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LIMSUP DEVIATIONS ON TREES*

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Abstract

The vertices of an infinite locally finite tree T are labelled by a collection of i.i.d. real random variables $\{X_\sigma\}_{\sigma\in T}$ which defines a tree indexed walk $S_\sigma = \sum_{\emptyset<\tau\le\sigma} X_\tau$. We introduce and study the oscillations of the walk:

$$\mathrm{OSC}_\Phi(\xi) = \overline{\lim}_{\sigma\to\xi\in\partial T}\frac{X_\sigma}{\Phi(|\sigma|)},$$

where $\Phi(n)$ is an increasing sequence of positive numbers. We prove that for each Φ belonging to a certain class of sequences of different orders, there are ξ's depending on Φ such that $0 < \mathrm{OSC}_\Phi(\xi) < \infty$. Exact Hausdorff dimension of the set of such ξ's is calculated. An application is given to study the local variation of Brownian motion. A general limsup deviation problem on trees is also studied.

Key words *limsup deviation, tree-indexed walk, oscillation, Hausdorff dimension, Brownian motion, Percolation, random covering, indexed martingale, Peyrière measure.*

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1 Introduction

Consider an infinite and locally finite tree T. Let $\emptyset$ be the root of T and let ∂T be the boundary of T, which is the set of infinite paths emanating from the root and going through no vertex more than once. Suppose that we are given a family of real valued i.i.d. random variables

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$\{X_\sigma\}_{\sigma\in T}$ indexed by the vertices of T. The process defined by

$$S_\sigma = \sum_{\emptyset<\tau\le\sigma} X_\tau$$

is called a walk indexed by the tree T, where $\tau \le \sigma$ means that τ is on the shortest path from σ to the root. When $T = \mathbb{N}$, we recover the usual walks.

In this paper, we study the oscillations of such a walk. For a boundary point $\xi \in \partial T$, we use the symbol $\sigma \to \xi$ to mean that the vertex σ tends to the infinity through the path ξ. Let $\Phi = \{\Phi(n)\}_{n\ge1}$ be a sequence of increasing positive real numbers. We define the Φ-oscillation of the walk by

$$\mathrm{OSC}_\Phi(\xi) = \limsup_{\sigma\to\xi} \frac{X_\sigma}{\Phi(|\sigma|)}, \qquad \xi \in \partial T,$$

where $|\sigma|$ denotes the length of σ which is the number of edges on the shortest path from σ to the root.

We would like to study the following problem. Given a sequence Φ, are there points ξ such that $\mathrm{OSC}_\Phi(\xi)$ is non-zero and finite? We shall prove that in most cases, there is an infinite number of such sequences Φ of different orders (two sequences Φ and Ψ have the same order if $\dfrac{\Phi(n)}{\Psi(n)}$ are between two constants). Therefore, we may conclude that the oscillations of the walk along different paths are different.

More precisely, the following sets will be studied and their Hausdorff dimensions will be calculated. For a real number $a \in \mathbb{R}$, denote

$$E^*(\Phi, a) = \{\xi \in \partial T : \ \mathrm{OSC}_\Phi(\xi) \le a\},$$

$$E_*(\Phi, a) = \{\xi \in \partial T : \ \mathrm{OSC}_\Phi(\xi) \ge a\},$$

$$E(\Phi, a) = E_*(\Phi, a) \bigcap E^*(\Phi, a).$$

To get the feeling, let us state our results for the special case (general results are stated in §5) where T is a tree such that its branching number $\mathrm{br}T > 1$ and equals to its upper growth rate $\overline{\mathrm{gr}}T$ and $X = |Z|$ with $Z \sim N(0,1)$, the normal law of mean 0 and variance 1 (see §2.2 for the definitions of $\mathrm{br}T$ and $\overline{\mathrm{gr}}T$). Let A be the solution of $P(X \le A) = \dfrac{1}{\mathrm{br}T}$. We use "a.s." to abbreviate "almost surely" and use $\dim_H$ to denote the Hausdorff dimension relative to the natural metric of ∂T, i.e., $d(\xi,\eta) = e^{-|\xi\wedge\eta|}$, where $\xi \wedge \eta$ denotes the common path of ξ and η, and $|\xi \wedge \eta|$ denotes the number of vertices on $\xi \wedge \eta$.

Theorem A. *Keep the above assumption and notation. Then*

(1) *a.s. for all* $\xi \in \partial T$,

$$A \le \mathrm{OSC}_1(\xi), \qquad \mathrm{OSC}_{\sqrt{n}}(\xi) \le \sqrt{2\log \mathrm{br}T}.$$

(2) *If* $B > A$, *then a.s.*

$$\dim_H E^*(1, B) = \log \mathrm{br}T - \log \frac{1}{P(X \le B)}.$$

(3) *If* $a < \sqrt{2 \log \mathrm{br}T}$, *then a.s.*

$$\dim_H E(\sqrt{n}, a) = \dim_H E_*(\sqrt{n}, a) = \log \mathrm{br}T - \frac{a^2}{2}.$$

(4) *If* $0 < \gamma < \frac{1}{2}$ *and* $c > 0$, *then a.s.*

$$\dim_H E(n^\gamma, c) = \dim_H E_*(n^\gamma, c) = \dim_H E^*(n^\gamma, c) = \log \mathrm{br}T.$$

Notice that $\log \mathrm{br}T = \dim_H \partial T$ where $\dim_H$ denotes the Hausdorff dimension.

These results may be translated into local variation properties of Brownian motion. Let $B(t)$ be a linear Brownian motion. We define its symmetric variations at $t \in [0, 1]$ as

$$\Delta_n B(t) = 2B(t_n^{(c)}) - B(t_n^{(g)}) - B(t_n^{(d)}),$$

where $t_n^{(c)}, t_n^{(g)}$ and $t_n^{(d)}$ are respectively the center, left-end and right-end points of the dyadic interval of length 2^{-n} containing t. Let A be the number such that

$$P(X \le A) = \frac{1}{2}.$$

We have

(1) a.s. for all $t \in [0, 1]$

$$A \le \limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2^{-n}}}, \quad \limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2 \cdot 2^{-n} \log 2^n}} \le 1.$$

(2) If $B > A$, then a.s.

$$\dim \left\{ t \in [0, 1] : \limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2^{-n}}} \le B \right\} = 1 - \log_2 \frac{1}{P(X \le B)}.$$

(3) If $0 < \alpha < 1$, then a.s.

$$\dim \left\{ t \in [0, 1] : \limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2 \cdot 2^{-n} \log 2^n}} = \alpha \right\} = 1 - \alpha^2.$$

(4) If $0 < \beta < 1$ and $c > 0$, then a.s.

$$\dim \left\{ t \in [0, 1] : \limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2 \cdot 2^{-n} \log^\beta 2^n}} = c \right\} = 1,$$

here dim denotes the Hausdorff dimension relative to the Euclidean metric.

Actually we will study a more general problem (more abstract to some extent). Given an infinite and locally finite tree T and a sequence $\mathbf{q} = \{q_n\}_{n\geq 1}$ of positive numbers such that $0 < q_n < 1$. We label each vertex (or equivalently the edge preceding the vertex) good or bad independently with probability q_n where n is the distance from the root to the vertex. A path is said to be good (more exactly $\mathbf{q}$- good) if it has infinitely many good vertices. Otherwise, it is said to be bad (more exactly $\mathbf{q}$–bad). We denote by $\mathcal{G}(\mathbf{q})$ the set of good paths and by $\mathcal{B}(\mathbf{q})$ the set of bad paths. It is clear that $\partial T = \mathcal{G}(\mathbf{q}) \bigcup \mathcal{B}(\mathbf{q})$. We address the following questions.

Question 1. when is there almost surely a good path?

Question 2. what is the Hausdorff dimension of the set of good paths $\mathcal{G}(\mathbf{q})$ (when it is non-empty)?

Question 3. when is there almost surely a bad path ?

Question 4. what is the Hausdorff dimension of the set of bad paths $\mathcal{B}(\mathbf{q})$ (when it is non-empty)?

Let us state our answers to these questions for Markov trees (see §§3-4 for the general case). The boundary of a homogeneous tree may be considered as the product space $\{0, 1, \cdots, m-1\}^{\mathbb{N}}$ ($m \geq 2$ being an integer) if we consider every sequence as a path and make the convention that every path is joint to a common root. A Markov tree is a subtree of a homogeneous tree with boundary $\{0, 1, \cdots, m-1\}^{\mathbb{N}}$ ($m \geq 2$ being an integer). Let A be a $m \times m$-matrix with entries 0 or 1. We assume that there is an integer $M > 0$ such that $A^M > 0$ (i.e., all entries are strictly positive). Let

$$\Sigma_A = \{(x_n)_{n\geq 1} : \ A_{x_k, x_k} = 1, \ \forall k \geq 1\}.$$

We call Σ_A the Markov tree with incident matrix A (we should say Σ_A is the boundary of a Markov tree). In the theory of dynamical systems, Σ_A is called subshift of finite type. Let $\rho(A)$ be the spectral radius of A. For $n \geq 1$, let T_n be the set of vertices having a distance n to the root and let $|T_n|$ denote the cardinal of T_n. It is well known that $|T_n| \sim c\rho(A)^n$ (as $n \to \infty$) with some constant $c > 0$.

Theorem B. *Keep the above assumption and notation.*

(1) *a.s.* $\mathcal{G}(\mathbf{q}) \neq \emptyset$ *if and only if*

$$\sum_{n=1}^{\infty} q_n |T_n| = \infty.$$

(2) *a.s.* $\mathcal{B}(\mathbf{q})) \neq \emptyset$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{(1-q_1)\cdots(1-q_n)|T_n|} < \infty.$$

(3) *If* $\limsup\limits_{n\to\infty} \frac{1}{n} \log \frac{1}{q_n} < \dim_H \partial T$, *then a.s.*

$$\dim_H \mathcal{G}(\mathbf{q}) = \dim_H \partial T - \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{q_n}.$$

(4) *If* $\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log \frac{1}{1-q_n} < \dim_H \partial T$, *then a.s.*

$$\dim_H \mathcal{B}(\mathrm{q}) = \dim_H \partial T - \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log \frac{1}{1-q_n}.$$

We may qualify good path and bad path respectively as limsup deviation path and liminf deviation path. Tree-indexed walks were first studied by Joffe and Moncayo[JoM] where they were interested in the limit distribution of $\frac{S_\sigma}{\sqrt{|\sigma|}}$ when $|\sigma|$ tends to the infinity. Many subsequent works have been done[BP,LP,LPP] (see also [Bi, Fal1, Fal2, GMW, H, Liu]). A work closely related to Brownian motion is due to Y. Peres' intersection equivalence [P]. Our consideration is motivated, on one hand, by the study on local properties of Brownian motion, and on the other hand, by the author's previous work with J.P. Kahane[FK] on random dyadic covering which itself is motivated by the Dvoretzky covering problem [D1].

One tool for our study is the multiplicative chaos, developed in a general setting by Kahane [K4] (see also [WW]). A prototype is the Mandelbrot's random cascades model [KP,Man] which goes back to Kolmogoroff's log-normal model for turbulence [Kol] (see also [Fri]). The main gradient that we need is the Peyrière's probability measure (see [K4]). Another tool is a version of Frostman theorem which, in the case of tree, is a consequence of the max-flow min-cut theorem due to Ford and Fulkerson[FF]. We also use a capacity criterion of percolation due to Fan[F1,F2] and Lyons[L1,L2].

The materials are organized as follows. The section §2 starts with a recall of notation and basic notions concerning trees, then it develops the needed tools mentioned above. The bad sets and good sets are respectively studied in §3 and §4. In §5, we apply the results obtained in §§3-4 to study the oscillations of random walks. In §6, we examine the Gaussian case and translate the results into local properties of Brownian motion.

