funtion defined on R^n. Then the multilinear singular integral operator T_A is defined by

$$T_A f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A; x, y) f(y) \mathrm{d}y,$$

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where $R_m(A;x,y)$ denotes the m-th remainder of Taylor series of A at x about y. More precisely,

$$R_m(A;x,y) = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma,$$

where $D^\gamma A \in \dot{\Lambda}_{A\&B}(|\gamma| = m-1)$, and for $\beta > 0$, the homogeneous Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{x,h\in R^n, h\neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty,$$

where $\Delta_h^1 f(x) = f(x+h) - f(x), \Delta_h^{k+1} f(x) = \Delta_h^k f(x+h) - \Delta_h^k f(x), k \geq 1$. When $D^\gamma A \in L^r(R^n)(1 < r \leq \infty)$ or $D^\gamma A \in$BMO(R), it is well known that the multilinear operators have been widely studied by many authors([1]-[4]). In 1982, using the rotation method, Cohen and Gosselin[1] proved that T_A is bounded on L^p. For $m = 1, T_A$ is obviously the commutator operator, $[A,T]f(x) = Tf(x) - T(Af)(x)$, where T is the following singular integral operator:

$$Tf(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\mathrm{d}y.$$

In 1995, Paluszynski[2] proved that if $A \in \dot{\Lambda}_\beta(0 < \beta < 1), \Omega \in C^\infty(s^{n-1})$ and

$$\int_{S^{n-1}} \Omega(\theta)\mathrm{d}\theta = 0,$$

then $[A,T]$ is a bounded operator from $L^p(R^n)$ to $L^q(R^n)(1 < p < \frac{n}{\beta}), \frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$.

In 2001, Chen Wgngu, Hu Guoen[3] obtained the weak type (H^1, L^1) estimate for a multilinear operator T_A with $D^{A\&A}A \in$BMO.

In 2003, Lu Shanzhen, Wu Huoxiong, Zhang Pu[3] proved that if $D^\gamma A \in \dot{\Lambda}_\beta(0 < \beta < 1)$, then T_A is weak bounded. That is, they proved.

Theorem A. *Let $0 < \beta < 1, \Omega$ be homogeneous of degree zero on R^n and $\Omega(x) \in L^s(s^{n-1})(s \geq \frac{n}{n-\beta})$. If $D^\gamma A \in \dot{\Lambda}_\beta(|\gamma| = m-1)$, then*

$$|\{x \in R^n : |T_A f(x)| > \lambda\}| \leq C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \frac{\|f\|_1}{\lambda} \right)^{\frac{n}{n-\beta}}, \qquad \forall\lambda > 0.$$

Theorem B. *Under the assumptions of Theorem A, we have*

$$|\{x \in R^n : |M_A f(x)| > \lambda\}| \leq C \left(\sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \frac{\|f\|_1}{\lambda} \right)^{\frac{n}{n-\beta}}, \qquad \forall\lambda > 0.$$

In this note, we will discuss the case for weighted weak boundedness with power weight. Now we can state our main results as follows.

Theorem 1. *Let $-1 < \alpha < 0, 0 < \beta < 1$, and Ω be homogeneous of degree zero on $R^n, \Omega(x) \in L^s(s^{n-1}), s \geq \frac{n}{n-\beta}$. If $D^\gamma A \in \dot{\Lambda}_\beta, |\gamma| = m-1$, then T_A is a bound operator from $L^1(|x|^{\alpha(n-\beta)/n})$ to $L^{n/(n-\beta),\infty}(|x|^\alpha)$. That is, there is a constant $C > 0$ such that for any $\lambda > 0$ and $f \in L^1(|x|^{\alpha(n-\beta)/n})$,*

$$\int_{\{x:|T_A f(x)|>\lambda\}} |x|^\alpha \mathrm{d}x \leq C \left(\frac{1}{\lambda} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \int_{R^n} |f(x)||x|^{\alpha(n-\beta)/n} \mathrm{d}x \right)^{n/(n-\beta)}$$

