# HAUSDORFF MEASURES OF A CLASS OF SIERPINSKI CARPETS*

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## Abstract

*In this paper, a lemma as a new method to calculate the Hausdorff measure of fractal is given. And then the exact values of Hausdorff measure of a class of Sierpinski sets which satisfy balance distribution and dimension $\leq 1$ are obtained.*

**Key words** *fractal, sierpinski carpet, Hausdorff measure, balance distribution*

**AMS(2000)subject classification** 28A80, 26A39

## 1 Introduction

Advocated by Zhou Zuoling, some authors present the Hausdorff measure of special fractal in recent years. Because it is a very difficult problem, no common serve method has been obtained up to now, we have to analyse specific example with specific method.

Just as the various fractal set of generalized Sierpinski in [2], there include cantor fractal set. We can caculate its exact value of Hausdorff measure for cantor set is discrete point set. Zhou Zuoling discuss Sierpinski carpet firstly, thereafter, [4],[5] caculate the Hausdorff measure of Sierpinski carpet which satisfy balance distribution and dimension=1. Furhtermore, paper [6]

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gives another caculating method and extends the result of [3] on the condition of dimension¡1.

In this paper, we obtain an important lemma by extending the lemma 1 of paper [1] which Zhou Zuoling gives. Using it and Combining the method in [3]-[6], we discuss a class of Sierpinski sets which satisfy balance distribution and dimension $\leq 1$. Furthermore, we caculate the exact values of their Hausdorff measures. The notations in this paper are the same as the [5] and [6].

## 2 Construction and Conclusion

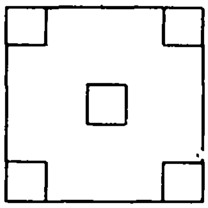
Let $S_0$ be a unit square in $R^2, S_n$ is composed of $m^n$ smaller squares and the length of their edge is $\frac{1}{m^n}$ in $S_{n-1}$. One of these squares is called elementary square of $S_n$ and denoted by $\square_n$. Let $n \to \infty$, we get a class of Sierpinski carpets such that their dimension=1 and $S = \lim_{n\to\infty} S_n$. Obviosuly, every $\square_n$ can form a Sierpinski carpet and each is similar to $S$ with similaity ratio $\frac{1}{m^n}$.

If we define distributed function $\mu$ on $S_0$, let

$$\begin{cases} \mu(S_0) = \sqrt{2}, \\ \mu(\square_m) = \dfrac{\sqrt{2}}{n^m}, \\ \mu(S_0 - S) = 0, \end{cases}$$

then $\mu$ is a measure on $S_0$ and $S$ is a quality distribution when $\mu$ restricte on $S$.

Paper [5] shows that $m = 5$ is a Sierpinski carpet that satisfies balance distribution. The construction can be seen from figure 1: The unit square $\square_1$ in $S_1$, one lies in the centre, the other distribute on the 4 angles of the square. Now, our aim is to extend the result of [5], that is, we let the edge length of $\square_n$ is $\frac{1}{m}(m \geq 5)$. It is easy to prove Hausdorff dimensions of these Sierpinski carpets (denoted by $s$) are



$$s = \dim_H S = \frac{\log^5}{\log^m} \leq 1 \qquad \text{(see [2])}. \tag{1}$$

We define distributed function $\mu$ on $S_0$, let

$$\begin{cases} \mu(S_0) = 2^{\frac{s}{2}}, \\ \mu(\Box_1) = \frac{1}{m}2^{\frac{s}{2}}, \\ \cdots\cdots \\ \mu(\Box_n) = \dfrac{1}{m^n}2^{\frac{s}{2}}, \\ \cdots\cdots \\ \mu(S_0 - S) = 0. \end{cases}$$

Our main result in this paper is as follows:

**Theorem 1.** *$H^s(S) = 2^{\frac{s}{2}}$ with* $s = \dfrac{\log 5}{\log m}$.

## 3 Proof of the Theorem

Let $S_n$ be a covering of $S$, we may obtain the upper estimate for Hausdorff measure of $S$:

$$H^s(S) \leq 2^{\frac{s}{2}},$$

so, the main proof of the theorem is to show

$$H^s(S) \geq 2^{\frac{s}{2}}. \tag{2}$$

For any plane point set $u$ if its diameter $|u| > 0$, we denote the member of $\Box_n$ that intersected with $U$ by $\alpha_n(u)$.

