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The Asymptotic Analyses of Nonlinear Waves in Rate-Dependent Media.

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Summary. — High- and low-frequency wave processes are analysed in order to obtain the evolution equations for a rather general nonlinear rate-type (viscoelastic) medium. Moreover, a comparison with the results obtained by Engelbrecht for the standard viscoelastic solid is given. Finally an example of a high-frequency process in a particular nonlinear-linear medium is considered. Such an analysis may be used as a mathematical approach to point out the main features of wave propagation either in certain soft tissues or in certain polymers.

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1. – Introduction.

Several asymptotic approaches have been used to analyse the main features of wave propagation in rate-dependent nonlinear media provided that, for the mathematical model describing the material, there exists an associated hyperbolic system governing the wave process in the first approximation. Thus the corresponding evolution equations can be constructed along the characteristics either of the nonlinear [1,2] or of the linear associated system [3,4]. The

(1) M. P. MORTELL and B. R. SEYMOUR: *SIAM J. Appl. Math.*, **22**, 209 (1972).
(2) B. R. SEYMOUR and E. VARLEY: *Proc. R. Soc. London, Ser. A*, **314**, 387 (1970).
(3) J. K. ENGELBRECHT and U. NIGUL: *Nonlinear Deformation Waves* (Moscow, 1981)

solution may then be obtained by either implicit or explicit expressions, respectively. In the last case the evolution equation, however, is an integro-differential equation in which the kernel function describes the rate dependence in the most general form (see ([5])). Moreover, we remark that an averaging method is used in ([6]) to investigate the development of plane shock waves in materials which display (linear) viscoelastic and nonlinear elastic constitutive behaviour. As well a constitutive equation similar to the one used in ([6]) is considered to investigate the standard viscoelastic solid in ([7]), where also the low- and high-frequency processes have been described.

Recently, within the framework of the wave theory developed in ([8,9]) (see also references quoted there), a rather general asymptotic approach has been proposed in ([10]) to obtain the evolution equations for a nonhomogeneous (*i.e.* involving a source term) quasi-linear first-order hyperbolic system. In this case, by using special stretching of variables, the evolution equations for low- and high-frequency multidimensional processes have been constructed. It is important to remark that the mathematical model considered in ([10]) describes several nonlinear media characterized by nonconservative field balance equations. Among others this is the case of a large class of inelastic materials with very general differential constitutive laws. Hence it is of interest to apply the approach given in ([10]) to obtain the evolution equations in a general rate-dependent (quasi-linear) medium in order to compare the results so obtained with the ones deduced by different methods of approach, especially with those got in ([7]) for one-dimensional wave motions.

We consider a one-dimensional wave process in a rate-dependent visco-elastic medium described by the following system of equations ([11]):

$$\frac{\partial v}{\partial t} - \frac{1}{\varrho}\frac{\partial \sigma}{\partial x} = 0\,, \tag{1.1a}$$

$$\frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} = 0\,, \tag{1.1b}$$

$$\frac{\partial \sigma}{\partial t} - \Phi(\sigma, \varepsilon)\,\frac{\partial \varepsilon}{\partial t} = \Psi(\sigma, \varepsilon)\,, \tag{1.1c}$$

(in Russian).

([4]) A. A. Lokshin and J. V. Suvorova: *Mathematical Theory of Wave Propagation in Media with Memory*, Moscow State University (Moscow, 1982) (in Russian).

([5]) G. B. Whitham: *Linear and Nonlinear Waves* (New York, N. Y., 1974).

([6]) R. W. Lardner: *Proc. R. Soc. London, Ser. A*, **347**, 329 (1976).

([7]) J. Engelbrecht: *Wave Motion*, **1**, 65 (1979).

([8]) P. Germain: *Progressive Waves* (Jber DGLR, 1971), p. 11 (Köln, 1972).

([9]) G. Boillat: *Ann. Mat. Pura Appl.*, **4**, 31 (1976).

([10]) D. Fusco: *Meccanica*, **17**, 128 (1982).

