AN APPROXIMATION METHOD TO ESTIMATE THE HAUSDORFF MEASURE OF THE SIERPINSKI GASKET*

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Abstract

In this paper, we firstly define a decreasing sequence $\{P^n(S)\}$ *by the generation of the Sierpinski gasket where each* $P^n(S)$ *can be obtained in finite steps. Then we prove that the Hausdorff measure* $H^s(S)$ *of the Sierpinski gasket* S *can be approximated by* $\{P^n(S)\}$ *with* $P^n(S)/(1+1/2^{n-3})^s \le H^s(S) \le P^n(S)$. *An algorithm is presented to get* $P^n(S)$ *for* $n \le 5$. *As an application, we obtain the best lower bound of* $H^s(S)$ *till now:* $H^s(S) \ge 0.5631$.

Key words *Hausdorff measure, sierpinski gasket, approximation method*

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1 Introduction

It is one of basic problems to calculate Hausdorff measure of sets in fractal geometry. It is well known that the Hausdorff measure of the Cantor middle-third set is 1. In [AS99], Ayer and Strichartz gave an algorithm for computing the Hausdorff measure of a class of Cantor sets in finite steps. Zhou et al got some estimates of Hausdorff measure of fractal sets including the

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Sierpinski gasket, the Koch curve and so on (see [Z97A, ZF00]). However, there is no precise value of Hausdorff measure obtained for any classical self-similar set with Hausdorff dimension larger than 1. Even there is no algorithm to approximate the value arbitrary near in finite step.

The Sierpinski gasket S is one of the classical self-similar set. Its Hausdorff dimension $\dim_H(S) = s = \log_2 3$. Marion [M87] showed that $H^s(S) \le \frac{1}{6}3^s \approx 0.9508$ and conjectured that this upper bound is the actual Hausdorff measure. Zhou[Z97A] pointed out that the conjecture is not true by showing that

$$H^s(S) \le \frac{1}{24}7^s \approx 0.9105.$$

In [Z97B] and [ZF00], the upper bound was improved to

$$H^s(S) \le \frac{25}{22}\left(\frac{6}{7}\right)^s \approx 0.8900$$

and

$$H^s(S) \le \frac{1927233}{1509380}\left(\frac{61}{80}\right)^s \approx 0.8308.$$

Using the result of [Z97A], Wang[W99] obtained the best upper bound till now: $H^s(S) \le 0.8179$.

The lower bound is more difficult to be estimated. Recently, Jia et al[JZZ] proved that $H^s(S) \ge 0.5$ by using mass distribution principle.

In this paper, we will present one method to approximate the Hausdorff measure $H^s(S)$ of the Sierpinski gasket by a decreasing sequence $\{P^n(S)\}$ with

$$P^n(S)/(1 + 1/((\sqrt{3}\cdot 2^{n-4}))^s \le H^s(S) \le P^n(S)$$

where $P^n(S)$ can be obtained in finite steps.

The paper is arranged as follows. In section 2, we define the sequence $\{P^n(S)\}$ by the generation of the Sierpinski gasket and present the main theorem. The proof of the main theorem is given in section 3. Finally, in section 4, we give an effective algorithm to calculate $P^n(S)$ for $n \le 5$ and obtain the best lower bound estimate of $H^s(S)$ till now.

2 Main Results

The generation of the Sierpinski gasket S can be discribed as follows.

Choose an equilateral triangle ΔABC with side of length 1. Call it S_0. Join the midpoints of sides with lines and remove the open inverted equilateral triangle. Call the remaining set S_1. For each of the three remaining triangles, join the midpoints of sides with lines and remove the open inverted equilateral triangle. Call the remaining set S_2 (see Figure 1). Repeating this process, we obtain $S_0 \supset S_1 \supset S_2 \supset \cdots \supset S_n \supset \cdots$. The non-empty set $S = \cap_{n\ge 0} S_n$ is called the Sierpinski gasket.

It is well known that the Hausdorff dimension of S equals $\log_2 3$. In the sequel of this paper, s will always denote the Hausdorff dimension of S.

From the generation of the Sierpinski gasket S, we can see that for each $n \geq 0$, S_n consists of 3^n equilateral triangles with side length $1/2^n$. We denote these by $\Delta_1^n, \Delta_2^n, \ldots, \Delta_{3^n}^n$. Each Δ_i^n is called a n-level triangle.