Theorem 2. *Under the assumptions of Theorem 1, we have for any $\gamma > 0$ and $f \in L^1(|x|^{\alpha(n-\beta)/n})$*

$$\int_{\{x:|M_A f(x)|>\lambda\}} |x|^\alpha \mathrm{d}x \leq C \left(\frac{1}{\lambda} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \int_{R^n} |f(x)||x|^{\alpha(n-\beta)/n} \mathrm{d}x \right)^{n/(n-\beta)}.$$

2 Lemmas and Proof of Theorems

Lemma 1[6]. *Let $0 < \beta < n, -1 < \alpha < 0$, and $\Omega \in L^s(s^{n-1}), s \geq \frac{n}{n-\beta}$. Then $T_{\Omega,\beta}$ is a bound operator from $L^1(|x|^{\alpha(n-\beta)/n})$ to $L^{n/(n-\beta),\infty}(|x|^\alpha)$. That is, there is a constant $C > 0$ such that for any $\lambda > 0$ and $f \in L^1(|x|^{\alpha(n-\beta)/n})$*

$$\int_{\{x:|T_{\Omega,\beta} f(x)|>\lambda\}} |x|^\alpha \mathrm{d}x \leq C \left(\frac{1}{\lambda} \int_{R^n} |f(x)||x|^{\alpha(n-\beta)/n} \mathrm{d}x \right)^{n/(n-\beta)}.$$

Lemma 2[1]. *Let $A(x)$ be a function on R^n with m-th order derivatives in $L^l_{\mathrm{loc}}(R^n)$ for some $l > n$. Then*

$$|R_m(A;x,y)| \leq C \left(|x-y|^m \sum_{|r|=m} \frac{1}{|Q_x^y|} \int_{Q_x^y} |D^\gamma A(x)|^l \mathrm{d}z \right)^{\frac{1}{l}},$$

where Q_x^y is the cube centered at x and having diameter $5\sqrt{n}|x-y|$.

Lemma 3[2]. *For $0 < \beta < 1, 1 \leq q < \infty$, we have*

$$\begin{aligned} \|f\|_{\dot{\Lambda}_\beta} &= \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - m_Q(f)| \mathrm{d}x \\ &\approx \sup_Q \frac{1}{|Q|^{\beta A\#/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - m_Q(f)|^q \mathrm{d}x \right)^{\frac{1}{q}}. \end{aligned}$$

Lemma 4[5]. *Let $Q^* \subset Q, g \in \dot{\Lambda}_\beta (0 < \beta < 1)$. Then*

$$|m_{Q^*}(g) - m_Q(g)| \leq C|Q|^{\beta/n}\|g\|_{\dot{\Lambda}_\beta}.$$

Recently, Lu Shanzhen, Wu huoxiong, Zhang Pu[3] studied the multilinear operator T_A by closely linked with a class of rough fractional integral operator $T_{\Omega,\alpha}$,

$$T_{\Omega,\alpha}f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\mathrm{d}y.$$

They obtained the following importment Lemma.

Lemma 5. *Under the assumptions of Theorem 1 or Theorem 2, for any $x \in R^n$, we have*

$$|T_A f(x)| \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \overline{T}_{\Omega,\beta} f(x),$$

where

$$\overline{T}_{\Omega,\beta}f(x) = \int_{R^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\beta}} |f(y)|\mathrm{d}y.$$

Remark. Lemma 1 is also established for $\overline{T}_{\Omega,\alpha}$.

Proof of Lemma 5. For any fixed $x \in R^n$, given $r > 0$, we have

$$|T_A f(x)| \leq \sum_{k=-\infty}^{\infty} \int_{2^{k-1}r \leq |x-y| < 2^k r} \frac{|\Omega(x-y)|}{|x-y|^{n+m-1}} |R_m(A;x,y)||f(y)|\mathrm{d}y := \sum_{k=-\infty}^{\infty} I_k.$$

For each I_k, let Q be a cube centered at x and having diameter $r, Q_k = 2^k Q$ and set

$$A_{Q_k}(y) = A(y) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} m_{Q_k}(D^\gamma A) y^\gamma.$$