**Lemma 1.** *For any suficiently large positive integer $n$, we have*

$$\frac{\alpha_n(U)}{5^n}2^{\frac{s}{2}} \leq |U|^s \tag{3}$$

*and then (2) holds.*

The imilar proof can be seen from [6].

Similar to [4], we denote

$$\alpha_n(U) = i_n(U) + j_n(U) + k_n(U)$$

($k_n(U)$ is the number of $\Box_n$ that lied in the center and intersected with $U$). When $|U| \geq \sqrt{2}$, we easily get to know (3) holds for every $n$. So, let $0 < |U| < \sqrt{2}$, then, for a given $u$, there exists a positive integer $k$ such that

$$\frac{\sqrt{2}}{m^{k+1}} \leq |u| < \frac{\sqrt{2}}{m^k}.$$

Using the self-similarity, we only discuss the case of a certain $k$. So, we always assume without loss of generality that

$$\frac{\sqrt{2}}{m} \leq |U| < \sqrt{2}.$$

The five cases $\alpha_1(U) = 1, 2, 3, 4, 5$ can be discussed respectively as following:

**Case 1**: If $\alpha_1(U) = 1$, then $\alpha_n(U) \leq \frac{1}{5}5^n$ and $\frac{\alpha_n(U)}{5^n}2^{\frac{s}{2}} \leq \frac{1}{5}2^{\frac{s}{2}} \leq |U|^s$. So, (3) holds obviously.

**Case 2**: $\alpha_1(U) = 2$.

Now, the two unit squares that intersected with $U$ are $I_1$ (or $J_1$) and $K_1$. We only need to prove

$$\frac{\alpha_3(U)}{5^3}2^{\frac{s}{2}} \leq |U|^s. \quad (4)$$

Since $\{\frac{\alpha_n(u)}{5^n}\}$ is a decreasing sequence, so, (3) holds.

To prove (4), we notice that $\alpha_2(U) \leq 10$ and consider three cases:

(i) $\alpha_2(U) \leq 5$

From $\alpha_2(U) \leq 5$ and $\alpha_3(U) \leq 5 \times 5$, then

$$\frac{\alpha_3(U)}{5^3}2^{\frac{s}{2}} \leq \frac{1}{5}2^{\frac{s}{2}}.$$

Furthermore, $|U| \geq \frac{\sqrt{2}}{2}(1 - \frac{3}{m}) \geq \frac{\sqrt{2}}{5}$, thus $|U|^s \geq \frac{1}{5}2^{\frac{s}{2}}$. So, (4) holds.

(ii) $6 \leq \alpha_2(U) \leq 9$

From $\alpha_2(U) \leq 9$ and $\alpha_3(U) \leq 5 \times 9$, then

$$\frac{\alpha_3(U)}{5^3}2^{\frac{s}{2}} \leq \frac{9}{5^2}2^{\frac{s}{2}}.$$

Also from $\alpha_2(U) \geq 6$, we obtain

$$|U| \geq (1 - \frac{3}{m} - \frac{1}{m^2})\sqrt{2} \geq (\frac{2}{5} - \frac{1}{5^2})\sqrt{2} = \frac{9}{5^2}\sqrt{2}.$$

Thus $|U|^s \geq \frac{9}{5^2}2^{\frac{s}{2}}$. So, (4) holds.

