On Singularity of Function Family*

Sun Daochun

(Department of Mathematics, Wuhan University, Wuhan 430072, China)

Abstract We define a notion of singularity for a family of meromorphic functions. Some sufficient and necessary conditions for this singularity are estabilished.

Key words normal family, covering surface, characteristic, island

Suppose that V is the Riemann sphere of unit diameter, F^* is a connected domain on V, whose boundary ∂F^* consists of q circles $\{B_i\}$ $(1\leqslant i\leqslant q)$, so that for any two circles $B_i, B_j (i\neq j)$, the spherical distance $d(B_i, B_j)>b\in(0, 0.5]$. Let F be a finite covering surface of F^*, whose boundary ∂F consists of a finite number of analytic Jordan curves. We call the part of ∂F, which lies in the interior of F^*, the relative boundary of F, and we denote its length by L.

Let D be a domain in F^*, which is bounded by a finite number of analytic Jordan curves, and $F(D)$ be the part of F which lies above D. We denote the area of F^*, F, $F(D)$ by $|F^*|$, $|F|$, $|F(D)|$ respectively. Set

$$S^* = \frac{|F|}{|F^*|},\quad S = \frac{|F|}{|V|},\quad S(D) = \frac{|F(D)|}{|D|}$$

In [1], we have proved the following result:

Theorem A For any finite covering surface F of F^*, we have

$$|S^* - S(D)| > \frac{\pi^2 L}{|D|b}$$

In [2], we have proved

Theorem B For any finite connected covering surface F, we have

$$\rho^+(F) > \rho(F^*)S - 32\frac{\pi^2 L}{b^3}$$

where $\rho(F)$ is the characteristic of F, $\rho^+=\max(0,\rho)$.

Let $\{B_i\}_{i=1}^q$ be $q\geqslant 3$ disjoint circles on V, and that for any two $B_i, B_j (i\neq j)$, the sphercial distance $d(B_i, B_j)>b\in(0, 0.5]$. Set

$$F^* = V\setminus\bigcup_{i=1}^{q} B_i$$

Then

$$\rho(F^*) = q - 2$$

Now $F(B_i)$ consist of a finite number of connected surfaces

$$F(B_i) = U_j F^k_{ij} + U_j F^z_{ij}$$

Where F^k has no relative boundary with respect to B_i, and is called an island, while F^z has such one, which is called a peninsula.

Theorem 1 Let $n(i)$ be the number of simply connected islands in $F(B_i)$, then

---

* Supported by the National Natural Science Foundation of China
Received Aug. 23, 1995. Sun Daochun: born in July, 1943, Professor

$$\max\{0, \sum_{i=1}^{q} n(i) - 1\} > (q-2)S - 480\frac{L}{b^3}$$

where L is the length of the boundary of F.

Proof We take off all peninsulas $\{F_{ij}^{z}\}$ above $\{B_i\}$ from F, and let F' be the remaining surface:

$$F' = F \backslash \bigcup_{i=1}^{q} U_j F_{ij}^{z}$$

Suppose F' consists of $N(F')$ simply connected surfaces $\{F'(t)\}$, then

$$\rho(F') = \sum_{t} \rho(F'(t)) = -N(F')$$

Set $F''(t) = F'(t) \backslash \bigcup_{i=1}^{q} U_j F_{ij}^{k}$. Suppose that $F''(t)$ consists of a finite number of connected surfaces $\{F''(t,u)\}$. Since islands do not touch the boundary of F', the characteristic do not change, so that we have

$$-N(F') = \rho(F') = \sum_{i=1}^{q}\sum_{j} \rho(F_{ij}^{k}) + \sum_{t}\sum_{u} \rho(F''(t,u)) \quad (1)$$

Since $F'(t)$ is simply connected, there is only one $F''(\cdot) \in \{F''(t,u)\}$, such that $\partial F'(t) \cap \partial F''(\cdot) \neq \emptyset$, and other surfaces $F''(t,u)$ have relative boundary with length $L=0$. From Theorem B, one sees that $\rho^{+}(F''(t,u)) > S > 0$. Thus $F''(t,u)$ is not simply connected. This implies that the number $N(F''(t))$ of simply connected surfaces in $F''(t)$ is not bigger than $N(F')$. We have that

