IL NUOVO CIMENTO VOL. 80 B, N. 1 11 Marzo 1984

Variational Principle for a Prototype Rastall Theory of Gravitation.

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(ricevuto il 27 Giugno 1983; manoscritto revisionato ricevuto il 22 Dicembre 1983)

Summary. — A prototype of Rastall's theory of gravity, in which the divergence of the energy-momentum tensor is proportional to the gradient of the scalar curvature, is shown to be derivable from a variational principle. Both the proportionality factor and the unrenormalized gravitational constant are found to be covariantly constant, but not necessarily constant. The prototype theory is, therefore, a gravitational theory with variable gravitational constant.

PACS. 04.20. - General relativity.

A serious criticism of Rastall's theory (1) of gravitation is that it is not a Lagrangian-based theory (2,3). LEE, LIGHTMAN and NI (3) have put forth theorems in which they claim that the matter response equations, the zero divergence of the energy-momentum tensor of any Lagrangian-based, generally covariant metric theory of gravity, are a consequence of the gravitational-field equation if, and only if, the theory contains no absolute variables. They further conjecture that the conservation of energy-momentum is equivalent to the existence of a Lagrangian formulation. With regard to this latter conjecture, it has been shown that Rastall's theory is a conservative theory, but no

(1) P. RASTALL: *Phys. Rev. D*, **6**, 3357 (1972).
(2) K. S. THORNE, D. L. LEE and A. P. LIGHTMAN: *Phys, Rev. D.* **7**, 3563 (1973).
(3) D. L. LEE, A. P. LIGHTMAN and W.-T. NI: *Phys. Rev. D*, **10**, 1685 (1974).

Lagrangian formulation is known. But, since Rastall's theory contains no absolute variables, the variational principle described below will, in practice, allow one to see how to incorporate Rastall's theory into a more general theorem concerning gravitational theories which have conservative energy-momentum— not just zero divergence of the energy-momentum tensor. In addition, the lack of a Lagrangian base has posed for Rastall's theory and especially for its generalizations certain problems in obtaining field equations consistent with the Bianchi identities ([4]). In another example, COLEY has shown that, in a dust solution for a particular Rastall-like theory, the motion of the fluid flow is irrotational ([5]). Restrictions such as these can occur even for perfect fluids in general relativity. RAY has shown that this difficulty can easily be circumvented through use of a particle number constraint in the case of a perfect fluid ([6]), the so-called Lin constraint used successfully for understanding the physics of liquid helium ([7]). The correction can easily be applied to Rastall's theory provided one has a Lagrangian formulation available.

Rastall's modification of the Einstein field equations is motivated by the observation that the zero divergence of the energy-momentum tensor is not theoretically necessary. As an example, the divergence of $T^{\mu\nu}$ is assumed proportional to the gradient of the scalar curvature R:

$$T^{\mu\nu}{}_{;\nu} = \lambda R^{;\mu} . \tag{1}$$

The consistent field equations are then

$$R^{\mu\nu} - \tfrac{1}{2}(2\lambda\varkappa + 1)g^{\mu\nu}R = -\varkappa T^{\mu\nu} , \tag{2}$$

where $R^{\mu\nu}$ is the Ricci tensor, $g^{\mu\nu}$ is the metric (with signature $(-1, 1, 1, 1)$), λ is the proportionality factor and $\varkappa$ is the unrenormalized gravitational constant. A complete post-Newtonian approximation yields the Einstein results provided $\lambda \sim O(v^2)$, where v is the velocity ([8]).

Let us now consider the Lagrangian density

$$\mathscr{L} = -\frac{\sqrt{-g}}{2\varkappa'} R \exp[2\sqrt{-g}\,\lambda\varkappa'] + \sqrt{-g}\,L_m , \tag{3}$$

where L_m is the matter Lagrangian and λ' and $\varkappa'$ are two constant parameters. Note, however, that the Lagrangian given by eq. (3) is not the standard form

