A TIDAL THREE-DIMENSIONAL NONLINEAR MODEL WITH VARIABLE EDDY VISCOSITY (I)——A DYNAMIC MODEL

Feng Shizuo and Sun Wenxin

(Department of Physical Oceanography, Shandong College of Oceanology)

A three-dimensional non-linear model of tides was proposed by one of the authors (1977)[1]. Later the model was applied to modeling two principal first-order constituents M_4 and MS_4 of shallow water in the Bohai Sea (1981)[2]. As pointed out in the reference[1], however, the model has physically a main weakpoint in the hypothesis about the eddy viscosity ν, i.e., $\nu=\nu\,(x,y)$ was considered to be irrelative to both the nondimensional vertical coordinate z and the tidal current structure, where (x, y) are nondimensional horizontal coordinates. No doubt, the non-linear model of tides, which only involves the nonlinear effects of both the convection acceleration in the equation of motion and the kinematic boundary condition at the free surface, is not completely nonlinear one. The effect of variable eddy viscosity on the currents in shallow area may be especially noteworthy[2]. Therefore, extending of the three-dimensional nonlinear model of tides mentioned above to a tidal three-dimensional nonlinear model with variable eddy viscosity for attaining a completely nonlinear model becomes a matter of interest. In this short article, we try to accomplish this extension.

The problem of tides in nondimensional form is expressed as follows[1]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

$$\frac{\partial u}{\partial t}+\kappa\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)-\Omega v=-\frac{\partial \zeta}{\partial x}+\frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right)+\omega_x,$$

$$\frac{\partial v}{\partial t}+\kappa\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)+\Omega u=-\frac{\partial \zeta}{\partial y}+\frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right)+\omega_y;$$

The boundary conditions are

at the sea surface $z=\kappa\zeta$:

$$w=\frac{\partial \zeta}{\partial t}+\kappa\left(u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}\right), \tag{1}$$

$$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=0;$$

at the sea bottom $z=-h$:

$$u=v=w=0;$$

along the shore boundary C_1:

$$\cos\alpha_x\int_{-h}^{\kappa\zeta} udz+\cos\alpha_y\int_{-h}^{\kappa\zeta} vdz=0;$$

along the open boundary C_2:

$$\zeta=S;$$

where

$$\kappa=R/h_0;$$

Symbols introduced but not defined here are the same as those defined in the reference [1].

Taking κ as a small parameter and assuming that the boundary condition can be expressed as

$$S=\sum_{j=0,1,\cdots}\kappa^j S_j,\qquad S_j=\begin{cases}S & (j=0)\\ 0 & (j=1,\ 2,\ \cdots)\end{cases}$$

we have the solutions

$$\begin{Bmatrix}\zeta\\ u\\ v\\ w\end{Bmatrix}=\sum_{j=0,1,\cdots}\kappa^j\begin{Bmatrix}\zeta_j\\ u_j\\ v_j\\ w_j\end{Bmatrix}\tag{2}$$

We suppose that the eddy viscosity can also be expanded as

$$\nu=\sum_{j=0,\ 1,\cdots}\kappa^j\nu_j\tag{3}$$

and thus ζ_j, u_j, v_j and w_j satisfy the following problem:

$$\frac{\partial u_j}{\partial x}+\frac{\partial v_j}{\partial y}+\frac{\partial w_j}{\partial z}=0,$$

$$\frac{\partial u_j}{\partial t}-\Omega v_j=-\frac{\partial\zeta_j}{\partial x}+\frac{\partial}{\partial z}\left(\nu_0\frac{\partial u_j}{\partial z}\right)+\begin{cases}\omega_x & (j=0)\\ -{}_xE_{j-1}+{}_x\Xi_{j-1}, & (j=1,\ 2,\ \cdots)\end{cases}$$