Obviously, $R_m(A;x,y) = R_m(A_{Q_k};x,y)$. By Lemma 2, we get

$$\begin{aligned} |R_m(A_{Q_k};x,y)| &\leq |R_{m-1}(A_{Q_k};x,y)| + C \sum_{|\gamma|=m-1} |D^\gamma A_{Q_k}(y)||x-y|^{m-1} \\ &\leq C|x-y|^{m-1} \sum_{|\gamma|=m-1} \left(\frac{1}{|Q_x^y|} \int_{Q_x^y} |D^\gamma A_{Q_k}(z)|^l \mathrm{d}z \right)^{\frac{1}{l}} + \\ &\quad C|x-y|^{m-1} \sum_{|\gamma|=m-1} |D^\gamma A_{Q_k}(y)|, \end{aligned}$$

where Q_x^y is the cube centered at x and having diameter $5\sqrt{n}|x-y|$. Note that, if $|x-y| < 2^k r$, then $Q_x^y \subset 4nQ_k$.

By Lemma 3 and Lemma 4, we have

$$\begin{aligned}\left(\frac{1}{|Q_x^y|}\int_{Q_x^y}|D^\gamma A_{Q_k}(z)|^l\mathrm{d}z\right)^{\frac{1}{l}} &= \left(\frac{1}{|Q_x^y|}\int_{Q_x^y}|D^\gamma A(z)-m_{Q_k}(D^\gamma A)|^l\mathrm{d}z\right)^{\frac{1}{l}}\\ &\le \left(\frac{1}{|Q_x^y|}\int_{Q_x^y}|D^\gamma A(z)-m_{Q_x^y}(D^\gamma A)|^l\mathrm{d}z\right)^{\frac{1}{l}}+\\ &\quad |m_{Q_x^y}(D^\gamma A)-m_{2nQ_k}(D^\gamma A)|+|m_{2nQ_k}(D^\gamma A)-m_{Q_k}(D^\gamma A)|\\ &\le C|Q_k|^{\beta/n}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\le C(2^kr)^\beta\|D^\gamma A\|_{\dot{\Lambda}_\beta},\end{aligned}$$

and

$$|D^\gamma A_{Q_k}(y)|=|D^\gamma A(y)-m_{Q_k}(D^\gamma A)|\le C|Q_k|^{\beta/n}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\le C(2^kr)^\beta\|D^\gamma A\|_{\dot{\Lambda}_\beta}.$$

Hence

$$|R_m(A_{Q_k};x,y)|\le C(2^kr)^\beta|x-y|^{m-1}\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}.$$

Therefore

$$\begin{aligned}I_k &\le \sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\int_{2^{k-1}r\le|x-y|<2^kr}\frac{(2^kr)^\beta}{|x-y|^n}|\Omega(x-y)||f(y)|\mathrm{d}y\\ &\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\int_{2^{k-1}r\le|x-y|<2^kr}\frac{|\Omega(x-y)|}{|x-y|^{n-\beta}}|f(y)|\mathrm{d}y.\end{aligned}$$

It follows that

$$\begin{aligned}|T_Af(x)| &\le \sum_{k=-\infty}^{\infty}\left(C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\int_{2^{k-2}r\le|x-y|<2^kr}\frac{|\Omega(x-y)|}{|x-y|^{n-\beta}}|f(y)|\mathrm{d}y\right)\\ &\le C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\sum_{k=-\infty}^{\infty}\int_{2^{k-1}r\le|x-y|<2^kr}\frac{|\Omega(x-y)|}{|x-y|^{n-\beta}}|f(y)|\mathrm{d}y\\ &= C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\int_{R^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\beta}}|f(y)|\mathrm{d}y\\ &= C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot{\Lambda}_\beta}\overline{T}_{\Omega,\beta}f(x).\end{aligned}$$

Lemma 6[4]**.** *Under the assumptions of Theorem 1, we have* $\overline{T}_Af(x)\ge M_Af(x)$*, where*

$$\overline{T}_Af(x)=\int_{R^n}\frac{|\Omega(x-y)|}{|x-y|^{n+m-1}}|R_m(A;x,y)||f(y)|\mathrm{d}y.$$

Proof of Theorem 1. The proof will be depened on the power weight bounded ness of the fractional integral operator $T_{\Omega,\beta}$. By Lemma 1 and Lemma 5, we have

$$\int_{\{x:|T_A f(x)|>\lambda\}} |x|^\alpha \mathrm{d}x \le \int_{\{x:C\sum\limits_{|\gamma|=m-1}\|D^\gamma A\|_{\dot\Lambda_\beta}|\overline{T}_{\Omega,\beta}f(x)|>\lambda\}} |x|^\alpha \mathrm{d}x$$

$$\le C\left(\frac{1}{\lambda}\sum_{|\gamma|=m-1}\|D^\gamma A\|_{\dot\Lambda_{\beta}}\int_{R^n}|f(x)||x|^{\frac{\alpha(n-\beta)}{n}}\mathrm{d}x\right)^{\frac{n}{(n-\beta)}}.$$

This completes the proof of Theorem 1.