([11]) N. Cristescu: *Dynamic Plasticity* (Amsterdam, 1967).

where x identifies the reference position of the particle at reference time $t = 0$, σ is the stress, ε the strain, v the particle velocity and ϱ the reference mass density. Moreover, $\Phi(\sigma, \varepsilon)$ and $\Psi(\sigma, \varepsilon)$ are smooth response functions. Inelastic media described by a constitutive law like (1.1c) are called « Maxwellian materials ».

The system of equations (1.1) may be written also in the following matrix form:

$$\frac{\partial V}{\partial t} + A(V)\frac{\partial V}{\partial x} = B\,, \tag{1.2a}$$

$$V = \left\|\begin{matrix} v \\ \varepsilon \\ \sigma \end{matrix}\right\|, \quad A(V) = \left\|\begin{matrix} 0 & 0 & -\dfrac{1}{\varrho} \\ -1 & 0 & 0 \\ -\Phi & 0 & 0 \end{matrix}\right\|, \quad B(V) = \left\|\begin{matrix} 0 \\ 0 \\ \Psi \end{matrix}\right\|. \tag{1.2b}$$

In this paper we shall look for an asymptotic solution of system (1.2) and we shall show that the constitutive equation (1.1c), in a rather general form, leads to evolution equations in the high- and low-frequency domains. For the evolution equation characterizing the wave process in the high-frequency domain, we point out the possibility of the wave breaking at a finite time (critical time), as usual in nonlinear hyperbolic-wave processes. Thus, because of the « attenuation effects » (see [10]), present in the basic mathematical model, an analysis similar to the one related to the critical strain gradient for shock waves (see [7,12]) occurs. In our paper such an analysis is suitably extended to a high-frequency process in a particular nonlinear-linear medium which may be a mathematical model either of certain soft tissues or of certain polymers. Thus our method of approach can be used to investigate the main features of wave propagation in such a class of materials.

In sect. **2**, following the method of approach proposed in [10], we shall obtain the evolution equations by a convenient stretching of independent variables. We shall explain also the physical background of the process.

In sect. **3**, the comparison of the results obtained in sect. **2** with the analysis of a particular case of standard viscoelastic solid [7] is presented.

Section **4** deals with an example of a high-frequency process in a particular nonlinear-linear medium of physical interest. Mainly we point out the role played in the wave breaking process by the threshold between the nonlinear behaviour and the linear one in the material.

2. – General asymptotic analysis.

2·1. *High-frequency process.* – Let us look for asymptotic solutions of eq. (1.2) exibiting the feature of progressive waves [8,9], *i.e.* let us assume the following

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[12] P. J. CHEN and M. E. GURTIN: *Arch. Ration. Mech. Anal.*, **36**, 33 (1970).

asymptotic expansion:

$$V \sim V_0 + \delta V_1(x, t, \xi) + \delta^2 V_2(x, t, \xi) + \dots, \tag{2.1}$$

where V_0 is a known constant solution of (1.2) such that the condition

$$B(V_0) = 0 := \Psi(\sigma_0, \varepsilon_0) \tag{2.2}$$

is satisfied; ξ is a «fast» variable defined as $\xi = \delta^{-1} f(x, t)$, $f(x, t)$ is a phase variable to be determined further and δ is a real small parameter defined by

$$\delta = T\tau_a^{-1} \ll 1, \tag{2.3}$$

where T represents a characteristic time scale of the input and τ_a is the attenuation time (see further). Hence, according to the theory of rate-dependent media [2], the parameter δ, defined by expression (2.3), characterizes a «high-frequency» process.

By considering the Taylor expansion of the matrix coefficients A and B in the neighbourhood of V_0 and taking into account expressions (2.1) and (2.2), we get

$$A(V) = A(V_0) + (\nabla A)_0 V_1 + O(\delta^2), \tag{2.4a}$$

$$B(V) = (\nabla B)_0 V_1 + O(\delta^2), \tag{2.4b}$$

where $\nabla = \partial/\partial V = (\partial/\partial v, \partial/\partial\varepsilon, \partial/\partial\sigma)$ and the subscript $_0$ means that a certain field-dependent quantity is evaluated at $V = V_0$.