2 Preliminaries

In this section, necessary notation is introduced and basic notions on trees are recalled ([Bo,C,N]). Several known results are stated for later use. These concern capacity, Frostman theorem, dimension of measure, percolation, random covering and multiplicative chaos.

2.1 Basic Notions on Trees and Notation

Let $\mathbb{N}$ be the set of positive integers and $\mathcal{T} = \bigcup_{k=0}^{\infty} \mathbb{N}^k$ (with convention $\mathbb{N}^0 = \{\emptyset\}$), the family of all finite sequences. There is a binary operation on $\mathcal{T}$, called juxtaposition, defined by

$$\tau * \sigma = (\tau_1, \cdots, \tau_n, \sigma_1, \cdots, \sigma_m), \quad \text{for } \tau = (\tau_j)_{j=1}^n, \ \sigma = (\sigma_k)_{k=1}^m.$$

(We may also write $\tau\sigma = \tau * \sigma$). $\mathcal{T}$ is partially ordered by $\tau \leq \sigma$, which means $\sigma = \tau * \tau'$ for some $\tau' \in \mathcal{T}$ (σ is then called an extension of τ). The length of a sequence $\sigma \in \mathcal{T}$ will be denoted by $|\sigma|$.

A tree T is a subset of $\mathcal{T}$ satisfying the two conditions

(i) $\emptyset \in T$,

(ii) $\sigma \in T \Longrightarrow \tau \in T \quad (\forall \tau \leq \sigma)$.

The points in a tree T are called vertices. $\emptyset$ is called the root. The couples $(\tau, \tau * j)$ belonging to $T \times T$ with $|j| = 1$ are called edges and then $\tau * j$ is called a (direct) descendent of τ which is called the parent of $\tau * j$. Let N_σ be the number of descendants of $\sigma \in T$. We shall always assume that

(iii) $\quad 1 \leq N_\sigma < \infty \quad (\forall \sigma \in T)$.

That is to say, the tree T has no leaves and is locally finite. The number $N_\sigma + 1$ is called the degree of σ if $\sigma \neq \emptyset$.

Given an infinite sequence $\xi = (\xi_k)_{k \geq 1} \in \mathbb{N}^\infty$. For any $n \geq 1$, we write $\xi|_n = (\xi_1, \cdots, \xi_n)$, which is called the k-curtailment of ξ. If all k-curtailments of ξ belong to a tree T, we say ξ is a boundary point of T. We denote by ∂T the set of all boundary points of T. As a subspace of $\mathbb{N}^\infty$, ∂T is a compact metric space because of the local finiteness of T. We shall use the following metric

$$d(\xi, \eta) = e^{-|\xi \wedge \eta|}, \quad \xi, \eta \in \partial T,$$

where $\xi \wedge \eta$ is the longest common curtailment of ξ and η. A boundary point $\xi \in \partial T$ is also called an (infinite) path of T.

For $\xi \in \partial T$ and $\tau \in T$, $\tau \leq \xi$ means the path ξ passes by τ or τ is a curtailment of ξ. We denote $B_\tau = \{\xi \in \partial T : \tau \leq \xi\}$. It is the ball of radius $e^{-|\tau|}$ centered at any path passing by τ.

2.2 Growths, Dimensions and Capacities

Let T_n be the set of vertices of length n (considered as finite sequences). The lower and upper growth rates of T are defined as

$$\underline{\mathrm{gr}}T = \liminf_{n \to \infty} |T_n|^{1/n}, \quad \overline{\mathrm{gr}}T = \lim_{n \to \infty} |T_n|^{1/n},$$

where $|T_n|$ means the cardinality of T_n. When the limit exists, their common value gr$T$ is called the growth rate . A more important quantity br$T$, called branching number, was introduced by R. Lyons [L1]. It accounts more for the structure of the tree. It is defined as follows

$$\mathrm{br}T = \sup\left\{\lambda \geq 1 : \inf_\pi \sum_{\sigma \in \pi} \lambda^{-|\sigma|} > 0\right\},$$

where π is an arbitrary cutset of T, that means a set of vertices such that any path $\xi \in \partial T$ passes one and only one vertex in π.

These three notions are just the exponentials of three kinds dimension of ∂T. In fact, the boundary being a metric space, for any subset $F \subset \partial T$ we can define, as usual, its Hausdorff dimension $\dim_H F$, lower box dimension $\underline{\dim}_B F$ and upper box dimension $\overline{\dim}_B F$. We have the relation

$$\mathrm{br} T = e^{\dim_H \partial T}, \quad \underline{\mathrm{gr}} T = e^{\underline{\dim}_B \partial T}, \quad \overline{\mathrm{gr}} T = e^{\overline{\dim}_B \partial T}.$$

See [Mat] for a general account on dimensions. Actually, the logarithm of the branching number of a tree was first introduced by H. Furstenberg just as the "dimension" of the tree [Fur].

In order to study the Hausdorff dimension, a useful tool is the capacity. Let $\alpha > 0$. The α-Riesz kernel is defined by

$$R_\alpha(\xi, \eta) = d(\xi, \eta)^{-\alpha} = e^{\alpha|\xi \wedge \eta|}, \quad \xi, \eta \in \partial T.$$

More general kernels may be defined as follows. A function Ψ on T is increasing if $\Psi(\tau) \leq \Psi(\sigma)$ for $\tau \leq \sigma$. Any such a function defines a kernel $K(\xi, \eta) = \Psi(\xi \wedge \eta)$ (with the convention that $\Psi(\xi) = \lim_{\sigma \to \xi} \Psi(\sigma)$).

Given a Borel probability measure μ on ∂T, we define its energy relative to a kernel K by

$$I_K^\mu = \int\int K(\xi, \eta) \mathrm{d}\mu(\xi) \mathrm{d}\mu(\eta).$$

The K- capacity of a compact set $F \subset \partial T$, denoted $\mathrm{Cap}_K F$, is defined by

$$(\mathrm{Cap}_K F)^{-1} = \inf\{I_K^\mu : \mu \in M_1^+(F)\},$$

where $M_1^+(F)$ is the space of all probability measures concentrated on F. The K- capacity of a Borel set $B \subset \partial T$ is defined by $\sup_F \mathrm{Cap}_K F$ where the supremum is taken over all compact subsets of B. When $K = R_\alpha$, we shall write $I_K^\mu = I_\alpha^\mu$ and $\mathrm{Cap}_K F = \mathrm{Cap}_\alpha F$. The capacity dimension of $F \subset \partial T$ is defined as

$$\dim_C F = \sup\{\alpha : \mathrm{Cap}_\alpha F > 0\} = \sup\{\alpha : \exists \mu \in M_1^+(F),\ I_\alpha^\mu < \infty\}.$$

Proposition 2.1. $\dim_H F = \dim_C F$ for any $F \subset \partial T$.

This result allows us to estimate Hausdorff dimensions of sets through energy integrals of measures. Such a result was first obtained by Frostman [Fro] in Euclidean spaces and was generalized to homogeneous spaces by Assouad [A]. However ∂T is not homogeneous unless the degrees of T are uniformly bounded. But, in the case of trees, the last proposition is just a consequence of the max-flow min-cut theorem on networks due to Ford and Fulkerson [FF]. We point out that R. Kaufman [Kau] has recently generalized the equality $\dim_H F = \dim_C F$ to every complete separable metric space, including ∂T for any infinite and locally finite tree T. We shall refer to the last proposition as Frostman theorem, despite of its different sources.

Recall now the (lower) dimension of a measure μ defined by

$$\dim \mu = \inf\{\dim F : \mu(F) > 0\}.$$

(See [F1,F3] for more information). The following proposition, a consequence of Frostman theorem, provides a way to estimate the Hausdorff dimension of a set by estimating the dimensions of measures.

Proposition 2.2. *Let $F \subset \partial T$ be a Borel set F and μ be a measure. Then*
(1) $\dim F \geq \dim \mu$ *if* $\mu(F) > 0$.
(2) $\dim \mu \geq \tau$ *if* $I_\tau^\mu < \infty$.

2.3 Percolation and Covering

Let $\mathbf{p} = (p_n)_{n\geq 1}$ be a sequence of positive numbers such that $0 < p_n \leq 1$. We remove edges at random from a tree T, keeping each edge in the n–th generation with probability p_n and making decisions independently for all different edges of all generations. This procedure is called $\mathbf{p}$–Bernoulli percolation. If, with a positive probability, an infinite path emanating from the root remains in the tree, we say that the percolation occurs. Let

$$K(t,s) := K_{\mathbf{p}}(t,s) := \prod_{n=1}^{|t\wedge s|} \frac{1}{p_n}. \tag{2.1}$$

There is a complete solution, in term of the above kernel K, to this $\mathbf{p}$–Bernoulli percolation problem, due to A. H. Fan([F1,F2]) and R. Lyons([L1,L2]).

Proposition 2.3. *A necessary and sufficient condition for the* $\mathbf{p}$*–Bernoulli percolation to occur is the* $\mathrm{Cap}_K \partial T > 0$ *where K is the kernel defined by (2.1).*

This result was first stated in [F1,F2] for the special case of trees of uniformly bounded degrees as a solution to a random covering problem. But the proof is the same for the general case stated above. A different proof involving the electrical network technique was used in [L2] and the dependence on the vertices of the probability is allowed.

Actually, the above result may be interpreted from random covering point of view in the following way. In stead of saying that an edge is removed, we say it is covered. So that an edge of n–th generation is covered with probability $q_n = 1 - p_n$. We say a path $\xi \in \partial T$ is covered if one of the edges (or equivalently vertices) on ξ is covered. Let $J_{\mathbf{p}}$ ($\subset \partial T$) be the (random) set of all covered paths. We say a path ξ is infinitely covered if an infinite number of edges on ξ are covered. Let $J_{\mathbf{p}}^{\mathrm{inf}}$ be the set of all infinitely covered paths. It is clear that

$$\mathcal{G}(\mathbf{q}) = J_{\mathbf{p}}^{\mathrm{inf}}, \quad \mathcal{B}(\mathbf{q}) = \partial T \setminus J_{\mathbf{p}}^{\mathrm{inf}}$$

with $\mathbf{q} = (1 - p_n)_{n\geq 1}$.

2.4 Multiplicative Chaos on Trees

The general theory of multiplicative chaos was developed by J.P. Kahane[K4]. The theory holds on any metric space. But we recall it here just for the case of trees. The key part for us is the Peyrière probability measure. Let (P_n) be a sequence of non-negative independent random functions defined on ∂T such that $\mathbb{E}P_n(\xi) = 1$ ($\forall \xi \in \partial T$). They are called weights. Consider the finite products

$$Q_N(\xi) = \prod_{n=1}^{N} P_n(\xi).$$

We call $Q_N(\xi)$ an indexed martingale because it is a martingale for each $\xi \in \partial T$. For any $\mu \in M_1^+(\partial T)$, it was proved in [K4] that a.s. the random measures $Q_N(\xi)d\mu(\xi)$ converge weakly to a (random) measure that we denote by Q or Q^μ. The random measure Q is called a multiplicative chaos. If the following L^2 condition

$$\int\int \prod_{n=1}^{\infty} \mathbb{E}P_n(\xi)P_n(\eta)\mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta) < \infty \tag{2.2}$$

is satisfied, the measure Q is not vanished and a probability measure $\mathcal{Q}$ on $\Omega \times \partial T$, called Peyrière measure, may be defined by the relation

$$\int_{\Omega\times\partial T} \varphi(\omega,\xi)\mathrm{d}\mathcal{Q}(\omega,\xi) = \mathbb{E}\int_{\partial T} \varphi(\omega,\xi)\mathrm{d}Q(\xi)$$

(for all bounded measurable functions φ). If the distribution of the variable $P_n(\xi)$ is independent of $\xi \in \partial T$, the weight P_n is said to be homogeneous. The following fact will be useful to us.