([4]) L. L. SMALLEY: *Phys. Rev. D*, **12**, 376 (1975).
([5]) A. A. COLEY: *Nuovo Cimento B*, **69**, 89 (1982).
([6]) J. R. RAY: *J. Math. Phys.* (N. Y.), **13**, 1451 (1972).
([7]) C. LIN: *Proc. S.I.F.*, Course XXI, edited by G. CARERI (New York, N. Y., 1963).
([8]) L. L. SMALLEY: *Found. Phys.*, **8**, 59 (1978).

one usually faces in field theory. First of all, $\mathscr{L}$ is not now a scalar density because of the exponential factor. In fact, if the exponential factor is expanded (if such an expansion makes sense, and it may not), then one has a density series of terms of increasing weight. If one insists that $\mathscr{L}$ be a scalar density, then we can progress no further. Let us now instead look at evidence which might be in favor of allowing the Lagrangian proposed by eq. (3).

In Rastall's theory, the energy-momentum tensor does not, in general, have zero divergence except in asymptotically flat space ([1]). The zero divergence is not a requirement ([9]), but it has nevertheless been generally accepted ([10]) that the zero divergence implies that the theory would be a conservative theory, and, therefore, considerable effort has been expended to show that this consequence implies globally conserved energy-momentum and angular momentum ([11]). Nevertheless, this requirement is too strong and is not even carried over to a more general definition of a metric theory by THORNE, LEE, and LIGHTMAN ([2]).

Indeed we have proven, at least through level of the post-Newtonian approximation, that Rastall's theory and similar theories are conservative theories ([12]). The proof was somewhat surprising, but seems to provide a reasonably strong argument for allowing theories with nonzero divergence of $T^{\mu\nu}$. The proof has not been extended to post-post-Newtonian or higher approximation, and no exact calculation is known.

Next look at the problem from a mathematical point of view. According to very general theorems, the nonzero divergence of $T^{\mu\nu}$ implies specific properties for the Lagrangian. These arguments have been given by WEINBERG ([13]). In summarizing these arguments, he states « Thus the energy-momentum tensor... is conserved if and only if the matter action is a scalar ». (Here, WEINBERG uses the word « conserved » to mean zero divergence. This use of the word « conserved » should not be confused with its use for « conservative » theories ([11,12]).) The « if and only if » means that one *cannot* have a scalar density and obtain Rastall's field equations which obviously imply nonzero divergence of the energy-momentum tensor. If the theorem is true, then one can *never* arrive at a Rastall-type theory, described by eqs. (1) and (2), through the use of a scalar Lagrangian. We suspect that the theorem is true and, therefore, motivate our generalization of the variational concept for physical systems

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([9]) K. S. THORNE, C. M. WILL and W.-T. NI: in *Proceedings of the Conference on Experimental Tests of Gravitational Theories* edited by R. W. DAVIES (NASA-JPL Tech. Memo 33-449, unpublished), p. 10.

([10]) K. S. THORNE and C. M. WILL: *Astrophys. J.*, **163**, 595 (1971).

([11]) C. M. WILL and K. NORDVEDT jr.: *Astrophys. J.*, **177**, 757 (1972); K. NORDVEDT jr. and C. M. WILL: *Astrophys. J.*, **177**, 775 (1972).

([12]) L. L. SMALLEY: *Phys. Lett. A*, **57**, 300 (1976).

([13]) S. WEINBERG: *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (New York, N. Y., 1972), p. 363.

to include Lagrangians of the form given by eq. (3). We look now at the consequences of that action.

The variation of $\mathscr{L}$ with respect to the metric then becomes

$$\delta\mathscr{L} = \left\{\frac{\sqrt{-g}\exp\left[2\sqrt{-g}\,\lambda'\varkappa'\right]}{2\varkappa'}\left[R^{\mu\nu} - \frac{1}{2}(2\lambda'\varkappa'\sqrt{-g} + 1)g^{\mu\nu}R\right] + \frac{1}{2}\sqrt{-g}\,T^{\mu\nu}\right\}\delta g_{\mu\nu}, \tag{4}$$

where total divergences have been dropped as usual. The equations of motion then become

$$R^{\mu\nu} - \tfrac{1}{2}(2\lambda'\varkappa'\sqrt{-g} + 1)g^{\mu\nu}R = -\varkappa'\exp\left[-2\lambda'\varkappa'\sqrt{-g}\right]T^{\mu\nu} \tag{5}$$

and the divergence of the energy-momentum tensor takes the form

$$T^{\mu\nu}{}_{;\nu} = \sqrt{-g}\,\lambda'\exp\left[2\lambda'\varkappa'\sqrt{-g}\right]g^{\mu\nu}R_{;\nu}. \tag{6}$$