By Lemma 6 and Theorem 1, the proof of Theorem 2 is directly deduced.

3 Further Results

Let us give the definition of the multilinear fractional integral operator as follows

$$T_{A_1,A_2,\cdots,A_k}f(x)=\int_{R^n}\frac{\Omega(x-y)}{|x-y|^{n+M-k}}\prod_{j=1}^k R_{m_j}(A_j;x,y)f(y)\mathrm{d}y;$$

$$M_{A_1,A_2,\cdots,A_k}f(x)=\sup_{r>0}\frac{1}{r^{n+m-k}}\int_{|x-y|<r}|\Omega(x-y)|\prod_{j=1}^k|R_{m_j}(A_j;x,y)||f(y)|\mathrm{d}y,$$

where

$$R_{m_j}(A_j;x,y)=A_j(x)-\sum_{|\alpha|<m_j}\frac{1}{\alpha!}D^\gamma A_j(y)(x-y)^\alpha,\quad j=1,2,\cdots,k,\quad M=\sum_{j=1}^k m_j.$$

Then we have the following theorems:

Theorem 3. *Let* $-1<\alpha<0, 0<\beta<1, \Omega$ *be homogenous of degree zero on* R^n *and* $\Omega(x)\in L^s(s^{n-1})(s\ge\frac{n}{n-\beta})$. *If* $D^\gamma A_j\in\dot\Lambda_\beta(|\gamma|=m_j-1)$, *then* $T_{A_1,A_2,\cdots,A_k}$ *is a bound operator from* $L^1(|x|^{\alpha(n-\beta)/n})$ *to* $L^{n/(n-\beta),\infty}(|x|^\alpha)$. *That is, there is a constant* $C>0$ *such that for any* $\lambda>0$ *and* $f\in L^1(|x|^{\alpha(n-\beta)/n})$

$$\int_{\{x:|T_{A_1,A_2,\cdots,A_k}f(x)|>\lambda\}}|x|^\alpha\mathrm{d}x\le C\left(\frac{1}{\lambda}\prod_{j=1}^k\left(\sum_{|\gamma|=m-1}\|D^\gamma A_j\|_{\dot\Lambda_\beta}\right)\int_{R^n}|f(x)||x|^{\alpha(n-\beta)/n}\mathrm{d}x\right)^{n/(n-\beta)}$$

Theorem 4. *Under the same conditions as in Theorem 3, we have for any* $\lambda>0$ *and* $f\in L^1(|x|^{\alpha(n-\beta)/n})$,

$$\int_{\{x:|M_{A_1,A_2,\cdots,A_k}f(x)|>\lambda\}}|x|^\alpha\mathrm{d}x\le C\left(\frac{1}{\lambda}\prod_{j=1}^k\left(\sum_{|\gamma|=m-1}\|D^\gamma A_j\|_{\dot\Lambda_\beta}\right)\int_{R^n}|f(x)||x|^{\alpha(n-\beta)/n}\mathrm{d}x\right)^{n/(n-\beta)}$$

Lemma 7. *Under the assumption of Theorem 3 or Theorem 4, for any $x \in R^n$, we have*

$$|T_{A_1,A_2,\cdots,A_k}f(x)| \le C\prod_{j=1}^{k}\left(\sum_{|\gamma|=m_j-1}\|D^\gamma A_j\|_{\dot{\Lambda}_\beta}\right)\overline{T}_{\Omega,\beta}f(x).$$

Remark. By Lemma 1 and Lemma 7, we can obtain the proof of Theorem 3 and Theorem 4 similar to that of Theorem 1 and Theorem 2.

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Department of Mathematics

Lishui Teachers College

Lishui, Zhejiang, 323000

P. R. China

e-mail: jiachenglan@163.com