ON A CLASS OF BESICOVITCH FUNCTIONS TO HAVE EXACT BOX DIMENSION: A NECESSARY AND SUFFICIENT CONDITION*

Zhou Songping

(*Zhejiang Institute of Science and Technology, China Ningbo University, China*)

He Guolong

(*Zhejiang Normal University, China*)

Received Sept. 20, 2003

Abstract

This paper summarized recent achievements obtained by the authors about the box dimensions of the Besicovitch functions given by

$$B(t) := \sum_{k=1}^{\infty} \lambda_k^{s-2} \sin(\lambda_k t),$$

where $1 < s < 2, \lambda_k > 0$ *tends to infinity as* $k \to \infty$ *and* λ_k *satisfies* $\lambda_{k+1}/\lambda_k \geq \lambda > 1$. *The results show that*

$$\lim_{k\to\infty} \frac{\log \lambda_{k+1}}{\log \lambda_k} = 1$$

is a necessary and sufficient condition for Graph$(B(t))$ *to have same upper and lower box dimensions. For the fractional Riemann-Liouville differential operator* D^u *and the fractional integral operator* D^{-v}, *the results show that if* λ *is sufficiently large, then a necessary and sufficient condition for box dimension of Graph*$(D^{-v}(B)), 0 < v < s-1$, *to be* $s - v$ *and box dimension of Graph*$(D^u(B)), 0 < u < 2-s$, *to be* $s + u$ *is also* $\lim_{k\to\infty} \frac{\log \lambda_{k+1}}{\log \lambda_k} = 1$.

Key words *Weierstrass function, Besicovitch function, fractal dimension, box dimension, Hard-*

---

* Research supported by national Natural Science Foundation of China (10141001) and Zhejiang Provincial Natural Science Foundation 9100042 and 1010009).

mard condition

AMS(2000)subject classification 28A80

1 Introduction

Fractal structure of functions is an intriguing aspect of fractal geometry, Besicovitch functions

$$B(t) = \sum_{k=1}^{\infty} \lambda_k^{s-2} \sin(\lambda_k t)$$

for $1 < s < 2$, $\lambda_k > 0$, and $\lim_{k\to\infty} \lambda_k = \infty$ are typically interested. It can be regarded as the generalization of the classical Weierstrass type functions[5,6,12]

$$W(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t), \qquad 1 < s < 2, \lambda > 1.$$

This type of functions is highly irregular in the point of view of classical analysis and is hard to be characterized and treated with by ordinary calculus.

They are usually served as counterexamples to illustrate the complexity of functions and for a long time seem beyond anyone's interest for the other purposes in classical analysis. However, as we know, in the continuous function space, these highly irregular functions (especially the continuous and nowhere differentiable functions) consist a residual set in the sense of Baire's category theory. Along with the development of fractal geometry, because of these functions' typical fractal structure, they gradually attract more and more attention and are investigated widely and extensively.

In fractal geometry, fractal dimension is one of the most important properties to describe fractals. In particular, Hausdorff dimension and box counting dimension are widely and extensively used.[5,6,12].

Definition 1. Let F be a nonempty and bounded subset of $R^2, N_\delta(F)$ is the least number of sets whose union covers F and diameters does not exceed given $\delta > 0$, then upper and lower box dimensions of F is defined respectively by

$$\overline{\dim}_B F = \limsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}, \quad \underline{\dim}_B F = \liminf_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}.$$

If these two limits are equal, the common value is called box dimension of F, and denoted by $\dim_B F$.

There are several useful equivalent definitions for box dimension[5,6,12], for example, while consider the $\delta-$mesh of R^2,

$$\{[i\delta,(i+1)\delta]\times[j\delta,(j+1)\delta]:i,j\in Z\},$$

then $N_\delta(F)$ in definition 1 can be replaced by the number of $\delta-$squares that intersect F.