Substituting (2.1) and (2.4) into (1.2), we get

$$(A_0 - \lambda I)\frac{\partial V_1}{\partial\xi} = 0, \tag{2.5a}$$

$$(A_0 - \lambda I)\frac{\partial V_2}{\partial\xi} + f_x^{-1}\{\partial_t V_1 + A_0\partial_x V_1\} + \{(\nabla A)_0 V_1\}\frac{\partial V_1}{\partial\xi} = f_x^{-1}(\nabla B)_0 V_1, \tag{2.5b}$$

where $\lambda = -f_t/f_x$. Here the following notations are taken into account:

$$\partial/\partial x^\alpha = \partial_{x^\alpha} + \delta^{-1} f_\alpha \partial/\partial\xi, \qquad \partial_{x^\alpha} = \partial/\partial x^\alpha|_\xi, \qquad \alpha = 0, 1, \tag{2.6a}$$

$$x^0 = t, \qquad x^1 = x, \qquad f_\alpha = \partial f/\partial x^\alpha. \tag{2.6b}$$

From (2.5a) the characteristic polynomial follows:

$$\lambda(\lambda^2 - \Phi\varrho^{-1}) = 0. \tag{2.7}$$

Considering the velocity $\lambda = \lambda^+ = (\Phi\varrho^{-1})^{\frac{1}{2}}$ with $\Phi > 0$ [11] and determining the left and right eigenvectors of A by

$$l = \|\varrho\lambda \quad 0 \quad 1\|\,, \qquad r^{\mathrm{T}} = \|\lambda \quad -1 \quad -\Phi\|\,, \tag{2.8}$$

where superscript T means transposition, we obtain

$$V_1(x, t, \xi) = u(x, t, \xi)\, r_0 + h(x, t) \tag{2.9}$$

as a solution of eq. (2.5a). Here $u = -\varepsilon_1$ is the amplitude factor to be determined and $h(x, t)$ is an integration constant which, according to the initial conditions, can be chosen to be zero (*e.g.*, see [8]). The phase $f(x, t)$ is determined by

$$f_t + \lambda_0 f_x = 0 \tag{2.10}$$

and, if $f(x, 0) = x$, then $f(x, t) = x - \lambda_0 t$.

Multiplying now (2.5b) by l, we obtain along the characteristics curves associated to (2.10) the following evolution equation for u:

$$\frac{\partial u}{\partial \tau} + a_0 u \frac{\partial u}{\partial \xi} = b_0 u\,, \tag{2.11}$$

where $\partial/\partial\tau = \partial_t + \lambda_0 \partial_x$ and

$$a = \{l((\nabla A) r) r\}(l\cdot r)^{-1} = \nabla\lambda\cdot r = -\frac{1}{2}(\varrho\Phi)^{-\frac{1}{2}}\left(\frac{\partial\Phi}{\partial\varepsilon} + \Phi\frac{\partial\Phi}{\partial\sigma}\right), \tag{2.12a}$$

$$b = \{l(\nabla B) r\}(l\cdot r)^{-1} = \frac{1}{2}\Phi^{-1}\left(\frac{\partial\Psi}{\partial\varepsilon} + \Phi\frac{\partial\Psi}{\partial\sigma}\right). \tag{2.12b}$$

Now we define the following physical parameters [2]:

i) static Young modulus

$$e = -\left(\frac{\partial\Psi}{\partial\varepsilon}\right)_0\left(\frac{\partial\Psi}{\partial\sigma}\right)_0^{-1} = \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon}\right|_{\Psi=0},$$

ii) stress relaxation time

$$\tau_1 = -\left(\frac{\partial\Psi}{\partial\sigma}\right)_0^{-1},$$

iii) strain relaxation time

$$\tau_2 = \Phi_0\left(\frac{\partial\Psi}{\partial\varepsilon}\right)_0^{-1}.$$

By using the above quantities, expression (2.12*b*) yields

$$b_0 = \tfrac{1}{2}(\tau_1 - \tau_2)(\tau_1 \tau_2)^{-1} = \tfrac{1}{2}\tau_1^{-1}(e\Phi_0^{-1} - 1) \tag{2.13}$$

and $e\Phi_0^{-1} < 1$, *i.e.* $\tau_1 < \tau_2$, implies $b_0 < 0$ [2]. This means also that the comments made in [10] hold. Among others we remark that the dissipative mechanisms connected with the right-hand side of (1.1*c*) (source term) produce a delay in the wave breaking with respect to the corresponding case related to the absence of memory effects (purely elastic case, $\Psi \equiv 0$). Now the evolution equation (2.11) may be rewritten in the form

$$\frac{\partial u}{\partial \tau} + a_0 u \frac{\partial u}{\partial \xi} + \frac{u}{\tau_*} = 0\,, \tag{2.14}$$

where

$$\tau_* = - b_0^{-1} \tag{2.15}$$

is the attenuation time characterizing the medium [2,10]. This completes also expression (2.3) used earlier to determine the character of the process.