Proposition 2.4. *If the L^2-condition (2.2) is satisfied and if the weights P_n ($n \geq 1$) are homogeneous, then $P_n(\omega,\xi)$'s, considered as random variables on $\Omega \times \partial T$, are $\mathcal{Q}$-independent. Furthermore, we have the formula*

$$\mathbb{E}_{\mathcal{Q}}h(P_n) = \mathbb{E}h(P_n)P_n$$

(for any Borel function h).

Remark that if a property relative to (ω,ξ) holds $\mathcal{Q}$-almost everywhere, then almost surely the property holds for Q-almost every ξ. In the sequel, "almost surely" will be shortened to "a.s." and "almost everywhere" to "a.e.". The first is referred to the random sampling and the second to the boundary point of a tree.

3 Bad Set $\mathcal{B}(\mathbf{q})$

3.1 General Case

For any $N \geq 1$, consider the set $J_{\mathbf{p}}^N$ of paths which are covered at least N times. The following relations are obvious

$$J_{\mathbf{p}} = \bigcup_{N=1}^{\infty} J_{\mathbf{p}}^N, \quad J_{\mathbf{p}}^{\inf} = \bigcap_{N=1}^{\infty} J_{\mathbf{p}}^N. \tag{3.1}$$

If $J_{\mathbf{p}} = \partial T$ a.s., we simply say that ∂T is covered. Thus we have the following interpretation. The percolation occurs if and only if ∂T is not covered with positive probability. Or equivalently, the percolation doesn't occurs if and only if ∂T is covered with probability 1. Now we state the necessary and sufficient condition for the bad set $\mathcal{B}(\mathbf{q})$ to be not empty.

Theorem 3.1. *Consider the* p*-Bernoulli percolation described above associated to the kernel K defined by (2.1). We have*

$$\mathrm{Cap}_K \partial T = 0 \Longrightarrow P(\partial T = J_{\mathbf{p}}^{\inf}) = 1,$$

$$\mathrm{Cap}_K \partial T > 0 \Longrightarrow P(\partial T = J_{\mathbf{p}}^{\inf}) = 0.$$

In other words, $\mathcal{B}(\mathbf{q}) \neq \emptyset$ a.s. if and only if $\mathrm{Cap}_K \partial T > 0$, where $q_n = 1 - p_n$.

Proof. Suppose $\mathrm{Cap}_K \partial T = 0$. For any $v \in T$, let T^v be the subtree rooted at v. It is clear that $\mathrm{Cap}_K \partial T^v = 0$ (∂T^v may be identified with a subset, a ball in ∂T). By Proposition 2.3, ∂T^v is a.s. covered so that $\bigcup_{v \in T_n} \partial T^v$ is a.s. covered for any $n \geq 1$. Therefore ∂T is a.s. infinitely covered, i.e. $J_{\mathbf{p}}^{\inf} = \partial T$ a.s. because

$$\{J_{\mathbf{p}}^{\inf} = \partial T\} = \bigcap_{n=1}^{\infty} \left\{ \bigcup_{v \in T_n} \partial T^v \text{ is covered} \right\}.$$

Thus the first assertion is proved.

Suppose now $\mathrm{Cap}_K \partial T > 0$. Then, by Proposition 3.1, we have

$$P(\partial T = J_{\mathbf{p}}^{\inf}) \leq P(\partial T = J_{\mathbf{p}}) < 1.$$

However, the above argument shows that $\{\partial T = J_{\mathbf{p}}^{\inf}\}$ is a tail event. Therefore we must have

$$P(\partial T = J_{\mathbf{p}}^{\inf}) = 0.$$

Let $\mathbf{p}' = (p'_n)$ be a sequence similar to $\mathbf{p}$. By $\mathbf{pp}'$-Bernoulli percolation we mean a $\mathbf{p}$-Bernoulli percolation followed by an independent $\mathbf{p}'$-Bernoulli percolation. We shall use this device to estimate the size of the set of path which are not infinitely covered, i.e., the set $\mathcal{B}_{\mathbf{q}}$ with $\mathbf{q} = 1 - \mathbf{p}$ (the notation should be self-understood). This idea was used in [K3] for studying Dvoretzky covering. The following result gives estimates on the size of the bad set $\mathcal{B}(\mathbf{q}) = \partial T \backslash J_{\mathbf{p}}^{\inf}$ in terms of capacity.

Theorem 3.2. *Consider a* p*-Bernoulli percolation with kernel K defined by (??). Consider also a similar kernel K'. We have*

$$\mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{\inf}) > 0 \ \text{ a.s.} \quad \Longleftrightarrow \quad \mathrm{Cap}_{KK'} \partial T > 0.$$

Proof. Suppose $\mathrm{Cap}_{KK'} \partial T = 0$. It is clear that KK' is the kernel of the $\mathbf{pp}'$-Bernoulli percolation. By Theorem 3.1, $P(\partial T = J_{\mathbf{pp}'}^{\inf}) = 1$. That is to say, ∂T is infinitely covered by one of these two covering procedures corresponding to the **p**–Bernoulli percolation and the **p**′–Bernoulli percolation. Notice that the σ-stability of capacity implies

$$\mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{\mathrm{inf}}) = \sup_{N\geq 1} \mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{N}), \tag{3.2}$$

(see the formula (3.1)). Notice also that $\partial T \setminus J_{\mathbf{p}}^{N}$ may be regarded as the boundary of the subtree, obtained by cutting the tails of the paths which are covered at least N times. More precisely, if $\xi = (\xi_k)_{k\geq 1} \in \partial T$ is covered at least N times and if ξ_n is the N-th covered vertex, then $\{\xi_k\}_{k\geq n}$ is cut. Since the paths in $\partial T \setminus J_{\mathbf{p}}^{N}$ are not infinitely covered by the covering procedure corresponding to the **p**–Bernoulli percolation, they must be infinitely covered by the covering procedure corresponding to the **p**′–Bernoulli percolation. Apply once more Theorem 3.1, $\mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{N}) = 0$ a.s. This, together with (3.2), leads to $\mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{\mathrm{inf}}) = 0$ a.s.

Suppose $\mathrm{Cap}_{KK'}\partial T > 0$. We claim that with positive probability we have

$$\mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{1}) > 0.$$

Otherwise, $\partial T \setminus J_{\mathbf{p}}^{1}$ is a.s. covered by the covering procedure corresponding to the **p**′–Bernoulli percolation. So, ∂T is a.s. covered by the covering procedure corresponding to **pp**′-Bernoulli percolation, which contradicts $\mathrm{Cap}_{KK'}\partial T > 0$. Thus with positive probability we have

$$\mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{\mathrm{inf}}) \geq \mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{1}) > 0.$$

Since $\{\mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{\mathrm{inf}}) > 0\}$ is a tail event, we have a.s. $\mathrm{Cap}_{K'}(\partial T \setminus J_{\mathbf{p}}^{\mathrm{inf}}) > 0$.

It follows the Hausdorff dimension formula

$$\dim_H \mathcal{B}(\mathbf{q}) = \sup\left\{\alpha \geq 0 : \mathrm{Cap}_{R_\alpha K}\partial T > 0\right\} \quad \text{a.s.}, \tag{3.3}$$

where

$$K(t,s) = \prod_{n=1}^{|t\wedge s|} \frac{1}{1-q_n}, \quad R_\alpha(t,s) = e^{\alpha|t\wedge s|}.$$

3.2 Markov Trees and Spherically Symmetrical Trees

We examine two examples: the Markov trees and the spherically symmetrical trees. The preceding results take more explicit forms. For two functions u and v, if there is a constant $C > 0$ such that $C^{-1}v \leq u \leq Cv$, we write $u \approx v$.

Lemma 3.1. *Let ν be the Parry measure on Σ_A. We have*

$$U^{\nu}(t) \approx \sum_{n=1}^{\infty} \frac{1}{\rho(A)^n p_1 p_2 \cdots p_n}, \qquad \forall t \in \Sigma_A, \tag{3.4}$$

where the kernel K is defined by (2.1), $\rho(A)$ is the spectral radius of A. Consequently we have

$$\int\!\!\int K(t-s)\mathrm{d}\nu(t)\mathrm{d}\nu(s) \approx \sum_{n=1}^{\infty} \frac{1}{\rho(A)^n p_1 p_2 \cdots p_n}. \tag{3.5}$$

Proof. Let u and v be positive left and right eigenvector of A associated to $\rho := \rho(A)$, i.e. $u^tA = \rho u^t$ and $Av = \rho v$. Suppose that u and v are normalized so that

$$\pi = (\pi_0, \pi_1, \cdots, \pi_{m-1})$$

with $\pi_i = u_i v_i$ is a probability vector. Let

$$p_{i,j} = A_{i,j}\frac{v_j}{\rho v_i}.$$

Then the Parry measure is the Markov measure with transition matrix $P = (p_{i,j})$ and initial probability π. That means

$$\nu(I_n(x_1, x_2, \cdots, x_n)) = \pi_{x_1} p_{x_1,x_2} \cdots p_{x_{n-1}x_n}.$$

Notice that

$$\nu(I_n(x_1, x_2, \cdots, x_n)) = \rho^{-(n-1)} u_{x_1} A_{x_1,x_2} \cdots A_{x_{n-1}x_n} v_{x_n}.$$

Let $I_n(t)$ be the n-cylinder containing t. Since $I_0(t) = \Sigma_A$, we have

$$\begin{aligned} U^\nu(t) &= \sum_{n=0}^\infty \frac{1}{p_1p_2\cdots p_n}\int_{I_n(t)\setminus I_{n+1}(t)} \mathrm{d}\nu(x) \\ &= \sum_{n=0}^\infty \frac{1}{p_1p_2\cdots p_n}\sum_{s_{n+1}\neq t_{n+1}} \nu(I_{n+1}(t_1,\cdots,t_n,s_{n+1})) \\ &= \sum_{n=0}^\infty \frac{1}{p_1p_2\cdots p_n\rho^n} u_{t_1}A_{t_1,t_2}\cdots A_{t_{n-1},t_n}\sum_{s_{n+1}\neq t_{n+1}} A_{t_n,s_{n+1}}v_{s_{n+1}} \\ &\approx \sum_{n=0}^\infty \frac{1}{p_1p_2\cdots p_n\rho^n}. \end{aligned}$$

Theorem 3.3. *Consider a Markov tree Σ_A defined by the primitive matrix A (whose spectral radius is denoted $\rho(A)$). For a given sequence $\mathrm{q} = (q_n)_{n\geq 1}$ with $0 < q_n \leq 1$. Then*

(a) *a.s. $\mathcal{B}(\mathrm{q}) \neq \emptyset$ if and only if*

$$\sum_{n=1}^\infty \frac{1}{\rho(A)^n(1-q_1)(1-q_2)\cdots(1-q_n)} < \infty. \tag{3.6}$$

(b) *Furthermore, for any kernel K' defined by a sequence (p'_n), we have $\mathrm{Cap}_{K'}(\mathcal{B}(\mathrm{q})) > 0$ a.s., if and only if*

$$\sum_{n=1}^\infty \frac{1}{\rho(A)^n p'_1(1-q_1)p'_2(1-q_2)\cdots p'_n(1-q_n)} < \infty. \tag{3.7}$$

(c) *In particular, under the condition (3.3), we have a.s.*

$$\dim_H \mathcal{B}(\mathrm{q}) = \log\rho(A) - \limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log\frac{1}{1-q_n}. \tag{3.8}$$

Proof. Lemma 3.1 tells us that the potential $U_K^\nu(t)$ of the Parry measure is nearly a constant on Σ_A, a property shared by the equilibrium measure of Σ_A. Suppose the series in (3.6) is divergent, i.e. the potential is every everywhere infinite. Then the argument used in [F1,F2] shows that Σ_A is covered by the covering according to the **p**–Bernoulli percolation with $p_n = 1 - q_n$. That means $\mathcal{B}(\mathbf{q}) = \emptyset$ a.s. Suppose now the series in (3.6) is convergent. Then $\mathrm{Cap}_K(\Sigma_A) > 0$ by (3.5) in Lemma 3.1. So, according to Theorem [NFL], $\mathcal{B}(\mathbf{q}) \neq \emptyset$ a.s. Thus (a) is proved.