The following result on the classical Weierstrass functions is well known:

Theorem 1.[5,12] *Let $\lambda>1, 1<s<2, I=[0,1]$, $W(t)$ be the classical Weierstrass functions on I:*

$$W(t)=\sum_{k=1}^{\infty}\lambda^{(s-2)k}\sin(\lambda^k t).$$

Then for sufficiently large $\lambda>1$, it holds that

$$\dim_B \mathrm{Graph}(W,I)=s.$$

There are several incomplete results on Besicovitch functions (see [12]). For example, If $\lambda_{k+1}/\lambda_k\ge\lambda>1$, $1\le s\le 2$, then

$$\begin{aligned}\dim_H \mathrm{Graph}(B,I) &\le \underline{\dim}_B \mathrm{Graph}(B,I)\\ &\le 1+\liminf_{k\to\infty}\frac{(s-1)\log\lambda_n}{(s-1)\log\lambda_n+(2-s)\log\lambda_{n+1}},\end{aligned}$$

where $\dim_H \mathrm{Graph}(B,I)$ stands for the Hausdorff dimension of the graph of the function $B(t)$ on I.

Generally speaking, it is far beyond settled to get exact fractal dimensions for Besicovitch functions in most cases. In fact, even for Werierstrass functions, it is still an open question whether the Hausdorff dimension is s or not.

At the same time, fractional calculus is a subject to deal with generalization of ordinary integration and differentiation to fractional orders. Some important and detailed works were achieved in [8], [11]. It proves suitable for dealing with irregular functions and fractals since they may have no ordinary derivatives at all and must keep their fractal structures.

There are numbers of ways to define fractional integrals and derivatives, the most natural and important one is introduced by Riemann-Liouville as follows.

Definition 2.[4,9,10] Let f be a function piecewisely continuous on $(0,\infty)$ and integrable on any finite subinterval of $(0,\infty)$. Then

$$D^{-v}f(t)=\frac{1}{\Gamma(v)}\int_0^t(t-\xi)^{v-1}f(\xi)\mathrm{d}\xi \tag{1}$$

is called the fractional integral of f of order v for $t > 0$ and $\mathrm{Re}(v) > 0$.

Definition 3.[4,9,10] Let f be a function piecewisely continuous on $(0, \infty)$ and integrable on any finite subinterval of $(0, \infty)$. Then

$$D^u f(t) = \frac{\mathrm{d}}{\mathrm{d}t} D^{-(1-u)} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\Gamma(1-u)} \int_0^t (t-\xi)^{-u} f(\xi) \mathrm{d}\xi$$

is called the fractional derivative of f of order u for $t > 0$ and $0 < u < 1$.

We begin with the fractional integrals of two basic trigonometric functions $\sin\lambda t$ and $\cos\lambda t$. By the definition, for $t > 0$ and $0 < v < 1$,

$$D^{-v} \sin \lambda t = \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1} \sin \lambda\xi \mathrm{d}\xi =: S_t(v, \lambda),$$

$$D^{-v} \cos \lambda t = \frac{1}{\Gamma(v)} \int_0^t (t-\xi)^{v-1} \cos \lambda\xi \mathrm{d}\xi =: C_t(v, \lambda).$$

For $1 < s < 2$, $0 < v < 1$, define

$$g(t) := D^{-v}(B(t)) = \sum_{k=1}^{\infty} \lambda_k^{s-2} S_t(v, \lambda_k)$$

to be the fractional integrals of Besicovitch functions $B(t)$ of order v.

And for $0 < u < 1$,

$$D^u \sin \lambda t = \frac{d}{dt} D^{-(1-u)} \sin \lambda t = \lambda C_t(1-u, \lambda).$$

For $1 < s < 2$, $0 < u < 1$, define

$$\widetilde{g}(t) := D^u(B(t)) = \sum_{k=1}^{\infty} \lambda_k^{s-1} C_t(1-u, \lambda_k)$$

to be the fractal derivatives of Besicovitch functions $B(t)$ of order u.

Very recently, some interesting and important work was done in Zhou, Yao and Su[13] which investigated the exact box dimension of the fractal integrals of the Weierstrass functions.