As is well known [8-10], if

$$u|_{\tau=0} = F(x_0, \xi_0)\,, \qquad x_0 = x|_{\tau=0}\,, \qquad \xi_0 = \delta^{-1} f|_{\tau=0}\,,$$

then a finite time (critical time), at which an irregularity in the solution of (2.14) may occur, will exist if the following conditions hold (see also [13,14]):

$$F_0' a_0 < 0\,, \qquad -F_0' a_0 > \tau_*^{-1}\,, \qquad F_0' = \partial F / \partial \xi_0\,.$$

At the critical time a shock wave formation may occur.

Let us remark that in [7], for the standard viscoelastic solid, a comparison has been stated between the possibility of the formation of discontinuous solutions of eq. (2.14) and the analysis related to the so-called « critical strain gradient » for shock waves [12].

However, if the response function $\Phi(\sigma, \varepsilon)$ satisfies the following relation [15]:

$$\frac{\partial \Phi}{\partial \varepsilon} + \Phi \frac{\partial \Phi}{\partial \sigma} = 0 \Rightarrow \Phi = P(\sigma - \Phi\varepsilon)\,, \tag{2.16}$$

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(13) J. D. Murray: *SIAM J. Appl. Math.*, **19**, 273 (1970).

(14) A. Donato and D. Fusco: *Atti Accad. Peloritana Pericolanti, Cl. Sci. Fis. Mat. Nat.*, **59**, 149 (1981).

(15) D. Fusco: *Int. J. Non-Linear Mech.*, **16**, 459 (1981).

where P is an arbitrary function, then the «exceptionality condition» [16] holds

$$\nabla\lambda\cdot r = 0 = a \tag{2.17}$$

and the wave never evolves into a nonlinear shock after a finite time has elapsed.

2'2. *Low-frequency process.* Let us consider the following stretching:

$$x' = \hat{\delta}^2 x\,, \qquad t' = \hat{\delta}^2 t\,, \tag{2.18}$$

where $\hat{\delta} \ll 1$ is a small parameter. Omitting $'$ in the new independent variables, system (1.2) holds with the exception that

$$B^{\mathrm{T}} = \|0 \;\; 0 \;\; \hat{\delta}^{-2}\Psi\|\,. \tag{2.19}$$

Let T be a characteristic time scale of the input and τ_* be the attenuation time, considered above. According to [2] we define the characteristic length $L = (\Phi_0/\varrho)^{\frac{1}{2}}T$ and the attenuation length $L_* = (\Phi_0/\varrho)^{\frac{1}{2}}\tau_*$. If the condition

$$\hat{\delta}^2 = \tau_* T^{-1} = L_* L^{-1} \ll 1 \tag{2.20}$$

is satisfied, then, in terms of [2], we are considering a low-frequency process, *i.e.* times and distances considered are large in comparison to the attenuation time and the attenuation length. System (1.2) in variables (2.18) and with (2.19) represents a small perturbation of an equilibrium (fully relaxed) state characterized by eqs. (1.1a), (1.1b) and

$$\Psi(\sigma, \varepsilon) = 0 \tag{2.21}$$

This system is actually the reduced system of the theory developed in [10]. The asymptotic solution is now sought in the form

$$V \sim V_0 + \hat{\delta} V_1(x, t, \zeta) + \hat{\delta}^2 V_2(x, t, \zeta) + \dots\,, \tag{2.22}$$

where V_0 is the same constant vector considered in subsect. **2'1**, *i.e.* such that $\Psi(\sigma_0, \varepsilon_0) = 0$ (and certainly a solution of the reduced system), $\zeta = \hat{\delta}^{-1}\hat{f}(x, t)$ and $\hat{f}(x, t)$ is the phase function to be determined further. It is worth noticing that $V(x, t, \zeta) - \{V_0 + \hat{\delta} V_1\} = O(\hat{\delta}^2)$, *i.e.* of the same order as the right-hand side of system (1.2) with (2.18) and (2.19). However, even if the same symbols are used, only the first term V_0 is the same in expansions (2.1) and (2.22).