(i) If there is no island in $F' = \{F'(t)\}$, i.e. $\sum_{i} n(i) = 0$, then

$$N(F''(t)) = N(F')$$

(ii) If there is $F'(t_0) \in F'$ contanning at least one island, then for any $F''(t_0,u) \in F''(t_0) = \{F''(t_0, u)\}$, $F''(t_0,u)$ is not simply connected. That is,

$$N(F''(t)) \geqslant N(F') - 1$$

Combine with (1), we get

$$-N(F') = \rho(F') = \sum_{i=1}^{q}\sum_{j} \rho^{+}(F_{ij}^{k}) - \sum_{i=1}^{q} n(i) + \sum_{t}\sum_{u} \rho^{+}(F''(t,u)) - N(F'')$$

$$0 \geqslant \sum_{t}\sum_{u} \rho^{+}(F''(t,u)), \qquad \sum_{i=1}^{q} n(i) = 0$$

and

$$\sum_{i=1}^{q} n(i) \geqslant \sum_{t}\sum_{u} \rho^{+}(F''(t,u)) + 1 + \sum_{i=1}^{q}\sum_{j} \rho^{+}(F_{ij}^{i})$$

$$\geqslant \sum_{t}\sum_{u} \rho^{+}(F''(t,u)) + 1, \quad \sum_{i=1}^{q} n(i) \neq 0 \quad (2)$$

Since every $F''(t,u)$ is a covering surface of F^{*}, by Theorem B, one has

$$\sum_{t}\sum_{u} \rho^{+}(F''(t,u)) > (q-2)\sum_{t}\sum_{u} S''(t,u) - 352\sum_{t}\sum_{u} \frac{L''(t,u)}{b^3}$$

$$\geqslant (q-2)S'' - 352\frac{L}{b^3} \quad (3)$$

where $S''(t,u) = \dfrac{|F''(t,u)|}{|F|}$, $S'' = \sum_{t}\sum_{u} S''(t,u)$, and $L''(t,u)$ is the relative boundary of $F''(t,u)$.

From Theorem A, we have

$$|S'' - S| < \frac{\pi^2 L}{b|F^{*}|} \quad (4)$$

Note that $q < 4|F^{*}|/b^2$, and combine with (2), (3), (4), we have

$$\max\{0, \sum_{i=1}^{q} n(i) - 1\} \geqslant \sum_{t}\sum_{u} \rho^{+}(F''(t,u)) \geqslant (q-)S'' - 352\frac{L}{b^3}$$

$$\geqslant (q-2)(S-\frac{\pi^2 L}{b|F^*|})-352\frac{L}{b^3}\geqslant (q-2)S-(q\frac{\pi^2}{b|F^*|}+\frac{352}{b^3})L$$

$$\geqslant (q-2)S-(4\frac{\pi^2}{b^3}+\frac{352}{b^3})L\geqslant (q-2)S-480\frac{L}{b^3} \qquad Q.E.D.$$

Combining our method with Theorem 1, Theorem VI in[3], we have:

Theorem 2 If $\{D_i\}_{i=1}^q$ is a set of q disjoint simply connected domains on V, each of which is bounded by ananalytic Jordon curve. Let $n(i)$ be the number of simply connected islands in $F(D_i)$, then

$$\max\{0,\sum_{i=1}^q n(i)-1\}>(q-2)S-hL$$

where $h>0$ is a constant, depending only on $\{D_i\}$ and L is the length of the boundary of F.

Theorem 3 Suppose that F is a simply connected finite covering surface of V, and that every connected island in $F(B_i)$ has at least $m(i)$ $(i=1,2,\cdots,q;\ \ q\geqslant 3)$ sheets, where $m(i)=\infty$ means that there is no simply connected island in $F(B_i)$ (except one from all islands), then

$$\sum_{i=1}^q (1-1/m(i))\leqslant 2+(480+\frac{510}{|B|})\frac{L}{Sb^3}$$

where $|B|=\min\{|B_i|;i=1,2,\ldots,q\}$.