$$\frac{\partial v_j}{\partial t}+\Omega u_j=-\frac{\partial\zeta_j}{\partial y}+\frac{\partial}{\partial z}\left(\nu_0\frac{\partial v_j}{\partial z}\right)+\begin{cases}\omega_y & (j=0)\\ -{}_yE_{j-1}+{}_y\Xi_{j-1}; & (j=1,\ 2,\ \cdots)\end{cases}$$

at the free surface $z=0$:

$$w_j=\frac{\partial\zeta_j}{\partial t}+\begin{cases}0 & (j=0)\\ F_{j-1} & (j=1,\ 2,\ \cdots),\end{cases}\tag{4}$$

$$\frac{\partial u_j}{\partial z}=\begin{cases}0 & (j=0)\\ {}_x\Gamma_{j-1} & (j=1,\ 2,\ \cdots),\end{cases}$$

$$\frac{\partial v_j}{\partial z}=\begin{cases}0 & (j=0)\\ {}_y\Gamma_{j-1} & (j=1,\ 2,\ \cdots);\end{cases}$$

at the sea bottom $z=-h$:

$$u_j=v_j=w_j=0;$$

at the shore boundary C_1:

$$\cos\alpha_x\left[\int_{-h}^{0}u_jdz+\begin{Bmatrix}0\\ {}_xH_{j-1}\end{Bmatrix}\right]+\cos\alpha_y\left[\int_{-h}^{0}v_jdz+\begin{Bmatrix}0\\ {}_yH_{j-1}\end{Bmatrix}\right]=0;\ \begin{matrix}(j=0)\\ (j=1,\ 2,\ \cdots)\end{matrix}$$

along the open boundary C_2:

$$\zeta_j=S_j,$$

where

$$\left\{\begin{matrix} {}_x\Xi_{j-1} \\ \\ {}_y\Xi_{j-1} \end{matrix}\right\} = \sum_{m=0}^{j-1} \left\{\begin{matrix} \frac{\partial}{\partial z}\left(\nu_{j-m}\frac{\partial u_m}{\partial z}\right) \\ \\ \frac{\partial}{\partial z}\left(\nu_{j-m}\frac{\partial v_m}{\partial z}\right). \end{matrix}\right.$$

The problem (4) should be satisfied by the j-th order of constituents of tides. Since (4) is a linear one, any tidal constituent with the nondimensional frequency σ can be solved respectively; thus we introduce the harmonic factor exp $(-i\sigma t)$ into the field quantities $u, v, w, \zeta, \omega_x, \omega_y$ and S, i.e., let

$$\left\{\begin{matrix} u \\ v \\ w \\ \zeta \\ \omega_x \\ \omega_y \\ s \end{matrix}\right\} = \left\{\begin{matrix} u' \\ v' \\ w' \\ \zeta' \\ \omega_x' \\ \omega_y' \\ s' \end{matrix}\right\} \cos(\sigma t) + \left\{\begin{matrix} u'' \\ v'' \\ w'' \\ \zeta'' \\ \omega_x'' \\ \omega_y'' \\ s'' \end{matrix}\right\} \sin(\sigma t) = \mathrm{Re}\left[\left\{\begin{matrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{\zeta} \\ \bar{\omega}_x \\ \bar{\omega}_y \\ \bar{s} \end{matrix}\right\} e^{-i\sigma t}\right]; \tag{5}$$

where and in the following, the subscript j has been and will be omitted for brevity, respectively.

And further, we assume

$$\nu_j = \begin{cases} \nu_0(x, y) & (j=0) \\ \nu_j'(x,y,z)\cos(\sigma t) + \nu_j''(x,y,z)\sin(\sigma t) = \mathrm{Re}[\bar{\nu}_j(x,y,z)e^{-i\sigma t}] & (j=1,2,\cdots), \end{cases} \tag{6}$$

which implies that the variable eddy viscosity is characterized by the three-dimensional nonlinear waves with the same periods as those of different orders of tidal constituents. Thus, the effect of variable eddy viscosity is considered in terms of the simpler assumption described by (3) and (6). It should be emphasized that $\bar{\nu}_j$ $(j=1,2,\cdots)$ introduced in the expression (6) might be a physically acceptable arbitrary function not only of space coordinates but also of lower orders of tidal currents. The specific models of $\bar{\nu}_j$ will be developed in the next article, only the formulation of the tidal model is proposed in this article.