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[16] P. D. LAX: *Commun. Pure Appl. Math.*, **10**, 537 (1957).

Substituting (2.22) into the basic system with variables (2.18), developing the matrix coefficients in the usual manner and solving the corresponding set of equations according to the approach stated in [10], we obtain

$$\sigma_1 = -\varepsilon_1\left(\frac{\partial\Psi}{\partial\varepsilon}\right)_0\left(\frac{\partial\Psi}{\partial\sigma}\right)_0^{-1} = e\varepsilon_1\,, \quad v_1 = -\hat{\lambda}_0\varepsilon_1\,, \quad \hat{\lambda}_0^2 = e\varrho^{-1}\,, \quad \hat{\lambda}_0 = \hat{\lambda}_0^+\,, \tag{2.23}$$

with $\hat{f}(x, t) = x - (e\varrho^{-1})^{\frac{1}{2}}t$ and the wave amplitude $\hat{u} = -\varepsilon_1(x, t, \zeta)$ satisfies the following transport (evolution) equation:

$$\frac{\partial\hat{u}}{\partial s} + \hat{a}_0\hat{u}\frac{\partial\hat{u}}{\partial\zeta} = \hat{a}_2\frac{\partial^2\hat{u}}{\partial\zeta^2}\,, \tag{2.24}$$

where $\partial/\partial s = \partial_t + \hat{\lambda}_0\partial_x$ and

$$\hat{a}_0 = -\frac{1}{2}(e\varrho)^{-1}\left(\frac{\partial e}{\partial\varepsilon} + e\frac{\partial e}{\partial\sigma}\right)_0\,, \tag{2.25}$$

$$\hat{a}_2 = \frac{1}{2}\tau_1\varrho^{-1}(\Phi_0 - e) = \tau_1^2\Phi_0(\varrho\tau_*)^{-1}\,. \tag{2.26}$$

The low-frequency process in the rate-dependent medium with the constitutive equation (1.1*c*) is governed by eq. (2.24), *i.e.* by Burgers' equation.

3. – The standard viscoelastic medium.

Here we follow the approach described in [3] and used for viscoelastic media in [7,17]. The constitutive equation (1.1*c*) holds with [8]

$$\Phi = \Phi(\varepsilon) = (1+\gamma_1)\{\bar{\lambda} + 2\bar{\mu} + 6(\nu_1 + \nu_2 + \nu_3)\varepsilon\}\,, \tag{3.1a}$$

$$\Psi(\sigma, \varepsilon) = \tau_1^{-1}\{(\bar{\lambda} + 2\bar{\mu})\varepsilon + 3(\nu_1 + \nu_2 + \nu_3)\varepsilon^2 - \sigma\}\,, \tag{3.1b}$$

where $\bar{\lambda}$ and $\bar{\mu}$ are Lamé constants, ν_i, $i = 1, 2, 3$, the third-order elastic moduli and τ_1, γ_1 are the parameters of viscosity: τ_1, as in sect. **2**, is the stress relaxation time and γ_1 is the dimensionless parameter determining the difference between the equilibrium velocity $\hat{\lambda}$ and the instantaneous velocity λ: $\hat{\lambda}^2 = e\varrho^{-1} = (\bar{\lambda} + 2\bar{\mu})\varrho^{-1}$, $\lambda^2 = \Phi_0\varrho^{-1} = (1+\gamma_1)(\bar{\lambda} + 2\bar{\mu})\varrho^{-1}$. The condition $\gamma_1 > 0$ is satisfied always due to thermodynamics, hence $\lambda > \hat{\lambda}$. As shown in [7],

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(17) A. JEFFREY and J. ENGELBRECHT: *Waves in non-linear relaxing media*, in *Wave Propagation in Viscoelastic Media*, edited by F. MAINARDI, *Research Notes in Mathematics*, No. 52 (London, 1982).

there is no principal difference in choosing one or another velocity as a basis for the phase.