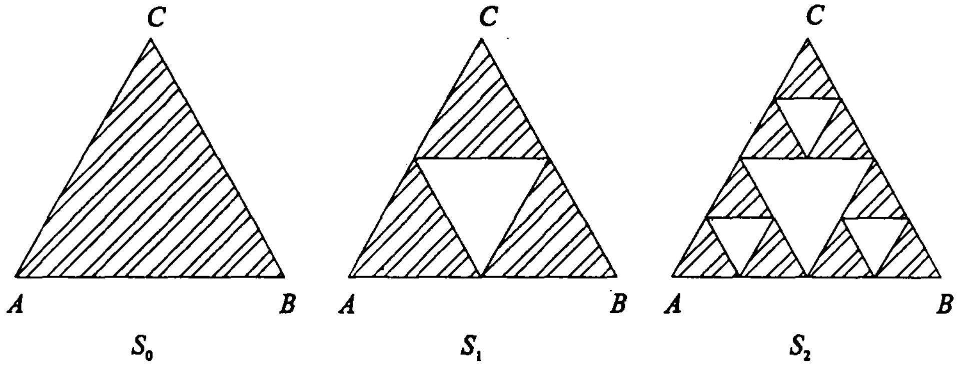

Figure 1 The Generation of the Sierpinski Gasket

We define a measure ν on $\mathbb{R}^2$ such that

$$\begin{cases} \nu(S_0) = 1, & \\ \nu(\Delta_i^n) = \frac{1}{3^n}, & \text{for any } i = 1, 2, \ldots, 3^n. \\ \nu(\mathbb{R}^2 \setminus S) = 0. & \end{cases} \tag{1}$$

We call ν the Sierpinski distribution.

Remark 2.1. It is clear that

$$\nu(U) = \frac{H^s(U)}{H^s(S)}$$

for any Borel set $U \subset \mathbb{R}^2$.

Definition 2.1. Let t be an integer satisfying $1 \leq t \leq 3^n$. Let $i_j \in \{1, 2, \ldots, 3^n\}$ for all $j = 1, 2, \ldots, t$ and $i_j \neq i_k$ for $j \neq k$. We define

$$P(\{\Delta_{i_j}^n\}_{j=1}^t) = \frac{3^n |\bigcup_{j=1}^t \Delta_{i_j}^n|^s}{t} \equiv \frac{|\bigcup_{j=1}^t \Delta_{i_j}^n|^s}{\nu(\bigcup_{j=1}^t \Delta_{i_j}^n)}.$$

Definition 2.2. Let t be an integer satisfying $1 \leq t \leq 3^n$, we define

$$P_t^n(S) = \min\{P(\{\Delta_{i_j}^n\}_{j=1}^t)\},$$

where the minimum is over all possible subsets $\{\Delta_{i_j}^n\}_{j=1}^t$ of $\{\Delta_i^n\}_{i=1}^{3^n}$.

Definition 2.3. Define $P^n(S) = \min\{P_t^n(S) : 1 \leq t \leq 3^n\}$.

We will show that $H^s(S)$ can be approximated by $P^n(S)$ as

Theorem 2.1. (i) $\{P^n(S)\}_{n=1}^\infty$ *is a decreasing sequence.*

(ii) $$\frac{P^n(S)}{(1 + 1/\sqrt{3} \cdot 2^{n-4})^s} \leq H^s(S) \leq P^n(S).$$

(iii) $H^s(S) = \lim_{n\to\infty} P^n(S)$ *and*

$$|H^s(S) - P^n(S)| \le \left(1 - (1 + 1/(\sqrt{3}\cdot 2^{n-4}))^{-s}\right)P^n(S) \le 1 - (1 + 1/(\sqrt{3}\cdot 2^{n-4}))^{-s}.$$

3 Proof of Theorem 2.1

In order to prove the theorem, we need following lemmas.

Lemma 3.1. (Mass distribution principle, see [F90]) *Let $E \subset \mathbb{R}^n$, let μ be a measure with support contained in E such that $0 < \mu(E) < \infty$. Suppose that there exist $t \ge 0$, $c > 0$ and $\delta > 0$ such that*

$$\mu(U) \le c|U|^t$$

for all sets U with $|U| \le \delta$, then $H^t(E) \ge \mu(E)/c$.