Noting that

$$\begin{aligned} {}_x\Xi_{j-1} = & \sum_{m=0}^{j-1}\sum_{p,q}\mathrm{Re}\left[\frac{1}{2}\frac{\partial}{\partial z}\left(\bar{\nu}_{j-m,p}\frac{\partial \bar{u}_{m,q}}{\partial z}\right)\exp\{-i(\sigma_{j-m,p}+\sigma_{m,q})t\}\right] - \\ & -\sum_{m=0}^{j-1}\sum_{p,q}\mathrm{Re}\left[\frac{1}{2}\frac{\partial}{\partial z}\left(\bar{\nu}_{j-m,p}\frac{\partial \bar{u}^*_{m,q}}{\partial z}\right)\exp\{-i(\sigma_{j-m,p}-\sigma_{m,q})t\}\right], \end{aligned} \tag{7}$$

$$\begin{aligned} {}_y\Xi_{j-1} = & \sum_{m=0}^{j-1}\sum_{p,q}\mathrm{Re}\left[\frac{1}{2}\frac{\partial}{\partial z}\left(\bar{\nu}_{j-m,p}\frac{\partial \bar{v}_{m,q}}{\partial z}\right)\exp\{-i(\sigma_{j-m,p}+\sigma_{m,q})t\}\right] - \\ & -\sum_{m=0}^{j-1}\sum_{p,q}\mathrm{Re}\left[\frac{1}{2}\frac{\partial}{\partial z}\left(\bar{\nu}_{j-m,p}\frac{\partial \bar{v}^*_{m,q}}{\partial z}\right)\exp\{-i(\sigma_{j-m,p}-\sigma_{m,q})t\}\right], \end{aligned}$$

and introducing ξ_1 and ξ_2 into ${}_x\Xi_{j-1}$ and ${}_y\Xi_{j-1}$, respectively, where the expressions under $\sum\sum$ in (7) are abbreviated as ξ_1 and ξ_2, the nondimensional problem (4) will be then reduced to that of (8), which is satisfied by a certain constituent with the nondimensional frequency σ:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0,$$

$$\nu_0\frac{\partial^2\bar{u}}{\partial z^2}+i\sigma\bar{u}+\Omega\bar{v}=\frac{\partial\bar{\zeta}}{\partial x}+\bar{\psi}_1-\bar{\xi}_1,$$

$$\nu_0\frac{\partial^2\bar{v}}{\partial z^2}+i\sigma\bar{v}-\Omega\bar{u}=\frac{\partial\bar{\zeta}}{\partial y}+\bar{\psi}_2-\bar{\xi}_2; \tag{8}$$

at the free surface $z=0$:

$$\bar{w}=-i\sigma\bar{\zeta}+\bar{f},\qquad \frac{\partial\bar{u}}{\partial z}=\bar{\gamma}_1,\qquad \frac{\partial\bar{v}}{\partial z}=\bar{\gamma}_2;$$

at the bottom $z=-h$:

$$\bar{u}=\bar{v}=\bar{w}=0;$$

along the shore boundary C_1:

$$\cos\alpha_x\left(\int_{-h}^{0}\bar{u}dz+\bar{\eta}_1\right)+\cos\alpha_y\left(\int_{-h}^{0}\bar{v}dz+\bar{\eta}_2\right)=0;$$

along the open boundary C_2:

$$\bar{\zeta}=\bar{S}.$$

For the zeroth-order model, in the problem (8), we have

$$\bar{f}=\bar{\gamma}_1=\bar{\gamma}_2=\bar{\eta}_1=\bar{\eta}_2=0 \quad \text{and}$$

$$(\bar{\psi}_1-\bar{\xi}_1)=\bar{\omega}_x,\ (\bar{\psi}_2-\bar{\xi}_2)=\bar{\omega}_y \ \text{or}\ \bar{\omega}_x=\bar{\omega}_y=0.$$