(iii) $\alpha_2(U) = 10$

From $\alpha_3(U) \leq 5 \times 10$, then $\frac{\alpha_3(U)}{5^3}2^{\frac{s}{2}} \leq \frac{10}{5^2}2^{\frac{s}{2}}$. Furthermore,

$$|U| \geq (1 - \frac{3}{m})\sqrt{2} \geq \frac{2\sqrt{2}}{5} = \frac{10}{5^2}\sqrt{2},$$

thus $|U|^s \geq \frac{10}{5^2}2^{\frac{s}{2}}$. So, (4) holds.

Therefore, in the case 2 of $\alpha_1(U) = 2$, (3) holds.

**Case 3**: $\alpha_1(U) = 3$.

Now, the three unit squares that intersected with $U$ are $I_1(I_1')$ and $J_1(J_1')$ and $K_1$.

Concerning $\alpha_2(U) \leq 15$, we consider four cases to prove (4) holds:

(i) $\alpha_2(U) \leq 10$

From $\alpha_2(U) \leq 10$ and $\alpha_3(U) \leq 5 \times 10$, then

$$\frac{\alpha_3(U)}{5^3} 2^{\frac{s}{2}} \leq \frac{2}{5} 2^{\frac{s}{2}}.$$

Since $\alpha_1(U) = 3$, we obtain $|U| \geq 1 - \frac{2}{m} \geq \frac{3}{5} > \frac{2}{5}\sqrt{2}$, thus $|U|^s \geq \frac{2}{5} 2^{\frac{s}{2}}$. So, (4) holds.

(ii) $11 \leq \alpha_2(U) \leq 12$

From $\alpha_2(U) \leq 12$, and $\alpha_3(U) \leq 5 \times 12$, then $\frac{\alpha_3(U)}{5^3} 2^{\frac{s}{2}} \leq \frac{12}{5^2} 2^{\frac{s}{2}}$, also from $11 \leq \alpha_2(U)$, we obtain

$$|U| \geq (1 - \frac{1}{m} - \frac{1}{m^2}) \geq \frac{4}{5} - \frac{1}{5^2} = \frac{19}{5^2} > \frac{12}{5^2}\sqrt{2}.$$

Thus $|U|^s \geq \frac{12}{5^2} 2^{\frac{s}{2}}$. So, (4) holds.

(iii) $13 \leq \alpha_2(U) \leq 14$

From $\alpha_2(U) \leq 14$, and $\alpha_3(U) \leq 5 \times 14$, then

$$\frac{\alpha_3(U)}{5^3} 2^{\frac{s}{2}} \leq \frac{14}{5^2} 2^{\frac{s}{2}}.$$

Also from $13 \leq \alpha_2(U)$, we obtain

$$|U| \geq (1 - \frac{2}{m} + \frac{2}{m^2} + \frac{4}{m^2}) \geq \frac{4}{5} + \frac{1}{5^2} = \frac{21}{5^2} > \frac{14}{5^2}\sqrt{2}.$$

Thus $|U|^s \geq \frac{14}{5^2} 2^{\frac{s}{2}}$. So, (4) holds.

(iv) $\alpha_2(U) = 15$

From $\alpha_3(U) \leq 5 \times 15$, then

$$\frac{\alpha_3(U)}{5^3} 2^{\frac{s}{2}} \leq \frac{15}{5^2} 2^{\frac{s}{2}}.$$

Furthermore

$$\begin{aligned} |U| &> \sqrt{(1 - \frac{2}{m^2})^2 + (\frac{1}{m} - \frac{2}{m^2})^2} \\ &\geq \sqrt{(1 - \frac{2}{5^2})^2 + (\frac{1}{5} - \frac{2}{5^2})^2} = \frac{\sqrt{269}}{5^2}\sqrt{2} > \frac{16}{5^2}\sqrt{2}, \end{aligned}$$

thus $|U|^s \geq \frac{16}{5^2} 2^{\frac{s}{2}}$. So, (4) holds.