The assertion (b) is a direct consequence of Theorem 3.2, because (3.7) means $\mathrm{Cap}_{KK'}\Sigma_A > 0$ by Lemma 3.1 and the proof of (a).

For (c), consider $p'_n = e^{-\alpha}$ ($\forall n \geq 1$). The corresponding kernel is the Riesz kernel R_α. Let $x = e^\alpha$ and consider the following Taylor series

$$\sum_{n=1}^{\infty} \frac{x^n}{\rho(A)^n(1-q_1)(1-q_2)\cdots(1-q_n)}.$$

By (b) and the formula (3.3), $e^{\dim_H \mathcal{B}(\mathbf{q})}$ is equal to the radius of convergence of the Taylor series. Thus we get the formula (3.8) by using the Cauchy-Hadamard formula.

Suppose that T is a spherically symmetrical tree with numbers of descendants $\{m_n\}_{n\geq 1}$. We can consider ∂T as the infinite group $\bigotimes_{n=1}^{\infty} \mathbb{Z}/m_n\mathbb{Z}$. The natural measure on T is the Haar measure ν: any n–cylinder has a measure $(m_1m_2\cdots m_n)^{-1}$. This measure is invariant under the group translation. It is easier to prove the following lemma than Lemma 3.1 by using the translation invariance.

Lemma 3.2. *Suppose that T is a spherically symmetrical tree with numbers of descendants $\{m_n\}_{n\geq 1}$. Let ν be the Haar measure on T. We have*

$$U_K^\nu(t) \approx \sum_{n=1}^{\infty} \frac{1}{p_1p_2\cdots p_n m_1m_2\cdots m_n}, \qquad \forall t \in \partial T, \tag{3.9}$$

where the kernel K is defined by (2.1). Consequently we have

$$\int\int K(t-s)\mathrm{d}\nu(t)\mathrm{d}\nu(s) \approx \sum_{n=1}^{\infty} \frac{1}{p_1p_2\cdots p_n m_1m_2\cdots m_n}. \tag{3.10}$$

We can prove

Theorem 3.4. *Consider a spherically symmetrical tree T with numbers of descendants $\{m_n\}_{n\geq 1}$. Let $\mathbf{q} = (q_n)_{n\geq 1}$ be a sequence of numbers with $0 < q_n < 1$. Denote $|T_n| = m_1m_2\cdots m_n$. Then*

(a) *a.s. $\mathcal{B}(\mathbf{q}) \neq \emptyset$ iff*

$$\sum_{n=1}^{\infty} \frac{1}{|T_n|(1-q_1)(1-q_2)\cdots(1-q_n)} < \infty. \tag{3.11}$$

(b) *Furthermore, for any kernel K' defined by a sequence (p'_n), we have* $\mathrm{Cap}_{K'}(\mathcal{B}(\mathbf{q})) > 0$ *a.s. iff*

$$\sum_{n=1}^{\infty} \frac{1}{|T_n| p'_1(1-q_1)p'_2(1-q_2)\cdots p'_n(1-q_n)} < \infty. \tag{3.12}$$

(c) *In particular, under the condition (3.11), we have a.s.*

$$\dim_H \mathcal{B}(\mathbf{q}) = \liminf_{n\to\infty} \frac{1}{n}\left[\log|T_n| - \sum_{j=1}^{n} \log\frac{1}{1-q_n}\right]. \tag{3.13}$$

4 Good Set $\mathcal{G}(\mathbf{q})$

4.1 Existence of Good Path

We start with a simple fact. Given any measure μ on ∂T. By the notation $A = \partial T$ μ-a.e., we mean that the symmetrical difference $(A \setminus \partial T) \cup (\partial T \setminus A)$ has null μ-measure. If

$$\sum_{n=1}^{\infty} q_n = \infty$$

then a.s. $\mathcal{G}(\mathbf{q}) = \partial T$ μ-a.e. If the above series converges, $\mathcal{B}(\mathbf{q}) = \emptyset$ μ-a.e. In fact, it suffices to apply the Borel-Cantelli lemma and the Fubini theorem.

The above condition (the divergence of the series) is much stronger than the real condition for the existence of good path. Recall that for any vertex σ, we denote by T^σ the subtree emanating from σ. The following is a criterion for the good set $\mathcal{G}(\mathbf{q})$ to be non empty. It is actually a criterion for $\mathcal{G}(\mathbf{q})$ to be dense in ∂T.

Theorem 4.1. *The good set $\mathcal{G}(\mathbf{q}) = \emptyset$ a.s. if*

$$\sum_{n=1}^{\infty} q_n |T_n| < \infty. \tag{4.1}$$

The good set $\mathcal{G}(\mathbf{q}) \neq \emptyset$ a.s. if for any vertex $\sigma \in T$ we have

$$\sum_{n=1}^{\infty} q_{|\sigma|+n} |T_n^\sigma| = \infty. \tag{4.2}$$

In this case, the good set is a.s. dense in ∂T.

Proof. Suppose the series in (4.1) is convergent. By the Borel-Cantelli lemma, there is a.s. only a finite number of good edges on the tree, a fortiori $\mathcal{G}(\mathbf{q}) = \emptyset$ a.s. Suppose now the divergence of the series in (4.2). The same Borel-Cantelli lemma says that on any subtree there is a.s. an infinite number of good edges. So, a.s. there is an infinite number of good edges on any subtree T^σ, because such subtrees are countable. We claim that a.s. the tree contains a good path. Take a sampling and construct a good path in the following manner. An edge joins a parent vertex to a descendent vertex. In stead of saying the edge is good, we say the (descendent) vertex is good. So, a good path is one passing an infinite number of good vertices. Since there is an infinite number of good vertices on the tree T, we choose a good one σ_1. Since there is yet an infinite number of good vertices on the subtree T^{σ_1}, we choose a second good vertex σ_2, different from σ_1, on T^{σ_1}. Now consider the subtree T^{σ_2} and so on. In this way we get a sequence of good vertices $\{\sigma_n\}_{n\geq 1}$. The path passing these vertices is a good path. This proves $\mathcal{G}(\mathbf{q}) \neq \emptyset$. In the same way, we can construct a good path on any subtree. Thus we prove the density of $\mathcal{G}(\mathbf{q})$

Remark 4.1. In general, the condition

$$\sum_{n=1}^{\infty} q_n |T_n| = \infty$$

is not sufficient for ensuring $\mathcal{G}(\mathbf{q}) \neq \emptyset$ a.s.

Let us consider the Markov tree (not primitive) defined by

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and the probabilities $q_n = n^{-2}$. We have $|T_n| = n + 1$ so that

$$\sum_{n=1}^{\infty} q_n |T_n| = \infty.$$

However, any path is a.s.bad because

$$\sum_{n=1}^{\infty} q_n < \infty.$$

Since there are only a countable paths, a.s. any path is bad, i.e. $\mathcal{G}(\mathbf{q}) = \emptyset$ a.s.

Remark 4.2. But for trees sharing some symmetry or periodicity, the condition $\sum_{n=1}^{\infty} q_n |T_n| = \infty$ is necessary and sufficient for $\mathcal{G}(\mathbf{q}) \neq \emptyset$ a.s.

We call a tree T growth–periodic if for any vertex $\sigma \in T$ there is a constant $C = C(\sigma) > 0$ such that

$$\frac{|T_n^{\sigma}|}{|T_{|\sigma|+n}|} \geq C(\sigma), \quad \forall n \geq 1.$$

Corollary 4.1. *Suppose T is a growth–periodic tree. Then the good set $\mathcal{G}(\mathbf{q}) \neq \emptyset$ a.s. iff*

$$\sum_{n=1}^{\infty} q_n |T_n| = \infty. \tag{4.3}$$

In this case, the good set is a.s. dense in ∂T.

Proof. We have only to show that the condition (4.3) implies the condition (4.2). It is true because

$$\sum_{n=1}^{\infty} q_{|\sigma|+n}|T_n^\sigma| \geq C(\sigma) \sum_{k=|\sigma|+1} q_k|T_k|$$

where $C(\sigma)$ is the constant involved in the definition of the growth–periodicity.

Corollary 4.2. *If T be a (primitive) Markov tree Σ_A or a spherically symmetrical tree $\bigotimes_{n=1}^{\infty} \mathbb{Z}/m_n\mathbb{Z}$, then it is growth–periodic. Consequently, either $\mathcal{G}(\mathrm{q})$ is a.s. dense in ∂T if*

$$\sum_{n=1}^{\infty} q_n|T_n| = \infty$$

or $\mathcal{G}(\mathrm{q})$ is a.s. empty if the series converges, where $|T_n| \approx \rho(A)^n$ for the Markov tree or $|T_n| = m_1 m_2 \cdots m_n$ for the spherically symmetrical tree.

Proof. For the spherically symmetrical tree, it suffices to notice that

$$m_1 \cdots m_{|\sigma|}|T_n^\sigma| = |T_{n+|\sigma|}|.$$

We may take $C(\sigma) = (m_1 \cdots m_{|\sigma|})^{-1}$.

For the Markov tree, it suffices to notice that

$$|T_n^\sigma| = \sum A_{|\sigma|,x_1} A_{x_1,x_2} \cdots A_{x_{n-1},x_n} \approx \rho(A)^n = \rho(A)^{-|\sigma|}|T_{|\sigma|+n}|.$$

The condition (4.3) of non-existence of good path is implied by

$$\liminf_{n\to\infty} \frac{1}{n} \log \frac{1}{q_n} > \overline{\dim}_B \partial T.$$

The condition of existence of good path for growth–periodic tree is implied by

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{q_n} < \underline{\dim}_B \partial T.$$

We are now going to estimate the Hausdorff dimension of the good set.