Relation (1.1*c*) with (3.1) may be given also in an integral form

$$\sigma = (\bar{\lambda} + 2\bar{\mu})\varepsilon + 3(\nu_1 + \nu_2 + \nu_3)\varepsilon^2 + \gamma_1(\bar{\lambda} + 2\bar{\mu})\int_{-\infty}^{t} \exp\left[-\frac{t-\eta}{\tau_1}\right]\frac{\partial\varepsilon}{\partial\eta}\,\mathrm{d}\eta \tag{3.2}$$

that was the starting expression for the analysis in [7]. The corresponding evolution equation has the form

$$\frac{\partial u_1}{\partial\tau} + a_1 u_1 \frac{\partial u_1}{\partial\xi_e} - \frac{\gamma_1}{2\gamma}\frac{\partial}{\partial\xi_e}\int_0^{\xi_e}\exp\left[-\frac{\xi_e - y}{\hat{\lambda}\tau_1}\right]\frac{\partial u_1}{\partial y}\,\mathrm{d}y = 0\,. \tag{3.3}$$

Here γ is a small parameter usually given in connection of the Mach number [3] and

$$a_1 = \tfrac{3}{2}(1 + m_0)(\gamma\hat{\lambda})^{-1}\,, \tag{3.4a}$$

$$m_0 = 2(\nu_1 + \nu_2 + \nu_3)(\bar{\lambda} + 2\bar{\mu})^{-1}\,. \tag{3.4b}$$

The reader is referred to [7] for details. However, it must be pointed out that

$$u_1 = \partial U_1/\partial t = -\hat{\lambda}\,\partial U_1/\partial x\,, \tag{3.5}$$

$$\xi_e = \hat{\lambda}t - x\,, \qquad \tau = \gamma x\,. \tag{3.6}$$

Here U_1 denotes the displacement.

The dimensionless parameter (in our notation)

$$Z = \tau_1\lambda L^{-1} \tag{3.7}$$

introduced in [7] permits us to estimate the character of the process. If $Z \gg 1$ is satisfied, then with fixed τ_1 and λ it corresponds to a high-frequency process. If $Z \ll 1$ is satisfied, then with the same fixed τ_1 and λ it corresponds to a low-frequency process. Conditions (2.3) and (2.20) have the same meaning, respectively.

3·1. *High-frequency process.* – If $Z \gg 1$ is satisfied (cf. condition (2.3)), then the exponential function in the integral of eq. (3.3) changes slowly and its exact expansion into a series may be used instead of the complete function. The corresponding evolution equation in two terms of the series has the form

$$\frac{\partial u_1}{\partial\tau} + a_1 u_1 \frac{\partial u_1}{\partial\xi_e} - \frac{\gamma_1}{2\gamma}\frac{\partial u_1}{\partial\xi_e} + \frac{\gamma_1}{2\gamma\hat{\lambda}\tau_1}u_1 = 0\,. \tag{3.8}$$

By making use of the transformation

$$\xi = \xi_e + (\gamma_1/2\gamma)\,\tau\,, \tag{3.9}$$

eq. (3.8) yields

$$\frac{\partial u_1}{\partial \tau} + a_1 u_1 \frac{\partial u_1}{\partial \xi} + \frac{\gamma_1}{2\gamma\hat{\lambda}\tau_1}\, u_1 = 0\,. \tag{3.10}$$

Equation (3.10) has the same structure as eq. (2.11), while in our case $e = \bar{\lambda} + 2\bar{\mu}$, $\Phi_0 = (1+\gamma_1)(\bar{\lambda} + 2\bar{\mu})$ and thus from (2.13) we obtain

$$b_0 = -\frac{\gamma_1}{2(1+\gamma_1)\,\tau_1}\,. \tag{3.11}$$

Note also that (3.5) and (3.6) hold and transformation (3.9) gives the change in the velocity. Such a result was derived first in (1) by a straightforward analysis and used later for standing waves in bounded media (18).