From (8), we derive the boundary-value problem (9) of the elliptic differential equation for the elevation $\bar{\zeta}$, the analytical expression (10) for the vertical distribution of the currents $\bar{u}$, $\bar{v}$ and the formula (11) for the vertical velocity, $\bar{w}$, as follows:

(i)

$$\frac{1}{\nu_0}\frac{A+\hat{A}}{2}\Delta\bar{\zeta}+\left[\frac{\partial}{\partial x}\left(\frac{1}{\nu_0}\frac{A+\hat{A}}{2}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\nu_0}\frac{A-\hat{A}}{2i}\right)\right]\frac{\partial\bar{\zeta}}{\partial x}+$$

$$+\left[\frac{\partial}{\partial y}\left(\frac{1}{\nu_0}\frac{A+\hat{A}}{2}\right)-\frac{\partial}{\partial x}\left(\frac{1}{\nu_0}\frac{A-\hat{A}}{2i}\right)\right]\frac{\partial\bar{\zeta}}{\partial y}+i\sigma\bar{\zeta}=\bar{f}-\bar{F}; \tag{9}$$

along the shore boundary C_1:

$$\bar{Q}_n=\bar{\eta}_n;$$

along the open boundary C_2:

$$\bar{\zeta}=\bar{S};$$

where

$$\bar{F}=F-\frac{\partial}{\partial x}\int_{-h}^{0}\left[\frac{B+\hat{B}}{2}\bar{\xi}_1-\frac{B-\hat{B}}{2i}\bar{\xi}_2\right]dz-\frac{\partial}{\partial y}\int_{-h}^{0}\left[\frac{B-\hat{B}}{2i}\bar{\xi}_1+\frac{B+\hat{B}}{2}\bar{\xi}_2\right]dz,$$

$$\bar{Q}_n=\frac{1}{\nu_0}\frac{A+\hat{A}}{2}\frac{\partial\bar{\zeta}}{\partial n}+\frac{1}{\nu_0}\frac{A-\hat{A}}{2i}\frac{\partial\bar{\zeta}}{\partial\tau}+\int_{-h}^{0}\left[\frac{B+\hat{B}}{2}(\bar{\psi}_n-\bar{\xi}_n)+\right.$$

$$\left.+\frac{B-\hat{B}}{2i}(\bar{\psi}_\tau-\bar{\xi}_\tau)\right]dz+\frac{C+\hat{C}}{2}\bar{\gamma}_n+\frac{C-\hat{C}}{2i}\bar{\gamma}_\tau,$$

$$B=B(k,z)=\frac{1}{\nu_0 k^2}\left\{1-\frac{\mathrm{ch}(kz)}{\mathrm{ch}(kh)}\right\}, \qquad \hat{B}=B(\hat{k},\ z),$$

$$k=(1+i)\sqrt{(\Omega-\sigma)/(2\nu_0)},$$

$$\hat{k}=(1-i)\sqrt{(\Omega+\sigma)/(2\nu_0)},$$

$$\bar{\xi}_n=\bar{\xi}_1\cos\alpha_x+\bar{\xi}_2\cos\alpha_y,$$

$$\bar{\xi}_\tau=-\bar{\xi}_2\cos\alpha_x+\bar{\xi}_1\cos\alpha_y;$$

(ii)

$$\bar{u}=\frac{1}{2}(q+\hat{q}), \qquad \bar{v}=\frac{1}{2i}(q-\hat{q}); \tag{10}$$