Lemma 3.2. (see [ZF00]) *Let $E \subset \mathbb{R}^n$ be a self-similar set satisfying the open set condition and $\mathcal{B}(\mathbb{R}^n)$ be the Borel σ-algebra on $\mathbb{R}^n$. Let t be the Hausdorff dimension of E. For any $U \subset \mathbb{R}^n$, we define*

$$F_E(U) = [H^t(E)/H^t(E \cap U)]|U|^t$$

if $H^t(E\cap U) \ne 0$, $F_E(U) = +\infty$ if $H^t(E\cap U) = 0$. Then

$$H^t(E) = \inf\{F_E(U) : U \in \mathcal{B}(\mathbb{R}^n)\}.$$

We will prove the following lemma by Lemma 3.1.

Lemma 3.3. *Let s be the Hausdorff dimension of the Sierpinski gasket S. Let ν be the Sierpinski distribution. Suppose that there exists $c > 0$ such that*

$$\nu(U) \le c|U|^s$$

for all Borel sets $U \subset S$ with $1 \ge |U| \ge \sqrt{3}/8$, then $H^s(S) \ge \nu(S)/c = 1/c$.

Proof. First we will prove that for all Borel sets $U \subset S$,

$$\nu(U) \le c|U|^s. \tag{2}$$

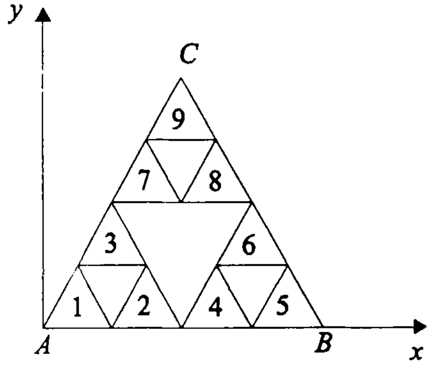

Figure 2 S_2 and Δ_i^2, $i = 1, 2, \ldots, 9$.

If U contains only one point in S, then $\nu(U) = |U|^s = 0$ so that (2) is satisfied. Thus we can assume that $|U| > 0$.

We establish a rectangular coordinate system and arrange $\Delta_i^2, i = 1, 2, \ldots, 9$ as Figure 2. Let $\Delta_1^1 = \Delta_1^2 \cup \Delta_2^2 \cup \Delta_3^2$, $\Delta_2^1 = \Delta_4^2 \cup \Delta_5^2 \cup \Delta_6^2$ and $\Delta_3^1 = \Delta_7^2 \cup \Delta_8^2 \cup \Delta_9^2$.

Δ_i^2 and Δ_j^2 are called adjacent if they have a common vertex. Otherwise, they are called seperated. It is obvious that when Δ_i^2 and Δ_j^2 are seperated,

$$d(x,y) \geq \frac{\sqrt{3}}{2} \cdot \frac{1}{4} = \frac{\sqrt{3}}{8}, \quad \text{if } x \in \Delta_i^2 \text{ and } y \in \Delta_j^2. \tag{3}$$

Case 1. If $U \cap \Delta_i^1 \neq \emptyset$ for $i = 1, 2, 3$, we can choose $x, y, z \in U$ such that $x \in \Delta_1^1, y \in \Delta_2^1$ and $z \in \Delta_3^1$. If $d(x,y) < \sqrt{3}/8$, then $x \in \Delta_2^2$ by (3). Using (3) again, we can see that $d(x,z) \geq \sqrt{3}/8$. Hence $|U| \geq \sqrt{3}/8$ so that $\nu(U) \leq c|U|^s$.

Case 2. If there exist only two 1-level triangles Δ_i^1 satisfying $U \cap \Delta_i^1 \neq \emptyset$. Without loss of generality, we can assume that $U \cap \Delta_1^1 \neq \emptyset, U \cap \Delta_2^1 \neq \emptyset$ and $U \cap \Delta_3^1 = \emptyset$.

Case 2.1. If $U \cap \Delta_1^2 \neq \emptyset$ or $U \cap \Delta_3^2 \neq \emptyset$, then for any $y \in \Delta_2^1 \cap U$, we have $d(x,y) \geq \sqrt{3}/8$. Thus $|U| \geq \sqrt{3}/8$ so that $\nu(U) \leq c|U|^s$.

Case 2.2. If $U \cap \Delta_5^2 \neq \emptyset$ or $U \cap \Delta_6^2 \neq \emptyset$, we can prove $|U| \geq \sqrt{3}/8$ and $\nu(U) \leq c|U|^s$ by the same method as in case 2.1.