The identification

$$\varkappa'\exp\left[-2\lambda'\varkappa'\sqrt{-g}\right] \to \varkappa, \tag{7}$$

$$\sqrt{-g}\,\lambda'\exp\left[2\lambda'\varkappa'\sqrt{-g}\right] \to \lambda \tag{8}$$

reproduces a Rastall-like theory given by eqs. (1) and (2) but with the parameters λ and $\varkappa$ now covariantly constant ([14]).

We must, however, be cautious about the resulting field equations, since the theory is only manifestly covariant in the strict limit that both λ and $\varkappa$ are constant. This is certainly possible, but this limit would restrict the usefulness of the theory, at present, to a calculational tool. In order to emphasize, on the other hand, how close these theories really are for the covariantly constant case, we have calculated the post-Newtonian approximation of eqs. (5) and (6) directly. We obtain the same PPN parameters ([8]). We also find the same conserved integral (global) momentum and angular momentum ([15]), providing we always make the identification given by eqs. (7) and (8) (to the appropriate order) for the renormalized gravitational constant. Thus, since the scale of the gravitational constant is unimportant, the Lagrangian-based theory given by eq. (3) is identical in the post-Newtonian approximation to the original Rastall theory. This is not just a coincidence, but is strongly related

([14]) J. A. SCHOUTEN: *Ricci Calculus*, 2nd edition (Berlin, 1954), p. 125.

([15]) L. L. SMALLEY: in *Scientific Applications of Lunar Laser Ranging* (Dordrecht, 1977), p. 91.

to the very weak corrections due to the higher-order effects of the density, $\sqrt{-g}$, on $\varkappa$ and λ. Just as in the original Rastall theory, deviations would necessarily be expected in higher orders, although the theory still retains viability at present. We will, however, present a specific example of a deviation from general relativity after the next paragraph.

We have now established that the Lagrangian given by eq. (3) leads to a physically, *i.e.* experimentally acceptable theory. The generalization of the variational principle to include Lagrangian densities of this form seems to indicate that a wider set of physical systems may be possible than ever before realized. On the other hand, constraining $\mathscr{L}$ to be only a scalar density does not alone ensure that the field equations will be physically acceptable. The examples of nonphysical, Lagrangian-based theories litter the gravitational landscape ([16]). The lesson is a simple one and has been routinely espoused at the most rudimental level. For example, when GOLDSTEIN describes fields by variational principles ([17]), he states « Indeed, we may use any expression for $\mathscr{L}$ which leads to the desired field equations ». One should realize that in this quote (taken somewhat out of context) GOLDSTEIN is not referring to covariant Lagrangian formulations, and, in some sense, we are not either. Then with the proper constraints on the variables ([18]) the variational principle ensures consistency and experiment decides the physical acceptability. In our case, the utility of Rastall's field equations has been discussed elsewhere ([8,15,19]). Our generalization here should further enhance our understanding of the structure of the gravitational field.

As an example of one immediate result we note that, if the identification given by eq. (7) is substituted into the Lagrangian, eq. (3), we see now that the origin of the prototype Rastall's theory is a modification of general relativity with a covariantly constant gravitational constant $\varkappa$. Thus we could, for example, have a time-varying gravitational constant G. From eq. (3), we see then that $G \sim \varkappa' \exp[-2\sqrt{-g}\lambda'\varkappa']$, so that

$$\frac{\dot{G}}{G} = -2\partial_0(\sqrt{-g})\,\lambda'\varkappa' \approx -2\partial_0(\sqrt{-g})\,G\lambda' . \tag{9}$$

But one finds from the post-Newtonian approximation that $\partial_0(\sqrt{-g}) \sim -2\partial_0 U$. Since $\lambda' \sim O(v^2)$, $U \sim O(v^2)$ and $\partial_0 \sim O(v)$, then

$$\frac{\dot{G}}{G} \sim O(v^5)\,G . \tag{10}$$

([16]) W.-T. NI: *Phys. Rev. D*, **7**, 2880 (1973).