Proof Let $n(i)$ be the number of simply connected islands in $F(B_i)$. If $m(i)\neq\infty$, then $S(B_i)\geqslant m(i)n(i)$. For B_{i_0} which has a exception, $S(B_{i_0})\geqslant m(i_0)[n(i_0)-1]+1$. In this case, $n(i_0)>0$, so that by Theorem 1 and A,

$$(q-2)S-480\frac{L}{b^3}\leqslant\sum_{i=1}^q n(i)-1\leqslant\sum_{i=1}^q\frac{S(B_i)}{m(i)}-\frac{1}{m(i_0)}$$

$$\leqslant\sum_{i=1}^q\frac{S(B_i)}{m(i)}-\frac{1}{m(i_0)}+\frac{1}{m(i_0)}\sum_{i=1}^q\frac{\pi^2 L}{b|B_{i_0}|}$$

$$\leqslant\sum_{i=1}^q\frac{S(B_i)}{m(i)}-\frac{1}{m(i_0)}+\frac{q\pi^2 L}{b|B_{i_0}|}$$

For $n(i)=0$, the above inequality holds too. By noting that $q<\frac{4\pi}{b^2}$, we obtain Theorem 3. Q.E.D.

Theorem 4 Let $w=f(z)$ be a meromorphic function on $B=\{|z|<R\}$. Assume that the number of zeros of $\prod_{i=1}^q(f(z)-a(i))$ $(q\geqslant 3)$ in B is smaller than n, where multiple zeros has been counted only once. If the spherical distance of any two points in $\{a(i)\}$ are bigger than b, then

$$(q-2)S(r)<n+2048\frac{\pi^6 R}{(R-r)b^6},\qquad r\in[0,R)$$

where

$$S(r)=\frac{1}{\pi}\iint_{|z|<r}(\frac{|w'(z)|}{1+|w(z)|^2})^2 r\,dr\,dt,\qquad z=re^{i\theta}$$

Proof We take off q points $\{a(i)\}$ from the w-sphere V, and let F^* be the remaining surface. Then $\rho=\rho(F^*)=q-2$. We take off the zeros of $\prod_{i=1}^q(f(z)-a(i))$ from B, and let D be the remaining domain. Set

$$D(r)=D\cap\{|z|<r<R\},\quad F(r)=\{f(z);z\in D(r)\}$$

Then $F(r)$ is a covering surface of F^*. By Theorem B,

$$\rho^+(r)\geqslant(q-2)S(r)-hL(r)$$

where $h=\frac{32\pi^2}{b^3}$, $\rho(r)$ is the characteristic of $F(r)$, and $L(r)$ is the length of the image of $\partial D(r)$ on V. Thus

$$(q-2)S(r)-n\leqslant(q-2)S(r)-\rho^+(r)\leqslant hL(r)$$

If $(q-2)S(t)-n>0$ for all $t\in[r,R)$, then by Schwarz inequality, we have

$$((q-2)S(t)-n)^2 \leqslant h^2(L(t))^2 \leqslant 2\pi^2 h^2 t \frac{\mathrm{d}S(t)}{\mathrm{d}t}$$

so that

$$\frac{R-r}{R} \leqslant \int_r^R \frac{\mathrm{d}t}{t} \leqslant 2\pi^2 h^2 \int_r^R \frac{\mathrm{d}S(t)}{((q-2)S(t)-n)^2} \leqslant \frac{2\pi^2 h^2}{(q-2)S(r)-n}$$

$$(q-2)S(r) \leqslant n + 2\pi^2 h^2 \frac{R}{R-r} = n + 2048 \frac{\pi^6 R}{(R-r)b^6}$$