Case 2.3. If $U \subset \Delta_2^2 \cup \Delta_4^2$. Let $U^* = \{(x - 1/4, y) \ : \ (x,y) \in U\}$. Then $\nu(U^*) = \nu(U)$ and $U^* \subset \Delta_1^1$. We will prove $\nu(U^*) \leq c|U^*|^s$ in case 3.

Case 3. If U is contained in some Δ_i^1. Without loss of generality, we can assume that $U \subset \Delta_1^1$.

Let $U_1 = 2U$. From the generation of S and the definition of Hausdorff measure, we can easily see that $\nu(U_1) = 3\nu(U)$. Thus, we have

$$\frac{\nu(U_1)}{|U_1|^s} = \frac{\nu(U)}{|U|^s}.$$

It is clear that $U_1 \subset S$.

If U_1 is a set satisfying one of case 1, case 2.1 and case 2.2, then $|U_1| \geq \sqrt{3}/8$ so that $\nu(U_1) \leq c|U_1|^s$. Hence $\frac{\nu(U)}{|U|^s} = \frac{\nu(U_1)}{|U_1|^s} \leq c$.

If U_1 is a set satisfying one of case 2.3 and case 3, we can find U_2 such that

$$|U_2| = 2|U_1|, \quad U_2 \subset S, \text{ and } \frac{\nu(U_2)}{|U_2|^s} = \frac{\nu(U_1)}{|U_1|^s} = \frac{\nu(U)}{|U|^s}.$$

By induction, there exitsts U_n such that

$$|U_n| \geq \sqrt{3}/8, \quad U_n \subset S, \text{ and } \frac{\nu(U_n)}{|U_n|^s} = \frac{\nu(U)}{|U|^s}.$$

From $|U_n| \geq \sqrt{3}/8$, we have $\frac{\nu(U_n)}{|U_n|^s} \leq c$. Thus $\nu(U) \leq c|U|^s$.

Combining case 1, case 2 and case 3, we have $\nu(U) \leq c|U|^s$ for all Borel sets $U \subset S$. Thus, from the definition of ν, we have

$$\nu(U) \leq \nu(\overline{U}) = \nu(\overline{U} \cap S) \leq c|\overline{U} \cap S|^s \leq c|\overline{U}|^s = c|U|^s$$

for all sets $U \subset \mathbb{R}^2$. By Lemma 3.1, $H^s(S) \geq \nu(S)/c = 1/c$.

Proof of Theorem 2.1.

(i) It is clearly true.

(ii) By Remark 2.1 and Lemma 3.2, $H^s(S) \leq |U|^s/\nu(U)$ for all $U \in \mathcal{B}(\mathbb{R}^2)$, where we define $|U|^s/\nu(U) = +\infty$ if $\nu(U) = 0$. Thus we have $H^s(S) \leq P^n(S)$ for all $n \in \mathbb{N}$ by the definition of $P^n(S)$.

Now we will prove $H^s(S) \geq \frac{P^n(S)}{(1+1/(\sqrt{3}\cdot 2^{n-4}))^s}$.

Let $U \subset S$ is a set with $|U| \geq \sqrt{3}/8$. Define $\mathcal{A} = \{i | \Delta_i^n \cap U \neq \emptyset\}$. Let $V = \cup_{i\in\mathcal{A}} \Delta_i^n$, then $U \subset V$, $\nu(U) \leq \nu(V)$ and $\sqrt{3}/8 \leq |U| \leq |V|$. For any $x, y \in V$, there exist $x^*, y^* \in U$ such that x and x^*, y and y^* are contained in same n-level triangle, respectively. Thus $|x-y| \leq |x-x^*| + |x^*-y^*| + |y^*-y| \leq |U| + 1/2^{n-1}$. Hence $|V| \leq |U| + 1/2^{n-1}$.

From $|U| \geq 1/4$ and the definition of $P^n(S)$, we have

$$\begin{aligned}\nu(U) \leq \nu(V) \leq\ & \frac{\nu(V)}{|V|^s}(|U| + 1/2^{n-1})^s \leq \frac{\nu(V)}{|V|^s}\Big(|U| + \frac{|U|}{\sqrt{3}/8}\cdot\frac{1}{2^{n-1}}\Big)^s \\ \leq\ & \frac{\nu(V)}{|V|^s}|U|^s(1+1/(\sqrt{3}\cdot 2^{n-4}))^s \leq (P^n(S))^{-1}(1+1/(\sqrt{3}\cdot 2^{n-4}))^s|U|^s.\end{aligned}$$