3'2. *Low-frequency process.* – If $Z \ll 1$ is satisfied (cf. condition (2.20)), then the exponential function in the integrand of eq. (3.3) changes faster than the derivative of u_1 with respect to y and, therefore, the last may be expanded into a series. Keeping only the first term in the series, we come actually to Voigt material. At some distance from the front $\xi_e = 0$ (for details see (17)), it is possible to get once again the celebrated Burgers' equation

$$\frac{\partial u_1}{\partial \tau} + a_1 u_1 \frac{\partial u_1}{\partial \xi_e} = \frac{\gamma_1 \hat{\lambda}\tau_1}{2\gamma}\,\frac{\partial^2 u_1}{\partial \xi_e^2}\,. \tag{3.12}$$

Once more, from (2.26) we obtain

$$\hat{a}_2 = \tfrac{1}{2}\,\tau_1\gamma_1\hat{\lambda}^2 \tag{3.13}$$

and, bearing also in mind expressions (3.5) and (3.6), one can easily conclude that (2.24) and (3.12) coincide. Such an evolution equation is obtained also in (18) for Voight material only and a similar result may be obtained for thermoelastic damping (3).

4. – The nonlinear-linear medium.

Several media behave themselves in the following way: for small strains the stress-strain relation is strongly nonlinear, but, if the strain is bigger than a

(18) M. P. Mortell: *Z. Angew. Math. Phys.*, **28**, 33 (1977).

certain threshold ε^*, the stress-strain relation is linear. One of the best examples of this kind are soft tissues [19], where the small stress causes the sliding of long moleculcs and, therefore, the average stress-strain relation is strongly nonlinear. If viscous effects are not considered, then

$$(4.1a) \qquad \Phi = \Phi(\varepsilon) = \bar{\lambda} + 2\bar{\mu} + 6(\nu_1 + \nu_2 + \nu_3)\varepsilon\,, \qquad 0 \leqslant \varepsilon \leqslant \varepsilon^*,$$

$$(4.1b) \qquad \Phi = \bar{\lambda}_1 + 2\bar{\mu}_1\,, \qquad \varepsilon^* \leqslant \varepsilon,$$

and

$$(4.2) \qquad \bar{\lambda}_1 + 2\bar{\mu}_1 = \bar{\lambda} + 2\bar{\mu} + 6(\nu_1 + \nu_2 + \nu_3)\,\varepsilon^*\,.$$

If viscous effects are added, then, according to model (3.2), expression (4.1a) must be changed to (3.1a), expression (4.1b) to

$$(4.3) \qquad \Phi = (1 + \gamma_1)(\bar{\lambda}_1 + 2\bar{\mu}_1)\,, \qquad \varepsilon^* \leqslant \varepsilon,$$

and $\Psi'(\sigma, \varepsilon)$ must be calculated (3.1b) by taking into account the linearized version for $\varepsilon > \varepsilon^*$. It is obvious that the threshold ε^* plays an important role in wave propagation, especially in shock wave propagation. For the high-frequency process, as shown above, either (2.14) or (3.10) holds. This is the evolution equation with critical strain gradient [7]. For model (3.1) (or (3.2)), it is determined by the expression

$$(4.4) \qquad \lambda^* = \frac{\gamma_1}{2\gamma\hat{\lambda}\tau_1}\,.$$

For the usual nonlinear viscoelastic medium the situation is known: if the real strain gradient is smaller than the critical strain gradient, then the dissipative effects are strong enough to avoid the shock wave formation and, if the real strain gradient is bigger than the critical one, then the nonlinear effects take over and the shock wave may form [7]. Here the situation is more complicated because, beside the critical strain gradient λ^*, the threshold strain ε^* governs the shock wave formation. There is no explicit possibility to compare λ^* and ε^* between themselves, but it is clear that, generally speaking, for $\varepsilon > \varepsilon^*$ the condition $\lambda_r > \lambda^*$ may be fulfilled, but no shock will form (here $\lambda_r = \partial\varepsilon/\partial\xi$ is the real gradient). The only possibility is to find the regions on the physical plane ξ, τ for fixed τ (*e.g.* for $\tau = 0$), where both estimates may be compared. The possible situations are shown in fig. 1. Case a) does not permit any shock formation, while $\lambda^* > \lambda_r$ always, and ε^* may be arbitrary. Case b) permits shock formation because there are regions along the ξ-axis where

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[19] Y. C. Fung: *Biomechanics. Mechanical Properties of Living Tissues* (Berlin, 1980).