If $(q-2)S(t)-n \leqslant 0$ for some $t \in [r,R)$, then

$$(q-2)S(r)-n \leqslant (q-2)S(t)-n \leqslant 0$$

which implies that our theorem holds in general. *Q.E.D.*

Theorem 5 Let $w=f(z)$ be a meromorphic function in $B=\{|z|<R\}$. Let F be generated by $w=f(z)$ on w-sphere V. Let $\{B_j\}$ be q disks on V, and denoted by n the total number of simply connected islands in $\{F(B_j)\}_{j=1}^q$. If the spherical distance of any two different disks in $\{B_j\}$ are bigger than b, then

$$(q-2)S(r) < n + 2048 \frac{\pi^6 R}{(R-r)b^6}, \qquad r \in [0,R)$$

Proof Theorem 5 follows from Theorem 1 and the method of the proof of Theorem 4. *Q.E.D.*

Let $M(D)$ be a family of meromorphic functions in a domain D. A subfamily $F=\{f\}\subset M(D)$ of meromorphic functions is said to be normal in D, if every infinite sequence of functions from F, contains a subsequence which converges locally uniformly on D. A family $F\subset M(D)$ is said to be normal at a point $e\in D$, if there is a neighborhood U ($e\in U\subset D$), such that F is normal in U.

Definition 1 If a family $F=\{f\}\subset M(D)$ is not normal at a point $e\in D$, then e is called a singularity of F.

Definition 2 We say that there exists a sequence of fulling circles in a family $F=\{f\}\subset M(D)$ at some neighborhood of $e\in D$, if for any $b,t>0$, there exists $f\in F$ such that $f(\{|z-e|<t\})$ covers the sphere surface V except for some point in at most two disks with radius b.

Theorem 6 Given a family $F=\{f\}\subset M(D)$, and a point $e\in D$, the following statements are equivalent:

1) The point e is a singularity of F;

2) There exists a sequence of filling circles at point e;

3) For any $t>0$, $\sup\{\frac{|f'(z)|}{(1+|f(z)|)^2}; f\in F, \quad d(z,e)<t\}=\infty$;

4) There exist two different finite or infinite complex numbers p and q such that for any $t>0$, there exists $f\in F$, such that $p,q\in f(\{|z-e|<t\})$;

5) For any three different finite or infinite complex numbers p,q and g, and any $t>0$, there exists $f\in F$ such that the total amount which f takes values on p,q or g in $\{|z-e|<t\}$ at least two times;

6) For any $b,t>0$, there exists $f\in F$, such that for any three different finite or infinite p,q and g, for which the spherical distances $d(p,q)$, $d(q,g)$ and $d(g,p)$ are bigger than b, f takes values on p,q or g at least two times in $\{|z-e|<t\}$;

7) For any $b,t>0$, there exists $f\in F$, such that for any three disks A,B and C with the spherical distances $d(A,B)$, $d(B,C)$, $d(A,C)\geqslant b$, the total number of simply connected islands of $f(\{|z-e|<t\})$ is no less than two on A,B and C;

8) For any $b,t>0$, there exists $f\in F$, such that for any $q\geqslant 3$ disks $\{B_i\}$ whose spherical distances between any two of them are larger than b, then the q positive integrals $\{m(i)\}$ satisfy:

$$\sum_{i=1}^{q}(1-\frac{1}{m(i)}) \leqslant 2$$

where every simply connected island of $f(\{|z-e|<t\})$ which lying above B_i has at least $m(i)$ sheets (the total amount of exceptional points is one at most), and $m(i)=\infty$ means that there is no simply connected island in $f(\{|z-e|<t\})$.

Proof 1)$\Leftrightarrow$3): follows immediately from Marty normal criterion in [4].

4)$\Rightarrow$3): Suppose that 3)is not true, then there is a neighborhood U of the point e and a positive constant $M>0$, such that for all $f\in F, z\in U$,

$$\frac{|f'(z)|}{1+|f(z)|^2}<M$$

Let $b=d(p,q)$. We take $t\in(0,\frac{b}{4M})$, such that $Q=\{|z-e|<t\}\subset U$. Then for any two points u, v in Q, and any $f\in F$, we have

$$d(f(u),f(v))<\int_0^R \frac{|f'(u+re^{i\theta})|}{1+|f(u+re^{i\theta})|^2}dr<RM<2tM<\frac{b}{2}$$

where $Re^{i\theta}=v-u$, i. e. the diameter of the surface $f(Q)$ is uniformly smaller than b with respect to F. Thus $f(Q)$ does not cover p and q. This contradicts with 4).