where

$$q=q(k,\ G,\ \psi,\ \gamma,\ \xi,\ z)=\frac{G}{k^2}\cdot\frac{1}{\nu_0}\left\{\frac{\mathrm{ch}(kz)}{\mathrm{ch}(kh)}-1\right\}+$$

$$+\frac{1}{k}\mathrm{sh}(kz)\int_0^z\frac{\psi-\xi}{\nu_0}\mathrm{ch}(k\bar{z})d\bar{z}-\frac{1}{k}\mathrm{ch}(kz)\left\{\int_{-h}^{z}\frac{\psi-\xi}{\nu_0}\mathrm{sh}(k\bar{z})d\bar{z}+\right.$$

$$\left.+\mathrm{th}(kh)\int_{-h}^{0}\frac{\psi-\xi}{\nu_0}\mathrm{ch}(kz)dz\right\}+\frac{\gamma}{k}\frac{\mathrm{sh}[k(z+h)]}{\mathrm{ch}(kh)},$$

$$\hat{q}=q(\hat{k},\ \hat{G},\ \hat{\psi},\ \hat{\gamma},\ \hat{\xi},\ z),$$

$$\xi=\bar{\xi}_1+i\bar{\xi}_2, \qquad \hat{\xi}=\bar{\xi}_1-i\bar{\xi}_2;$$

(iii)

$$\bar{w}=-i\sigma\bar{\zeta}+\bar{f}-\frac{\partial}{\partial x}\left[\frac{\bar{P}(z)+\hat{\bar{P}}(z)}{2}\right]-\frac{\partial}{\partial y}\left[\frac{\bar{P}(z)-\hat{\bar{P}}(z)}{2i}\right]; \tag{11}$$

where

$$\bar{P}(z)=\bar{P}(k,\ G,\ \psi,\ \gamma,\ \xi,\ z)=\frac{G}{k^2}\frac{1}{\nu_0}\left\{\frac{1}{k}\frac{\mathrm{sh}(kz)}{\mathrm{ch}(kh)}-z\right\}-$$

$$-\frac{1}{k^2}\int_0^z\frac{\psi-\xi}{\nu_0}d\bar{z}+\frac{1}{k^2}\mathrm{ch}[k(z+h)]\int_0^z\frac{\psi-\xi}{\nu_0}\cdot\frac{\mathrm{ch}(k\bar{z})}{\mathrm{ch}(kh)}d\bar{z}-$$

$$-\frac{1}{k^2}\frac{\mathrm{sh}(kz)}{\mathrm{ch}(kh)}\int_{-h}^{z}\mathrm{sh}[k(\bar{z}+h)]\frac{\psi-\xi}{\nu_0}d\bar{z}+\frac{\gamma}{k^2}\left[\frac{\mathrm{ch}[k(z+h)]}{\mathrm{ch}(kh)}-1\right],$$

$$\hat{\bar{P}}(z)=\bar{P}(\hat{k},\ \hat{G},\ \hat{\psi},\ \hat{\gamma},\ \hat{\xi},\ z).$$

Finally, it should be emphasized that the tidal three-dimensional nonlinear model with variable eddy viscosity proposed in this article is not mathematically more complicated than the original model presented in the reference [1] though the former is essentially improved in comparison with the latter. However, the simple hypothesis about $\nu_0=\nu_0(x,y)$ involved in the expression (6) is still a physically principal weak point of the model developed in this article. A numerical study on the model will be exhibited in the next article.

References

[1] Feng Shih-zao (Feng Shizuo), 1977. A three-dimensional non-linear model of tides. *Scientia Sinica* **20** (4): 436—446.

[2] Sun Wenxin, Chen Zongyong and Feng Shizuo, 1981. Numerical simulation of three-dimensional non-linear tidal waves (I)——A numerical study for M_4 and MS_4 waves in the Bohai Sea. *J. Shandong College of Oceanology* **11** (1): 23—31.