Therefore, in the case of $\alpha_1(U) = 3$, (3) holds.

**Case 4**: $\alpha_1(U) = 4$.

Now, the four unit squares that intersected with $U$ are $I_1, I_1'$, and $J_1(J_1')$ and $K_1$.

Concerning $\alpha_2(U) \leq 20$, we consider three cases to prove (4) holds:

(i) $\alpha_2(U) \leq 15$

By $\alpha_2(U) \leq 15$ and $\alpha_3(U) \leq 5 \times 15$, then $\frac{\alpha_3(U)}{5^3}2^{\frac{s}{2}} \leq \frac{3}{5}2^{\frac{s}{2}}$.
Also from $|U| \geq (1 - \frac{2}{m})\sqrt{2} \geq \frac{3}{5}\sqrt{2}$, we obtain $|U|^s \geq \frac{3}{5}2^{\frac{s}{2}}$. So, (4) holds.

(ii) $16 \leq \alpha_2(U) \leq 19$

From $\alpha_2(U) \leq 19$, and $\alpha_3(U) \leq 5 \times 19$, then

$$\frac{\alpha_3(U)}{5^3}2^{\frac{s}{2}} \leq \frac{19}{5^2}2^{\frac{s}{2}}.$$

Also from $16 \leq \alpha_2(U)$, we obtain

$$|U| \geq (1 - \frac{2}{m} + \frac{4}{m^2})\sqrt{2} \geq (\frac{3}{5} + \frac{4}{5^2})\sqrt{2} = \frac{19}{5^2}\sqrt{2},$$

thus $|U|^s \geq \frac{19}{5^2}2^{\frac{s}{2}}$. So, (4) holds.

(ii) $\alpha_2(U) = 20$

From $\alpha_3(U) \leq 5 \times 20$, then $\frac{\alpha_3(U)}{5^3}2^{\frac{s}{2}} \leq \frac{20}{5^2}2^{\frac{s}{2}}$. Meanwhile,

$$|U| \geq (1 - \frac{2}{m} + \frac{4}{m^2} + \frac{4}{m^2})\sqrt{2} \geq (\frac{3}{5} + \frac{8}{5^2})\sqrt{2} = \frac{23}{5^2}\sqrt{2},$$

thus $|U|^s \geq \frac{23}{5^2}2^{\frac{s}{2}}$. So, (4) holds.

Therefore, in the case of $\alpha_1(U) = 4$, (3) holds.

**Case 5**: $\alpha_1(u) = 5$

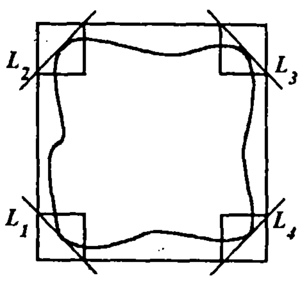
We need the following lemma:

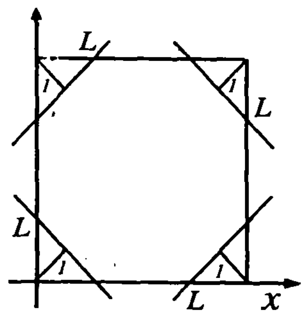
**Lemma 2.** *We assume that $L$ is a straight line which parrellas to one of diagonal lines of $S_0$ and intersects with $S_0$. Denote*

$$d((x, y), L) = l, \quad (x, y) = \{(0,0), (1,0), (0,1), (1,1)\}, \quad 0 \leq l \leq \frac{1}{m}\sqrt{2},$$

*also denote the traingular that surrounded by $L$ and the edge of $S_0$ by $\Delta L$. If we let $l = l'\sqrt{2}$, then we obtain*

$$\mu(\Delta L \cap S_0) \geq \frac{l'}{2}2^{\frac{s}{2}}, \qquad 0 \leq l' < \frac{1}{m}.$$




*Proof.* We consider three cases:

(i) $\frac{1}{2m} \le l' < \frac{1}{m}$

By $\frac{1}{2m} \le l' < \frac{1}{m}$, then $\mu(\Delta L \cap S_0) \ge \frac{1}{2}\mu(\Box_1) = \frac{2^{\frac{s}{2}}}{2 \times 5} > \frac{1}{2m}2^{\frac{s}{2}} > \frac{l'}{2}2^{\frac{s}{2}}$.