The implications 5)$\Rightarrow$2), 2)$\Rightarrow$3) and 8)$\Rightarrow$7)$\Rightarrow$6)$\Rightarrow$5) are obvious.

5)$\Rightarrow$4): Take four different complex numbers x, y, u, v, and let $b=\min\{d(x,y), d(y,u), d(u,v), d(v,x), d(x,u), d(y,v)\}$. It follows from 5), that there exists a sequence $\{f_n\}\subset F$, such that there are at least two numbers in $f_n(|z-e|<\frac{1}{n})\cap\{x,y,u,v\}$. Thus there are $p,q\in\{x,y,u,v\}$ such that

$$\#\{f_n; p,q\in f_n(|z-e|<\frac{1}{n})\}=\infty$$

This proves 4).

3)$\Rightarrow$7): Suppose that 7) is not true. Then there exist $b,t>0$, such that for any $f\in F$, there exist three disks A, B and C with respect to f with spherical distances $d(A,B), d(B,C), d(A,C)\geqslant b$, and the total number of simply connected islands of $f(|z-e|<t)$ is less than two on A, B and C. We take $R<t$, so that $\{|z-e|<R\}\subset D$, and $Y\in U=\{|z-e|<\frac{R}{2}\}$. Let $X=(Y-e)\in\{|z|<\frac{R}{2}\}$ and

$$M=M(Y,f)=\frac{|f'(Y)|(R^2-|X|^2)}{(1+|f(Y)|^2)R} \tag{5}$$

For $M>1$, the mapping

$$h=h(z)=\frac{MR(z-X)}{R^2-z\overline{X}}$$

transforms $\{|z|<R\}$ into $\{|h|<M\}$, and $h(X)=0$. Its inverse mapping is

$$z=z(h)=\frac{R(Rh+MX)}{MR+h\overline{X}}$$

Let $f(z+e)=f(z(h)+e)=g(h)$. Note $g(0)=f(X+e)=f(Y)$ (6)

$$g'(h)=f'(z(h)+e)z'(h)=f'(z(h)+e)\frac{MR^3-MR|X|^2}{(MR+h\overline{X})^2}$$

$$|g'(0)|=|f'(Y)\frac{R^2-|X|^2}{MR}|=1+|f(Y)|^2=1+|g(0)|^2 \tag{7}$$

It follows from 6) that for any $r\in(0,M)$, since $g(|h|<r)\subset f(D\cap\{|z-e|<t\})$, the total times of $g(|h|<r)$ covering A, B and C is less than two. Set

$$S(r)=\frac{1}{\pi}\iint_{|h|<r}\frac{|g'(h)|^2}{(1+|g(h)|^2)^2}t\mathrm{d}t\mathrm{d}\theta,\quad L(r)=\iint_{|h|=r}\frac{|g'(h)|}{1+|g(h)|^2}r\mathrm{d}\theta$$

where $h=te^{i\theta}$. It follows from Theorem 1 that

$$0=\max\{0,\sum_{j=1}^{3}n(j)-1\}>S(r)-GL(r)$$

where G is a constant depending only on t. By Schwarz inequality,

$$S^2(r)<G^2L^2(r)\leqslant 2G^2\pi^2 r\frac{\mathrm{d}S(r)}{\mathrm{d}r}$$

$$\log M = \int_1^M \frac{\mathrm{d}r}{r} < 2G^2\pi^2 \int_1^M \frac{\mathrm{d}S(r)}{S^2(r)} \leqslant \frac{2G^2\pi^2}{S(1)}$$

By the Theorem VI,11 of [1] and (7),we know that there is a disk whose radius is a positive constant c in $g(|h|<1)$. Thus

$$\log M < \frac{2G^2\pi^3}{c^2} \quad (>0) \tag{8}$$

For $M \leqslant 1$, (8) is obvious. Then for any $Y \in U$, any $f \in F$, by (5), (8) and $|X| < \frac{R}{2}$,

$$\frac{|f'(Y)|}{1+|f(Y)|^2} = \frac{RM}{R^2 - |X|^2} < 2e^{\frac{2G^2\pi^3}{c^2}}$$

This contradicts to 3).