4.2 Hausdorff Dimension of $\mathcal{G}(\mathrm{q})$

Theorem 4.2. *Let*

$$\alpha = \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{q_n}.$$

Suppose $\dim_H \partial T > \alpha$. *Then the good set* $\mathcal{G}(\mathbf{q})$ *is of positive Hausdorff dimension and*

$$\dim_H \partial T - \alpha \le \dim_H \mathcal{G}(\mathbf{q}) \le \overline{\dim}_B \partial T - \alpha \quad \text{a.s.} \tag{4.4}$$

In stead of saying good edges, we say good vertices. Fix $\epsilon > 0$. Consider the random subset of the n-level cutset T_n

$$\mathcal{C}_n = \{\sigma \in T_n : \sigma \text{ is good}\}$$

and the set of paths covered by balls B_σ with $\sigma \in \mathcal{C}_n$, i.e.

$$G_n = \bigcup_{\sigma \in \mathcal{C}_n} B_\sigma.$$

For a good path ξ there is an infinite number of G_n's which contain ξ. In other words,

$$\mathcal{G}(\mathbf{q}) \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} G_n.$$

Using the σ-stability of the Hausdorff dimension and the fact $\dim_H F \le \underline{\dim}_B F$, we have

$$\dim_H \mathcal{G}(\mathbf{q}) \le \sup_{N \ge 1} \underline{\dim}_B \bigcap_{n=N}^{\infty} G_n.$$

However

$$\underline{\dim}_B \bigcap_{n=N}^{\infty} G_n \le \liminf_{n \to \infty} \frac{\log \mathrm{Card}\mathcal{C}_n}{n}.$$

We are then led to estimate $\mathrm{Card}\mathcal{C}_n$. Notice that

$$\mathbb{E}\mathrm{Card}\mathcal{C}_n = |T_n| q_n.$$

Let $\ell = \liminf_n (|T_n| q_n)^{1/n}$. Then for any $\ell'' > \ell' > \ell$ there is a subsequence n_k such that $|T_{n_k}| q_{n_k} \le \ell'^{n_k}$. Next consider the event $E_n = \{\mathrm{Card}\mathcal{C}_n \le \ell''^n\}$. By the Chebyshev inequality, we have

$$P\left(E_{n_k}^c\right) \le \frac{\mathbb{E}\mathrm{Card}\mathcal{C}_{n_k}}{\ell''^{n_k}} \le \left(\frac{\ell'}{\ell''}\right)^{n_k} \le \frac{\ell'}{\ell''} < 1.$$

It follows that

$$\sum_{n=1}^{\infty} P(E_n) \ge \sum_{k=1}^{\infty} P\left(E_{n_k}\right) \ge \sum_{k=1}^{\infty} \left(1 - \frac{\ell'}{\ell''}\right) = \infty.$$

Notice that E_n are independent. By the Borel-Cantell lemma, we have a.s.

$$\liminf_{n\to\infty} \frac{\log \mathrm{Card} C_n}{n} \leq \log \ell''.$$

Since $\ell'' > \ell$ is arbitrary, we have

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log \mathrm{Card} C_n}{n} &\leq \liminf_{n\to\infty} \frac{1}{n} \log(|T_n| q_n) \\ &\leq \overline{\dim}_B \partial T - \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{q_n}.\end{aligned}$$

In order to get the lower bound we consider a family of independent Bernoulli variables indexed by the tree $\{\epsilon_v\}_{v\in T}$ such that

$$P(\epsilon_v = 1) = q_{|v|} = 1 - P(\epsilon_v = 0).$$

Then consider the multiplicative process with weights

$$P_n(\xi) = \frac{\exp(b_n \epsilon_{\xi|_n})}{1 - q_n + q_n e^{b_n}},$$

where the sequence of numbers $b = (b_n)$ will be chosen later.

Take β and D such that $\alpha < \beta < D < \dim_H \partial T$. By the Frostman theorem, there is a probability measure $\mu \in M_1^+(\partial T)$ with

$$I_D^\mu < \infty. \tag{4.5}$$

Consider the multiplicative chaos measure Q_b^μ associated to μ. Notice that, by the definition of α, there are constants $n_0 > 1$ such that

$$\log \frac{1}{q_n} \leq \beta n, \qquad \forall n \geq n_0. \tag{4.6}$$

For any $\kappa \geq 0$, it may be calculated that

$$\begin{aligned}\mathbb{E} I_\kappa^{Q_b^\mu} &= \lim_{N\to\infty} \int\int R_\kappa(\xi,\eta) \mathbb{E}\ Q_N(\xi) Q_N(\eta) \mathrm{d}\mu(\xi) \mathrm{d}\mu(\eta) \\ &= \int\int R_\kappa(\xi,\eta) \prod_{1\leq n\leq|\xi\wedge\eta|} \exp[c_n(2b_n) - 2c_n(b_n)] \mathrm{d}\mu(\xi) \mathrm{d}\mu(\eta),\end{aligned}$$

where $c_n(x) = \log(1 - q_n + q_n e^x)$. In virtue of the elementary inequality

$$(1 - a + ay^2) \leq y(1 - a + ay)^2$$

holding for $0 \le a \le 1$ and $y \ge 1$, we have

$$c_n(2b_n) - 2c_n(b_n) \le b_n. \tag{4.7}$$

Take now an integer $\Delta \ge 2$ and define $\{b_n\}$ as follows:

$$b_n = 0 \quad \text{if} \quad n \ne \Delta^j, \qquad b_{\Delta^j} = \log \frac{1}{q_{\Delta^j}}. \tag{4.8}$$

Then, by the inequality (4.7) and the choice of b_n's, $\mathbb{E} I_\kappa^{Q_b^\mu}$ is bounded by

$$\begin{aligned} &\int\int R_\kappa(\xi,\eta) \exp \sum_{1 \le \Delta^j \le |\xi \wedge \eta|} b_{\Delta^j} \mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta) \\ &\quad = \int\int R_\kappa(\xi,\eta) \exp \sum_{1 \le \Delta^j \le |\xi \wedge \eta|} (-\log p_{\Delta^j}) \mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta) \end{aligned}$$

Using (4.6), we see that $\mathbb{E} I_\kappa^{Q_b^\mu}$ is bounded up to a multiplicative constant by

$$\begin{aligned} &\int\int R_\kappa(\xi,\eta) \exp \left(\beta \sum_{1 \le \Delta^j \le |\xi \wedge \eta|} \Delta^j \right) \mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta) \\ &\quad \le \int\int R_\kappa(\xi,\eta) \exp \left(\beta \frac{\Delta}{\Delta - 1} |\xi \wedge \eta| \right) \mathrm{d}\mu(\xi)\mathrm{d}\mu(\eta) \\ &\quad = I^\mu_{\kappa + \beta \frac{\Delta}{\Delta-1}}. \end{aligned}$$

The last energy integral of μ is bounded by I_D^μ then is finite (see (4.5)) if κ verifies

$$\kappa + \beta \frac{\Delta}{\Delta - 1} \le D.$$

For $\kappa = 0$ and for large Δ, the finiteness of $I^\mu_{\beta + \Delta/(\Delta-1)}$ implies that the L^2-condition (2.2) is fulfilled. Then with a positive probability $Q_b^\mu \ne 0$. Notice that since $P_n(\xi)$ are strictly positive, the event $Q_b^\mu \ne 0$ is a tail event. We conclude that a.s. $Q_b^\mu \ne 0$. It follows (Proposition 2.2 (2)) that a.s.

$$\dim Q_b^\mu \ge D - \beta \frac{\Delta}{\Delta - 1}. \tag{4.9}$$

On the other hand, by the formula in Proposition 2.4, we have

$$\mathbb{E}_{Q_b^\mu} 1_{\{\xi|_n = 1\}} = \frac{\mathbb{E} 1_{\{\xi|_n=1\}} \exp(b_n 1_{\{\xi|_n=1\}})}{1 - q_n + q_n e^{b_n}} = \frac{q_n e^{b_n}}{1 - q_n + q_n e^{b_n}}.$$

Thus

$$\sum_{n=1}^\infty Q_b^\mu \left(\epsilon_{\xi|_n} = 1 \right) = \sum_{n=1}^\infty \frac{e^{b_n} q_n}{1 - q_n + e^{b_n} q_n} \ge \sum_{j=1}^\infty \frac{1}{2 - q_{\Delta^j}} = \infty.$$

Therefore, by the Borel-Cantelli lemma, a.s. $\mathcal{G}(\mathbf{q})$ is of full Q_b^μ–measure. So, using (4.9) and Proposition 2.2(1), we have a.s.

$$\dim_H \mathcal{G}(\mathbf{q}) \geq \dim Q_b^\mu \geq D - \beta \frac{\Delta}{\Delta - 1}.$$

The RHS approaches to $\dim \partial T - \alpha$ if we let $D \to \dim_H \partial T$, $\Delta \to \infty$ and then $\beta \to \alpha$.

Remark 4.3. *The inequalities in (4.4) become equalities if the tree is regular in the sense that* $\dim_H \partial T = \overline{\dim}_B \partial T$. *It is the case for primitive Markov trees. Thus Theorem B in the introduction is proved.*

5 Oscillations

From now on, we concentrate on growth–periodic trees T such that $\dim_H \partial T > 0$ (or equivalently $\mathrm{br} T > 1$) and $\dim_H \partial T = \overline{\dim}_B \partial T$. For sake of simplicity, these assumptions will be not repeated in the statements of theorems.

We can consider random walks a little more general than those mentioned in the introduction. Namely, we only require that the n–th level variables $\{X_\sigma\}_{\sigma \in T_n}$ are identically distributed (of course, all variables are independent). We use X_n to denote the common law of the n–th level variables defining our tree-indexed walk. Recall that $\Phi = \Phi(n)$ is a sequence of positive numbers and

$$\mathrm{OSC}_\Phi(\xi) = \limsup_{\sigma \to \xi} \frac{X_\sigma}{\Phi(|\sigma|)}, \qquad \xi \in \partial T.$$

5.1 Uniform Lower and Upper Bounds

Theorem 5.1. *We have* $\sup_{\xi \in \partial T} \mathrm{OSC}_\Phi(\xi) = \gamma_{\max}$ *a.s. where*

$$\gamma_{\max} = \inf\left\{\gamma \in \mathbb{R} : \sum_{n=1}^{\infty} |T_n| P\left(X_n > \gamma \Phi(n)\right) < \infty\right\}. \tag{5.1}$$

We have $\inf_{\xi \in \partial T} \mathrm{OSC}_\Phi(\xi) = \gamma_{\min}$ *a.s. where*

$$\gamma_{\min} = \sup\left\{\gamma \in \mathbb{R} : \mathrm{Cap}_{K_\gamma}(\partial T) > 0\right\}, \tag{5.2}$$

where Cap_{K_γ} *refers to the capacity relative to the kernel*

$$K_\gamma(t, s) = \prod_{n=1}^{|t \wedge s|} \frac{1}{P(X_n \leq \gamma \Phi(n))}.$$

Proof. If $\gamma > \gamma_{\max}$, the condition $\sum_n q_n|T_n| < \infty$ is satisfied with $q_n = P(X_n > \gamma\Phi(n))$. By Corollary 4.1, $\mathcal{G}(\mathbf{q}) = \emptyset$ a.s. It follows that a.s. for all $\xi \in \partial T$ we have $X_{\xi|_n} \le \gamma\Phi(n)$ for all but a finite number of n. Then $\sup_\xi \mathrm{OSC}_\Phi(\xi) \le \gamma$ a.s. So that $\gamma_{\max}$ is a uniform upper bound. If $\gamma < \gamma_{\max}$, we have $\sum_n q_n|T_n| = \infty$. Then, by Corollary 4.1, $\mathcal{G}(\mathbf{q}) \ne \emptyset$ a.s. It follows that a.s. there is a point $\xi \in \partial T$ such that $X_{\xi|_n} > \gamma_n\Phi(n)$ for an infinite number of n. For this ξ, we have $\mathrm{OSC}_\Phi(\xi) \ge \gamma$. So, $\sup_\xi \mathrm{OSC}_\Phi(\xi) \ge \gamma_{\max}$.

Notice that $K_\gamma(t,s)$ is decreasing as function of γ so that $\mathrm{Cap}_{K_\gamma}\partial T$ is decreasing as function of γ. Suppose $\gamma < \gamma_{\min}$. Since $\mathrm{Cap}_{K_\gamma}\partial T = 0$, by Theorem ?? we have $\mathcal{B}(\mathbf{q}) = \emptyset$ a.s. with $q_n = (X_n > \gamma\Phi(n))$. That is to say, a.s. for all $\xi \in \partial T$ we have $X_{\xi|_n > \gamma\Phi(n)}$ for an infinite number of n. It follows that $\inf_\xi \mathrm{OSC}_\Phi(\xi) \ge \gamma$. So, $\gamma_{\min}$ is a uniform lower bound. Suppose $\gamma > \gamma_{\min}$. We have $\mathrm{Cap}_{K_\gamma}\partial T > 0$, then $\mathcal{B}(\mathbf{q}) \ne$ a.s. This implies $\inf_\xi \mathrm{OSC}_\Phi(\xi) \le \gamma$.