Theorem 2. *Let $1 < s < 2, 0 < v < 1, s > 1 + v, G(t)$ be the fractional integral function of $W(t)$ of order v as defined as*

$$G(t) := D^{-v}(W(t)) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} S_t(v, \lambda^k),$$

then for sufficiently large $\lambda > 1$, it holds that

$$\dim_B \mathrm{Graph}(G, I) = s - v.$$

Definiton 4. The sequence $\{\lambda_k\}_{k=1}^{\infty}$ is called to satisfy the Hardmard condition, if there exists a number $\lambda > 1$ such that $\frac{\lambda_{k+1}}{\lambda_k} \geq \lambda$ for $k = 1, 2, \cdots$.

By the definition of Besicovitch functions, the Hardmard condition is a convenient assumption to ensure the convergency. In the sequel, we suppose $\{\lambda_k\}_{k=1}^{\infty}$ satisfies this condition, and λ always stands for the constant in the Hardmard condition.

We first investigate the intrinsic relationship between box dimension of graphs of Besicovich functions and the asymptotic behavior of coefficients sequence $\{\lambda_k\}_{k=1}^{\infty}$. We show that the upper box dimension does not exceed s in general, and equals to s while the constant λ in Hardmard condition is sufficiently large. If the sequence $\{\lambda_k\}_{k=1}^{\infty}$ grows in sense faster than any geometrically rate, the lower box dimensionof Graph(B) will be strictly less than s. Then a necessary and sufficient condition is obtained for this type of Besicovitch functions to have exact Box dimension. In [12], Wen pointed that the asymptotic behavior of $\lim_{n\to\infty} \frac{\log\lambda_{n+1}}{\log\lambda_n}$ can influence the graph's dimension. In [14], Sun and Wen investigated the Bouligand dimension (which is identical to box dimension) for trigonometric lacunary series and get the following.

Theorem 3. *Let*

$$f(x) = \sum_{j\geq 1} a_j \cos(\lambda_j x + \theta_i),$$

where $\{a_j\}$ is absolutely convergent and $\{\lambda_j\}$ is positive and monotonically tends to infinity, $\theta_i \in [0, 2\pi]$, if for any positive integer $n, r_{n+1}\lambda_{n+1}/r_n\lambda_n \geq 2.5$ and $r_n/r_{n+1} \geq 2.4$ holds ($r_j = |a_j|$) then

$$\overline{\dim}_B \mathrm{Graph}(B, I) = 2 + \limsup_{n\to\infty} \frac{\log r_n}{\log\lambda_n};$$

$$\underline{\dim}_B \mathrm{Graph}(B, I) = 1 + \liminf_{n\to\infty} \frac{\log r_n\lambda_n}{\log r_n\lambda_n - \log r_{n+1}}.$$

The above result implied that for the case of Besicovitch functions, a sufficient and necessary condition can be characterized by

$$\lim_{n\to\infty} \frac{\log\lambda_{n+1}}{\log\lambda_n} = 1.$$

We establish the following explicit result by a quite different constructive technique.

Theorem 4. *Let $\{\lambda_k\}_{k=1}^{\infty}$ satisfy the Hardmard condition, $1 < s < 2$. Then for sufficiently large λ, $\overline{\dim}_B \mathrm{Graph}(B, I) = \underline{\dim}_B \mathrm{Graph}(B, I) = s$ holds if and only if*

$$\lim_{n\to\infty} \frac{\log\lambda_{n+1}}{\log\lambda_n} = 1.$$

For fractional integrals of Besicovitch functions, we obtain the following result by some novel ideas and innovated techniques:

Theorem 5. *If* $\{\lambda_k\}_{k=1}^{\infty}$ *satisfies the Hardmard condition,* $1 < s < 2, 0 < v < 1$, *then*

(1) *If* $s \le 1 + v$, *then* $\dim_H \mathrm{Graph}(g, I) = \dim_B \mathrm{Graph}(g, I) = 1$;

(2) *If* $s > 1 + v$, *then* $\dim_H \mathrm{Graph}(g, I) \le \overline{\dim}_B \mathrm{Graph}(g, I) \le s - v$;

(3) *If* $s > 1 + v$, *then for sufficiently large* $\lambda, \overline{\dim}_B \mathrm{Graph}(g, I) = s - v$.