3)⇒8): Suppose that 8) is not true, then there exist $b, t > 0$, such that for any $f \in F$, there exist $q > 2$ disks $\{B_i\}$ with respect to f such that the spherical distance between any two of $\{B_i\}$ is larger than b. And $m(i)$ satisfies

$$\sum_{i=1}^{q} \left(1 - \frac{1}{m(i)}\right) > 2$$

Since $m(i)$ are positive integers, we have

$$\sum_{i=1}^{q} \left(1 - \frac{1}{m(i)}\right) - 2 \geqslant \frac{1}{42}$$

where the minimum value $\frac{1}{42}$ is attained, when $m(1)=2, m(2)=3, m(3)=7, m(4)=m(5)=\cdots=m(q)=1$. By Theorem 3,

$$2 + \frac{1}{42} \leqslant \sum_{i=1}^{q} \left(1 - \frac{1}{m(i)}\right) < 2 + c\frac{L(t)}{S(t)}$$

Thus one has

$$S(t) < 42cL(t)$$

The rest part is the same as in 3)⇒7). *Q. E. D.*

Definition 3 Given a family $F=\{f\} \subset M(D)$, and a point $e \in D$, for any $t > 0$,

$$\sup\left\{\iint_{|z-e|<t} \frac{|f'(z)|}{(1+|f(z)|^2)^2} r\mathrm{d}r\mathrm{d}\theta; f \in F\right\} = \infty, \quad (z = re^{i\theta})$$

then the point e is called a transcendental singularity of F.

Definition 4 We say that there exists a sequence of transcendental fulling cercles for a family $F=\{f\} \subset M(D)$ at some neighborhood of $e \in D$, if for any $b, t, N > 0$, there exists $f \in F$ such that for any finite or infinite complex number a, the number of times that f takes value a in $\{|z-e|<t\}$ is bigger than N, except for some points in at most two disks with radius b.

Definition 5 We say that there exists a sequence of transcendental filling circles in a family $F=\{f\} \subset M(D)$ in the domain D, if for any $b, N > 0$, there exists $f \in F$ such that for any finite or infinite complex number a, the number of times that f takes value a in D is more than N, except for some points in at most two disks with radius b.

Theorem 7 Given a family $F=\{f\} \subset M(D)$, and a point $e \in D$. If for any $f \in F$, $f(D)$ does not contain p, q $(p \neq q)$, then the following statements are equivalent:

1) The point e is a singularity of F;

2) The point e is a transcendental singularity of F;

3) There exists a sequence of transcendental fulling circles at a neighborhood of e;

4) For any $b, t, N > 0$, there exists $f \in F$, such that for any finite or infinite complex number z with spherical distances $d(z,p), d(z,q) > b$, f takes values z more than N times in $\{|z-e|<t\}$;

5) For any $t, N > 0$, and simply connected closed domain B in $V \setminus \{p,q\}$ bounded by an analytic Jordan curve, there exists $f \in F$ such that the number of simply connected islands in $f(|z-e|<t)$ is more

than N on B;

6) (In fact, we can go further): For any $b, t, N>0$, there exists $f\in F$, such that for any simply connected closed domain A bounded by an analytic Jordan curve with the spherical distances $d(A,p), d(A,q)<b$, the number of simply connected islands in $f(|z-e|<t)$ is larger than N on A.