By Lemma 3.3, we have

$$H^s(S) \geq \frac{P^n(S)}{(1+1/(\sqrt{3}\cdot 2^{n-4}))^s}.$$

(iii) Since $P^1(S) = 1$ from the definition, using (i) and (ii), we have

$$|H^s(S) - P^n(S)| \leq \big(1 - (1+1/(\sqrt{3}\cdot 2^{n-4}))^{-s}\big)P^n(S) \leq 1 - (1+1/(\sqrt{3}\cdot 2^{n-4}))^{-s}.$$

4 Numerical Analysis

It is easy to calculate $P^1(S)$ and $P^2(S)$ by exhaust algorithm. Then, we design an efficient algorithm to obtain $P^n(S)$ for $n \leq 5$ by using the symmetry of the Sierpinski gasket.

Definition 4.1. We denote three midlines of ΔABC by l_1, l_2 and l_3. We call Δ_i^n and Δ_j^n are symmetrical triangles if they satisfy one of the following cases:

- Δ_i^n and Δ_j^n are symmetrical with respect to some l_p, where $p \in \{1, 2, 3\}$.

- There exist Δ_k^n, l_p and l_q such that Δ_i^n and Δ_k^n are symmetrical with respect to l_p, Δ_k^n and Δ_j^n are symmetrical with respect to l_q, respectively, where $p, q \in \{1,2,3\}$ and $k \in \{1,2,\dots,3^n\}$.

Note that Δ_i^n is a symmetrical triangle of itself from the definition. Using the symmetrical relationship, we can devide $\{\Delta_i^n\}_{i=1}^{3^n}$ to different eqivalent classes $K_1^n, K_2^n, \dots, K_{n_\nu}^n$ with $K_p^n = \{\Delta_{p(1)}^n, \Delta_{p(2)}^n, \dots, \Delta_{p(up)}^n\}$, where $1 \le p(1) < p(2) < \cdots < p(up) \le 3^n$ and $p(1) < q(1)$ for $1 \le p < q \le n_\nu$. $\Delta_{p(1)}^n$ is called the first triangle in K_p^n. We define $r(\Delta_i^n) = p$ if $\Delta_i^n \in K_p^n$.

Example 4.1. *In Figure 2, we can devide $\{\Delta_i^2\}_{i=1}^9$ to K_1^2 and K_2^2 with $K_1^2 = \{\Delta_1^2, \Delta_5^2, \Delta_9^2\}$ and $K_2^2 = \{\Delta_2^2, \Delta_3^2, \Delta_4^2, \Delta_6^2, \Delta_7^2, \Delta_8^2\}$.*

Definition 4.2. Let $M > 0$ be a given real number. Let $t \in \{1,2,\dots,3^n\}$ be a given integer. We call Δ_i^n is a suitable triangle of K_p^n with (t, M) if $r(\Delta_i^n) \ge p$ and $|\Delta_i^n \cup \Delta_{p(1)}^n|^s \le \frac{t}{3^n} \bullet M$. Denote the set of all suitable triangles of K_p^n with (t, M) by $G(n,t,p,M)$.

Remark 4.1. $\Delta_{p(1)}^n \in G(n,t,p,M)$ for any $M > 0$.

Definition 4.3. Let $p \in \{1,2,\dots,n_\nu\}$. If $\#G(n,t,p,M) < t$, we define $P_{t,p,M}^n = M$; otherwise we define $P_{t,p,M}^n = \min\{M, R_{t,p,M}^n\}$ with $R_{t,p,M}^n = \min\{P(\{\Delta_{i_j}^n\}_{j=1}^t)\}$, where the minimum is over all possible subsets $\{\Delta_{i_j}^n\}_{j=1}^t$ of $G(n,t,p,M)$.

Remark 4.2. $\min\{M, P_t^n(S)\} \le P_{t,p,M}^n \le M$.

Theorem 4.1. *Let $M_1 > 0$ be any given real number, $M_{p+1} = P_{t,p,M_p}^n$, $p = 1,2,\dots,n_\nu$. Then $M_{n_\nu+1} = \min\{M_1, P_t^n(S)\}$.*

Proof. It is easy to see that $P_{t,p,M_p}^n \ge \min\{M_1, P_t^n(S)\}$ for any $p \in \{1,2,\dots,n_\nu\}$ from Remark 4.2. Thus $M_{n_\nu+1} \ge \min\{M_1, P_t^n(S)\}$.