Remark 5.1. The condition $\dim_H \partial T = \overline{\dim}_B \partial T$ is not required by the last theorem. On primitive Markov trees and spherically symmetrical trees, we have a formula for $\gamma_{\min}$ simpler than (5.2):

$$\gamma_{\min} = \sup\left\{\gamma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{1}{|T_n|} \prod_{k=1}^{n} \frac{1}{P(X_k \le \gamma\Phi(k))} < \infty\right\}. \tag{5.3}$$

We may compare it to (5.1).

Corollary 5.1. *Suppose (X_n) is an i.i.d. sequence. Let $F(x) = P(X \le x)$ be their common distribution and F^* be the inverse of F, defined as*

$$F^*(y) = \sup\{a \in \mathbb{R} : F(a) \le y\}.$$

Then we have a.s.

$$\inf_\xi \mathrm{OSC}_1(\xi) = F^*(1/\mathrm{br}T), \qquad \sup_\xi \mathrm{OSC}_1(\xi) = \mathrm{ess\ sup}X.$$

Proof. Notice that

$$K_\gamma(t,s) = \frac{1}{P(X \le \gamma)^{|t\wedge s|}} = R_{\log(1/P(X\le\gamma))}(t,s).$$

So, by the Frostman theorem, we have

$$\begin{aligned}\gamma_{\min} &= \inf\{\gamma \in \mathbb{R} : \log(1/P(X \le \gamma)) < \dim_H \partial T\} \\ &= \inf\{\gamma \in \mathbb{R} : P(X \le \gamma)) > 1/\mathrm{br}T\} \\ &= \sup\{\gamma \in \mathbb{R} : P(X \le \gamma)) \le 1/\mathrm{br}T\} = F^*(1/\mathrm{br}).\end{aligned}$$

The essential bounded is of course a upper bound of $\lim_n X_{\xi|_n}$. If $\gamma <$ ess sup$X$, we will have $P(X > \gamma) > 0$ so that $\sum_{n=1} |T_n| P(X_n > \gamma) = \infty$. This implies that a.s. there is a point $\xi \in \partial T$ such that $\limsup_n X_{\xi|_n} \geq \gamma$.

If a suitable normalization sequence $\Phi(n)$ is found so that $-\infty < \gamma_{\min} < \infty$, we would like to ask if, for a given number $\gamma > \gamma_{\min}$, there are points ξ such that $\gamma_{\min} \leq \mathrm{OSC}_\Phi(\xi) \leq \gamma$. Such points, if exist, will be called slow points (or slow paths). For the case where all the variables X_σ are identically distributed, we can choose the constant sequence $\Phi(n) \equiv 1$ as normalization sequence, as is shown by the last corollary. Usually, the good choice of the normalization sequence depends on the distributions of X_n through the following quantities

$$\beta(\Phi, \gamma) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{n} \frac{1}{P(X_k \leq \gamma \Phi(k))}. \tag{5.4}$$

Roughly speaking, the sequence Φ may be chosen as a normalization if $\beta(\Phi, \gamma) > \dim \partial T$ for some γ and $\beta(\Phi, \gamma) < \dim \partial T$ for some other γ.

There is also a similar question of finding a normalization sequence $\Phi(n)$ in order to get $-\infty < \gamma_{\max} < \infty$. Suppose such sequence is found. For a given number $\gamma < \gamma_{\max}$, points ξ such that $\gamma \leq \mathrm{OSC}_\Phi(\xi) \leq \gamma_{\max}$, if exist, will be called quick points (or quick paths). Concerning this maximum normalization, even for the case where all the variables X_σ are identically distributed, the determination of normalization sequence depends on the distribution of the variable X. Consider the following quantities

$$\alpha(\Phi, \gamma) = \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{P(X_n > \gamma \Phi(n))}. \tag{5.5}$$

Roughly speaking, the sequence Φ may chosen as a maximum normalization if $\alpha(\Phi, \gamma) > \dim \partial T$ for some γ and $\alpha(\Phi, \gamma) < \dim \partial T$ for some other γ. We will see that we can choose $\Phi(n) = \sqrt{n}$ as a maximum normalization sequence for the random walk on dyadic tree determined by a gaussian variable X.

Suppose that we have found a minimum normalization $\Phi_{\min}$ and a maximum normalization $\Phi_{\max}$. Usually we have $\Phi_{\min} \prec \Phi_{\max}$ (we mean $\dfrac{\Phi_{\min}(n)}{\Phi_{\max}(n)}$ tends to 0 as $n \to \infty$). In most cases, there are sequences Φ with the property $\Phi_{\min} \prec \Phi \prec \Phi_{\max}$ such that $\mathrm{OSC}_\Phi(\xi)$ is finite and non zero for some ξ. Such points will be called intermediate points (or intermediate paths). Of cause, for intermediate points ξ we have $\mathrm{OSC}_{\Phi_{\min}}(\xi) = \pm\infty$ and $\mathrm{OSC}_{\Phi_{\max}}(\xi) = 0$.

Before going further in the study of different kinds of point that we have classified, let us look at two examples and examine their possible normalizations.

Suppose that the common law X is uniformly distributed in $[0,1]$. In this case, it is natural to take $\Phi_{\min}(n) = \Phi_{\max}(n) \equiv 1$. Then we have $\gamma_{\min} = \frac{1}{\mathrm{br}T}$ and $\gamma_{\max} = 1$. Notice that the oscillations of random walk in this case are all of the same order along any path. But paths can be classified by $\mathrm{OSC}_\Phi(\xi) = \alpha$ according to $\frac{1}{\mathrm{br}T} < \alpha < 1$, as we shall see.

Suppose now that the common law X is an exponential law with mean value 1. In this case, we will take $\Phi_{\min}(n) = 1$ and $\Phi_{\max}(n) = n$. Then we have $\gamma_{\min} = \log \frac{\mathrm{br}T-1}{\mathrm{br}T}$ and $\gamma_{\max} = \log \underline{\mathrm{gr}}T$. In this case, the oscillations are of all possible orders $\Phi(n) = n^s$ with $0 < s < 1$.

We will examine in more details the case of gaussian random walks (§ 6).

5.2 Slow Oscillations

Recall that $\gamma_{\min}$ is the best lower bound. Now we study the set of slow paths $\xi \in \partial T$ such that $\gamma_{\min} \le \mathrm{OSC}_\Phi(\xi) \le \gamma$ for some $\gamma > \gamma_{\min}$.

Theorem 5.2. *Suppose $\gamma_{\min} < \infty$. Let $\gamma > \gamma_{\min}$. Then we have a.s.*

$$\dim_H \{\xi \in \partial T : \mathrm{OSC}_\Phi(\xi) \le \gamma\} = \sup\left\{\alpha \ge 0 : \mathrm{Cap}_{R_\alpha K_\gamma} \partial T > 0\right\}.$$

Proof. Let α_0 be the number at the right hand side. Let S_γ be the set of all points $\xi \in \partial T$ such that $\mathrm{OSC}_\Phi(\xi) \le \gamma$. For any $\alpha > \alpha_0$, we have $\mathrm{Cap}_{R_\alpha K_\gamma} \partial T = 0$ or $\mathrm{Cap}_{R_\beta R_{\alpha-\beta} K_\gamma} \partial T = 0$ for any $\beta \in (\alpha_0, \alpha)$. So, by Theorem 3.2, $\mathrm{Cap}_\beta \mathcal{B}(\mathbf{q}') = 0$ a.s. with

$$q'_n = e^{-(\alpha-\beta)} P(X_n > \gamma\Phi(n)) = P(X_n > \gamma_n \Phi(n)),$$

where $\gamma_n > \gamma + \epsilon$ for some $\epsilon > 0$. So, $\dim_H \mathcal{B}(\mathbf{q}') \le \beta$ a.s. We claim that $S_\gamma \subset \mathcal{B}(\mathbf{q}')$, because $\xi \in S_\gamma$ implies $X_{\xi|_n} \le (\gamma+\epsilon)\Phi(n)$ $(\forall \epsilon > 0, \forall n \ge n(\epsilon))$ while $\xi \in \mathcal{B}(\mathbf{q}')$ means $X_{\xi|_n} \le \gamma_n \Phi(n)$ for all but a finite number of n. Thus we have proved $\dim S_\gamma \le \alpha_0$.

For any $\alpha < \alpha_0$, we have $\mathrm{Cap}_{R_\alpha K_\gamma} \partial T > 0$. So, by Theorem 3.2, $\mathrm{Cap}_\alpha \mathcal{B}(\mathbf{q}) > 0$ a.s. with $q_n = P(X_n > \gamma\Phi(n))$. So, $\dim_H \mathcal{B}(\mathbf{q}) \ge \alpha$ a.s. We claim that $S_\gamma \supset \mathcal{B}(\mathbf{q})$.

Corollary 5.2. *Suppose T be a primitive Markov tree or a spherically symmetrical* *tree. For $\gamma > \gamma_{\min}$, we have a.s.*

$$\dim_H \{\xi \in \partial T : \mathrm{OSC}_\Phi(\xi) \le \gamma\} = \liminf_{n\to\infty} \frac{1}{n} \left[\log |T_n| + \sum_{k=1}^{n} \log P(X_n \le \gamma\Phi(n))\right].$$

Proof. The proof is the same as that of Theorem 3.3(c).

5.3 Quick Oscillations

For a fixed sequence Φ, we introduce the following function of $\gamma \in \mathbb{R}$

$$\alpha(\Phi,\gamma) = \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{P(X_n > \gamma\Phi(n))}.$$

Theorem 5.3. *Suppose that $\alpha(\Phi,\gamma)$ is left-continuous at $\gamma > 0$ and that $\alpha(\Phi,\gamma) < \dim_H \partial T$. Then a.s.*

$$\dim_H \partial T - \alpha(\Phi,\gamma) \le \dim_H E_*(\Phi,\gamma) \le \overline{\dim}_B \partial T - \alpha(\Phi,\gamma). \tag{5.6}$$

If, furthermore, there is a sequence $\epsilon_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} \frac{P(X_n \ge (1+\epsilon_n)\gamma\Phi(n))}{P(X_n \ge \gamma\Phi(n))} < \infty, \tag{5.7}$$

then a.s.

$$\dim_H \partial T - \alpha \le \dim_H E(\Phi, 1) \le \overline{\dim}_B \partial T - \alpha. \tag{5.8}$$

Proof. Without loss of generality, we assume that $\gamma = 1$. For the first assertion, it suffices to apply Theorem 4.2 and to notice the following relation

$$\mathcal{G}(\mathbf{q}^{(0)}) \subset E_*(\Phi, 1) \subset \mathcal{G}(\mathbf{q}^{(\epsilon)}), \qquad \forall \epsilon > 0,$$

where $q_n^{(\epsilon)} = P(X_{\xi|_n} > (1-\epsilon)\Phi(n))$ (even for $\epsilon = 0$).