Proof 1)⇒5): We connect p,q by an analytic Jordan curve l in $V\backslash B$. Denote the length of l by $|l|$. Set

$$P=1-\frac{1}{2N},\qquad b=\min\{|p,q|,|B,l|,\frac{|l|}{\pi},\frac{\pi}{8N|l|}\}>0$$

$$A=\{x\in V;|x,p|\leqslant\frac{b}{3}\},\qquad E=\{x\in V;|x,q|\leqslant\frac{b}{3}\}$$

$$G=\{x\in V;|x,l|\leqslant\frac{b}{3}\},\qquad H=V-G\supset B$$

Then

$$|H|>\pi-2b|l|-2b^2\pi\geqslant\pi-4b|l|\geqslant\pi-\frac{\pi}{2N}\tag{9}$$

If 5) is not true, i. e. there exits $t>0$, such that for any $f\in F$, the number of simply connected islands in $f(|z-e|<t)$ is not larger than N on A. Then the total number of simply connected islands in $f(|z-e|<t)$ is not bigger than N on A, E and H. Since $S\geqslant\frac{N|H|}{\pi}$, one then has

$$SP-(N-1)\geqslant\frac{N|H|(2N-1)}{2\pi N}-N+1\geqslant N(\pi-\frac{\pi}{2N})\frac{2N-1}{2\pi N}-N+1$$
$$=\frac{(2N-1)^2}{4N}-N+1=\frac{1}{4N}>0\tag{10}$$

By Theorem 2, for $q=3$, one gets

$$N-1\geqslant\max\{0,\sum_{i=1}^{3}n(i)-1\}>S-hL$$

where h is a constant which depends on A, E and H. By (10), we have

$$(1-P)S<hL$$

The rest is the same as in the Proof of 3)⇒7) of Theorem 6. We would obtain that F is normal at the point e. This contradicts to 1).

5)⇒4)⇒3) and 2)⇒1) are obvious.

4)⇒2) and 3)⇒2): If 4) or 3) is valid, $S(|z-e|<t;f)$ is not bounded for f. Thus 2) holds. *Q. E. D.*

Theorem 8 If the point e is a singularity of a family of analytic functions $\{f(z)\}$ on a simply connected domain D, then e is also a singularity of the family

$$\{\int_0^z f(z)\mathrm{d}z\}$$

Proof Let

$$F(z)=\int_0^z f(z)\mathrm{d}z,$$

then $F(0)=0$. If the conclusion is not true, then there exists $t>0$, such that every infinite sequence of functions from $\{F\}$, contains a subsequence $\{F_n\}$ satisfying that

$$F_n(z)=\int_0^z f_n(z)\mathrm{d}z\longrightarrow F(z)$$

uniformly on $\{|z-e|<t\}$. Thus $\{f_n\}$ converges uniformly to f. This is a contradiction. *Q. E. D.*

Theorem 9 Given a meromorphic function family $F=\{f\}\subset M(D)$, the following statements are equivalent:

1) There exist at least one transcendental singularity of F in D;

2) There exists a closed domain $E\subset D$ such that

$$\sup\{S(E,f) = \iint_E \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 r\,dr\,d\theta, \quad f \in F\} = \infty$$

3) There exists a sequence of transcedental filling circles in F in D;

4) For any $b,t,N>0$, there exists $f\in F$, such that for any q different finite or infinite complex numbers $C_1,C_2,\ldots,C_q$ for which the spherical distances $d(C_i,C_j)(i\neq j)$ are all larger than b, f takes values at C_j $(j=1,2,\ldots,q)$ at least N times, with at most two exceptions in $\{C_i\}$;

5) For any $b,t,N>0$, there exists $f\in F$, such that for any q disks $B_1,B_2,\ldots,B_q$, such that the sphercal distances $d(B_i,B_j)\geqslant b$ $(i\neq j)$, the number of simply connected islands of $f(|z-e|<t)$ are larger than N for all $B_j(j=1,2,\ldots,q)$ with at most two exceptions in $\{B_i\}$.

Proof The implications 5) ⇒4) ⇒3) ⇒2) ⇒1) are obvious. By Theorem 5, one gets 1) ⇒5).

Q. E. D.

References

1 Sun Daochun. Improvement of Ahlfors inequality. *Journal of Wuhan University(Natural Science Edition)*, 1992(3):1~8(in Chinese with English abstract)

2 Sun Daochun. Main theorem on covering sufaces. *Acta Math Sci*, 1994, **14**(2): 213~225

3 Tsuji M. *Potential theory in modern function theory.* Tokyo: Maruzen, 1959

4 Hayman W K. *Meromorphic functions.* Oxford: Clarendon Press, 1964