(ii) $\frac{m-1}{2}\frac{1}{m^2} \le l' < \frac{1}{2m}$

Then there exists positive a integer $k \in \{2, 3, \cdots\}$ such that

$$\frac{m-1}{2}\sum_{i=2}^{k}\frac{1}{m^i} + \frac{1}{m^{k+1}} \le l' \le \frac{m-1}{2}\sum_{i=2}^{k+1}\frac{1}{m^i} = \frac{m-1}{2m^2}(1 + \frac{1}{m} + \cdots + \frac{1}{m^{k-1}}).$$

Thus

$$\mu(\Delta L \cap S_0) \ge \sum_{i=2}^{k+1}\mu(\Box_i) = \sum_{i=2}^{k+1}\frac{2^{\frac{s}{2}}}{5^i} = \frac{2^{\frac{s}{2}}}{5^2}(1 + \frac{1}{5} + \cdots + \frac{1}{5^{k-1}}).$$

We only have to prove

$$\frac{m-1}{4m^2} \le \frac{1}{5^2}. \tag{5}$$

If $m = 5$, (5) is identity. Let $f(x) = \frac{x^2}{x-1}$, then $f'(x) = \frac{2x}{x-1} - \frac{x^2}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$.

If $x > 2$, then $f'(x) > 0$ and $f$ monoton increasing such that

$$f(m) = \frac{m^2}{m-1} \ge f(5) = \frac{5^2}{4}, \qquad m \ge 5.$$

So, in this case, we have proved $\mu(\Delta L \cap S_0) \ge \frac{l'}{2}2^{\frac{s}{2}}$.

(iii) $\frac{1}{m^2} \le l' < \frac{m-1}{2}\frac{1}{m^2}$

Meanwhile

$$\mu(\Delta L \cap S_0) \ge \frac{1}{2}\mu(\Box_2) = \frac{1}{5^2}2^{\frac{s}{2}}.$$

In view of (5), we also get $\mu(\Delta L \cap S_0) \ge \frac{l'}{2}2^{\frac{s}{2}}$.

Now, from the argument above, we can use the conclusion of lemma 2 to prove.

Case 5: $\mu(u) \le |u|^s$. In fact, we can get from figure:

$$|u| \ge (1 - \frac{2}{m})\sqrt{2} + (\frac{\sqrt{2}}{m} - l_1) + (\frac{\sqrt{2}}{m} - l_3) = (1 - l_1' - l_3')\sqrt{2},$$
$$|u| \ge (1 - l_2' - l_4')\sqrt{2}.$$

Thus

$$|u| \ge (1 - \frac{1}{2}\sum_{i=1}^{4} l_i')\sqrt{2},$$
$$|u|^s \ge (1 - \frac{1}{2}\sum_{i=1}^{4} l_i')^s 2^{\frac{s}{2}} \ge (1 - \frac{1}{2}\sum_{i=1}^{4} l_i')2^{\frac{s}{2}}.$$

On the other hand

$$\begin{aligned}\mu(u) &\le \frac{\alpha_2(u)}{5}2^{\frac{s}{2}} - \sum_{i=1}^{4}\mu(\Delta L_i \cap S_0)\\ &= 2^{\frac{s}{2}} - \frac{1}{2}\sum_{i=1}^{4} l_i' 2^{\frac{s}{2}} = (1 - \frac{1}{2}\sum_{i=1}^{4} l_i')2^{\frac{s}{2}} \le |u|^s.\end{aligned}$$

Consequently, by using lemma 1, theorem holds.

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