([17]) H. GOLDSTEIN: *Classical Mechanics* (Reading, Mass., 1959), p. 365.

([18]) F. W. HEHL, E. A. LORD and L. L. SMALLEY: *Gen. Rel. Grav.*, **13**, 1037 (1981).

([19]) L. L. SMALLEY: *J. Phys. A*, **16**, 2179 (1983).

Thus Rastall's theory along with solar-system experiments should put severe restrictions on the value of $\dot{G}/G$.

In closing we mention one possible solution to the general covariance problem for the Lagrangian density described by eq. (3). Elsewhere we have shown that, for the covariant derivative of the energy-momentum tensor, we could replace eq. (1) with other forms for the divergence which specifically reflect that $\lambda \sim O(v^2)$ ([20]). One solution was of the form

$$T_{\mu\ ;\nu}^{\ \nu} = \delta(UR)_{;\mu}, \tag{11}$$

where U is the gravitational potential and δ is now a parameter of $O(1)$. We were also able to show that this is a conservative theory (through the post-Newtonian level) and that it leads to the constraint on the Nordtvedt parameter η given by ([21])

$$\eta = 4\beta - \gamma - 3 + \tfrac{2}{3}\delta, \tag{12}$$

where β and γ are the usual Robertson parameters.

CAMPBELL *et al.* have shown that the parameter Γ from orbital precession $\Gamma = (2 + 2\gamma - \beta)/3$ has the value $\Gamma = 0.987 \pm 0.006$, which is about 2 standard deviations from the prediction $\Gamma = 1$ of general relativity ([22]). If, for the sake of definiteness, we assume that $\gamma = 1$ and that the Nordtvedt parameter $\eta = 0$, this leads to the value $\delta = -0.054$.

With this motivation, the solution that we propose involves replacing λ in the exponential part of the Lagrangian by a field $\varphi(x)$. The field φ is then closely related to the gravitational potential in the post-Newtonian limit. Then, when $\mathscr{L}$ is subjected to a co-ordinate transformation, φ is now required at the same time to undergo a gauge transformation that then ensures covariance of the field equations. In order to complete this example, we must add to $\mathscr{L}$ a Lagrangian for the $\varphi(x)$ field. We have not yet completed this task and leave it for future work.

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Grateful acknowledgement is accorded the referees who stressed the need for clarification of our application of the quotes of Profs. H. GOLDSTEIN and S. WEINBERG and also to the limitation of the covariance of the resulting field equations. It is also a pleasure to thank Prof. J. R. RAY for helpful discussions.

(20) L. L. SMALLEY: *Scientific Applications of Lunar Laser Ranging* (Dordrecht, 1977), p. 91.

(21) K. NORDTVEDT jr.: *Phys. Rev.*, **169**, 1014 (1968); **170**, 1186 (1968); *Phys. Rev. D*, **7**, 2347 (1973).

(22) L. CAMPBELL J. C. MC DOW, J. W. MOFFAT and D. VINCENT: *Nature (London)*, **305**, 508 (1983).

● RIASSUNTO (•)

Si mostra che un prototipo della teoria di Rastall, nel quale la divergenza del tensore energia-impulso è proportionale al gradiente della curvatura scalare, è derivabile da un principio variazionale. Si trova che sia il fattore di proporzionalità che la costante gravitazionale non rinormalizzata sono covariantemente costanti, ma non necessariamente costanti. La teoria prototipo è perciò una teoria gravitazionale con costante gravitazionale variabile.

(•) *Traduzione a cura della Redasione.*

Вариационный принцип для прототипа теории гравитации Растолла.

Резюме (•). — Показывается, что прототип теории гравитации Растолла, в которой расходимость тензора энергии-импульса пропорциональна градиенту скалярной кривизны, может быть получен из вариационного принципа. Получается, что множитель пропорциональности и неперенормируемая гравитационная постоянная являются ковариантно постоянными. Следовательно, прототип теории представляет гравитационную теорию с переменной гравитационной константой.

(•) *Переведено редакцией.*