In order to prove the second assertion, we go back to the proof of Theorem 4.2 where a random measure Q_b^μ is constructed by a family of Bernoulli variables ϵ_v. We now define $\epsilon_v = 1$ or 0 according $X_v > \Phi(|v|)$ or $X_v \le \Phi(|v|)$. Thus we have only to show that a.s. for Q_b^μ-almost every ξ we have

$$\limsup_{n\to\infty} \frac{X_{\xi|_n}}{\Phi(n)} = 1.$$

We first show that a.s. for Q_b^μ-almost every ξ, the limit is bounded by one from above. This is true because, according to the choice of b_n (see (4.8)), we have

$$\begin{aligned}
\sum_{n=1}^{\infty} & Q_b^\mu(X_{\xi|_n} \geq (1+\epsilon_n)\Phi(n)) \\
= & \sum_{n=1}^{\infty} \frac{\mathbb{E}1_{\{X_{\xi|_n} > (1+\epsilon_n)\Phi(n)\}} e^{b_n 1_{\{X_{\xi|_n} > \Phi(n)\}}}}{\mathbb{E}1_{\{X_{\xi|_n} > \Phi(n)\}}} \\
= & \sum_{n=1}^{\infty} \frac{e^{b_n} P(X_n > (1+\epsilon_n)\Phi(n))}{1 - q_n + e^{b_n} q_n} \\
= & \sum_{n \neq \Delta^j} P(X_n > (1+\epsilon_n)\Phi(n)) + \sum_{n=\Delta^j} \frac{1}{2-q_n} \frac{P(X_n \geq (1+\epsilon_n)\Phi(n))}{P(X_n \geq \Phi(n))} \\
\leq C & \sum_{n=1}^{\infty} \frac{P(X_n \geq (1+\epsilon_n)\Phi(n))}{P(X_n \geq \Phi(n))} < \infty,
\end{aligned}$$

where $q_n = P(X_n > \Phi(n))$. We apply once more the Borel-Cantelli lemma. The extra condition (5.7) was used. It is easier to show that the limit is bounded by one from below without using the extra condition. In fact, we have

$$\sum_{n=1}^{\infty} Q_b^\mu(X_{\xi|_n} > \Phi(n)) = \sum_{n=1}^{\infty} \frac{e^{b_n}}{1 - q_n + e^{b_n} q_n} \geq \sum_{j=1}^{\infty} \frac{1}{2 - q_{\Delta^j}} = \infty.$$

5.4 Intermediate Oscillations

Roughly speaking, if the tail probability $P(X_n \geq \Phi(n))$ decays exponentially, the paths such that $0 < \mathrm{OSC}_\Phi(\xi) < \infty$ are rapid; if the tail probability is bounded from below, the paths such that $0 < \mathrm{OSC}_\Phi(\xi) < \infty$ are slow. We shall see that sub-exponential decay of the tail probability corresponds to intermediate oscillations. The following result is a particular case of the last theorem. But we state it as a theorem because it will provide us important intermediate points.

Theorem 5.4. *Suppose that the function $\alpha(\Phi,\gamma)$ is left-continuous at $\gamma > 0$ and $\alpha(\Phi,\gamma) = 0$. Suppose furthermore*

$$\sum_{n=1}^{\infty} \frac{P(X_n \geq (1+\epsilon_n)\gamma\Phi(n))}{P(X_n \geq \gamma\Phi(n))} < \infty, \quad \text{with some } \epsilon_n \downarrow 0.$$

Then a.s. $\dim_H E(\Phi,\gamma) = \dim_H \partial T$.

For intermediate points to exist, it suffices to find an intermediate sequence Φ (i.e. $\Phi_{\min} \prec \Phi \prec \Phi_{\max}$) such that $\alpha(\Phi,\gamma) = 0$ for some γ's.

Let us finish our discussion by mentioning ordinary paths. A natural probability measure on ∂T is defined as follows: for any descendent τ of σ

$$\lambda(B_\tau) = \frac{\lambda(B_\sigma)}{N_\sigma},$$

where N_σ is the degree of σ. We may consider λ as the Lebesgue measure of the tree.

Theorem 5.5. *Suppose that for any* $0 < \epsilon < 1$ *we have*

$$\sum_{n=1}^{\infty} P(X \geq (1+\epsilon)\Phi(n)) < \infty, \qquad \sum_{n=1}^{\infty} P(X \geq (1-\epsilon)\Phi(n)) = \infty.$$

Then a.s. $\mathrm{OSC}_\Phi(\xi) = 1$ *for* λ*-a.e.* $\xi \in \partial T$.

Proof. Fix $\xi \in \partial T$, the conditions implies that a.s. $1-\epsilon \leq \mathrm{OSC}_\Phi(\xi) \leq 1+\epsilon$ for all $\epsilon > 0$. So, a.s. $\mathrm{OSC}_\Phi(\xi) = 1$. Then the Fubini theorem implies the desired result.

The points in the conclusion of the last theorem are called ordinary points. Actually, the result in the last Theorem remains true if λ is replaced by any measure on ∂T. So, if the condition in the theorem is satisfied, in the support of any probability measure, there are ordinary points and moreover almost all points are ordinary points with respect to the given measure. The interest of the present paper is just to discover unusual behaviors along non ordinary paths.

6 Gaussian Walks and Brownian Motion

We examine a special gaussian walks by checking the conditions in the preceding theorems. Then we translate the results into local properties of Brownian motion.

6.1 Gaussian Walks

By gaussian walks we mean the tree-indexed walks determined by a gaussian variable $Z \sim N(0,1)$. For the purpose of the study on Brownian motion, we consider the absolute value of the gaussian variable $X = |Z|$. The associated walk is still said to be gaussian.

The main property needed of $X = |Z|$ is the following. For $u \geq 1$, we have

$$P(X \geq u) = 2P(Z \geq u) = \sqrt{\frac{2}{\pi}} \int_0^u e^{-\frac{x^2}{2}} \mathrm{d}x \approx \frac{e^{-\frac{u^2}{2}}}{u}. \tag{6.1}$$

In particular, take $u = an^\gamma$ with $a > 0$ and $\gamma > 0$. We get

$$q_n := P(X \geq an^\gamma) \approx n^{-\gamma} e^{-\frac{a^2}{2}n^{2\gamma}}. \tag{6.2}$$

Let A be defined by

$$\sqrt{\frac{2}{\pi}} \int_0^A e^{-\frac{x^2}{2}} \mathrm{d}x = \frac{1}{\mathrm{brT}}.$$

Theorem 6.1. *Suppose T is a growth-periodic tree such that $\dim_H \partial T = \overline{\dim}_B \partial T > 0$ and $X = |Z|$ where $Z \sim N(0,1)$. For the random walk on T defined by X, we have*

(1) *a.s.* $\mathrm{OSC}_1(\xi) \geq A$ *for all* $\xi \in \partial T$.

(2) *a.s.* $\mathrm{OSC}_{\sqrt{n}}(\xi) \leq \sqrt{2 \log \mathrm{br}T}$ *for all* $\xi \in \partial T$.

(3) *If $B > A$, then a.s.*

$$\dim_H E^*(1, B) = \dim_H \partial T + \log \left(\sqrt{\frac{2}{\pi}} \int_0^B e^{-\frac{x^2}{2}} \mathrm{d}x \right).$$

(4) *If $a < \sqrt{2 \log \mathrm{br}T}$, then a.s.*

$$\dim_H E(\sqrt{n}, a) = \dim_H E_*(\sqrt{n}, a) = \dim_H \partial T - \frac{a^2}{2}.$$

(5) *For any $0 < \gamma < \frac{1}{2}$ and any $c > 0$, a.s.*

$$\dim_H E(n^\gamma, c) = \dim_H \partial T.$$

(6) *a.s.* $\mathrm{OSC}_{\sqrt{\log n}}(\xi) = \sqrt{2}$ *λ-a.e. where λ is the Lebesgue measure.*

Proof.

(1) Since $P(X \leq a)$ is continuous and strictly increasing and $\dfrac{1}{\mathrm{br}T} < 1$, the number $A = F^*(1/\mathrm{br}T)$ is the unique solution of $P(X \leq A) = \dfrac{1}{\mathrm{br}T}$. By Corollary 5.1, $A = \gamma_{\min}$ with $\Phi_{\min}(n) \equiv 1$.

(2) Apply the formula (5.1) to $\Phi(n) = \sqrt{n}$. By the estimate (6.2), we are led to consider the convergence of the series

$$\sum_{n=}^{\infty} |T_n| P(X_n > \gamma\sqrt{n}) \approx \sum_{n=}^{\infty} \frac{|T_n|}{\sqrt{n}} z^n, \qquad z = e^{-\frac{\gamma^2}{2}}.$$

So, if $\gamma_{\max}$ denotes de upper bound associated to $\Phi(n) = \sqrt{n}$, then $e^{-\frac{\gamma_{\max}^2}{2}}$ must be the convergence radius of the last Taylor series. That is to say

$$e^{\frac{\gamma_{\max}^2}{2}} = \limsup_{n \to \infty} |T_n|^{1/n},$$

(the factor $1/\sqrt{n}$ being not important). It follows that $\gamma_{\max} = \sqrt{2\overline{\dim}_B \partial T}$. But $\overline{\dim}_B \partial T = \log \mathrm{br} T$ by the hypothesis on the tree.

(3) It is a direct consequence of Corollary 5.2.

(4) Take $\Phi(n) = \sqrt{n}$. By the estimate (6.2) we have $\alpha(\Phi, \gamma) = \frac{\gamma^2}{2}$. This function is continuous even differentiable. On the other hand, if $\epsilon_n = \frac{1}{\sqrt{n}}$, then

$$\sum_{n=1}^{\infty} \frac{P(X \geq (1+\epsilon_n)a\sqrt{n})}{P(X \geq a\sqrt{n})} \approx \sum_{n=1}^{\infty} e^{-\sqrt{n}} < \infty.$$

Now we can apply Theorem 5.3.

(5) Take $\Phi(n) = n^\gamma$ with $0 < \gamma < 1/2$. We have $\alpha(\Phi, \gamma) \equiv 0$ for all γ. If $\epsilon_n = \frac{1}{n^\gamma}$, we have

$$\sum_{n=1}^{\infty} \frac{P(X \geq (1+\epsilon_n)cn^\gamma)}{P(X \geq cn^\gamma)} \approx \sum_{n=1}^{\infty} e^{-n^\gamma} < \infty.$$

Now it suffices to apply Theorem 5.4.

(6) Take $\Phi(n) = \sqrt{2\log n}$. We have

$$P(X \geq (1+\epsilon)\sqrt{\log n}) \approx \frac{1}{n^{(1+\epsilon)^2}\sqrt{\log n}},$$

$$P(X \geq (1-\epsilon)\sqrt{\log n}) \approx \frac{1}{n^{(1-\epsilon)^2}\sqrt{\log n}}.$$

Apply Theorem 5.5 to get the desired result.

6.2 Other Examples

Let us just state two other examples: the exponential variable and the uniform variable. The first one is unbounded but the second one is bounded.

Suppose X obeys the exponential law of parameter $\lambda > 0$. We have

$$P(X \geq u) = \lambda \int_u^\infty e^{-\lambda x} \mathrm{d}x = e^{-\lambda u}.$$

Suppose furthermore $\dim_H \partial T = \overline{\dim}_B \partial T > 0$. Then have

(1) a.s. $\mathrm{OSC}_1(\xi) \geq \lambda^{-1} \log \frac{\mathrm{brT}}{\mathrm{brT}-1}$ for all $\xi \in \partial T$.

(2) for $B > \lambda^{-1} \log \frac{\mathrm{brT}}{\mathrm{brT}-1}$, a.s.

$$\dim_H E^*(1, B) = \dim_H \partial T - \log \frac{1}{1 - e^{-\lambda B}}.$$

(3) a.s. $\mathrm{OSC}_n(\xi) \le \frac{\dim_H \partial T}{\lambda}$ for all $\xi \in \partial T$.

(4) for $a < \frac{\dim_H \partial T}{\lambda}$, a.s.

$$\dim_H E(n,a) = \dim_H \partial T - a\lambda.$$

(5) For any $0 < \gamma < 1$ and any $c > 0$, a.s.

$$\dim_H E(n^\gamma, c) = \dim_H \partial T.$$

Suppose X is uniform distributed in $[0,\ell]$ with $\ell > 0$. For $0 \le u \le \ell$, we have

$$P(X \ge u) = \frac{\ell - u}{\ell}.$$

Suppose furthermore $\dim_H \partial T = \overline{\dim}_B \partial T > 0$. Then have

(1) a.s. $\frac{1}{\mathrm{br}T} \le \mathrm{OSC}_1(\xi) \le 1$ for all $x \in \partial T$.