On the other hand, note that $\{M_p\}_{p=1}^{n_\nu+1}$ is a decreasing sequence, we have $M_{n_\nu+1} = M_1$ if $M_1 < P_t^n(S)$.

If $M_1 \ge P_t^n(S)$, then there exist $1 \le i_1 < i_2 < \cdots < i_t \le 3^n$ such that $P(\{\Delta_{i_j}^n\}_{j=1}^t) = P_t^n(S) \le M_1$. Let

$$\Delta_{i_k}^n \in K_p^n = \{\Delta_{p(1)}^n, \Delta_{p(2)}^n, \dots, \Delta_{p(up)}^n\}$$

satisfy $r(\Delta_{i_k}^n) \le r(\Delta_{i_j}^n)$ for all $j \ne k$, where $p(1) < p(2) < \cdots < p(up)$.

1. If $\Delta_{i_k}^n = \Delta_{p(1)}^n$, then from $P(\{\Delta_{i_j}^n\}_{j=1}^t) = P_t^n(S) \le M_1$, we can see that $\{\Delta_{i_j}^n\}_{j=1}^t$ is a subset of $G(n,t,p,M_p)$. Thus

$$M_{n_\nu+1} \le M_{p+1} \le P(\{\Delta_{i_j}^n\}_{j=1}^t) = P_t^n(S) \le M_{n_\nu+1}.$$

   Hence $M_{n_\nu+1} = P_t^n(S)$.

2. If $\Delta_{i_k}^n \ne \Delta_{p(1)}^n$ and there exists $q \in \{1,2,3\}$ such that $\Delta_{i_k}^n$ and $\Delta_{p(1)}^n$ are symmetrical with l_q, then we can define $\{\Delta_{m_j}^n\}_{j=1}^t$ such that $\Delta_{i_j}^n$ and $\Delta_{m_j}^n$ are symmetrical with l_q for $j \in \{1,2,\dots,t\}$. By the symmetry, we have$P(\{\Delta_{m_j}^n\}_{j=1}^t) = P(\{\Delta_{i_j}^n\}_{j=1}^t) = P_t^n(S)$.*Since*$\Delta_{m_k}^n = \Delta_{p(1)}^n$ and $r(\Delta_{m_k}^n) \le r(\Delta_{m_j}^n)$ for all $j \ne k$, we have $M_{n_\nu+1} = P_t^n(S)$ from (i).

3. If $\Delta^n_{i_k} \neq \Delta^n_{p(1)}$ and there exist $\Delta^n_{p_\alpha} \in K^n_p$, and $q_1, q_2 \in \{1,2,3\}$ such that $\Delta^n_{i_k}$ and $\Delta^n_{p_\alpha}$, $\Delta^n_{p_\alpha}$ and $\Delta^n_{p(1)}$ are symmetrical with l_{q_1} and l_{q_2}, respectively. We can prove $M_{n_\nu+1} = P^n_t(S)$ by the similar method as in (ii).

From Theorem 4.1, let

$$Q^n_t = \min_{1\le i\le t}\{P^n_t(S)\}, \qquad 1 \le t \le 3^n,$$

we can calculate $P^n(S)$ by the following algorithm.

Algorithm 4.1.

- Set $Q^n_0 = 1.0$.
- For $t = 1, 2, \ldots, 3^n$,
  - Set $M^*_1 = Q^n_{t-1}$.
  - For $p = 1, 2, \ldots, n_\nu$,
    - Determine $G(n, t, p, M^*_p)$.
    - Calculate $M^*_{p+1} = P^n_{t,p,M^*_p}$.
  - Set $Q^n_t = M^*_{n_\nu+1}$
- Set $P^n(S) = Q^n_{3^n}$.

Using Algorithm 4.1, we can get $P^n(S), n \le 5$ as follows.

$P^1(S) = 1,$

$P^2(S) = P^2_6(S) \approx 0.950754,$

$P^3(S) = P^3_{24}(S) \approx 0.910411,$

$P^4(S) = P^4_{60}(S) \approx 0.8697754,$

$P^5(S) = P^5_{186}(S) \approx 0.841718.$

From Theorem 2.1, we have

$$H^s(S) \ge P^5(S)/(1 + 1/2\sqrt{3})^s.$$

Hence we obtain the best below bound estimate of $H^s(S)$:

Corollary 4.1. $H^s(S) \ge 0.5631$.

Question 4.1. *Which are the values of $P^n(S)$ for $n \ge 6$?*

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