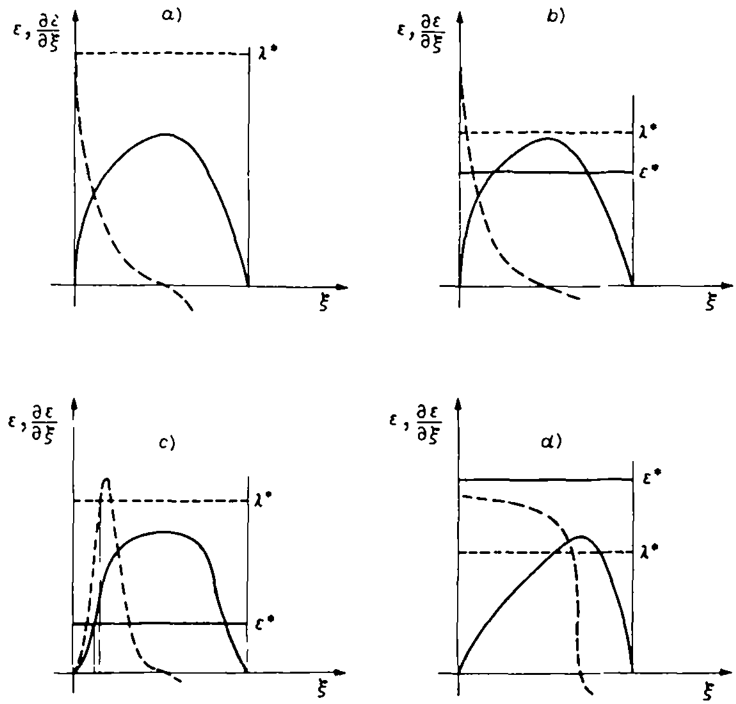

Fig. 1. – ε and $\partial\varepsilon/\partial\xi$ diagrams (solid and broken lines, respectively).

$\lambda_r > \lambda^*$ and $\varepsilon < \varepsilon^*$ are satisfied simultaneously. Case *c*) does not permit shock formation in the regions where $\lambda_r > \lambda^*$ is satisfied, but another condition $\varepsilon > \varepsilon^*$ indicates the linear model already. Case *d*) permits shock formation as in a usual nonlinear material because the condition $\varepsilon < \varepsilon^*$ is satisfied everywhere.

5. – Conclusions.

In sect. **2** a strict approach is given in order to obtain the evolution equations for high- and low-frequency processes in a rather general nonlinear rate-type (viscoelastic) medium modelled by a quasi-linear first-order hyperbolic system of partial differential equations. The results, the simple wave equation with a source term for the high-frequency process and Burgers' equation for the low-frequency process, hold for the general constitutive law (1.1*c*). In [2] the high-frequency process was analysed for the same constitutive equation and the corresponding evolution equation was obtained by the method of multiple scales for the case of a viscoelastic slab. Our result seems to be more general. The comparison with the special case of the standard viscoelastic medium [7] is given in which the same result is obtained by deriving, firstly the general integro-differential evolution equation and, secondly, the high- and low-frequency limits of it. The results may be used in a large range of visco-elastic media. As an example, a special nonlinear-linear medium is considered and the conditions giving the possibility of shock wave formation analysed.

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● RIASSUNTO

In questo lavoro sono caratterizzati i processi ondosi ad alta e bassa frequenza per una vasta classe di materiali non lineari, in cui siano presenti effetti di memoria, descritti da un'equazione costitutiva di tipo differenziale. Nei due casi sono dedotte le equazioni di evoluzione ed i risultati ottenuti sono confrontati con quelli dedotti da Engelbrecht per un particolare mezzo viscoelastico. Nella parte finale del lavoro si considera un processo ondoso ad alta frequenza in un particolare mezzo non lineare-lineare, che può essere assunto come modello matematico per descrivere certi tipi di tessuti biologici o certe classi di polimeri.

Резюме не получено.