(2) for $\frac{1}{\mathrm{br}T} < B \le 1$, a.s.

$$\dim_H E^*(1,B) = \dim_H \partial T - \log\frac{1}{B}.$$

6.3 Symmetric Variations of Brownian Motion

For $n \ge 1$ and $t \in [0,1]$, let $I_n(t) = [t_n^{(g)}, t_n^{(d)}]$ be the n-level dyadic interval containing t. Let $t_n^{(c)} = \frac{1}{2}(t_n^{(g)} + t_n^{(d)})$ be the middle point of the interval $I_n(t)$. For a continuous function $f \in C([0,1])$, we define its n-th (symmetric) variation at t by

$$\Delta_n f(t) = 2f(t_n^{(c)}) - f(t_n^{(g)}) - f(t_n^{(d)}).$$

The (symmetric) variation of f at a point t may be described by a suitable decreasing positive sequence $\varphi(n)$ such that

$$0 < \limsup_{n\to\infty} \frac{|\Delta_n f(t)|}{\varphi(n)} < \infty.$$

It is possible that different functions φ are needed for different points t. We shall see that it is the case for the trajectories of Brownian motion.

Let $B(t)$ be a linear Brownian motion. By using the Fourier-Franklin development, we can get the following expression for the n-th variation of $B(t)$:

$$\Delta_n B(t) = 2^{-n/2} Z_{t_1,\cdots,t_n}, \qquad t = \sum_{n=1}^{\infty} \frac{t_n}{2^n},$$

where $Z_{t_1,\cdots,t_n}$ ($n \geq 1; 1 \leq j \leq n; t_j = 0$ or 1) are i.i.d. standard normal variables (see [3]). Thus they define a tree-indexed walk, where the tree is the binary tree $\mathbb{D} = \{0,1\}^{\mathbb{N}}$.

On the binary tree, it is better to use the metric $\delta(t,s) = 2^{-|t\wedge s|}$ instead of $d(t,s) = e^{-|t\wedge s|}$ because the tree is usually identified with the interval $[0,1]$ on which we have the Euclidean metric. The Hausdorff dimension relative to the Euclidean metric coincides with the Hausdorff dimension relative to the metric δ. To distinguish, we use dim to denote the Hausdorff dimension relative to this new metric. Since $d(t,s) = \delta(t,s)^{\log 2}$, we have the relation $\dim E = \frac{1}{\log 2}\dim_H E$ for $E \subset \mathbb{D} \cong [0,1]$. The last theorem applied to the binary tree can be translated into the variations of Brownian motion as follows. That is what we stated as Theorem A in the introduction.

Theorem 6.2. *Let $B(t)$ be a linear Brownian motion. Then*

(1) *a.s. for all $t \in [0,1]$*

$$\limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2^{-n}}} \geq A = 0,608,$$

where A is defined in the last theorem with $\mathrm{br}T = 2$.

(2) *a.s. for all $t \in [0,1]$*

$$\limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2\cdot 2^{-n}\log 2^n}} \leq 1.$$

(3) *If $B > A$, a.s.*

$$\dim\left\{t\in[0,1] : \limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2^{-n}}} \leq B\right\} = -\log_2 \int_0^B e^{-x^2/2}\frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

(4) *If $0 < \alpha < 1$, a.s.*

$$\dim\left\{t\in[0,1] : \limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2\cdot 2^{-n}\log 2^n}} = \alpha\right\} = 1-\alpha^2.$$

(5) *If $0 < \beta < 1$ and $c > 0$, a.s.*

$$\dim\left\{t\in[0,1] : \limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2\cdot 2^{-n}\log^{\beta} 2^n}} = c\right\} = 1.$$

(6) *a.s. for a.e. $t \in [0,1]$*

$$\limsup_{n\to\infty} \frac{|\Delta_n B(t)|}{\sqrt{2\cdot 2^{-n}\log\log 2^n}} = 1.$$

It is clear that (1) implies that Brownian motion is nowhere differentiable. We call the points described in (3) symmetric slow points and the points described in (4) symmetric quick points. These points have their counterparts of the slow points discovered by J.P. Kahane [K1,K2]:

$$0 < \limsup_{h\to 0} \frac{|B(t+h)-B(t)|}{\sqrt{|h|}} < \infty, \tag{K}$$

and the quick points discovered by S. Orey and J. Taylor [OT]:

$$0 < \limsup_{h\to 0} \frac{|B(t+h)-B(t)|}{\sqrt{|h|\log(1/|h|)}} < \infty. \tag{OT}$$

The left inequality in (K) was earlier proved true for any t by A. Dvoretzky [D2] and the right inequality of (OT) was proved true for any t by P. Lévy [L]. They correspond to the lower and upper uniform bounds stated in (1) and (2). It is worthy to point out that symmetric rapid points are rapid points in the sense of Orey-Taylor. Therefore the result in (4) implies a well known result due to Orey and Taylor that the set of quick points in sense of Orey-Taylor is of Hausdorff dimension 1. However, the result in (2) on the symmetric slow points has no immediate consequence on the slow point in the sense of Kahane.

We call the point described in (5) symmetric β-points. There are many symmetric β-points ($0 < \beta < 1$) for the Brownian motion. It is proved in [F4] that there exist (asymmetric) β-points for the Brownian motion. By β-point t we mean

$$0 < \limsup_{h\to 0} \frac{|B(t+h)-B(t)|}{\sqrt{|h|\log^{\beta}(1/|h|)}} < \infty.$$

A earlier work of Kôno [Ko] also implies the existence of β-points.

Kahane's slow points may be called two-sided slow points. For $c > 0$, let

$$S_c = \{t : \limsup_{h\to 0} \frac{|B(t+h)-B(t)|}{\sqrt{|h|}} \le c\}.$$

B. Davis had considered one side slow points by introducing

$$S_c^+ = \{t : \limsup_{h\to 0+} \frac{|B(t+h)-B(t)|}{\sqrt{h}} \le c\}.$$

We may classify symmetrical slow points by introducing

$$S_c^{\mathrm{sym}} = \{t : \limsup_{n\to\infty} \frac{\Delta_n B(t)}{\sqrt{2^{-n}}} \le c\}.$$

B. Davis and E. Perkins had proved that (see [DP] and the reference therein)

$$S_c = \emptyset, \quad \forall c < \gamma_0, \quad S_c \neq \emptyset, \quad \forall c > \gamma_0, \quad \gamma_0 \approx 1.3069;$$
$$S_c^+ = \emptyset, \quad \forall c < 1, \quad S_c^+ \neq \emptyset, \qquad\qquad \forall c > 1.$$

It follows that there one-sided slow points which are not two-sided slow points. Theorem 6.2(3) means

$$S_c^{\mathrm{sym}} = \emptyset, \quad \forall c < A, \qquad S_c^{\mathrm{sym}} \neq \emptyset, \quad \forall c > A, \qquad A \approx 0.608.$$

So, there are symmetric slow points which are not one-sided slow points.

We finish this section by trying to find the gauge function of the set of symmetric β-points. For $0 < \beta < 1$, $c > 0$ and $-1 < \epsilon < 1$, denote

$$\varphi(s) = \varphi_{\beta,c,\epsilon}(s) = s \cdot \exp\left((1+\epsilon)\frac{c^2}{2}|\log s|^\beta\right).$$

Notice that for any $a > 0$, the function e^{ax^β} is larger than any power x^p (and smaller than any exponential $e^{\eta x}$) when x is large. It follows that we always have

$$\int_0^1 \frac{\mathrm{d}s}{\varphi(s)} < \infty$$

for the above choice of β, c and ϵ. Denote by $\mathcal{H}_\varphi$ the Hausdorff measure defined by the gauge function φ.

Theorem 6.3. *Let $E_{\beta,c}$ be the set discussed in Theorem 6.2(5) where* $0 < \beta < 1$ *and* $c > 0$. *We have a.s.*

$$\mathcal{H}_{\varphi_{\beta,c,\epsilon}}(E_{\beta,c}) = \infty, \quad \forall \epsilon > 0; \qquad \mathcal{H}_{\varphi_{\beta,c,\epsilon}}(E_{\beta,c}) = 0, \quad \forall \epsilon < 0.$$

Proof. We follow the proofs of Theorem 4.2 where we have estimated the expectation of the κ-energy integral of Q_b^μ. Now we work with the Lebesgue measure μ and write simply $Q_b = Q_b^\mu$.

First let $0 < \epsilon < 1$. Consider $I_{K_\epsilon}^{Q_b}$, the energy integral of Q_b with respect to the kernel

$$K_\epsilon(\xi, \eta) = \frac{1}{\varphi(\delta(\xi,\eta))}.$$

By similar calculation, we can control $\mathrm{E}I_{K_\epsilon}^{Q_b}$, up to a multiplicative constant, by

$$\int\int K_{\epsilon/2}(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta \leq \int_0^1 \frac{\mathrm{d}s}{\varphi_{\beta,c,\epsilon/2}(s)} < \infty.$$

It follows that a.s. $\text{Cap}_{K_\epsilon}(E_{\beta,c}) > 0$ then $\mathcal{H}_{\varphi_{\beta,c,2\epsilon}}(E_{\beta,c}) = \infty$. Since $0 < \epsilon < 1$ is arbitrary, we get the first assertion.

Now let $-1 < \epsilon < 0$. Take the same notation as in the proof of Theorem 4.2, but

$$\mathcal{C}_n = \{\sigma \in T_n : X_\sigma > (1+\epsilon)c|\sigma|^{\beta/2}\}.$$

We have

$$B_{\beta,c} \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} G_n.$$

In order to prove the second assertion, we have only to show that a. s.

$$\mathcal{H}_{\varphi_{\beta,c,2\epsilon}}\left(\bigcap_{n=N}^{\infty} G_n\right) = 0, \qquad \forall N \geq 1, -1 < 2\epsilon < 0.$$

Notice that if $n \geq N$, $\{B_\sigma\}_{\sigma \in \mathcal{C}_n}$ is a 2^{-n}-cover of $\bigcap_{n=N}^{\infty} G_n$. So, by the definition of the Hausdorff dimension

$$\mathcal{H}_{\varphi_{\beta,c,2\epsilon}}(B_{\beta,c}) \leq \liminf_{n\to\infty} \varphi_{\beta,c,2\epsilon}(2^{-n})\text{Card}\mathcal{C}_n.$$

By Fatou lemma, we have only to show that

$$\liminf_{n\to\infty} \varphi_{\beta,c,2\epsilon}(2^{-n})\mathbb{E}\text{Card}\mathcal{C}_n = 0.$$

However,

$$\mathbb{E}\text{Card}\mathcal{C}_n = O\left(2^n e^{-(1+\epsilon)\frac{c^2}{2}(n\log 2)\beta}\right).$$

Therefore

$$\varphi_{\beta,c,\epsilon/2}(2^{-n})\mathbb{E}\text{Card}\mathcal{C}_n = O\left(\exp\left(\frac{\epsilon c^2}{2}(n\log 2)^\beta\right)\right) = o(1).$$

7 Remarks

1. Notice that

$$\liminf_{\sigma\to\xi} \frac{X_\sigma}{\Phi(|\sigma|)} = -\limsup_{\sigma\to\xi} \frac{-X_\sigma}{\Phi(|\sigma|)}.$$

So, all the results on the (upper) oscillation OSC_Φ can be modified to results on the lower oscillation which is defined in a similar way, using the inferior limit.

2. We have required the positivity $\dim \partial T > 0$. Such trees T have rich branches (a parent has about $\text{br}T > 1$ descendants). We have also required the regularity $\text{br}T = \overline{\text{gr}}T$.

It is worthy to study the oscillations when the positivity and/or regularity are not satisfied.

3. If X takes as values non negative integers, we can express our results in terms of random covering (see [FK]).

4. The results on good and bad sets may be used to study random Fourier-Franklin series with independent coefficients of arbitrary law.

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