Moreover, we have

Theorem 6. *If* $\{\lambda_k\}_{k=1}^{\infty}$ *satisfies the Hardmard condition,* $1 < s < 2, 0 < v < 1, s > 1+v$, *then for sufficiently large* λ, *we have*

$$\overline{\dim}_B \mathrm{Graph}(g, I) = \underline{\dim}_B \mathrm{Graph}(g, I) = s - v$$

holds if and only if

$$\lim_{n\to\infty} \frac{\log\lambda_{n+1}}{\log\lambda_n} = 1.$$

We also have the following complete result for fractional derivatives of Besicovitch functions:

Theorem 7. *If* $\{\lambda_k\}_{k=1}^{\infty}$ *satisfies the Hardmard condition,* $1 < s < 2, 0 < u < 2 - s$, *then for sufficiently large* λ, *we have*

$$\overline{\dim}_B \mathrm{Graph}(\widetilde{g}, I) = \underline{\dim}_B \mathrm{Graph}(\widetilde{g}, I) = s + u$$

holds if and only if

$$\lim_{n\to\infty} \frac{\log\lambda_{n+1}}{\log\lambda_n} = 1.$$

References

[1] Bedford, T., On Weierstrass-Like Functions and Random Recurrent Sets, Math. Proc. Camb. Philo. Soc., 106(1989), 325-342.

[2] Berry, M.V. and Lewis, Z.V., On the Weierstrass-Mandelbrot Fractal Function. Proc. R. Soc. London, 370A(1980), 459-484.

[3] Besicovitch, A.S. and Ursell, H.D., Sets of Fractional Dimensions, V: On Dimensional Numbers of some Continues Curves, J. London Math. Soc., 12(1937), 18-25.

[4] Courant, R. and John, F., Introduction to Calculus and Analysis, Volume 2, Springer-Verlag, New York, 1989.

[5] Falconer, K., Fractal Geometry: Mathematical Foundations and Applications, John Wiley Sons, Inc., New York, 1990.

[6] Falconer, K., Techniques in Fractal Geometry, John Wiley Sons, Inc., New York, 1997.

[7] Hu, T.Y. and Lau, K.S., Fractal Dimensions and Singularities of the Weierstrass Type Function, Trans. Amer. Math. Soc., 335(1993), 649-655.

[8] Mandelbrot, B.B. and Vanness, J.W., Fractional Brownian Motions, Fractional Noise and Applications, SIAM Review, 10(1968), 422-437.

[9] Miller, K.S. and Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley Sons, Inc., New York, 1993.

[10] Oldham, K.B. and Spanier, J., The Fractional Calculus, Academic Press, New York, 1974.

[11] Tatom, F.B., The Relationship Between Fractional Calculus and Fractals, Fractal, 3(1995), 217-229.

[12] Wen, Z.Y., Mathematical Foundations of Fractal Geometry, Shanghai Sci. Tech. Edu. Publ. House, Shanghai, 2000, in Chinese.

[13] Zhou, S.P., Yao, K. and Su, W.Y., On the Fractional Integrals of the Weierstrass Functions: the Exact Box Dimension, Preprint.

[14] Sun, D.C. and Wen, Z.Y., The Bouligand Dimension of the Graphs of Lacunary Trigonometric Series, Chinese science bulletin, 38(1993), 487-490.

Zhou Songping

Faculty of Science

Zhejiang Institute of Science and Technology

Hangzhou 310018

P. R. China

e-mail: spz@mail.hz.zi.cn

He Guolong

Department of Matehematics

Zhejiang Normal University

Hangzhou, 321004

P. R. China

e-mail: hgl@mail.zjnu.net.cn